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NON-ALTERNATING HAMILTONIAN ALGEBRA $P(n, \mathbf{m})$ OF CHARACTERISTIC TWO

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ABSTRACT. Over a field F of characteristic $p = 2$, a class of Lie algebras $P(n, \mathbf{m})$, called non-alternating Hamiltonian algebras, is constructed, where n is a positive integer and $\mathbf{m}=(m_1, \cdots, m_n)$ is an n-tuple of positive integers. $P(n, \mathbf{m})$ is a graded and filtered subalgebra of the generalized Jacobson-Witt algebra $W(n, \mathbf{m})$ and bears resemblance to the Lie algebras of Cartan type. $P(n, m)$ is shown to be simple unless $\mathbf{m}=\mathbf{1}$ and $n < 4$. The dimension of $P(n, \mathbf{m})$ is $2^{|\mathbf{m}|} - 2$ if $\mathbf{m}=\mathbf{1}$, $2^{|\mathbf{m}|} - 1$ if $\mathbf{m} \neq \mathbf{1}$, where $|\mathbf{m}| = \sum_{i=1}^n m_i$. Different from the Lie algebras of Cartan type, all $P(n, \mathbf{m})$ are nonrestrictable. The derivation algebra of $P(n, \mathbf{m})$ is determined, and the natural filtration of $P(n, \mathbf{m})$ is proved to be invariant. It is then determined that $P(n, \mathbf{m})$ is a new class of simple Lie algebras if $(n, \mathbf{m})$ satisfies some condition.

§1. CONSTRUCTION

In the paper, we assume the ground field F to be of characteristic $p = 2$. If S is a subset of a linear space, $\langle S \rangle$ will denote the subspace spanned by S.

Let $\mathfrak{gl}(n)$ be the Lie algebra of all $n \times n$ matrices over F and E_{ij} the matrix in $\mathfrak{gl}(n)$ with (i, j)-entry 1 and other entries 0. Let $A(n)$ be the set of n-tuple of nonnegtive integers, $\varepsilon_i = (\delta_{1i}, \cdots, \delta_{ni}) \in A(n)$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in A(n)$, set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

If $\mathbf{m} = (m_1, \cdots, m_n)$ is an n-tuple of positive integers, we put $A(n, \mathbf{m}) = \{\alpha \in A(n) \mid 0 \leq \alpha \leq \pi\}$, where $\pi = (\pi_1, \cdots, \pi_n) := (2^{m_1} - 1, \cdots, 2^{m_n} - 1)$. Set $\mathfrak{A} = \mathfrak{A}(n)$ be the commutative associative F-algebra of all formal sums $\sum a_\alpha x^\alpha$ with multiplication

$$x^\alpha x^\beta = \binom{\alpha + \beta}{\alpha} x^{\alpha+\beta},$$

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where $\binom{\alpha+\beta}{\alpha} = \prod_{i=1}^{n} \binom{\alpha_i+\beta_i}{\alpha_i}$. Let $\mathfrak{A}_{[i]} = \langle x^\alpha \mid \alpha \in A(n),\ |\alpha| = i\rangle$, then $\mathfrak{A} = \sum \mathfrak{A}_{[i]}$ is a graded algebra. If $0 \neq f \in \mathfrak{A}_{[i]}$, write $\deg f = i$. Set $\mathfrak{A}(n, \mathbf{m}) = \langle x^\alpha \mid \alpha \in A(n, \mathbf{m})\rangle$, then $\mathfrak{A}(n, \mathbf{m})$ is a subalgebra of $\mathfrak{A}(n)$. Define derivations D_i:

$$D_i(x^\alpha) = x^{\alpha-\varepsilon_i},\ \alpha \in A(n),\ \ i = 1, \cdots, n.$$

Let $W(n) = \{\sum a_i D_i \mid a_i \in \mathfrak{A}(n)\}$. Then $W(n) = \sum W(n)_{[i]}$ is a graded Lie algebra, where $W(n)_{[i]} = \{\sum a_j D_j \mid a_j \in \mathfrak{A}(n)_{[i+1]}\}$. $W(n)$ is also a filtered Lie algebra with a filtration $\{W(n)_i\}$ associated with the gradation, and

$$W(n, \mathbf{m}) = \left\{ \sum a_i D_i \;\middle|\; a_i \in \mathfrak{A}(n, \mathbf{m}) \right\}$$

is a graded and filtered subalgebra of $W(n)$. Let $P_0 = \{A \in \mathfrak{gl}(n) \mid A = A^t\}$, then P_0 is a Lie subalgebra of $\mathfrak{gl}(n)$. Let $P(n)$ be the extention of P_0 in $W(n)$ (cf. [5, Definition 1.1]), that is

$$P(n) := \left\{ \sum a_i D_i \in W(n) \;\middle|\; \sum_{i,j} D_i(a_j) \otimes E_{i,j} \in \mathfrak{A} \otimes P_0 \right\}.$$

By [5, Theorem 1.1], $P(n)$ is a Lie subalgebra of $W(n)$ and an elementary computation shows that

$$P(n) = \left\{ \sum a_i D_i \in W(n) \;\middle|\; D_i(a_j) = D_j(a_i),\ i, j = 1, \cdots, n \right\}.$$

Define

$$P''(n, \mathbf{m}) := P(n) \cap W(n, \mathbf{m}),$$

then $P''(n, \mathbf{m})$ is a subalgebra of $W(n, \mathbf{m})$. We define $D_P : \mathfrak{A}(n) \longrightarrow W(n)$ by means of

$$D_P(f) := \sum_{j=1}^{n} D_j(f) D_j,\ f \in \mathfrak{A}(n).$$

Clearly, $D_P(\mathfrak{A}(n, \mathbf{m})) \subset W(n, \mathbf{m})$. Let $P'(n, \mathbf{m})$ denote the image of $\mathfrak{A}(n, \mathbf{m})$ under D_P. Note that $x^{\pi_i \varepsilon_i} D_i, 1 \leq i \leq n$, are elements of $P''(n, \mathbf{m})$ which do not lie in $P'(n, \mathbf{m})$. We put

$$P(n, \mathbf{m}) := P'(n, \mathbf{m})^{(1)}.$$

Lemma 1.1. *(1) The linear map D_P has degree -2.*

(2) $P'(n, \mathbf{m})$ is contained in $P''(n, \mathbf{m})$.

(3) $\ker D_P = F1$.

(4) Let $D = \sum f_j D_j$, $E = \sum g_j D_j$ be elements of $P''(n, \mathbf{m})$ (or $P(n)$); then

$$[D, E] = D_P\left(\sum_{i=1}^{n} f_i g_i\right). \tag{1.1}$$

(5) $P(n) = D_P(\mathfrak{A}(n))$.

Proof. The proof of (1)-(4) is very similar to that of [1, Chap. 4, Lemma 4.1].

(5) Similar to the proof of (2), we have $D_P(\mathfrak{A}(n)) \subset P(n)$. Given $D = \sum f_i D_i \in P(n)$, we have $D_i(f_j) = D_j(f_i), 1 \leq i, j \leq n$, thanks to [7, Lemma 1.2], there is $f \in \mathfrak{A}(n)$, such that $D_i(f) = f_i, 1 \leq i \leq n$. Hence $D = D_P(f)$ and $P(n) = D_P(\mathfrak{A}(n))$. ◻

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Proposition 1.2. *$P(n,\mathbf{m})$ is an ideal of $P''(n,\mathbf{m})$.*

Definition. *The Lie algebras $P(n,\mathbf{m})$ (resp. $P(n)$) are called the finite (resp. infinite) non-alternating Hamiltonian algebras.*

Lemma 1.3. *The following results hold:*

(1) $[D_P(f), D_P(g)] = D_P(D_P(f)(g)), \quad f,g \in \mathfrak{A}(n).$ (1.2)

(2) $[D_P(x^\alpha), D_P(x^\beta)] = \sum_{i=1}^n \binom{\alpha+\beta-2\varepsilon_i}{\alpha-\varepsilon_i} D_P(x^{\alpha+\beta-2\varepsilon_i}).$ (1.3)

(3) $[D_P(x^\alpha), D_P(x^{\varepsilon_i})] = D_P(x^{\alpha-\varepsilon_i}).$ (1.4)

(4) $[D_P(x^\alpha), D_P(x^{2\varepsilon_i})] = \alpha_i D_P(x^\alpha).$ (1.5)

(5) *$P(n,\mathbf{m})$ (resp. $P(n)$) is a graded and filtered subalgebra of $W(n,\mathbf{m})$ (resp. $W(n)$).*

Proposition 1.4. *Suppose that $\mathbf{m} \neq \mathbf{1} := (1,1,\cdots,1)$ or $\mathbf{m} = \mathbf{1}$ and $n \geq 3$, then we have*

(1) $P(n,\mathbf{1}) = \langle D_P(x^\alpha) \mid 0 < \alpha < \pi \rangle.$

(2) *If $\mathbf{m} \neq \mathbf{1}$, then $P(n,\mathbf{m}) = P'(n,\mathbf{m})$.*

(3) $P(n,\mathbf{m})_{[-1]} = W(n,\mathbf{m})_{[-1]}.$

(4) *The representation*

$$\varphi_P : P(n)_{[0]} \longrightarrow \mathfrak{gl}(\mathfrak{A}(n)_{[1]})$$

which is induced by the canonical representation of $W(n)_{[0]}$ in $W(n)_{[-1]}$, defines an isomorphism $P(n)_{[0]} \simeq P_0$, and $\varphi_P|_{P(n,1)_{[0]}}$ defines an isomorphism $P(n,1)_{[0]} \simeq P_0^{(1)}$.

Proof. (1) From Lemma 1.3 (3) it follows that $D_P(x^\alpha) \in P(n,\mathbf{1})$, for $0 < \alpha < \pi$. Note that if $\mathbf{m} = \mathbf{1}$, for $0 < \alpha, \beta \leq \pi, \alpha + \beta - 2\varepsilon_i \neq \pi, 1 \leq i \leq n$. Hence, by virtue of (1.3)(2), $[D_P(x^\alpha), D_P(x^\beta)] \in \langle D_P(x^\gamma) \mid 0 < \gamma < \pi \rangle$.

(2) If $\mathbf{m} \neq \mathbf{1}$, there exists $m_i > 1$, so $x^{2\varepsilon_i} \in \mathfrak{A}(n,\mathbf{m})$, (1.3)(4) shows that $D_P(x^\pi) = [D_P(x^\pi), D_P(x^{2\varepsilon_i})] \in P(n,\mathbf{m})$. Therefore $P(n,\mathbf{m}) = P'(n,\mathbf{m})$.

(3) Note that $D_P(x^{\varepsilon_i}) = D_i, 1 \leq i \leq n$. The assertion now follows from (1) and (2).

(4) $P(n)_{[0]} = \langle D_P(x^{\varepsilon_i+\varepsilon_j}) \mid 1 \leq i,j \leq n \rangle$. By Lemma 1.1 (3), we have $\dim P(n)_{[0]} = \frac{n}{2}(n+1) = \dim P_0$. We also note that for $1 \leq i < j \leq n$,

$$\varphi_P(D_P(x^{\varepsilon_i+\varepsilon_j})) = \varphi_P(x^{\varepsilon_i} D_j + x^{\varepsilon_j} D_i);$$

and

$$\varphi_P(D_P(x^{2\varepsilon_i})) = \varphi_P(x^{\varepsilon_i} D_i),$$

for $1 \leq i \leq n$. The matrices representating these endomorphisms with respect to the basis $\{x^{\varepsilon_1}, x^{\varepsilon_2}, \cdots, x^{\varepsilon_n}\}$ are given by $E_{ij} + E_{ji}$ in the former case, and E_{ii} in the latter case. These matrices belong to P_0. Consequently, $P(n)_{[0]} \simeq P_0$. Observe that if $n \geq 3$, $P(n,1)_{[0]} = P(n)_{[0]}^{(1)}$, thus, $P(n,1)_{[0]} \simeq P_0^{(1)}$. □

§2. Simplicity

Lemma 2.1. *(1) $P(n,\mathbf{m})_{[-1]}$ is an irreducible $P(n,\mathbf{m})_{[0]}$-module unless $\mathbf{m} = \mathbf{1}$ and $n < 3$.*

(2) $P(n)_{[-1]}$ is an irreducible $P(n)_{[0]}$-module. □

Theorem 2.2. *(1) Suppose that $\mathbf{m} \neq \mathbf{1}$, then $P(n,\mathbf{m})$ is simple and $\dim P(n,\mathbf{m}) = 2^{|\mathbf{m}|} - 1$.*

(2) Suppose that $\mathbf{m} = \mathbf{1}$, then $P(n,\mathbf{m})$ is simple if and only if $n \geq 4$ and $\dim P(n,\mathbf{1}) = 2^n - 2$.

Proof. The assertions concerning the dimension of $P(n,\mathbf{m})$ follow from Lemma 1.1 (3) and Proposition 1.4 (1), (2). The simplicity of $P(n,\mathbf{m})$ will be proven by applying [1, Chap. 3, Theorem 3.7]. The only work we have to do is to verify that the conditions (a)-(e) in the simplicity theorem hold.

As $P(n,\mathbf{m})_{[-1]}$ coincides with $W(n,\mathbf{m})_{[-1]}$, $P(n,\mathbf{m})$ is admissibly graded. The assumption that $\mathbf{m}=\mathbf{1}$ implies that $n\geq 4$ guarantees that condition (a) holds. (b) is trivially met. Thanks to Lemma 2.1 (c) is also met. Checking (d) is a small exercise. According to Lemma 1.3 (2),

$$D_P(x^{\pi-\varepsilon_i}) = [D_P(x^{\pi-\varepsilon_j}), D_P(x^{\varepsilon_i+\varepsilon_j})], \text{ for } i\neq j.$$

Therefore, (e) is fulfilled. Since $P(n,\mathbf{1})_{[1]}\neq 0$ implies that $n\geq 4$, $P(n,\mathbf{1})$ is not simple when $n<4$. Now the asserted results follow from the simplicity theorem. □

§3. Nonrestrictablity

Theorem 3.1. *Suppose that* $\mathbf{m}\neq\mathbf{1}$ *or* $\mathbf{m}=\mathbf{1}$ *and* $n\geq 4$, *then all algebras* $P(n,\mathbf{m})$ *are not restrictable.*

Proof. Let $\mathbf{m}\neq\mathbf{1}$, then there exists i, such that $m_i>1$. Hence $D_P(x^{3\varepsilon_i})\in P(n,\mathbf{m})$, and $(\mathrm{ad}D_i)^2(D_P(x^{3\varepsilon_i})) = D_i\neq 0$. Thus, $(\mathrm{ad}D_i)^2\neq 0$. But $(\mathrm{ad}D_i)^2$ is not an inner derivation. Consequently, $P(n,\mathbf{m})$ is not restrictable.

Let $\mathbf{m}=\mathbf{1}$ and $n\geq 4$. Suppose $(P(n,\mathbf{1}),[p])$ is restricted, then for any $D\in P(n,\mathbf{1})_{[0]}$, $D^{[2]}\in P(n,\mathbf{1})_{[0]}$. Choose $D=D_P(x^{\varepsilon_1+\varepsilon_2})$ and put $E=D^{[2]}$, then

$$(\mathrm{ad}E)(D_1)=D_1,\ (\mathrm{ad}E)(D_2)=D_2,\ (\mathrm{ad}E)(D_i)=0,\ 2<i\leq n.$$

But such element E does not exist. Therefore, $P(n,\mathbf{1})$ is not restrictable. □

§4. Derivation Algebra

In this section, we will determine the derivation algebra of $P(n,\mathbf{m})$.

Theorem 4.1. *If* $n\geq 4$, *then* $P(n,\mathbf{1})$ *is generated by* $P(n,\mathbf{1})_{[-1]}\oplus P(n,\mathbf{1})_{[1]}$.

Proof. Given $D_P(x^\alpha)\in P(n,\mathbf{1})_{[0]}$, there is i, such that $\alpha_i=0$. We obtain

$$D_P(x^\alpha) = [D_P(x^{\alpha+\varepsilon_i}), D_i],$$

and $P(n,\mathbf{1})_{[0]} = [P(n,\mathbf{1})_{[-1]}, P(n,\mathbf{1})_{[1]}]$. For $0\neq D_P(x^\alpha)\in P(n,\mathbf{1})_{[1+t]}$, where $t>0$, choose $i\neq j$, such that $\alpha_i=\alpha_j=1$. Since $\alpha<\pi$, there is k, such that $\alpha_k=0$. Now we have

$$D_P(x^\alpha) = [D_P(x^{\alpha+\varepsilon_k-\varepsilon_i-\varepsilon_j}), D_P(x^{\varepsilon_i+\varepsilon_j+\varepsilon_k})].$$

Hence, $P(n,\mathbf{1})_{[1+t]} = [P(n,\mathbf{1})_{[t]}, P(n,\mathbf{1})_{[1]}]$. By induction on t, we obtain

$$P(n,\mathbf{1})_{[1+t]} = P(n,\mathbf{1})_{[1]}^{(t)}.$$

The assertion holds. □

Let L be a Lie algebra, $\mathrm{Der}(L)$ the derivation algebra of L. If $L=\oplus_{i\in\mathbb{Z}}L_{[i]}$ is graded, then $\mathrm{Der}(L)=\oplus_{t\in\mathbb{Z}}\mathrm{Der}(L)_{[t]}$ is also graded, where

$$\mathrm{Der}(L)_{[t]} := \{\phi\in\mathrm{Der}(L)\mid \phi L_{[j]}\subset L_{[t+j]},\ \forall j\in\mathbb{Z}\}.$$

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An element ϕ is called a derivation of degree t if $\phi \neq 0$ and $\phi \in \mathrm{Der}(L)_{[t]}$.

Let M be a Lie algebra, L its subalgebra. $\mathrm{Nor}_M(L)$ denotes the normalizer of L in M.

Proposition 4.2. *Let $L = \oplus_{i=-1}^{s} L \cap W(n,\mathbf{m})_{[i]}$ be a graded subalgebra that contains $W(n,\mathbf{m})_{[-1]}$. Suppose that $\phi : L \longrightarrow L$ is a derivation of degree $t \geq 0$. Then there exists $E \in Nor_{W(n,\mathbf{m})}(L)$ such that $\phi = (\mathrm{ad}E)|_L$.*

Proof. When the base field F is of characteristic $p > 2$, this is Proposition 8.3 of [1, Chap. 4]. But we find that the assumption $p > 2$ is in fact unnecessary. $\square$

Lemma 4.3. *Suppose that L is a graded subalgebra of $W(n,\mathbf{m})$ and $L_{[-1]} = W(n,\mathbf{m})_{[-1]}$. Let $G = Der(L)$ be the derivation algebra of L. Let $t > 0$ and $\phi \in G_{[-t]}$. If $\phi L_{[t-1]} = 0$, then $\phi = 0$.*

Proof. Clearly, $\phi L_{[k]} = 0$, $\forall\ k \leq t-1$. Assume that $\phi L_{[k-1]} = 0$, for some $k > t-1$, then for any $D \in L_{[k]}$ and $1 \leq s \leq n$, $[D, D_s] \in L_{[k-1]}$. Hence $\phi[D, D_s] = 0$. Let $\phi(D) = \sum g_i D_i \in L_{[k-t]}$, then

$$0 = \phi[D, D_s] = [\phi(D), D_s] = \left[\sum g_i D_i, D_s\right] = \sum D_s(g_i) D_i.$$

Thus, $D_s(g_i) = 0$, $\forall\ 1 \leq i, s \leq n$. Consequently, $g_i \in F1$, $1 \leq i \leq n$. But $g_i \in \mathfrak{A}(n,\mathbf{m})_{[k-t+1]}$ and $k > t-1$. Hence, $k - t + 1 > 0$ and $g_i \in \mathfrak{A}(n,\mathbf{m})_1$. This yields $g_i = 0$, $1 \leq i \leq n$. Consequently, $\phi(D) = 0$ and $\phi L_{[k]} = 0$. By induction on k, we obtain $\phi = 0$. $\square$

In the following discussion, let $\mathbf{D}_1 = \mathrm{ad}_{P(n,\mathbf{m})}(P''(n,\mathbf{m}))$, $\mathbf{D}_2 = \langle (\mathrm{ad}D_i)^{2^{s_i}} \mid 0 < s_i < m_i,\ 1 \leq i \leq n \rangle$, $G = \mathrm{Der}(P(n,\mathbf{m}))$, and $I_k = \{i \mid m_i > k\}$, $k = 0, 1, 2, \cdots$.

Theorem 4.4. *Suppose that $\mathbf{m} \neq \mathbf{1}$ or $\mathbf{m} = \mathbf{1}$ and $n \geq 5$, then $G = \mathbf{D}_1 \oplus \mathbf{D}_2$.*

Proof. We divide the proof into several steps.

(1) For any $t > 0$, $G_{[t]} \subset \mathbf{D}_1$.

Given $\phi \in G_{[t]}$, by Proposition 4.2, there exists $E = \sum g_i D_i \in W(n,\mathbf{m})_{[t]}$, such that $\phi = (\mathrm{ad}E)|_{P(n,\mathbf{m})}$. We have $[D_i, E] \in P(n,\mathbf{m})$, $1 \leq i \leq n$. Thus, there are $f_1, f_2, \cdots, f_n \in \mathfrak{A}(n,\mathbf{m})$, such that $[D_i, E] = D_P(f_i)$, $1 \leq i \leq n$. Hence, $D_i(g_j) = D_j(f_i)$. But $D_P(D_i(f_j)) = D_P(D_j(f_i))$, According to Lemma 1.1 (3), $\ker D_P = F1$, so $D_i(f_j) + D_j(f_i) \in F1$. Because of $t > 0$, we have $D_i(g_j) = D_j(g_i)$. Consequently, $E \in P''(n,\mathbf{m})$.

(2) $G_{[0]} \subset \mathbf{D}_1$.

Given $\phi \in G_{[0]}$, according to Proposition 4.2, there is $E \in W(n,\mathbf{m})_{[0]}$, such that $\phi = \mathrm{ad}(E)|_{P(n,\mathbf{m})}$. Let $E = \sum a_{ij} x^{\varepsilon_i} D_j$, we want to prove that $E \in P''(n,\mathbf{m})$. If $n = 1$, this is obvious. Suppose $n \geq 2$. Set $h_i = x^{\varepsilon_i} D_i$, $i = 1, \cdots, n$. Rewrite E as following

$$E = \sum_i a_{ii} h_i + \sum_{i>j} a_{ij} D_P(x^{\varepsilon_i + \varepsilon_j}) + \sum_{i<j} (a_{ij} + a_{ji}) x^{\varepsilon_i} D_j. \tag{4.1}$$

Since the first two summands of the right hand side belong to $P''(n,\mathbf{m})$, we may harmlessly assume that $E = \sum_{i<j} b_{ij} x^{\varepsilon_i} D_j$. If $n = 2$, then $E = b_{12} x^{\varepsilon_1} D_2$. By the assumption $\mathbf{m} \neq \mathbf{1}$, we may assume that $m_1 > 1$. Then $[E, D_P(x^{2\varepsilon_1})] \in P(n,\mathbf{m})$, thus $b_{12} = 0$, and $E = 0$. If $n \geq 3$, put $b_{ij} = 0$, for $i \geq j$. Then for any $1 \leq k < l \leq n$, we have $\sum f_j D_j := [E, D_P(x^{\varepsilon_k + \varepsilon_l})] \in P(n,\mathbf{m})_{[0]}$, and

$$\sum_{j=1}^{n} f_j D_j = \sum_{i=1}^{n} b_{ik} x^{\varepsilon_i} D_l + \sum_{i=1}^{n} b_{il} x^{\varepsilon_i} D_k + \sum_{j=1}^{n} b_{lj} x^{\varepsilon_k} D_j + \sum_{j=1}^{n} b_{kj} x^{\varepsilon_l} D_j. \tag{4.2}$$

Hence, $D_i(f_j) = D_j(f_i)$, $1 \leq i,j \leq n$. Then for any $j \neq k,l$, $D_l(f_j) = b_{kj}, D_j(f_l) = b_{jk}$. Thus, $b_{kj} = b_{jk} = 0$. Similarly, it follows from $D_k(f_j) = D_j(f_k)$ that $b_{jl} = b_{lj}$. Consequently, $b_{ij} = 0$, $1 \leq i,j \leq n$, that is $E = 0$.

(3) $G_{[-1]} \subset \mathbf{D}_1$.

We first assume that $\mathbf{m} \neq \mathbf{1}$. We also assume that $n > 1$. Given $i \in I_1$ and $\phi \in G_{[-1]}$, we can show that $\phi(D_P(x^{2\varepsilon_i})) \in \langle D_i \rangle$. Thus, there is $\phi' = \phi - \sum_{i \in I_1} c_i(\mathrm{ad}D_i)$, where $c_i \in F$, such that $\phi'(D_P(x^{2\varepsilon_j})) = 0$, for any $j \in I_1$. Given $i \in I_1$ and $i \neq j$, we have $[D_P(x^{2\varepsilon_i}), D_P(x^{\varepsilon_i+\varepsilon_j})] = D_P(x^{\varepsilon_i+\varepsilon_j})$, which implies that $\phi'(D_P(x^{\varepsilon_i+\varepsilon_j})) \in \langle D_i \rangle$, thus $\phi'(D_P(x^{\varepsilon_i+\varepsilon_j})) = 0$, if $i,j \in I_1$. Fix some $i \in I_1$, and assume that $\phi'(D_P(x^{\varepsilon_i+\varepsilon_j})) = b_jD_i$, $\forall\, j \neq i$. Put

$$\psi := \phi' - \sum_{s \neq i} b_s(\mathrm{ad}D_s),$$

then $\psi(D_P(x^{\varepsilon_i+\varepsilon_j})) = 0$, for all j. Consequently, $\psi P(n,\mathbf{m})_{[0]} = 0$ if $n = 2$. If $n \geq 3$, then from the identity

$$D_P(x^{\varepsilon_j+\varepsilon_k}) = [D_P(x^{\varepsilon_i+\varepsilon_j}), D_P(x^{\varepsilon_i+\varepsilon_k})],$$

we have $\psi(D_P(x^{\varepsilon_j+\varepsilon_k})) = 0$. Hence $\psi P(n,\mathbf{m})_{[0]} = 0$. Now Lemma 4.3 yields $\psi = 0$. Consequently, $G_{[-1]} \subset \mathbf{D}_1$.

Now we suppose that $\mathbf{m} = \mathbf{1}$ and $n > 4$. For any $\phi \in G_{[-1]}$, we can prove that $\phi(D_P(x^{\varepsilon_i+\varepsilon_j})) \in \langle D_i, D_j \rangle$. Fix some l and let

$$\phi(D_P(x^{\varepsilon_i+\varepsilon_l})) = a_iD_l + b_iD_i,\ a_i,\ b_i \in F,\ 1 \leq i \leq n,\ i \neq l.$$

A direct computation shows that $b_i = b_j$, $\forall\, i,j \neq l$. Set $b = b_i$ and $D = \sum_{i \neq l} a_iD_i + bD_l$, then $(\mathrm{ad}D + \phi)P(n,1)_{[0]} = 0$. Also Lemma 4.3 yields $\mathrm{ad}D + \phi = 0$, that is $\phi = \mathrm{ad}D \in \mathbf{D}_1$.

(4) Suppose that $t \geq 2$. If $t = 2^u$, then

$$G_{[-t]} = \langle (\mathrm{ad}D_i)^t \mid i \in I_u \rangle.$$

Otherwise $G_{[-t]} = 0$.

Let $\phi \in G_{[-t]}$. Given $D_P(x^\alpha) \in L_{[t-1]}$, let $\phi(D_P(x^\alpha)) = \sum_{s=1}^n a_sD_s$.

(4-i) There exists i such that $\alpha_i \equiv 0 \pmod 2$, $\alpha_i > 0$ and $|\alpha| - \alpha_i \geq 2$.

In this case, we have

$$D_P(x^\alpha) = [D_P(x^{(\alpha_i+1)\varepsilon_i}), D_P(x^{\alpha+(1-\alpha_i)\varepsilon_i})], \tag{4.3}$$

where $D_P(x^{(\alpha_i+1)\varepsilon_i}) \in L_{[\alpha_i-1]}$, $D_P(x^{(\alpha+(1-\alpha_i)\varepsilon_i}) \in L_{[|\alpha|-\alpha_i-1]}$. Since $|(\alpha_i+1)\varepsilon_i|$, $|\alpha + (1-\alpha_i)\varepsilon_i| \geq 3$ and $|\alpha| = t+1$, $\alpha_i - 1$, $|\alpha| - \alpha_i - 1 < t-1$. But $\phi L_{[k]} = 0$ for any $k < t-1$, so we have

$$\phi(D_P(x^{(\alpha_i+1)\varepsilon_i})) = \phi(D_P(x^{\alpha+(1-\alpha_i)\varepsilon_i})) = 0. \tag{4.4}$$

(4.3) and (4.4) yield $\phi(D_P(x^\alpha)) = 0$.

(4-ii) For any s, $\alpha_s = 0$ or $\alpha_s \equiv 1 \pmod 2$, and there are $i \neq j$, such that $\alpha_i > 1$ and $\alpha_j > 0$.

In this case, we have $\alpha_i \geq 3$ and

$$D_P(x^\alpha) = [D_P(x^{\alpha_i\varepsilon_i}), D_P(x^{\alpha+(2-\alpha_i)\varepsilon_i})]. \tag{4.5}$$

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Since $|\alpha_i\varepsilon_i|$, $|\alpha+(2-\alpha_i)\varepsilon_i|\geq 3$, we obtain

$$\phi(D_P(x^{\alpha_i\varepsilon_i}))=\phi(D_P(x^{\alpha+(2-\alpha_i)\varepsilon_i}))=0. \tag{4.6}$$

(4.5) and (4.6) yield $\phi(D_P(x^\alpha))=0$.

(4-iii) All $\alpha_s\leq 1$.

Since $t\geq 2$ and $D_P(x^\alpha)\in L_{[t-1]}$, $|\alpha|\geq 3$. For any $i\neq j$ such that $\alpha_i=\alpha_j=1$, we have

$$[D_P(x^\alpha),D_P(x^{\varepsilon_i+\varepsilon_j})]=0.$$

Thus, $0=\phi[D_P(x^\alpha),D_P(x^{\varepsilon_i+\varepsilon_j})]=a_iD_j+a_jD_i$. Hence, $a_s=0$ for any s such that $\alpha_s=1$.

Suppose that $\mathbf{m}\neq\mathbf{1}$. If there is i, such that $\alpha_i=1$ and $m_i>1$, then $a_i=0$ and

$$D_P(x^\alpha)=[D_P(x^\alpha),D_P(x^{2\varepsilon_i})].$$

Hence, $\phi(D_P(x^\alpha))=a_iD_i=0$. Otherwise, there exists j, such that $\alpha_j=0$ and $m_j>1$. Then,

$$[D_P(x^\alpha),D_P(x^{2\varepsilon_j})]=0,$$

thus $a_j=0$. If there is $k\neq j$, such that $\alpha_k=0$, then for any such k, we have

$$[D_P(x^\alpha),D_P(x^{\varepsilon_j+\varepsilon_k})]=0,$$

which implies that $a_jD_k+a_kD_j=0$, that is, $a_k=0$. Consequently, $\phi(D_P(x^\alpha))=0$.

Suppose that $\mathbf{m}=\mathbf{1}$ and $n>4$. Then by virtue of Theorem 4.1, we have $G_{[-t]}=0$, for any $t>2$. Therefore we assume that $t=2$. Choose i,j such that $\alpha_i=0$ and $\alpha_j=1$. Since

$$\begin{aligned}\phi(D_P(x^{\alpha+\varepsilon_i-\varepsilon_j}))=&\phi[D_P(x^\alpha),D_P(x^{\varepsilon_i+\varepsilon_j})]\\=&a_iD_j+a_jD_i\\=&a_iD_j,\end{aligned}$$

we have

$$\phi(D_P(x^\alpha))=\phi[D_P(x^{\alpha+\varepsilon_i-\varepsilon_j}),D_P(x^{\varepsilon_i+\varepsilon_j})]=a_iD_i.$$

$|\alpha|=3$ and $n>4$ implies that we have at least two choices for i. Therefore $\phi(D_P(x^\alpha))=0$.

(4-iv) $\alpha=t\varepsilon_i+\varepsilon_j$, for some $i\neq j$, where $t\equiv 0 \pmod 2$.

If $t\geq 4$, let $t=2^u+v$, $0\leq v<2^u$. Thus, $u\geq 2$ and v is even. Suppose that $v\neq 0$, then $v\geq 2$ and

$$D_P(x^\alpha)=[D_P(x^{(v+1)\varepsilon_i+\varepsilon_j}),D_P(x^{(2^u+1)\varepsilon_i})]. \tag{4.7}$$

But $|(v+1)\varepsilon_i+\varepsilon_j|$, $|(2^u+1)\varepsilon_i|>3$, thus, we have

$$\phi(D_P(x^{(v+1)\varepsilon_i+\varepsilon_j}))=\phi(D_P(x^{(2^u+1)\varepsilon_i}))=0. \tag{4.8}$$

(4.7) and (4.8) yield $\phi(D_P(x^\alpha))=0$.

If $t=2^u$, where $u\geq 1$, then

$$[D_P(x^\alpha),D_P(x^{2\varepsilon_i})]=0$$

yields $a_i = 0$. If $n \geq 3$, then $\forall\, k \neq i, j$, and we get

$$[D_P(x^\alpha), D_P(x^{\varepsilon_i+\varepsilon_k})] = D_P(x^\beta),$$

where $\beta = (t-1)\varepsilon_i + \varepsilon_j + \varepsilon_k$ satisfies condition (ii) or (iii). Hence, $\phi(D_P(x^\beta)) = 0$ and $a_i D_k + a_k D_i = 0$, that is, $a_k = 0$. Consequently, $\phi(D_P(x^{2^u\varepsilon_i+\varepsilon_j})) \in \langle D_j \rangle$.

(4-v) $\alpha = (t+1)\varepsilon_i$, for some i, where $t \equiv 1 \pmod 2$.

In this case, $[D_P(x^\alpha), D_P(x^{2\varepsilon_i})] = 0$. Hence, $a_i = 0$. For any $j \neq i$,

$$[D_P(x^\alpha), D_P(x^{\varepsilon_i+\varepsilon_j})] = D_P(x^{t\varepsilon_i+\varepsilon_j}).$$

By virtue of (ii), $\phi(D_P(x^{t\varepsilon_i+\varepsilon_j})) = 0$. Hence, $a_i D_j + a_j D_i = 0$ and $a_j = 0$. Therefore, $\phi(D_P(x^\alpha)) = 0$.

(4-vi) $\alpha = (t+1)\varepsilon_i$ for some i, where $t \equiv 0 \pmod 2$.

Assume that $t \geq 4$. Let $t = 2^u + v$, where $0 \leq v < 2^u$. Then $u \geq 2$ and v is even. If $v > 0$, then we have

$$D_P(x^\alpha) = [D_P(x^{(2^u+1)\varepsilon_i}), D_P(x^{(v+2)\varepsilon_i})].$$

On the other hand, $|(2^u+1)\varepsilon_i|$, $|(v+2)\varepsilon_i| > 3$, hence

$$\phi(D_P(x^{(2^u+1)\varepsilon_i})) = \phi(D_P(x^{(v+2)\varepsilon_i})) = 0.$$

Consequently, $\phi(D_P(x^\alpha)) = 0$. If $t = 2^u$, for some $u \in \mathbb{N}$, then

$$D_P(x^\alpha) = [D_P(x^\alpha), D_P(x^{2\varepsilon_i})].$$

Hence, $\phi(D_P(x^{(2^u+1)\varepsilon_i})) \in \langle D_i \rangle$.

According to above (i)-(vi), we have

(4-vii) $\phi(D_P(x^\alpha)) = 0$, $\forall\, D_P(x^\alpha) \in L_{[t-1]}$, unless

(A) $\alpha = 2^u\varepsilon_i + \varepsilon_j$, for some $i \neq j$ and $u \in \mathbb{N}$. In this case, $\phi(D_P(x^\alpha)) \in \langle D_j \rangle$.

(B) $\alpha = (2^u+1)\varepsilon_i$, for some i and $u \in \mathbb{N}$. In this case, $\phi(D_P(x^\alpha)) \in \langle D_i \rangle$.

Hence, if $t \neq 2^u$, then $\phi L_{[t-1]} = 0$. By virtue of Lemma 4.3, we have $\phi = 0$. Consequently, $G_{[-t]} = 0$.

(4-viii) Let $\alpha = 2^u\varepsilon_i + \varepsilon_j$, $\beta = 2^u\varepsilon_i + \varepsilon_k$. According to (vii), let

$$\phi(D_P(x^\alpha)) = aD_j,\ \phi(D_P(x^\beta)) = bD_k,\ a, b \in F.$$

Since $D_P(x^\beta) = [D_P(x^\alpha), D_P(x^{\varepsilon_j+\varepsilon_k})]$, we have $bD_k = \phi[D_P(x^\alpha), D_P(x^{\varepsilon_j+\varepsilon_k})] = aD_k$, that is, $a = b$.

(4-ix) If $\alpha = 2^u\varepsilon_i + \varepsilon_j$ and $\beta = (2^u+1)\varepsilon_i$. According to (vii), let $\phi(D_P(x^\alpha)) = aD_j$, $\phi(D_P(x^\beta)) = bD_i$, $a, b \in F$. Then $aD_j = \phi(D_P(x^\alpha)) = \phi[D_P(x^\beta), D_P(x^{\varepsilon_i+\varepsilon_j})] = bD_j$, hence $a = b$.

(4-x) According to (vii)-(ix), if $t = 2^u$, we may assume that $\phi(D_P(x^{(t+1)\varepsilon_i})) = a_i D_i$ for $i \in I_u$, where $a_i \in F$. Put $\psi := \phi - \sum a_i(\mathrm{ad}D_i)^{2^u} \in G_{[-t]}$, then $\psi L_{[t-1]} = 0$. Thanks to Lemma 4.3, we obtain $\psi = 0$. Thus, $\phi = \sum a_i(\mathrm{ad}D_i)^{2^u} \in \langle (\mathrm{ad}D_i)^t \mid i \in I_u \rangle$.

It is easy to verify that $\mathbf{D}_1, \mathbf{D}_2 \subset G$. On the other hand, $\mathbf{D}_1 \cap \mathbf{D}_2 = \{0\}$ is clear. Therefore $\mathbf{D}_1 \oplus \mathbf{D}_2 \subset G$. According to (1)-(4), we have $G \subset \mathbf{D}_1 \oplus \mathbf{D}_2$. Consequently, $G = \mathbf{D}_1 \oplus \mathbf{D}_2$. □

Corollary 4.5. *If $\mathbf{m} \neq \mathbf{1}$, the dimension of the outer derivation algebra of $P(n, \mathbf{m})$ is $|\mathbf{m}|$.*

Theorem 4.6. *$\mathrm{Der}P(4, \mathbf{1}) = \mathrm{ad}_{P(4,\mathbf{1})}P''(4, \mathbf{1}) \oplus \langle \theta \rangle$, where θ is a homogeneous derivation of degree -2 defined by:*

$$\theta(D_P(x^{\pi-\varepsilon_i})) = D_i,\ 1 \leq i \leq 4.$$

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§5. Filtration

If L is a Lie algebra, $\phi \in \mathrm{Der}(L)$, let $I(\phi) = \dim(\mathrm{Im}\phi)$. Clearly, $I(\phi) = I(a\phi)$, $\forall a \in F^*$. If M is a subalgebra of $\mathrm{Der}(L)$, let $I(M) = \min_{0\neq\phi\in M} I(\phi)$ (cf. [6]). If L is a graded Lie algebra, $0 \neq x \in L$, let $\lambda(x)$ denote the nonzero homogeneous part of x with the least degree.

Lemma 5.1. *Let L be any graded Lie algebra, $w_1, w_2, \cdots, w_k \in L$. If $\{w_i\}$ is linearly dependent, then $\{\lambda(w_i)\}$ is also linearly dependent.* □

In the following discussion, we assume that $L = P(n,\mathbf{1})$ and $n \geq 5$. By virtue of Theorem 4.4, $\xi := \mathrm{ad}_L(D_P(x^\pi)) \in \mathrm{Der}(L)$.

Recall that $P(n,\mathbf{m})$ is a filtered algebra with filtration $\{P(n,\mathbf{m})_i\}$, where $P(n,\mathbf{m})_i = \sum_{j\geq i} P(n,\mathbf{m})_{[j]}$, for $i \geq -1$. we have

Theorem 5.2. *Let $n \geq 5$, then the following statements hold:*

(1) $I(\mathrm{Der}(P(n,\mathbf{1}))) = n$.

(2) $I(\phi) = n$ if and only if $0 \neq \phi \in \langle\xi\rangle$.

(3) If $\mathfrak{C} = \ker\xi$, then $\mathfrak{C} = P(n,\mathbf{1})_0$.

Proof. At first, a direct computation shows that if $0 \neq \phi \in \langle\xi\rangle$, then $I(\phi) = n$. We shall prove that if $\phi_0 \notin \langle\xi\rangle$, then $I(\phi_0) > n$.

Recall that $\lambda(\phi_0)$ is the nonzero homogeneous part of ϕ_0 with the least degree. According to Theorem 4.4

$$\mathrm{Der}(P(n,\mathbf{1})) = \mathrm{ad}P(n,\mathbf{1}) \bigoplus \langle \mathrm{ad}_{P(n,\mathbf{1})} h_i \mid 1 \leq i \leq n\rangle \bigoplus \langle\xi\rangle.$$

Hence,

$$\mathrm{Der}(P(n,\mathbf{1})) = \sum_{i=-1}^{n-2} \mathrm{Der}(P(n,\mathbf{1}))_{[i]}.$$

As $\phi_0 \notin \langle\xi\rangle$, $\lambda(\phi_0) \in \mathrm{Der}(P(n,\mathbf{1}))_{[i]}$, for some $i \leq n-3$. Let $\phi = \lambda(\phi_0)$. Then there exist $n+1$ homogeneous elements $E_1, \cdots, E_{n+1}$ of $P(n,\mathbf{1})$, such that $\{\phi(E_i)\}$ is linearly independent. We list the $n+1$ elements in the following

(1) Let $\phi = \mathrm{ad}(\sum a_i D_i)$, where $a_i \in F$.

By symmetry, we may assume that $a_1 \neq 0$. Put

$$E_j = D_P(x^{\varepsilon_1+\varepsilon_{j+1}}),\ 1 \leq j \leq n-1;\ E_n = D_P(x^{\varepsilon_1+\varepsilon_2+\varepsilon_3}),\ E_{n+1} = D_P(x^{\varepsilon_1+\varepsilon_2+\varepsilon_4}).$$

(2) Let $\phi = \mathrm{ad}D$, where $D \in L_{[t-2]}$, $t > 2$.

Let $D = D_P(x^\alpha) + \sum_{\beta\neq\alpha} a_\beta D_P(x^\beta)$. By symmetry, we may assume that $\alpha = \sum_{i=1}^t \varepsilon_i$, for some $2 < t < n$. Put

$$\begin{aligned} E_i &= D_i,\ 1 \leq i \leq t, \\ E_{t+j} &= D_P(x^{\varepsilon_1+\varepsilon_{t+1}+\varepsilon_{t+2}+\cdots+\varepsilon_{t+j+1}}),\ 1 \leq j \leq n-t-1, \\ E_n &= D_P(x^{\varepsilon_1+\varepsilon_n}), \\ E_{n+1} &= D_P(x^{\varepsilon_2+\varepsilon_n}). \end{aligned}$$

(3) Let $\phi = \mathrm{ad}_L(\sum a_i h_i) \in \mathrm{Der}(L)$.

It is harmless to assume that there exists $t \geq 1$, such that $a_i \neq 0$ for all $i \leq t$ and $a_j = 0$, for all $j > t$.

(3-i) $t = 1$. Put $E_1 = D_1$, $E_j = D_P(x^{\varepsilon_1+\varepsilon_j})$, $2 \leq j \leq n$, $E_{n+1} = D_P(x^{\varepsilon_1+\varepsilon_2+\varepsilon_3})$.

(3-ii) $1 < t < n$. In this case, put $E_i = D_i$, $1 \leq i \leq t$; $E_{ij} = D_P(x^{\varepsilon_i+\varepsilon_j})$, $1 \leq i \leq t, j > t$. As $t(n-t) \geq n+1-t$, we can choose $n+1-t$ different elements $E_{t+s} \in \{E_{ij}\}$, $1 \leq s \leq n+1-t$.

(3-iii) $t = n$. If $a_i = a_j$, $\forall\, i,j$, then put $E_i = D_i$, $1 \leq i \leq n$, $E_{n+1} = D_P(x^{\varepsilon_1+\varepsilon_2+\varepsilon_3})$. Otherwise, there are i,j, such that $a_i \neq a_j$. Put $E_s = D_s$, $1 \leq s \leq n$ and $E_{n+1} = D_P(x^{\varepsilon_i+\varepsilon_j})$.

(4) Let $\phi = \phi_1 + \phi_2$, where $\phi_1 = \mathrm{ad}_L(\sum a_i h_i)$, $0 \neq \phi_2 = \mathrm{ad}D$, $D = D_P(\sum a_\beta x^\beta) \in L_{[0]}$.

According to the assumption, there is β such that $a_\beta \neq 0$. We may assume that $\beta = \varepsilon_1 + \varepsilon_2$. Thus, put $E_1 = D_P(x^{\varepsilon_2+\varepsilon_3+\varepsilon_4})$, $E_2 = D_P(x^{\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5})$, $E_j = D_P(x^{\varepsilon_1+\varepsilon_j})$, $3 \leq j \leq n$, and $E_{n+1} = D_1$.

As $\phi(E_i) = \lambda(\phi_0)(E_i) = \lambda(\phi_0(E_i))$, $1 \leq i \leq n+1$, it follows from Lemma 5.1 that $\{\phi_0(E_i)\}$ is also linearly independent. Consequently, $I(\phi_0) > n$. Thus, (1) and (2) hold.

(3) is obvious. □

Corollary 5.3. *$P(n,1)_0$ is an invariant subalgebra of $P(n,1)$.* □

Theorem 5.4. *The natural filtration $\{P(n,1)_i\}$ of $P(n,1)$ is intrinsically determined.*

Proof. Let $\mathcal{L}_{-1} = P(n,1)$ and $\mathcal{L}_0 = \mathfrak{C}$. Following Kač and Weisfeiler we define

$$\mathcal{L}_i = \{D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\}, \text{ for } i \geq 1. \tag{5.1}$$

It is directly verified that $\mathcal{L}_i = P(n,1)_i$, $-1 \leq i \leq n-3$. Hence, the natural filtration

$$P(n,1) = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \cdots \supset \mathcal{L}_{n-3} \supsetneqq 0 \tag{5.2}$$

is intrinsically determined by Corollary 5.3, and the Theorem follows. □

Now we assume that

$$n > 1,\ \mathbf{m} = (m_1, \cdots, m_n),\ \text{and } m_i \geq 1 + \log_2(n+2),\ \forall\ i. \tag{5.3}$$

Let $L = P(n,\mathbf{m})$ and $\xi = \mathrm{ad}D_P(x^\pi)$.

Lemma 5.5. *if $0 \neq \phi \in \langle\xi\rangle$, then $I(\phi) = n+1$.*

Proof. It is evident that $\phi(L_1) = 0$. Moreover, $\phi(D_P(x^{\varepsilon_i+\varepsilon_j})) = 0$, $\forall\ i \neq j$. According to Lemma 1.3 (4), $0 \neq \phi(D_P(x^{2\varepsilon_i})) \in \langle D_P(x^\pi)\rangle$. Hence, $\{\phi(D_1), \cdots, \phi(D_n), D_P(x^\pi)\}$ consists of a basis of $\mathrm{Im}(\phi)$. Consequently, $I(\phi) = n+1$. □

Lemma 5.6. *If $0 \neq D \in L_{[t]}$, $-1 \leq t < |\pi| - 2$, then there exist $n+2$ homogeneous elements $E_1, \cdots, E_{n+2} \in L$, such that $\{[D,E_1],[D,E_2],\cdots,[D,E_{n+2}]\}$ is linearly independent.*

Proof. (1) $t = -1$. Let $D = \sum a_i D_i$, then there is j such that $a_j \neq 0$. Put $E_i = D_P(x^{(1+i)\varepsilon_j})$, $1 \leq i \leq n+2$. The assumption $m_j \geq 1 + \log_2(n+2)$ implies that $n+2 < 2^{m_j}$, thus $E_1, \cdots, E_{n+2} \in L$ and $\{[D,E_i]\}$ is linearly independent.

(2) $t > -1$. Let $D = \sum k_\alpha D_P(x^\alpha)$. Set $J = \{0 \neq \alpha \in A(n,\mathbf{m}) \mid k_\alpha \neq 0\}$, then $D = \sum_{\alpha \in J} k_\alpha D_P(x^\alpha)$.

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(a) There is $\alpha \in J$, such that $\alpha_j = 0$ for some j. By symmetry, we may assume that $\alpha_n = 0$ and $\alpha_1 > 0$. Put $E_i = D_P(x^{\varepsilon_1+2(i-1)\varepsilon_n})$, $1 \le i \le n+2$, then $\{E_i\} \subset L$ and $\{[D, E_i]\}$ is linearly independent.

(b) $\forall\ \alpha \in J$, $\alpha_1, \alpha_2, \cdots, \alpha_n \ne 0$. Put $E_i = D_i$, $1 \le i \le n$. If $\forall\ \alpha \in J$, $\alpha_i \equiv 0 \pmod 2$, $\forall\, i$, then $\alpha_i \ge 2$, $\forall\, i$. Fix some $\alpha \in J$. Put $E_{n+1} = D_P(x^{(\pi_1-\alpha_1+2)\varepsilon_1})$, $E_{n+2} = D_P(x^{(\pi_2-\alpha_2+2)\varepsilon_2})$. If there exists $\alpha \in J$ such that $\alpha_i \equiv 1 \pmod 2$ for some i. Put $E_{n+1} = D_P(x^{2\varepsilon_i})$. As $\alpha < \pi$, there is j such that $\alpha_j < \pi_j$. If $\alpha_j = 1$, put $E_{n+2} = D_P(x^{\pi_j\varepsilon_j})$. If $\alpha_j \ge 2$, put $E_{n+2} = D_P(x^{(\pi_j-\alpha_j+2)\varepsilon_j})$. Thus, $E_1, \cdots, E_{n+2} \in L$ and it is directly verified that $\{[D, E_i]\}$ is linearly independent. □

Theorem 5.7. *If $P(n,\mathbf{m})$ satisfies (5.3), then the following statments hold:*

(1) $I(\mathrm{ad}(P(n,\mathbf{m}))) = n+1$.

(2) $I(\mathrm{ad}D) = n+1$ if and only if $0 \ne D \in \langle D_P(x^\pi)\rangle$.

(3) If $0 \ne D \in \langle D_P(x^\pi)\rangle$, $\mathfrak{C}$ and $\mathfrak{N}$ are the centralizer and normalizer of D in $P(n,\mathbf{m})$ respectively, then $\dim \mathfrak{N}/\mathfrak{C} = 1$.

Proof. (1), (2). By direct computation we have if $0 \ne \mathrm{ad}D \in \langle \xi \rangle$, then $I(\mathrm{ad}D) = n+1$. We shall prove that if $D \in P(n,\mathbf{m})$ and $D \notin \langle D_P(x^\pi)\rangle$, then $I(\mathrm{ad}D) > n+1$. Clearly, $\lambda(D) \notin \langle D_P(x^\pi)\rangle$, thus, by virtue of Lemma 5.6, there are $n+2$ homogeneous elements $E_1, \cdots, E_{n+2} \in P(n,\mathbf{m})$, such that $\{[\lambda(D), E_i] \mid 1 \le i \le n+2\}$ is linearly independent. But $[\lambda(D), E_i] = \lambda([D, E_i])$, $1 \le i \le n+2$. Hence, $\{[D, E_i] \mid 1 \le i \le n+2\}$ is also linearly independent by Lemma 5.1. Therefore, $I(\mathrm{ad}D) \ge n+2 > n+1$.

(3). It is evident that

$$\mathfrak{C} = \langle D_P(x^\alpha) \mid \alpha \in A(n,\mathbf{m}), |\alpha| \ge 2 \text{ and } \alpha \ne 2\varepsilon_i\rangle \bigoplus \left\{ \sum_{i=1}^{n} a_i h_i \;\middle|\; \sum_{i=1}^{n} a_i = 0 \right\}$$

and $\mathfrak{N} = \mathfrak{C} \oplus \langle h_1 \rangle$. □

Corollary 5.8. *$\langle D_P(x^\pi)\rangle$ is an invariant subspace of $P(n,\mathbf{m})$.* □

Corollary 5.9. *$\mathfrak{C}$ and $\mathfrak{N}$ are invariant subalgebras of $P(n,\mathbf{m})$.* □

Theorem 5.10. *If $P(n,\mathbf{m})$ satisfies (5.3), then the natural filtration of $P(n,\mathbf{m})$ is intrinsically determined.*

Proof. Let $\mathcal{L}_{-1} = P(n,\mathbf{m})$, $\mathcal{L}_0 = \mathfrak{N}$. We define $\{\mathcal{L}_i\}$ as (5.1), then we have $\mathcal{L}_i = P(n,\mathbf{m})_i$, $\forall\, i$. Hence, the natural filtration

$$P(n,\mathbf{m}) = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \cdots \supset \mathcal{L}_s \supsetneq 0,$$

where $s = \sum_{i=1}^{n} 2^{m_i} - (n+2)$ is the length of $P(n,\mathbf{m})$, is intrinsically determined by Corollary 5.9. □

Theorem 5.11. *Let $P(n,\mathbf{m})$ and $P(n',\mathbf{m}')$ both satisfy (5.3). Then $P(n,\mathbf{m}) \simeq P(n',\mathbf{m}')$, if and only if $n = n'$ and $\{m_1, \cdots, m_n\} = \{m'_1, \cdots, m'_n\}$.*

Proof. Thanks to Theorem 5.7 (1), n is an invariant of $P(n,\mathbf{m})$, hence, the assumption $P(n,\mathbf{m}) \simeq P(n',\mathbf{m}')$ implies $n = n'$. Set $S = \{m_1, \cdots, m_n\}$. Let $V_t = (\mathrm{ad}P(n,\mathbf{m}) +$

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$(\mathrm{ad}P(n,\mathbf{m}))^2 + \cdots + (\mathrm{ad}P(n,\mathbf{m}))^{2^t})/\mathrm{ad}P(n,\mathbf{m})$, $t = 1, 2, \cdots$. Set $d_t = \dim V_t$, $t = 1, 2, \cdots$, which are invariants of $P(n,\mathbf{m})$. By Theorem 4.4 we have $d_t = \sum_{j=1}^{t} |S_j|$, where $S_j := \{x \in S \mid x > j\}$, $j = 1, 2, \cdots$. Thus, $|S_1| = d_1$, and $|S_t| = d_t - d_{t-1}$ for $t > 1$. Consequently, all $|S_t|$ are invariants. For $P(n', \mathbf{m}')$ we can define V'_t, d'_t, and S'_t analogously. Thus, $|S_t| = |S'_t|$, $t = 1, 2, \cdots$. Hence, $\{m_1, \cdots, m_n\} = \{m'_1, \cdots, m'_n\}$. □

Let $B_n = F[x_1, \cdots, x_n]$, $x_i^2 = 0$, be the truncated polynomial algebra over F, then $\mathfrak{A}(n, \mathbf{1}) \simeq B_n$. Set $y_i = 1 + x_i$, $i = 1, \cdots, n$, then $y_i^2 = 1$. For $\alpha \in A(n, \mathbf{1})$, put $y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. Then $\{y^\alpha \mid \alpha \in A(n, \mathbf{1})\}$ is a basis of B_n. It is easy to prove that $\{D_P(y^\alpha) \mid 0 < \alpha < \pi\}$ is a basis of $P(n, \mathbf{1})$. Before proving the following theorem, let's recall the definition of the first class of algebras $G(n)$ given by I. Kaplansky in [3].

Let $n \geq 4$, V an n-dimensional vector space over $\mathbb{Z}_2$ equipped with a symmetric inner product (,) which is nonsingular and nonalternate, and $e_1, \cdots, e_n$ an orthonormal basis of V. $G(n)$ is a Lie algebra over F with basis $\{x_\alpha \mid \alpha \in V,\ \alpha \neq 0,\ e_1 + e_2 + \cdots + e_n\}$ and Lie multiplication

$$[x_\alpha, x_\alpha] = 0, \quad [x_\alpha, x_\beta] = (\alpha, \beta) x_{\alpha+\beta},\ \alpha \neq \beta.$$

Theorem 5.12. *Let $n \geq 4$, then the Lie algebras $P(n, \mathbf{1})$ and $G(n)$ are isomorphic.*

Proof. Define a linear map $\eta : P(n, \mathbf{1}) \longrightarrow G(n)$ as follow:

$$\eta(D_P(y^\alpha)) = x_{\bar{\alpha}},\ \forall\, 0 < \alpha < \pi,$$

where $\bar{\alpha} = \sum_{i=1}^{n} \bar{\alpha_i} e_i$ and $n \mapsto \bar{n}$ is the canonical homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_2$. it is directly verified that η is an isomorphism of Lie algebras. □

Let $R(P(n, \mathbf{1}))$ be the subalgebra of $P(n, \mathbf{1})$ generated by "Kostrikin elements", i.e., these nonzero elements D with $(\mathrm{ad}D)^2 = 0$. Then we have

Theorem 5.13. *Let $n \geq 4$, then $R(P(n, \mathbf{1})) = P(n, \mathbf{1})$.*

Proof. It is easy to prove that $(\mathrm{ad}D_P(x^\alpha))^2 = 0$, $\forall\, 0 < \alpha < \pi$ and $\alpha \neq \varepsilon_i + \varepsilon_j$, thus $D_P(x^\alpha) \in R(P(n, \mathbf{1}))$, $\forall\, 0 < \alpha < \pi$ and $\alpha \neq \varepsilon_i + \varepsilon_j$. However,

$$D_P(x^{\varepsilon_i + \varepsilon_j}) = [D_k, D_P(x^{\varepsilon_i + \varepsilon_j + \varepsilon_k})],\ k \neq i, j.$$

Hence, $D_P(x^{\varepsilon_i + \varepsilon_j}) \in R(P(n, \mathbf{1}))$. Consequently, $R(P(n, \mathbf{1})) = P(n, \mathbf{1})$. □

Remark 5.14. According to Theorem 5.12 and Theorem 5.13, we correct an error occurring in [3, Remarks 2 (c)], where Kaplansky declared that $G(n)$ do not possess Kostrikin elements.

Remark 5.15. All the known simple Lie algebras over a field of characteristic 2 with dimension $2^N - 1$ are:

(1) $W(1, N)^{(1)}$ $(= P(1, N))$;

(2) some $K(n, \mathbf{m}, \mu_i)$ with $n = 2r + 1$ and $r \equiv 0 \pmod 2$, (for the definition, see [2]);

(3) the Lie algebra $L(N)$, which is one of the second class of Lie algebras defined in [3]);

(4) $P(n, \mathbf{m})$, for $n \geq 2$ and $|\mathbf{m}| = N$.

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According to [8] and Theorem 5.10, if N is big enough and $P(n, \mathbf{m})$ satisfies (5.3), then $P(n, \mathbf{m})$ is not isomorphic to $K(n, \mathbf{m}, \mu_i)$. Thus, for a fix n, there are infinitely many $\mathbf{m}$, such that $P(n, \mathbf{m})$ are new simple Lie algebras.

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