Chin. Ann. Math.
24B:3(2003),387–394.

ON THE VARIATIONS OF G_2***

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Abstract

In [1], Shen Guangyu constructed several classes of new simple Lie algebras of characteristic 2, which are called the variations of G_2. In this paper, the authors investigate their derivation algebras. It is shown that G_2 and its variations all possess unique nondegenerate associative forms. The authors also find some nonsingular derivations of V_iG for $i = 3, 4, 5, 6$, and thereby construct some left-symmetric structures on V_iG for $i = 3, 4, 5, 6$. Some errors about the variations of $\mathfrak{sl}(3, F)$ in [1] are corrected.

Keywords Variation, Derivation, Associative form, Left-symmetric structure
2000 MR Subject Classification 17B40, 17B50
Chinese Library Classification O152.5   **Document Code** A
Article ID 0252-9599(2003)03-0387-08

§1. Introduction

We know that over an algebraically closed field of characteristic $p > 7$, there are two types of finite-dimensional simple Lie algebras—classical and Cartan types. When the characteristic is small, especially, when characteristic is 2, the simple Lie algebras are more difficult to classify because in this case many classical results are no more true. Recently, various new classes of simple Lie algebras of characteristic 2 are obtained, for instances, Kaplansky's four classes of new Lie algebras in [5], Lin Lei's simple Lie algebras $K(2r + 1, \mathbf{n})$ and $P(m, \mathbf{n})$ in [5], etc. By imbedding G_2 into Cartan type Lie algebra $K(5)$, Shen Guangyu constructed several classes of new simple Lie algebras of characteristic 2 in [1], which are called the variations of G_2. In this paper, we investigate their properties. Throughout this paper F denotes an algebraically closed field of characteristic 2, and the Lie algebras considered are all finite-dimensional. The notations are the same as in [1].

It is known that the Cartan type Lie algebra $K(5)$ consists of the elements

$$D(f) := \sum_{i=1}^{2}((D_{2+i}f)D_i - ((D_i - x_{i+2}D_5)f)D_{2+i}) + \Big(f - \sum_{i=1}^{2}(D_{2+i}f)x_{2+i}\Big)D_5, \qquad f \in \mathfrak{A}(5),$$

Manuscript received November 22, 2001. Revised July 29, 2002.

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***Project supported by the National Natural Science Foundation of China (No.10271047), the Doctoral Programme Foundation of the Ministry of Education of China and the Shanghai Priority Academic Discipline.

where $\mathfrak{A}(5)$ is the algebra of divided power polynomials of 5 variables. To abbreviate the notations, in the following we shall write $D(f)$ simply as f, and we have (cf. [1])

$$[f,g]=(D_5f)\Big(g-\sum_{i=1}^{2}(D_{2+i}g)x_{2+i}\Big)-(D_5g)\Big(f-\sum_{i=1}^{2}(D_{2+i}f)x_{2+i}\Big)$$
$$+\sum_{i=1}^{2}((D_if)(D_{i+2}g)-(D_ig)(D_{i+2}f)),\qquad f,g\in K(5). \tag{1.1}$$

Now we introduce the G_2 and its variations.

Let $L=G_2$ or V_iG for $i=3,4,5,6,7$ with gradation $L=\bigoplus_{i=-2}^{2}L_i$, where $L_{-2}=\langle 1\rangle$, $L_{-1}=\langle x_1,x_2,x_3,x_4\rangle$, $L_0=\langle h_1,h_2,e_1,e_2\rangle$ (or e_3,e_4), $L_1=\langle f_1,f_2,f_3,f_4\rangle$, $L_2=\langle f_5\rangle$ satisfying

$$[f_i,f_j]=\delta_{ij'}f_5,\qquad i,j=1,2,3,4,$$

where $j'=j+2$ $(j=1,2)$ or $j-2$ $(j=3,4)$.

In $L=V_4G$,

$$\begin{aligned}
&h_1=x_1x_3+x_2x_4,\quad h_2=x_1x_3+x_5,\\
&e_1=x_1x_4+a_2x_2^{(2)}+a_3x_3^{(2)},\quad e_2=x_2x_3,\\
&f_1=a_2x_2^{(3)}+a_3x_2x_3^{(2)}+x_1x_2x_4+x_1x_5,\\
&f_2=x_1x_2x_3+x_2x_5,\quad f_3=x_3x_5,\\
&f_4=a_2x_2^{(2)}x_3+a_3x_3^{(3)}+x_4x_5,\\
&f_5=a_2x_2^{(3)}x_3+a_3x_2x_3^{(3)}+x_1x_3x_5+x_2x_4x_5,
\end{aligned}$$

where $a_2,a_3\neq 0$. Let $V_4G=V_4G(a_2,a_3)$. Then $V_3G=V_4G(a,0)$, $G_2=V_4G(0,0)$, where $a\neq 0$. We only introduce V_4G in detail.

By (1.1), we have

$$\begin{aligned}
&[1,h_2]=1;\quad [1,f_i]=x_i\ \text{ and }\ [x_i,f_5]=f_i,\quad i=1,2,3,4;\\
&[1,f_5]=h_1;\quad [x_i,x_j]=\delta_{ij'}1,\quad [e_1,t_i]=a_{i'}t_{i'},\quad i=1,4;\\
&[e_i,t_j]=t_i\ \text{ and }\ [e_i,t_{i'}]=t_{j'},\quad 1\le i\neq j\le 2,\quad [e_1,e_2]=h_1,
\end{aligned}$$

where t_i is one of x_i and f_i (denote that by $t_i=x_i,f_i$ as follows).

$H=\langle h_1,h_2\rangle$ is a Cartan subalgebra of V_4G. $[h_1,t_i]=t_i$, $i=1,2,3,4$, $t=x,f$; $[h_2,x_i]=x_i$, $i=2,3$; $[h_2,f_i]=f_i$, $i=1,4,5$; $[h_2,e_i]=e_i$, $i=1,2$.

$[x_i,f_{i'}]=h_2\,(i=1,4)$ or $h_1+h_2\,(i=2,3)$; $[x_i,f_{5-i}]=e_1\,(i=1,4)$ or $e_2\,(i=2,3)$; $[x_i,f_i]=a_{i'}e_2$, $i=1,4$.

Theorem 1.1.[1] *The Lie algebras V_3G and V_4G are simple and nonrestricted.* $\dim V_iG=14$, $i=3,4$.

By direct computation, we have

Theorem 1.2. *Let $L=V_3G$ or V_4G, $\varphi\in \mathrm{End}L$, such as $1\mapsto f_5$, $f_5\mapsto 1$, $x_i\mapsto f_i$, $f_i\mapsto x_i$, $i=1,2,3,4$, $e_i\mapsto e_i$, $i=1,2$, $h_1\mapsto h_1$, $h_2\mapsto h_1+h_2$. Then φ is an automorphism of L with $\varphi^2=\mathrm{id}_L$.*

About V_5G, V_6G, V_7G, we have

Theorem 1.3.[1] *The Lie algebras* V_iG, $i = 5, 6, 7$, *are nonrestricted simple Lie algebras.* $\dim V_5G = 15$, $\dim V_6G = 14$, $\dim V_7G = 16$.

§2. The Derivation Algebras of V_4G and V_5G

It is a classical result of Lie algebra theory that the derivations of a Lie algebra with nonsingular Killing form are inner. The theorem of Cartan then shows that nonmodular semisimple Lie algebras do not have outer derivations. In this section, we shall determine the derivation algebras of some of the variations of G_2, and thereby illustrate once more that simple modular Lie algebras may possess outer derivations.

2.1. The Derivation Algebra of V_4G

We know that the derivation algebra of the $\mathbb{Z}$-graded Lie algebra is $\mathbb{Z}$-graded. Let $L = V_4G$, $\mathfrak{D} = \mathrm{Der}_F L$. Then

$$L = \bigoplus_{i=-2}^{2} L_i, \quad \mathfrak{D} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{D}_i,$$

where $\mathfrak{D}_i = \{D \in \mathfrak{D} \mid DL_j \subset L_{i+j}, \ \forall j \in \mathbb{Z}\}$. The automorphism φ of L induces an automorphism Φ: $D \mapsto \varphi D \varphi^{-1}$ of $\mathfrak{D}$, and $\mathfrak{D}_{-i} = \Phi(\mathfrak{D}_i)$.

Lemma 2.1. $\mathfrak{D} = \bigoplus_{i=-2}^{2} \mathfrak{D}_i$.

Proof. By the above discussion, we only need to prove that $\mathfrak{D}_i = 0$, $\forall i \geq 3$. Clearly, $L = \bigoplus_{i=-2}^{2} L_i$ implies that $\mathfrak{D}_i = 0$ for $i > 4$. Let $D \in \mathfrak{D}_4$. By $L_{-2} = [L_{-1}, L_{-1}]$, we obtain $DL_{-2} = 0$, and $D = 0$. So $\mathfrak{D}_4 = 0$. Let $D \in \mathfrak{D}_3$. Then $DL_{-1} \subset L_2$. Suppose that $D(x_i) = k_i f_5$, $k_i \in F$, $i = 1, 2, 3, 4$, then

$$D(1) = D([x_1, x_3]) = D([x_2, x_4]) \qquad \text{for} \quad 1 = [x_1, x_3] = [x_2, x_4].$$

Then $k_1 f_3 + k_3 f_1 = k_2 f_4 + k_4 f_2$, and $k_i = 0$, $i = 1, 2, 3, 4$. So $DL_{-1} = 0$ and $DL_{-2} = 0$, which implies that $D = 0$. It follows that $\mathfrak{D}_3 = 0$.

Lemma 2.2. $\mathfrak{D}_2 = \mathrm{ad}L_2 \oplus \langle \mathrm{ad}_L(x_2x_3x_5), \mathrm{ad}_L\xi \rangle$, *where*

$$\xi = a_2 x_2^{(2)} x_5 + a_3 x_3^{(2)} x_5 + x_1 x_4 x_5 + a_2 x_1 x_2^{(2)} x_3 + a_3 x_1 x_3^{(3)}.$$

Proof. It is clear that $\mathrm{ad}(x_2x_3x_5)L \subset L$, $\mathrm{ad}\xi L \subset L$. So they are derivations of L, and belong to $\mathfrak{D}_2$. Since

$$\mathrm{ad}(x_2x_3x_5)(1) = e_2, \quad \mathrm{ad}\xi(1) = e_1, \quad \mathrm{ad}f_5(1) = h_1,$$

$\mathrm{ad}(x_2x_3x_5)$, $\mathrm{ad}\xi$ and $\mathrm{ad}f_5$ are linearly independent. So, we only need to prove that if $D \in \mathfrak{D}_2$, $D(1) = kh_2$, then $D = 0$. Clearly, $D(h_2) \in \langle f_5 \rangle$. Since $1 = [1, h_2]$, $[1, f_5] = h_1$, we have

$$kh_2 = D(1) = [D(1), h_2] + [1, D(h_2)] \in \langle h_1 \rangle.$$

So $k = 0$, which means $D(1) = 0$. Since $x_i = [1, f_i]$, $i = 1, 2, 3, 4$, we have $D(x_i) = 0$. So $DL_{-1} = 0$. But $L_0 = [L_{-1}, L_1]$, hence $DL_0 = 0$. That is $D = 0$.

Lemma 2.3. $\mathfrak{D}_1 = \mathrm{ad}L_1$.

Its proof is similar to the proof of Lemma 2.4 in [3].

Lemma 2.4. $\mathfrak{D}_0 = \mathrm{ad}L_0 \oplus \langle \mathrm{ad}_L(x_1x_2), \mathrm{ad}_L(x_3x_4) \rangle$.

Proof. It is clear that $\mathrm{ad}(x_1x_2)$, $\mathrm{ad}(x_3x_4)$ are derivations of L. Moreover

$$\mathrm{ad}(x_1x_2)L_{\pm 2}=0,\quad \mathrm{ad}(x_3x_4)L_{\pm 2}=0.$$

Let $D\in\mathfrak{D}_0$ and $D(1)=k\cdot 1$. Replacing D by $D-\mathrm{ad}(kh_2)$, we have

$$D(1)=0.$$

Let

$$D(x_2)=\sum_{i=1}^{4}k_ix_i,\quad D'=D-\mathrm{ad}(k_1e_1)-\mathrm{ad}(k_2h_1)-\mathrm{ad}(k_3x_3x_4),$$

then $D'(x_2)=k_4x_4$ and $D'(1)=0$. Writing D' again as D, we may suppose $D(1)=0$ and $D(x_2)=k_4x_4$. Next, we shall prove that

$$D\in\mathfrak{D}_0':=\mathrm{ad}L_0\oplus\langle\mathrm{ad}(x_1x_2),\ \mathrm{ad}(x_3x_4)\rangle.$$

By $[1,e_1]=[1,e_2]=[1,h_1]=0$, $[1,h_2]=1$, we have $[1,DL_0]=0$ and $DL_0\subset\langle e_1,\ e_2,\ h_1\rangle$. Let

$$D(e_i)=k_{i1}e_1+k_{i2}e_2+k_{i3}h_1;$$
$$D(h_i)=k_{i+2,1}e_1+k_{i+2,2}e_2+k_{i+2,3}h_1,\quad i=1,2.$$

As $0=[x_2,e_2]=[D(x_2),e_2]+[x_2,D(e_2)]=k_4x_3+k_{21}x_1+k_{23}x_2$, we obtain $k_4=k_{21}=k_{23}=0$ and $D(x_2)=0$.

As $[1,f_5]=h_1$, so $D(h_1)=k_{33}h_1$. But

$$x_2=[x_2,h_1],\ D(x_2)=[D(x_2),h_1]+[x_2,D(h_1)]=k_{33}x_2=0,$$

so $k_{33}=0$. Hence, $D(h_1)=0$ and $D(f_5)=0$. By calculation, we have

$$D=a_3^{-1}k_{12}\mathrm{ad}(x_1x_2)+k_{13}\mathrm{ad}e_2\in\mathfrak{D}_0'.$$

Hence $\mathfrak{D}_0=\mathfrak{D}_0'$.

By these lemmas, we have

Theorem 2.1. Der_FL=$\mathrm{ad}L\oplus\langle\mathrm{ad}_L(x_2x_3x_5),\mathrm{ad}_L\xi,\mathrm{ad}_L(x_1x_2),\mathrm{ad}_L(x_3x_4),\rho_1,\rho_2\rangle$, *where*

$$\xi=a_2x_2^{(2)}x_5+a_3x_3^{(2)}x_5+x_1x_4x_5+a_2x_1x_2^{(2)}x_3+a_3x_1x_3^{(3)},$$
$$\rho_1=\varphi(\mathrm{ad}_L)\xi\varphi^{-1},\quad \rho_2=\varphi(\mathrm{ad}_L(x_2x_3x_5))\varphi^{-1}$$

and φ *is as in Theorem* 1.2. *Let* $\mathfrak{L}=\mathrm{Der}L/\mathrm{ad}L$. *Then* $\mathfrak{L}$ *is a nilpotent Lie algebra of dimension* 6.

2.2. The Derivation Algebra of V_5G

Let $L=V_5G(a,a_1,b_1)$, $\mathfrak{D}=\mathrm{Der}_FL$. The results in Subsection 2.2 are similar to those in Subsection 2.1. Hence the proofs of this section will be brief or even omitted.

Proposition 2.1. *There is an order two automorphism* φ *of* V_5G *which is similar to that in Theorem* 1.2, *but with* $1\mapsto df_5$, $f_5\mapsto d^{-1}1$, $e_3\mapsto e_3$.

Lemma 2.5. The Derivation algebra $\mathfrak{D}$ of L has gradation $\mathfrak{D}=\bigoplus_{i=-2}^{2}\mathfrak{D}_i$, and $\mathfrak{D}_{-i}=\Phi(\mathfrak{D}_i)$, $i=1,2$, where Φ is an automorphism similar to that in V_4G.

Lemma 2.6. $\mathfrak{D}_2 = \mathrm{ad}L_2 \oplus \langle \mathrm{ad}_L\xi_1,\ \mathrm{ad}_L\xi_2\rangle$, where

$$\begin{aligned}\xi_1 &= da_1x_1x_4x_5 + dx_2^{(2)}x_5 + dax_3^{(2)}x_5 + a^2a_1x_4^{(3)}x_3 + a_1x_1^{(3)}x_2\\ &+ ax_2^{(3)}x_4 + aa_1x_1x_2x_4^{(2)} + aa_1x_1^{(2)}x_3x_4\\ &+ a_1b_1x_1x_2^{(2)}x_3 + aa_1b_1x_1x_3^{(3)} + a^2x_2x_3^{(2)}x_4,\\ \xi_2 &= db_1x_2x_3x_5 + dax_4^{(2)}x_5 + dx_1^{(2)}x_5 + ab_1x_1x_2x_3^{(2)} + a^2x_1x_3x_4^{(2)}\\ &+ ax_1^{(3)}x_3 + b_1x_1x_2^{(3)} + a_1b_1x_1^{(2)}x_2x_4 + ab_1x_2^{(2)}x_3x_4\\ &+ a^2b_1x_3^{(3)}x_4 + aa_1b_1x_2x_4^{(3)}.\end{aligned}$$

Similarly to the proof for V_4G, we can prove

Lemma 2.7. $\mathfrak{D}_1 = \mathrm{ad}L_1,\ \mathfrak{D}_0 = \mathrm{ad}L_0$.

By Lemma 2.5, Lemma 2.6, Lemma 2.7, Theorem 1.3 and Proposition 2.1, we obtain

Theorem 2.2. *The derivation algebra* $\mathrm{Der}_F V_5G$ *of* V_5G *is*

$$\mathrm{ad}L \oplus \langle \mathrm{ad}_L\xi_1,\ \mathrm{ad}_L\xi_2,\ \varphi(\mathrm{ad}_L\xi_1)\varphi^{-1},\ \varphi(\mathrm{ad}_L\xi_2)\varphi^{-1}\rangle,$$

where φ *is as in Proposition* 2.1, *and* ξ_1, ξ_2 *are as in Lemma* 2.6.

Theorem 2.3. *The outer derivation algebra of* V_5G *is a commutative Lie algebra of dimension* 4.

Remark 2.1. In fact, we have

$$\mathrm{ad}_L\xi_1 = (\mathrm{ad}f_3)^2, \quad \mathrm{ad}_L\xi_2 = (\mathrm{ad}f_4)^2,$$

where ξ_1, ξ_2 are as in Theorem 2.2. The derivation algebras of V_3G, V_6G, V_7G are determined in [8].

Theorem 2.4.[8] *The dimensions of derivation algebras of* V_iG, $i = 3, 6, 7$ *are* 20, 20 *and* 17 *respectively.*

§3. The Associative Forms of G_2 and Its Variations

We know that in most cases, simple modular Lie algebras do not admit nondegenerate trace forms. Nevertheless other associative forms may exist. The objective of this section is to show that the variations of G_2 which are graded simple Lie algebras all possess nonsingular symmetric associative bilinear forms.

It will be advantageous to view L as an $U(L)$-module and to express properties of the invariant forms in terms of the action of $U(L)$ on L.

Let S be the antipode map of $U(L)$, namely, it satisfies

(i) $S(x) = -x,\ \forall x \in L$.

(ii) $S(1) = 1$.

(iii) $S(xy) = S(y)S(x),\ \forall x, y \in U(L)$.

Proposition 3.1.[7] *Let* $L = \bigoplus_{i=-r}^{s} L_i$ *be a* $\mathbb{Z}$*-graded Lie algebra such that* $U(L^-)L_s = L$. *Suppose that there is an* L_0*-invariant form* $f : L \times L_s \to F$ *that satisfies*

(a) $f(L_i, L_s) = 0,\ \forall i \neq -r$.

(b) $f(u \cdot x, y) = f(S(u) \cdot y, x),\ \forall x, y \in L_s,\ u \in U(L^-)$.

Then there exists a uniquely determined associative form $\lambda : L \times L \to F$ *extending* f.

By Proposition 3.1, we have

Theorem 3.1. *The algebra V_4G possesses a nondegenerate symmetric associative bilinear form.*

To prove this theorem, first we have

Lemma 3.1. *Define $\lambda : L \times L_2 \to F$ such as $\lambda(1, f_5) = 1$, $\lambda(L_i, f_5) = 0$, $\forall i \neq -2$. Then λ is an L_0-invariant bilinear form.*

Proof. It can be directly checked that $\lambda(x \cdot 1, f_5) = \lambda(1, x \cdot f_5)$, $\forall x \in L_0$.

Lemma 3.2. *The form λ satisfies the condition* (b) *of Proposition* 3.1, *that is, $\lambda(u \cdot f_5, f_5) = \lambda(S(u) \cdot f_5, f_5)$, $\forall u \in U(L^-)$.*

Proof. For

$$c = (c_1, c_2, c_3, c_4, c_5), \quad \forall c_i \in \mathbb{N}_0, \quad i = 1, 2, 3, 4, 5,$$

let $u = x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} \cdot 1^{c_5}$. Lemma 3.1 ensures that it suffices to check the condition (b) for $u = x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} \cdot 1^{c_5}$ and $||c|| = 2$ (which means $c_1 + c_2 + c_3 + c_4 + 2c_5 = 4$). This can be checked directly.

The proof of Theorem 3.1 follows directly from Lemma 3.1, Lemma 3.2 and Proposition 3.1.

Remark 3.1. By direct calculation, we can easily obtain the value of the nonsingular associative form $\lambda : L \times L \to F$ on the basis:

$$\lambda(1, f_5) = 1, \quad \lambda(x_i, f_j) = \delta_{ij'}, \quad \lambda(e_1, e_2) = 1, \quad \lambda(h_1, h_2) = 1,$$

the others are 0.

Theorem 3.2. *Any of G_2 and its variations (i.e. V_iG, i = 3, 4, 5, 6, 7) possesses a nondegenerate associative form.*

Proof. The proof is similar to that of V_4G. We omit it.

Remark 3.2. The associative forms of G_2 and its variations on their basis are as follows:

(1) V_3G and G_2:

Their forms are the same as those of V_4G.

(2) V_5G:

$\lambda(1, f_5) = 1$, $\lambda(x_i, f_j) = \delta_{ij'}$, $\lambda(e_1, e_2) = 1$, $\lambda(h_1, h_2) = d^{-1}$, the others are 0.

(3) V_6G:

$\lambda(1, f_5) = 1$, $\lambda(x_1, f_3) = s^{-1}$, $\lambda(x_2, f_1) = 1$, $\lambda(x_3, f_4) = a^{-1}$, $\lambda(x_4, f_2) = s^{-1}$, $\lambda(e_1, e_{2'}) = 1$, $\lambda(h_1', h_2) = s^{-1}$, the others are 0.

(4) V_7G:

$\lambda(1, f_5) = 1$, $\lambda(x_i, f_j) = \delta_{ij'}$, $\lambda(e_1, e_2) = (c + 1)^{-1}$, $\lambda(e_3, e_4) = c^{-1}$, $\lambda(h_1, h_2) = 1$, the others are 0.

However, by calculation, we find that the Killing form on the above Lie algebras are singular. Therefore G_2 and its variations do not have nondegenerate Killing forms.

§4. Left-Symmetric Structure on the Variations of G_2

In this section, we shall show the existence of the nonsingular derivations of V_iG, $i = 3, 4, 5, 6$ and obtain some left-symmetric structures on them.

Let V be a vector space over an arbitrary field K. Consider a bilinear, distributive product $V \times V \to V$, denoted $(x, y) \mapsto x \cdot y$, which gives $(V, \cdot)$ the structure of a nonassociative algebra over K. Then V is said to be a left-symmetric algebra, or Koszul-Vinberg algebra, if

$$x\cdot(y\cdot z)-(x\cdot y)\cdot z=y\cdot(x\cdot z)-(y\cdot x)\cdot z,\qquad \forall x,y,z\in V. \tag{4.1}$$

If V is a left-symmetric algebra, then the operation

$$[x,y]=x\cdot y-y\cdot x \tag{4.2}$$

is skew-symmetric and satisfies Jacobi identity. Thus every left-symmetric algebra has a underlying Lie algebra structure. Conversely if L is a Lie algebra over K, then a left-symmetric operation satisfying (4.1), (4.2) on the vector space L will be called a compatible left-symmetric algebra structure on L, or simply a left-symmetric structure on L.

Proposition 4.1.[9] *If a Lie algebra L has a nonsingular derivation D, then it admits a left-symmetric structure which is given by*

$$x\cdot y=D^{-1}([x,D(y)]),\qquad \forall\, x,y\in L. \tag{4.3}$$

This left-symmetric structure is called an adjoint left-symmetric structure.

We are now going to seek nonsingular derivation in the variations of G_2. As before, F always denotes an algebraically closed field with characteristic 2.

Lemma 4.1. *Let $L=V_3G$, $\xi=ax_1x_2^{(2)}x_3+ax_2^{(2)}x_5+x_1x_4x_5$, $\rho=\varphi(\mathrm{ad}_L\xi)\varphi^{-1}$, where φ as in Theorem 1.2. Then*

(1) $\mathrm{ad}_L\xi$ *and ρ are outer derivations of V_3G.*

(2) $D'=\mathrm{ad}_L\xi+\rho+\mathrm{ad}_L(x_1+x_3)$ *is a nonsingular derivation of V_3G. Let $D=a^{-1/7}D'$, then $D^{14}=\mathrm{id}_L$.*

Lemma 4.2. *Let $L=V_4G$, $D=\mathrm{ad}_L\xi+\rho_1+\mathrm{ad}_Lx_1+\mathrm{ad}_Lx_3$, where ξ,ρ_1 are as in Theorem 2.1. Then D is a nonsingular derivation of V_4G.*

Lemma 4.3. *Let $L=V_5G$, $D=\mathrm{ad}_L\xi_1+\mathrm{ad}_L(x_1+x_2)$, where ξ_1 is as in Theorem 2.2. Then D is a nonsingular derivation of V_5G.*

Lemma 4.4. *Let*

$$L=V_6G,\quad \xi=x_1x_2x_3^{(2)}+a^{-1}x_1x_2^{(3)}+x_2^{(2)}x_3x_4+ax_3^{(3)}x_4+sx_1x_4x_5+x_2x_3x_5.$$

Then

(1) $\mathrm{ad}_L\xi$ *is an outer derivation of V_6G.*

(2) $D=\mathrm{ad}_L\xi+\mathrm{ad}_L(x_1+h_2)$ *is a nonsingular derivation of V_6G.*

With these lemmas and Proposition 4.1, we have

Theorem 4.1. *The nonrestricted simple Lie algebras V_3G, V_4G, V_5G and V_6G all admit adjoint left-symmetric structures, which are given by (4.3), where D is the nonsingular derivations of V_iG, $i=3$, 4, 5, 6 given in Lemmas 4.1–4.4.*

Notes: In fact, the Lie algebras V_iG, $i=3$, 4, 5, 6 have more nonsingular derivations. For example, V_3G has nonsingular derivations D^2, D^4, D^8 (where D is as in Lemma 4.1(2)). But the Lie algebra G_2 does not have nonsingular derivations (see [10]) and does not admit adjoint left-symmetric structures. There are Lie algebras which do not have nonsingular derivations, but admit left-symmetric structures. For example, the Cartan type Lie algebra $W(m,\mathbf{n})$ (cf. [2]).

Let $L=G_2$, $V'=\langle x_1, x_2, f_3, f_4\rangle\oplus L_0$, where $L_0=\langle e_1, e_2, h_1, h_2\rangle$ as in §1. Then V' is isomorphic to $\mathfrak{sl}(3,F)$. So we may use the same calculation for the derivation algebras of the variations of G_2 to prove that every derivation of $\mathfrak{sl}(3,F)$ is an inner derivation. It follows that $\mathfrak{sl}(3,F)$ do not have nonsingular derivations, and do not admit adjoint structures. But it has a left-symmetric structure (see [8]). Moreover, we can also show that $\mathfrak{sl}(3,F)$ has a unique nondegenerate associative form (not the Killing form).

Remark 4.1. By calculation, we find that the variations of $\mathfrak{sl}(3,F)$ (i.e. Lie algebra V_4', V_5', V_6') in [1] are all restricted, and are all isomorphic to $\mathfrak{sl}(3,F)$. So they are not new simple Lie algebras as asserted in [1].

Acknowledgement. The authors would like to thank Prof. Shen Guangyu for his instructions and help.

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