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LIE ALGEBRA $K(n,\mu_j,m)$ OF CARTAN TYPE OF CHARACTERISTIC $p=2$.

ZHANG YONGZHENG (张永正)* LIN LEI (林 磊)**

Abstract

Let $K(n, \mu_j, m)$, $n=2r+1$, denote the Lie algebra of characteristic $p=2$, which is defined in [4]. In the paper the restrictability of $K(n, \mu_j, m)$ is discussed and it is proved that, when $r\equiv 1 \pmod 2$ and $r>1$, $I(ad\, f)=n+1$ if and only if $0\neq f\in\langle x^\tau\rangle$. Then the invariance of some filtrations of $K(n, \mu_j, m)$ and the condition of isomorphism of $K(n, \mu_j, m)$ and $K(n', \mu_j', m')$ are obtained. Besides, the generators and the derivation algebra of $K(n, \mu_i, m)$ are discussed. The results also hold, when $r\equiv 0 \pmod 2$ and $r>0$.

§ 0. Introduction

Let F be a field of characteristic $p=2$, N be the set of nonnegative integers, $n=2r+1$ be a positive odd number. If $a=(a_1, a_2, \cdots, a_n)$, $b=(b_1, b_2, \cdots, b_n)\in N^n$, we define that $a\leqslant b \Leftrightarrow a_i\leqslant b_i$, $i=1, 2, \cdots, n$; $a<b\Leftrightarrow a\leqslant b$ and $a\neq b$. We let $\binom{a}{b}=\prod_{i=1}^{n}\binom{a_i}{b_i}$.

Let $A(n)$ consist of all formal sums of the indepent elements $\{x^a | a\in N^n\}$ over F and give it the structure of an associative algebra by defining

$$x^a x^b=\binom{a+b}{a}x^{a+b},\quad a, b\in N.$$

Let $m=(m_1, m_2, \cdots, m_n)$, where $m_1, \cdots, m_n$ are positive integers. We put $\tau=(2^{m_1}-1, \cdots, 2^{m_n}-1)$, $s_i=(\delta_{i1}, \cdots, \delta_{in})$, $\tau_i=(2^{m_i}-1)s_i$, where $i=1, \cdots, n$. Then $A(n, m)=\bigoplus_{a\leqslant\tau} Fx^a$ is an associative subalgebra of $A(n)$ (see [1]). Define special derivations $D_1, \cdots, D_n$ of $A(n, m)$ by

$$D_i(x^a)=x^{a-s_i},$$

where $x^b=0$, if $b\overline{\in} N^n$. Let μ_j, $j=1, 2, \cdots, 2r$, be $2r$ elements of F such that

$$\mu_j+\mu_{j'}=1,\ j=1, \cdots, 2r,$$

where

$$j'=\begin{cases} j+r, & \text{if } 1\leqslant j\leqslant r,\\ j-r, & \text{if } r<j\leqslant 2r.\end{cases}$$

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* Departme nt of Mathematics, Northeast Normal University, Changchun, Jilin 130024, China.
** Department of Mathematics, East China Normal University, Shanghai 200062, China.

In $A(n, \boldsymbol{m})$ we define Lie operation as following

$$[f, g]=\Big(I+\sum_{j=1}^{2r}\mu_j x^{\varepsilon_j}D_j\Big)(f)D_n(g)+\Big(I+\sum_{j=1}^{2r}\mu_j x^{\varepsilon_j}D_j\Big)(g)D_n(f)+\sum_{j=1}^{2r}D_j(f)D_{j'}(g).$$

Then $A(n, \boldsymbol{m})$ becomes a Lie algebra which is denoted by $K'(n, \mu_j, \boldsymbol{m})$ (see [4]).

Let $K(n, \mu_j, \boldsymbol{m})=K'(n, \mu_j, \boldsymbol{m})^{(1)}$. By Theorem 1 of [4] and (II) of [3] we know that $K(n, \mu_j, \boldsymbol{m})$ is a simple Lie algebra and

$$K(n, \mu_j, \boldsymbol{m})=\begin{cases} K'(n, \mu_j, \boldsymbol{m}), & \text{if } r\equiv 1(2),\\ \bigoplus\limits_{a<\tau} Fx^a, & \text{if } r\equiv 0(2),\end{cases}$$

where we abbreviate (mod 2) to (2).

Let $|a|=\sum_{i=1}^{n}a_i$, $\|a\|=|a|+a_n-2$, $K(n, \mu_j, \boldsymbol{m})_i=\langle\{x^a \mid \|a\|=i\}\rangle$. Then

$$K(n, \mu_j, \boldsymbol{m})=\bigoplus_{i=-2}^{s}K(n, \mu_j, \boldsymbol{m})_i$$

is a Z-graded Lie algebra. If $r\equiv 1(2)$, then $s=\|\tau\|$; if $r\equiv 0(2)$, then $s=\|\tau\|-1$.

§ 1. Intrinsic Property

Theorem 1. *$K(n, \mu_j, \boldsymbol{m})$ is a restricted Lie algebra if and only if $\boldsymbol{m}=\boldsymbol{1}$, where $\boldsymbol{1}=(1, 1, \cdots, 1)$.*

Proof Suppose $\boldsymbol{m}=\boldsymbol{1}$. We know that $K'(n, \mu_j, \boldsymbol{1})\simeq\{D\in W(n, \boldsymbol{1}) \mid D\omega\in A(n, \boldsymbol{1})\omega\}$, where $\omega=dx_n+\sum_{i=1}^{2r}\mu_i x_i dx_i$ (see [4]). Let $D\in K'(n, \mu_j, \boldsymbol{1})$ and $D\omega=u\omega$. Then $D^2\omega=D(u\omega)=(Du)\omega+u(D\omega)=(Du+u^2)\omega$. Since $W(n, \boldsymbol{1})=\mathrm{Der}\ A(n, \boldsymbol{1})$ is a restricted Lie algebra, $D^2\in W(n, \boldsymbol{1})$. Hence $D^2\in K'(n, \mu_j, \boldsymbol{1})$. Consequently $K'(n, \mu_j, \boldsymbol{1})$ is restricted. If $r\equiv 1(2)$, then $K(n, \mu_j, \boldsymbol{1})=K'(n, \mu_j, \boldsymbol{1})$ is restricted.

Let $r\equiv O(2)$ and $x^a\in K(n, \mu_j, \boldsymbol{1})$. Suppose $(x^a)^{[2]}=v+kx^\tau$, $k\in F$, $v\in K(n, \mu_j, \boldsymbol{1})$. Then

$$[(x^a)^{[2]}, 1]=[x^a, [x^a, 1]]=(x^{a-\varepsilon_n})^2+\sum_{j=1}^{2r}\mu_j a_j x^{\varepsilon_{j'}}x^{a-\varepsilon_j-\varepsilon_n}x^{a-\varepsilon_n}$$
$$+\sum_{j=1}^{2r}x^{a-\varepsilon_j}x^{a-\varepsilon_{j'}-\varepsilon_n}=\begin{cases}0, & \text{if } a\neq\varepsilon_n,\\ 1, & \text{if } a=\varepsilon_n.\end{cases}\qquad (1)$$

Also $[(x^a)^{[2]}, 1]=[v+kx^\tau, 1]=D_nv+kx^{\tau-\varepsilon_n}$. Hence the coefficient of $x^{\tau-\varepsilon_n}$ is k. By (1), $k=0$. Then $K(n, \mu_j, \boldsymbol{1})$ is restricted.

Conversely, suppose $K(n, \mu_j, \boldsymbol{m})$ is restricted. Then $(\mathrm{ad}1)^2$ is an inner derivation. If $(\mathrm{ad}1)^2\neq 0$, then the degree of homogeneous derivation $(\mathrm{ad}1)^2$ is equal to -4, because $1\in K(n, \mu_j, \boldsymbol{m})_{-2}$. Since the degree of any homogeneous inner derivation of $K(n, \mu_j, \boldsymbol{m})$ is greater than -3, $(\mathrm{ad}1)^2=0$. Hence

$$x^{\tau-\varepsilon_1-2\varepsilon_n}=(\mathrm{ad}1)^2(x^{\tau-\varepsilon_1})=0.$$

Then $m_n=1$.

Since the degree of homogeneous inner derivation $(\mathrm{ad}x^{\varepsilon_i})^2$, $1\leqslant i\leqslant 2r$, is equal to -2, $(\mathrm{ad}x^{\varepsilon_i})^2=\mathrm{ad}(\alpha 1)$, where $\alpha\in F$. Then

$$0=[\alpha 1, x^{\tau-\varepsilon_n}]=(\mathrm{ad}x^{\varepsilon_i})^2(x^{\tau-\varepsilon_n})=x^{\tau-2\varepsilon_{i'}-\varepsilon_n}$$

Therefore $m_{i'}=1$, $i'=1, \cdots, 2r$. Thus $\boldsymbol{m}=\mathbf{1}$.

Following [1] we let $\deg x^a=|a|+a_n$. If x is a linear combination of basis elements of the same degree k, then x is called a homogeneous element and we set $\deg x=k$.

Lemma 1. *Let $x=\sum c_b x^b\in K(n, \mu_j, \boldsymbol{m})$, where $c_b\in F$. Suppose $c_a x^a$ is a term of x.*

(i) *If $[x^{\varepsilon_n}, c_a x^a]\neq 0$, then $[x^{\varepsilon_n}, x]\neq 0$.*

(ii) *If $[x^{\varepsilon_j+\varepsilon_{j'}}, c_a x^a]\neq 0$, then $[x^{\varepsilon_j+\varepsilon_{j'}}, x]\neq 0$.*

Proof If $c_b x^b$ is another term of x, where $b\neq a$, it is easy to see that $[x^{\varepsilon_n}, c_a x^a]$ and $[x^{\varepsilon_n}, c_b x^b]$ cannot cancel. Hence $[x^{\varepsilon_n}, x]\neq 0$. The proof of (ii) is similar.

Lemma 2. *Let $x=\sum c_b x^b\in K(n, \mu_j, \boldsymbol{m})$. Suppose x^a is a term of x and $a_n=O(2)$.*

(i) *If $a_i\equiv O(2)$ and $[x^{\varepsilon_n+\varepsilon_i}, x^a]\neq 0$, then $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$.*

(ii) *If $a_i\equiv a_{i'}\equiv O(2)$, $a_j\equiv a_{j'}\equiv a_k\equiv a_{k'}\equiv O(2)$, $d=\varepsilon_j+\varepsilon_{j'}+\varepsilon_k+\varepsilon_{k'}$, then either $[x^{\varepsilon_n+\varepsilon_i}, x]$ or $[x^{\varepsilon_n+\varepsilon_{i'}}, x]$ is nonzero; either $[x^{d+\varepsilon_i}, x]$ or $[x^{d+\varepsilon_{i'}}, x]$ is nonzero.*

Proof (i) Obviously, $[x^{\varepsilon_n+\varepsilon_i}, x^a]=\alpha x^{a+\varepsilon_i}+x^{a-\varepsilon_{i'}+\varepsilon_n}$, where $\alpha\in F$. Suppose $\alpha x^{a+\varepsilon_i}\neq 0$. Let $c_b x^b$ be a term of x and $b\neq a$. Then $[x^{\varepsilon_n+\varepsilon_i}, c_b x^b]=\delta_1 x^{b+\varepsilon_i}+\delta_2 x^{b-\varepsilon_{i'}+\varepsilon_n}$, where $\delta_1, \delta_2\in F$ and $\delta_2=c_b\binom{b_n+1}{1}$.

If $b+\varepsilon_i=a+\varepsilon_i$, then $b=a$. It contradicts $b\neq a$. If $b-\varepsilon_{i'}+\varepsilon_n=a+\varepsilon_i$, then $b_n\equiv 1(2)$ because $a_n\equiv 0(2)$. Hence $\delta_2=0$. Then in $[x^{\varepsilon_n+\varepsilon_i}, x]$ the term $\alpha x^{a+\varepsilon_i}$ cannot be canceled. This implies $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$.

Suppose $\alpha x^{a+\varepsilon_i}=0$. Then $x^{a-\varepsilon_{i'}+\varepsilon_n}\neq 0$ because $[x^{\varepsilon_n+\varepsilon_i}, x^a]\neq 0$. In $[x^{\varepsilon_n+\varepsilon_i}, x]$ the only possible term to cancel $x^{a-\varepsilon_{i'}+\varepsilon_n}$ occurs in $[x^{\varepsilon_n+\varepsilon_i}, cx^{a-\varepsilon_i-\varepsilon_{i'}+\varepsilon_n}]$. By computation we see this term is zero. Hence $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$.

(ii) Since $a_{i'}\not\equiv 0(2)$ and $a_n\equiv 0(2)$, $x^{a-\varepsilon_{i'}+\varepsilon_n}\neq 0$ and $[x^{\varepsilon_n+\varepsilon_i}, x^a]=\alpha x^{a+\varepsilon_i}+x^{a-\varepsilon_{i'}+\varepsilon_n}\neq 0$. In $[x^{\varepsilon_n+\varepsilon_i}, x]$ the only possible term to cancel $x^{a-\varepsilon_{i'}+\varepsilon_n}$ occurs in $[x^{\varepsilon_n+\varepsilon_i}, cx^{a-\varepsilon_i-\varepsilon_{i'}+\varepsilon_n}]$ $=c(\delta-\mu_{i'})x^{a-\varepsilon_{i'}+\varepsilon_n}$, where $\delta\in F$. If $c(\delta-\mu_{i'})\neq 1$, then $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. If $c(\delta-\mu_{i'})=1$, then $c(\delta-\mu_i)\neq 1$ because $\mu_i\neq\mu_{i'}$. Thus we obtain $[x^{\varepsilon_n+\varepsilon_{i'}}, x]\neq 0$.

Using the above method we can also prove the remaining part of (ii).

Lemma 3. *Let $x=\sum c_b x^b\in K(n, \mu_j, \boldsymbol{m})$ in which every term $c_b x^b$ satisfies $b_n\equiv 1(2)$. Suppose $x^d\in K(n, \mu_j, \boldsymbol{m})$ and $d_n\equiv 1(2)$. If there exists some term $c_a x^a$ of x such that $[x^d, c_a x^a]\neq 0$, then $[x^d, x]\neq 0$,*

Imitating (i) of Lemma 1 we can prove this lemma.

Lemma 4. *Suppose $r=3$. Let g be a homogeneous element of $K(n, \mu_j, \boldsymbol{m})$ and $D_n(g)\neq 0$, $[g, x^{\varepsilon_i}]=0$, $[g, x^{\varepsilon_{i'}}]\neq 0$, $i=1, 2, 3$. Then there exists a basis element x^b, with* *deg* $x^b>3$, *such that* $[g, x^b]\neq 0$.

Proof Let $g=\sum c_d x^d$, $s=\max\{d_n|c_d\neq 0\}$. We write $g=x^a+\cdots$, where $a_n=s$.

Since $[x^a, x^{\varepsilon_i}]=(1-\mu_i)x^{\varepsilon_i}x^{a-\varepsilon_n}+x^{a-\varepsilon_{i'}}$ and $[g, x^{\varepsilon_i}]=0$, $a_{i'}=0$, $i=1, 2, 3$. If some a_t is odd, $1\leqslant t\leqslant 3$. Let $b=\sum_{i=1}^{3}\varepsilon_{i'}+\varepsilon_t$. Then $[x^a, x^b]=x^{a+b-\varepsilon_{t'}+\varepsilon_t}\neq 0$ and $[g, x^b]\neq 0$.

Suppose that a_t, $t=1, 2, 3$, are even numbers. If some a_t is nonzero, let $b=\sum_{i=1}^{6}\varepsilon_i-\varepsilon_t$, then $[x^a, x^b]\neq 0$ and $[g, x^b]\neq 0$.

If $a_t=0$, $t=1, 2, 3$, then $a=k\varepsilon_n$, $k\geqslant 1$. Let $b=\sum_{i=1}^{n}\varepsilon_i$. Then $[x^a, x^b]=x^{a+b-\varepsilon_n}\neq 0$. In $[g, x^b]$ the only possible term to cancel $x^{a+b-\varepsilon_n}$ occurs in $[cx^{(k+1)\varepsilon_n+\varepsilon_t+\varepsilon_{t'}}, x^b]$. By computation we know that $[cx^{(k-1)\varepsilon_n+\varepsilon_t+\varepsilon_{t'}}, x^b]=0$. Hence $[g, x^b]\neq 0$.

Lemma 5. *Suppose* $r\equiv 1(2)$. *If* x *is a nonzero homogeneous element of* $K(n, \mu_i, m)$, $x\overline{\in}\langle x^\tau\rangle$, *then there exist two basis elements* b_1, b_2, *with deg* $b_i>1$, $i=1, 2$, *such that* $[b_1, x]$ *and* $[b_2, x]$ *are linearly independent.*

Proof Let $x=\sum c_b x^b$, where $c_b\in F$.

(A) Assume there exists a nonzero term $c_b x^b$ such that $b_i\equiv 0(2)$. We can set $x=x^a+\cdots$, where $a_n\equiv 0(2)$. Let $\alpha=1+\sum_{i=1}^{2r}\mu_i a_i$.

1. $\alpha\neq 0$. Then $[x^{\varepsilon_n}, x^a]=\alpha x^a\neq 0$. By (i) of Lemma 1 we have $[x^{\varepsilon_n}, x]\neq 0$.

If $a_i\equiv 1(2)$, $i=1, 2, \cdots, 2r$, then $\alpha=1+r=0$. It contradicts $\alpha\neq 0$. Hence there exists some a_i $(i\leqslant 2r)$ such that $a_i\equiv 0(2)$. Then $[x^{\varepsilon_n+\varepsilon_i}, x^a]=\alpha x^{a+\varepsilon_i}+x^{a+\varepsilon_n-\varepsilon_{i'}}\neq 0$. By (i) of Lemma 2, $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. Because the degrees of $[x^{\varepsilon_n}, x]$ and $[x^{\varepsilon_n+\varepsilon_i}, x]$ are different, they are linearly independent.

2. $\alpha=0$.

(i) There exists some $a_i (i\leqslant 2r)$ such that $a_i\not\equiv a_{i'}(2)$. Without loss of generality, we set $a_i\equiv 0(2)$ and $a_{i'}\not\equiv 0(2)$. Then $[x^{\varepsilon_n+\varepsilon_i}, x^a]=x^{a-\varepsilon_{i'}+\varepsilon_n}\neq 0$. By (i) of Lemma 2, $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. Since $[x^{\varepsilon_i+\varepsilon_{i'}}, x^a]=x^a\neq 0$, by (ii) of Lemmal, $[x^{\varepsilon_i+\varepsilon_{i'}}, x]\neq 0$, $[x^{\varepsilon_i+\varepsilon_{i'}}, x]$ and $[x^{\varepsilon_n+\varepsilon_i}, x]$ are linearly tndependent.

(ii) $a_i\equiv a_{i'}(2)$, $i=1, 2, \cdots, 2r$.

(ii)-(a). Assume there exists some a_j $(j\leqslant 2r)$ such that $a_j\equiv 0$ (2). Then $a_{j'}\equiv 0(2)$. Because $\alpha=0$, there exists some $a_i(i\leqslant 2r)$ such that $a_i\not\equiv 0(2)$. Then $a_{i'}\not\equiv 0(2)$. By (ii) of Lemma 2, at least one of $[x^{\varepsilon_n+\varepsilon_i}, x]$ and $[x^{\varepsilon_n+\varepsilon_{i'}}, x]$ is nonzero. We can assume $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. Because $\alpha=0$ and $r\equiv 1(2)$, there exists also $K (k\neq j, j', k\leqslant 2r)$ such that $a_k\equiv 0(2)$. Let $d=\varepsilon_j+\varepsilon_{j'}+\varepsilon_k+\varepsilon_{k'}$. By (ii) of Lemma 2, at least one of $[x^{d+\varepsilon_i}, x]$ and $[x^{d+\varepsilon_{i'}}, x]$ is nonzero. It is linearly independent of $[x^{\varepsilon_n+\varepsilon_i}, x]$.

(ii)-(b). $a_i\not\equiv 0(2)$, $i=1, 2, \cdots, 2r$.

If some term $c_b x^b$ of x satisfies $b_n\equiv 0(2)$ and $b_i\equiv 0(2)$ for some $i(i\leqslant 2r)$, then we can set $x=x^b+\cdots$. This comes to the case of (ii)-(a). Hence we can assume that any term $c_b x^b$ of x with $b_n\equiv 0(2)$ satisfies $b_i\not\equiv 0(2)$, $i=1, 2, \cdots, 2r$.

Let $j\leqslant 2r$. Then $a_j\equiv a_{j'}\not\equiv 0(2)$. By (ii) of Lemma 2, without loss of generality, we suppose that $[x^{\varepsilon_n+\varepsilon_j}, x]\neq 0$. Since $a_n\equiv 0(2)$ and $a_j\equiv a_{j'}\not\equiv 0(2)$, we have $[x^{\varepsilon_n+\varepsilon_{j'}}, x^a]=x^{a-\varepsilon_j+\varepsilon_n}\neq 0$.

If $[x^{\varepsilon_n+\varepsilon_{j'}}, x]=0$, then x contains the nonzero term $cx^{a-\varepsilon_j-\varepsilon_{j'}+\varepsilon_n}$ so that in $[x^{\varepsilon_n+\varepsilon_{j'}}, x]$ the term $x^{a-\varepsilon_j+\varepsilon_n}$ can be cancelled. Then we affirm $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, x]\neq 0$. In fact, obviously $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, cx^{a-\varepsilon_j-\varepsilon_{j'}+\varepsilon_n}]=cx^{a+\varepsilon_n}\neq 0$. In $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, x]$ the only possible term to cancel $cx^{a+\varepsilon_n}$ occurs in $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, c_bx^b]$, where $b_n\equiv 0(2)$. By the assumption of (ii)-(b) we have $b_j\equiv b_{j'}\not\equiv 0$ (2). Then we obtain $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, c_bx^b]=0$ by direct computation. Hence the term $cx^{a+\varepsilon_n}$ cannot be cancelled and $[x^{\varepsilon_n+\varepsilon_j+\varepsilon_{j'}}, x]\neq 0$. It is linearly independent of $[x^{\varepsilon_n+\varepsilon_j}, x]$.

Suppose $[x^{\varepsilon_n+\varepsilon_{j'}}, x]\neq 0$. If it is linearly independent of $[x^{\varepsilon_n+\varepsilon_j}, x]$, we are through. If $[x^{\varepsilon_n+\varepsilon_j}, x]=k[x^{\varepsilon_n+\varepsilon_{j'}}, x]$, where $0\neq k\in F$, then $x^{a-\varepsilon_{j'}+\varepsilon_n}$ is also a term of $k[x^{\varepsilon_n+\varepsilon_{j'}}, x]$. We set $x=x^a+c_bx^a+\cdots$ and $k[x^{\varepsilon_n+\varepsilon_{j'}}, c_bx^b]$ contains the term $x^{a-\varepsilon_{j'}+\varepsilon_n}$.

We affirm that $b_n\not\equiv 0(2)$. In fact, if $b_n\equiv 0(2)$, we have $b_j\equiv b_{j'}\not\equiv 0(2)$ by the assumption of (ii)-(b). Then $k[x^{\varepsilon_n+\varepsilon_{j'}}, c_bx^b]=kc_bx^{b-\varepsilon_j+\varepsilon_n}$. Hence $b-\varepsilon_j+\varepsilon_n=a-\varepsilon_{j'}+\varepsilon_n$. Then $b_j\equiv O(2)$. It contradicts $b_j\not\equiv 0(2)$. The affirmation holds.

Since $b_n\not\equiv O(2)$, we have $k[x^{\varepsilon_n+\varepsilon_{j'}}, c_bx^b]=k\delta x^{b-\varepsilon_{j'}}$ where $\delta\in F$. Therefore $b+\varepsilon_{j'}=-a\varepsilon_{j'}+\varepsilon_n$. We have $b_{j'}=a_{j'}-2$, $b_n=a_n+1$ and $b_l=a_l$ $(l\neq j', n)$. Then

$$1+\sum_{l=1}^{2r}\mu_lb_l=1+\sum_{l=1}^{2r}\mu_la_l=\alpha=0.$$

Hence $[x^{\varepsilon_n}, c_bx^b]=c_bx^b\neq 0$. By (i) of Lemma 1, $[x^{\varepsilon_n}, x]\neq 0$. It is linearly independent of $[x^{\varepsilon_n+\varepsilon_j}, x]$.

(B) Every term c_bx^b of x satisfies $b_n\equiv 1(2)$. We set $x=x^a+\cdots$. Let $\beta=\sum_{l=1}^{2r}\mu_la_l$.

1. $\beta\neq 0$. Then $[x^{\varepsilon_n}, x^a]=\beta x^a\neq 0$. Hence $[x^{\varepsilon_n}, x]\neq 0$ by (i) of Lemma 1.

1-(a). Suppose there exist a_i and a_j $(i\neq j,\ i,\ j<n)$ such that $a_i\equiv a_j\equiv 0(2)$. Then $[x^{\varepsilon_n+\varepsilon_i}, x^a]=(\mu_i+\beta)x^{a+\varepsilon_i}$, $[x^{\varepsilon_n+\varepsilon_j}, x^a]=(\mu_j+\beta)x^{a+\varepsilon_j}$, $[x^{\varepsilon_n+\varepsilon_i+\varepsilon_j}, x^a]=(\mu_i+\mu_j+\beta)x^{a+\varepsilon_i+\varepsilon_j}$. It is easy to see that at least one of them is nonzero. We can suppose $[x^{\varepsilon_n+\varepsilon_i}, x^a]\neq 0$. By Lemma 3, $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. It is linearly independent of $[x^{\varepsilon_n}, x]$.

1-(b). Suppose there is only one a_i $(i\leqslant 2r)$ such that $a_i\equiv 0(2)$. Then $\mu_i+\sum_{l=1}^{2r}\mu_la_l=r\equiv 1(2)$. Hence $[x^{\varepsilon_n+\varepsilon_i}, x^a]=x^{a+\varepsilon_i}\neq 0$. By Lemma 3, $[x^{\varepsilon_n+\varepsilon_i}, x]\neq 0$. It is linearly independent to $[x^{\varepsilon_n}, x]$.

1-(c). $a_i\not\equiv 0(2)$. $i=1, 2, \cdots, 2r$.

If $a_n<2^{m_n}-1$, we let $h=2^{m_n}-a_n$. We know that

$$\binom{2^s-1}{t}\equiv 1(2), \tag{4}$$

where s, t are possitive integers and $1\leqslant t\leqslant 2^s-6$. Hence $[x^{h\varepsilon_n}, x^a]=x^{a+(h-1)\varepsilon_n}\neq 0$.

Then $[x^{h\epsilon_n}, x]\neq 0$. Since $a_n<2^{m_n}-1$ and $a_n\equiv 1(2)$, $h>1$. Then $[x^{\epsilon_n}, x]$ and $[x^{h\epsilon_n}, x]$ are linearly independent.

If $a_n=2^{m_n}-1$, since $x\notin\langle x^\tau\rangle$, we have $x^a\neq x^\tau$. Then there exists some $a_i (i\leqslant 2r)$ such that $a_i<2^{m_i}-1$. Let $h=2^{m_i}-1-a_i$. Using (Δ) we have $[x^{\epsilon_n+h\epsilon_i}, x^a]=x^{a+h\epsilon_i}\neq 0$. Hence $[x^{\epsilon_n+h\epsilon_i}, x]\neq 0$. It is linearly independent of $[x^{\epsilon_n}, x]$.

2. $\beta=0$.

2-(a). Suppose there exists some $a_i (i\leqslant 2r)$ such that $a_i\equiv a_{i'}\equiv 0(2)$. Then $[x^{\epsilon_n+\epsilon_i+\epsilon_{i'}}, x]=x^{a+\epsilon_i+\epsilon_{i'}}+\cdots\neq 0$. We also have $[x^{\epsilon_n+\epsilon_i}, x]=\mu_i x^{a+\epsilon_i}+\cdots$, $[x^{\epsilon_n+\epsilon_{i'}}, x]=\mu_{i'}x^{a+\epsilon_{i'}}+\cdots$. Obviously at least one of them is nonzero. It is linearly inde-pendent of $[x^{\epsilon_n+\epsilon_i+\epsilon_{i'}}, x]$.

2-(b). Suppose there is not any i which satisfies $a_i\equiv a_{i'}\equiv 0(2)$. Then there exists some a_j such that $a_j\equiv 0(2)$ and $\mu_j\neq 0$ (otherwise we have $\sum_{i=1}^{2r}\mu_i a_i=\tau\neq 0$, it contradicts $\beta=0$). Then $[x^{\epsilon_n+\epsilon_j}, x]=\mu_j x^{a+\epsilon_j}+\cdots\neq 0$.

Since $a_j\equiv 0(2)$, by the supposition of 2-(b), $a_{j'}\not\equiv 0(2)$. Then $[x^{\epsilon_j+\epsilon_{j'}}, x^a]=x^a\neq 0$. By (ii) of Lemma 1, $[x^{\epsilon_j+\epsilon_{j'}}, x]\neq 0$. It is linearly independent of $[x^{\epsilon_n+\epsilon_j}, x]$. The proof of this lemma is completed.

Lemma 6. *Let $\{g_{ia} | i\in I, a\in A\}\subseteq K(n, \mu_j, \boldsymbol{m})$, where I and A are finite sets. Suppose for every $a\in A$ there exists a linear transformation D_a such that $D_a(g_{ia})\neq 0$ for any $i\in I$ and $\{D_a(g_{ib}) | i\in I, b\in A, D_a(g_{ib})\neq 0\}$ are linerly independent. Then $\{g_{ia} | i\in I, a\in A\}$ are linearly independent.*

Proof Suppose $\sum_{i\in I, a\in A}\beta_{ia}g_{ia}=0$, where $\beta_{ia}\in F$. For any $b\in A$, we have a linear transformation D_b. Then $0=D_b(\sum_{i\in I, a\in A}\beta_{ia}g_{ia})=\sum_{i\in I, a\in A}\beta_{ia}D_b(g_{ia})=\sum_{i\in I}\beta_{ib}D_b(g_{ib})+\sum_{i\in I, a\in A\setminus\{b\}}\beta_{ia}D_b(g_{ia})$. Since $D_b(g_{ib})\neq 0$ for any $i\in I$ and $\{D_b(g_{ia}) | i\in I, a\in A, D_b(g_{ia})\neq 0\}$ are linearly independent, $\beta_{ib}=0$ for any $i\in I$. Hence the lemma holds.

Corollary 1. *Let $\{g_a | a\in A\}\subseteq K(n, \mu_j, \boldsymbol{m})$, where A is a finite set. Suppose for any $b\in A$ there exists a linear transformation D_a such that $D_a(g_a)\neq 0$ and $\{D_a(g_b) | b\in A, D_a(g_b)\neq 0\}$ are linearly independent. Then $\{g_a | a\in A\}$ are linearly independent.*

Following [1], we let $I(d)=\dim(\operatorname{Im} d)$, where $d\in \operatorname{Der}_F K(n, \mu_j, \boldsymbol{m})$. If M is asubset of $\operatorname{Der}_F(K(n, \mu_j, \boldsymbol{m}))$, we let $I(M)=\min_{0\neq d\in M} I(d)$.

Theorem 2. *Let $\tau\equiv 1(2)$ and $\tau>1$. If $0\neq f\in\langle x^\tau\rangle$, then $I(ad\ f)=n+1$; if $f\notin\langle x^\tau\rangle$, then $I(adf)>n+1$.*

Proof Let $0\neq cx^\tau\in\langle x^\tau\rangle$. Then $[cx^\tau, 1]$, $[cx^\tau, x^{\epsilon_n}]$, $[cx^\tau, x^{\epsilon_i}] i=1, 2, \cdots, 2r$, are linearly independent. If $\deg x^a>1$ and $a\neq\epsilon_n$, then $[cx^\tau, x^a]=0$. Hence $I(\operatorname{ad}(cx^\tau))=n+1$.

Let $f\notin\langle x^\tau\rangle$. We shall prove $I(\operatorname{ad} f)>n+1$. Let g be the nonzero homogeneous part of f with the least degree. It is sufficient to prove $I(\operatorname{ad} g)>n+1$. Let $V=\langle x^{\epsilon_n},$

$x^{\varepsilon_r}, \cdots, x^{\varepsilon_{2r}}\rangle$. Then $[x, y]\in F$, $\forall x, y\in V$. Hence V is a symplectic space (see [1]). Let $V_g=\{x\in V \mid [g, x]=0\}$. Suppose $\dim V_g=t$. By Lemma 1.5 of [1] and Witt's theorem we can directly assume that $\{x^{\varepsilon_1}, x^{\varepsilon_{1'}}, \cdots, x^{\varepsilon_u}, x^{\varepsilon_{u'}}, x^{\varepsilon_{u+1}}, \cdots, x^{\varepsilon_{t-u}}\}$ is a basis of V_g.

1. $[g, 1]=D_n(g)=0$. Then $[g, x^{\varepsilon_i}]=D_{i'}(g)$, $i=1, \cdots, 2r$.

(i) $t=2r$. Then $D_{i'}(g)=0$, $i'=1, \cdots, 2r$. Hence we can assume $g=1$. Since $[1, x^{\varepsilon_n}]$, $[1, x^{\varepsilon_n+\varepsilon_1+\varepsilon_{1'}}]$, $[1, x^{\varepsilon_n+\varepsilon_1+\varepsilon_{1'}+\varepsilon_i}]$, $[1, x^{\varepsilon_n+\varepsilon_i}]$, $i=1, \cdots, 2r$, are lindependent, $I(\mathrm{ad}\ g)>n+1$.

(ii) $t<2r$. Let $J=\{1, 1', \cdots, u, u', u+1, \cdots, t-u\}$, $J_0=\{1, 1', \cdots, u, u'\}$. $\bar{J}=\{1, 2, \cdots, n-1\}\setminus J$. We affirm that

(*). $\{D_{i'}(g) \mid i\in \bar{J}\}$ are linearly independent.

In fact, if $\sum_{i\in \bar{J}}\beta_i D_{i'}(g)=0$, then $\sum_{i\in \bar{J}}\beta_i[g, x^{\varepsilon_i}]=0$. Hence $[g, \sum_{i\in \bar{J}}\beta_i x^{\varepsilon_i}]=0$. Then $\sum_{i\in \bar{J}}\beta_i x^{\varepsilon_i}\in V_g$. Consequently $\beta_i=0$, $\forall i\in \bar{J}$. Then the affirmation holds.

(ii)-A. Let $T=\{\sum_{j\in J_0} k_j\varepsilon_j \mid k_j=0 \text{ or } 1\}$, $T_\lambda=\{a\in T \mid |a|=\lambda\}$, where $0\leqslant\lambda\leqslant 2u$. Let $g_{ia}=[g, x^{\varepsilon_i+a}]$, where $i\in \bar{J}$, $a\in T_\lambda$. Then $g_{ia}=D_{i'}(g)x^a$. Suppose $a=\varepsilon_{l_1}+\varepsilon_{l_2}+\cdots+\varepsilon_{l_\lambda}\in T_\lambda$. Then $l_1, l_2, \cdots, l_\lambda\in J_0$. Let $D_a=D_{l_1}D_{l_2}\cdots D_{l_\lambda}$. Since $D_l(g)=[g, x^{\varepsilon_{l'}}]=0$ for $l\in J_0$, $D_a(g)=0$. Hence $D_a(g_{ia})=D_a(D_{i'}(g)x^a)=D_{i'}(D_a(g))x^a+D_{i'}(g)=D_{i'}(g)$, where $i\in \bar{J}$, $a\in T_\lambda$. If $b\in T_\lambda$ and $b\neq a$, then $D_a(g_{ib})=D_a(D_{i'}(g)x^b)=0$. By (*) and Lemma 6, $\{g_{ia} \mid i\in \bar{J}, a\in T_\lambda\}$ are linearly independent.

Let $\lambda=0, 1, \cdots, 2u$. We get $(n-1-t)2^{2u}$ linearly independent elements in $\mathrm{Im}(\mathrm{ad}\ g)$.

(ii)-B. For $i\in \bar{J}$ we have $D_n[g, x^{\varepsilon_n+\varepsilon_i}]=D_n(D_{i'}(g)x^{\varepsilon_n}+gx^{\varepsilon_i}+\sum_{j=1}^{2r}\mu_j x^{\varepsilon_j}x^{\varepsilon_i}D_j(g))$ $=D_{i'}(g)$. By (*), $\{D_n[g, x^{\varepsilon_n+\varepsilon_i}] \mid i\in \bar{J}\}$ are linearly independent and so are $\{[g, x^{\varepsilon_n+\varepsilon_i}] \mid i\in \bar{J}\}$. We also get $n-1-t$ linearly indepent elements in $\mathrm{Im}\ (\mathrm{ad}\ g)$.

(ii)—C. Let $J_1=\{(u+1)', (u+2)', (t-u)'\}$. $H=\{\sum_{j\in J_1} k_j\varepsilon_j \mid k_j=0 \text{ or } 1\}$, $H_\lambda=\{a\in H \mid |a|=\lambda\}$, where $2\leqslant\lambda\leqslant t-2u$. Let $g_a=[g, x^{\varepsilon_n+a}]$ where $a\in H_\lambda$.

It is easy to prove that $\{D_n(g_a) \mid a\in H_\lambda\}$ are linearly independent. Consequently, $\{g_a \mid a\in H_\lambda\}$ are linearly independent. Let $\lambda=2, 3, \cdots, t-2u$. We get $2^{t-2u}-(t-2u)-1$ linearly independent elements in $\mathrm{Im}(\mathrm{ad}\ g)$.

It is easy to see that all elements we abtain in (ii)-A, (ii)-B and (ii)-C are linearly independent. Then $I(\mathrm{ad}\ g)\geqslant(n-1-t)2^{2u}+(n-1-t)+2^{t-2u}-(t-2u)-1$. Let $s=t-2u$. Since $r\equiv 1(2)$ and $r>1$, we have $n\geqslant 7$. Then $I(\mathrm{ad}\ g)\geqslant(n-1-s-2u)(2^{2u}+1)+2^s-s-1\geqslant(n-1-s)\times 2+2^s-s-1\geqslant n+2+(n-5+2^s-3s)>n+1$.

2. $[g, 1]\neq 0$.

(i) $V_g\neq 0$. If $x, y\in V_g$ and $[x, y]=1$, then $[g, 1]=[g, [x, y]]=[x, [g, y]]+[y, [g, x]]=0$. It contradicts $[g, 1]\neq 0$. Hence V_g is totally isotopic subspace of

V. Then $u=0$ and $\{x^{\varepsilon_1}, x^{\varepsilon_2}, \cdots, x^{\varepsilon_t}\}$ is a basis of V_g. Let $J=\{1, 2, \cdots, t\}$. $J'=\{1', 2', \cdots, t'\}$, $\bar{J}=\{1, 2, \cdots, n-1\}\backslash J$, $J_1=\bar{J}\backslash J'$. Let $H=\{\sum_{j\in J'} k_j\varepsilon_j+k_l\varepsilon_l \mid l\in J_1,\ k_j,\ k_l=0$ or $1\}$, $H_\lambda=\{a\in H \mid |a|=\lambda\}$, where $0\leqslant\lambda\leqslant t+1$. We shall prove by induction on λ that $\{[g, x^a] \mid a\in H_\lambda\}$ are linearly independent.

Since $[g, 1]\neq 0$, the conclusion is right for $\lambda=0$. The conclusion is also right for $\lambda=1$, because $\{[g, x^{\varepsilon_i}] \mid i\in\bar{J}\}$ are linearly independent. Let $\lambda>1$ and suppose $\{[g, x^a] \mid a\in H_{\lambda-1}\}$ are linearlyindependent.

Let $\sum_{a\in H_\lambda} k_a[g, x^a]=0$, where $k_a\in F$. Let $i\in J'$. Since $[g, x^{\varepsilon_{i'}}]=0$, we have

$$\begin{aligned}0&=\sum_{a\in H_\lambda} k_a[[g, x^a], x^{\varepsilon_{i'}}]=\sum_{a\in H_\lambda} k_a[[g, x^a], x^{\varepsilon_{i'}}]\\&=\sum_{a\in H_\lambda} k_a[g, [x^a, x^{\varepsilon_{i'}}]]=\sum_{a\in H_\lambda} k_a[g, x^{a-\varepsilon_i}].\end{aligned}$$

If $a-\varepsilon_i\not\geqslant 0$, then $x^{a-\varepsilon_i}=0$. Hence $\sum_{a\in H_\lambda, a-\varepsilon_i\geqslant 0} k_a[g, x^{a-\varepsilon_i}]=0$. If $a\in H_\lambda$ and $a-\varepsilon_i\geqslant 0$, then $a-\varepsilon_i\in H_{\lambda-1}$. Hence, by induction hypothesis, $k_a=0$. Let b be any element of H_λ. There exists some $i\in J'$ such that $b-\varepsilon_i\in H_{\lambda-1}$. By above proof, we have $k_b=0$. This implies $\{[g, x^a] \mid a\in H_\lambda\}$ are linearly independent. Let $\lambda=0, 1, \cdots, t+1$. We have

$$I(\text{ad } g)\geqslant 2^t+2^t(n-1-2t)=2^t(n-2t).$$

If $n>7$ or $n=7\,(t\neq 3)$, it is easy to see that $2^t(n-2t)>n+1$. If $n=7$ and $t=3$, by Lemma 4, $I(\text{ad } g)\geqslant 2^t(n-2t)+1>n+1$.

(ii) $V_g=0$. Then $[g, 1]$, $[g, x^{\varepsilon_i}]$, $i=1, \cdots, n-1$, are linearly independent. Since $f\in\langle x^\tau\rangle$, $g\in\langle x^\tau\rangle$. Using Lemma 5 we have $I(\text{ad }\ g)>n+1$. The theorem is proved.

Imitating the proof of Lemma 5, we have

Lemma 7. *Suppose $r\equiv 0(2)$ and $r\neq 0$. If x is a nonzero homogeneous element of $K(n, \mu_j, \boldsymbol{m})$, then there exists a basis element b, with $\deg b>1$, such that $[b, x]\neq 0$.*

Imitating the proof of Theorem 2, we get

Theorem 3. *Suppose $r\equiv 0(2)$ and $r\neq 0$. If $0\neq f\in\langle x^\tau\rangle$, then $I(ad\, f|_{K(n,\mu_j,m)})=n$. If $0\neq f\in K(n, \mu_j, \boldsymbol{m})$, then $I(ad\, f)>n$.*

Let $r\equiv 1(2)$ and $r>1$. Suppose R is the normalizer of $\langle x^\tau\rangle$ in $K(n, \mu_j, \boldsymbol{m})$. Then $R=\langle x^a|\deg x^a\geqslant 2\rangle$. Using Theorem 2, we have

Corollary 2. *Let $r\equiv 1(1)$ and $r>1$. Then $\langle x^\tau\rangle$ is an invariant subspace of $K(n, \mu_j, \boldsymbol{m})$ and R is an invariant subalgebra of $K(n, \mu_j, \boldsymbol{m})$.*

Following [1] we have the filtrations

$$K(n, \mu_j, \boldsymbol{m})=L_{-2}\supset L_{-1}\supset\cdots\supset L_s=0, \tag{1.1}$$

$$K(n, \mu_j, \boldsymbol{m})=\bar{L}_{-1}\supset\bar{L}_0\supset\cdots\supset\bar{L}_{s'}=0, \tag{1.2}$$

where $L_{-1}=V\oplus R$, $L_0=R$, $L_i=\{x\in L_{i-1} \mid [x, L_{-1}]\subset L_{i-1}\}$, $i\geqslant 1$; $\bar{L}_0=R$, $\bar{L}_i=\{x\in \bar{L}_{i-1} \mid [x, L_{-1}]\subset L_{i-1}\}$, $i\geqslant 1$; $s=\sum_{i=1}^{2r} 2^{m_i}+2^{m_{n+1}}-(n-2)$, $s'=\sum_{i=1}^{n} 2^{m_i}-(n+1)$.

Using Corollary 2 and imitating the proof of Theorem 3.1 of [1], we have

Theorem 4. *Let $r\equiv 1(2)$ and $r>1$. Then filtrations (1.1) and (1.2) are both intrinsically determined.*

Using Theorem 4 and imitating the correponding proofs of [1] and [5] we have

Theorem 5. *Let $r\equiv 1(2)$ and $r>1$. Then $K(n, \mu_j, \boldsymbol{m})$ and $K(n', \mu_j', \boldsymbol{m}')$ are isomorphic if and only if $n=n'$, $m_n=m'_{n'}$ and $\{\{m_1, m_{1'}\}, \cdots, \{m_r, m_{r'}\}\}=\{\{m'_1, m'_{1'}\}, \cdots, \{m'_r, m'_{r'}\}\}$.*

Remark. If $r=1$, then Theorem 2 becomes invalid. In fact, if $\boldsymbol{m}=\mathbf{1}$, then

$$[1, x^{\varepsilon_1}, x^{\varepsilon_{1'}}, x^{n}, x^{\varepsilon_1+\varepsilon_{1'}}, x^{\varepsilon_n+\varepsilon_1}, x^{\varepsilon_n+\varepsilon_{1'}}, x^{\varepsilon_n+\varepsilon_1+\varepsilon_{1'}}]$$

consists of basis of $K(3, \mu_j, \mathbf{1})$. It is easy to see that $I(\operatorname{ad} 1)=n+1$. Now Theorem 2 is not correct.

§ 2. Generators and Derivation Algebra

Let $\Delta=\{x^{\varepsilon_i+\varepsilon_j}, i, j=1, 2, \cdots, n; x^{2^s\varepsilon_i}, 0\leqslant s<m_i, i=1, \cdots, n\}$.

Theorem 6. *$K(n, \mu_j, \boldsymbol{m})$ is generated by Δ.*

Proof We only prove this theorem in the case of $r\equiv 1(2)$. When $r\equiv 0(2)$, the proof is essentially the same.

Let Y be the subalgebra generated by Δ. Then $1=[x^{\varepsilon_i}, x^{\varepsilon_{i'}}]\in Y$, $x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_i}, x^{\varepsilon_n+\varepsilon_{i'}}]\in Y$, $x^{\varepsilon_n+\varepsilon_i+\varepsilon_j}=[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{\varepsilon_i+\varepsilon_j}]\in Y$, $j<n$.

(1) $x^{k\varepsilon_i}\in Y$, $k\varepsilon_i\leqslant \tau_i$, $i=1, \cdots, 2r$.

We use induction on k. Let $k=2^jh$, where $h\equiv 1(2)$. We can suppose that $h>2$.

(a) $j=0$. Then $k\equiv 1(2)$. Hence $x^{k\varepsilon_i}=[x^{\varepsilon_n+\varepsilon_i}, x^{(k-1)\varepsilon_i}]\in Y$.

(b) $j>0$. By hypothesis of induction, $x^{2^j\varepsilon_i}$, $x^{(2^j+1)\varepsilon_i}\in Y$. Then $x^{(2^j+1)\varepsilon_i+\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{2^j\varepsilon_i}]\in Y$, $x^{\varepsilon_n+2^j\varepsilon_i}=[x^{(2^j+1)\varepsilon_i}, x^{\varepsilon_n+\varepsilon_{i'}}]-\mu_{i'}x^{(2^j+1)\varepsilon_i+\varepsilon_{i'}}\in Y$. Hence $x^{k\varepsilon_i}=[x^{\varepsilon_n+2^j\varepsilon_i}, x^{(k-2^j)\varepsilon_i}]\in Y$.

(2) $x^{k\varepsilon_n}\in Y$, $k\varepsilon_n\leqslant \tau_n$.

We use induction on $k=2^jh$, where $h\equiv 1(2)$. If $j=0$, then $x^{(k-1)\varepsilon_n+\varepsilon_{i'}}=[x^{(k-1)\varepsilon_n}, x^{\varepsilon_n+\varepsilon_{i'}}]\in Y$. Since $x^{(k-1)\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}=[x^{(k-1)\varepsilon_n}, x^{\varepsilon_n+\varepsilon_{i'}+\varepsilon_i}]\in Y$, $x^{k\varepsilon_n}=[x^{\varepsilon_n+\varepsilon_i}, x^{(k-1)\varepsilon_n+\varepsilon_{i'}}]-\mu_{i_1}x^{(k-1)\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}\in Y$.

If $j>0$, then $x^{k\varepsilon_n}=[x^{(2^j+1)\varepsilon_n}, x^{(k-2^j)\varepsilon_n}]\in Y$.

(3) $x^{k\varepsilon_i+l\varepsilon_{i'}}\in Y$, $k\varepsilon_i\leqslant \tau_i$, $l\varepsilon_{i'}\leqslant \tau_{i'}$.

(a) $k\varepsilon_i<\tau_i$, $l\varepsilon_{i'}<\tau_{i'}$. $x^{k\varepsilon_i+l\varepsilon_{i'}}=[x^{(k+1)\varepsilon_i}, x^{(l+1)\varepsilon_{i'}}]\in Y$.

(b) $k\varepsilon_i=\tau_i$, $l\varepsilon_{i'}<\tau_{i'}$.

(b)-(i). $\tau_i>\varepsilon_i$. Then $x^{2\varepsilon_i+\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_{i'}}, x^{2\varepsilon_i}]+x^{\varepsilon_n+\varepsilon_i}\in Y$. If $l\equiv 0(2)$, then $x^{\tau_i+l\varepsilon_{i'}}=[x^{(\tau_i-\varepsilon_i)+l\varepsilon_{i'}}, x^{2\varepsilon_i+\varepsilon_{i'}}]\in Y$. If $l\equiv 1(2)$, then $(l+1)\varepsilon_{i'}<\tau_{i'}$. Hence $x^{\tau_i+l\varepsilon_{i'}}=[x^{(\tau_i-\varepsilon_i)+(l+1)\varepsilon_{i'}}, x^{2\varepsilon_i}]\in Y$.

(b)-(ii). $\tau_i=\varepsilon_i$. If $l\equiv1(2)$, then $x^{\varepsilon_i+l\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{(l-1)\varepsilon_{i'}}]\in Y$. If $l\equiv0(2)$, then $x^{\varepsilon_i+(l+1)\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{l\varepsilon_{i'}}]\in Y$. Hence $x^{\varepsilon_i+l\varepsilon_{i'}}=[x^{\varepsilon_i}, x^{\varepsilon_i+(l+1)\varepsilon_{i'}}]\in Y$.

(c) $k\varepsilon_i=\tau_i$, $l\varepsilon_{i'}=\tau_{i'}$.

$$x^{\varepsilon_n+\tau_{i'}}=[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{\tau_{i'}}]\in Y. \tag{i}$$

Then $x^{\varepsilon_n+\varepsilon_i+\tau_{i'}}=[x^{\varepsilon_n+\tau_{i'}}, x^{m-\varepsilon_i}]\in Y$. Hence $x^{\tau_i+\tau_{i'}}=[x^{\tau_i-\varepsilon_i}, x^{\varepsilon_n+\varepsilon_i+\tau_{i'}}]\in Y$.

(4) $x^{k\varepsilon_n+\tau_i+\tau_{i'}}\in Y$, $k\varepsilon_n\leqslant\tau_n$, $k=1, \cdots, 2r$.

$$\text{Since}\quad x^{k\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}=[x^{\varepsilon_n+\varepsilon_{i'}}, x^{k\varepsilon_n}]\in Y,\ x^{k\varepsilon_n+\tau_i}=[x^{k\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{\tau_i}]\in Y. \tag{ii}$$

If $k\equiv1(2)$, by identity (i), $x^{k\varepsilon_n+\tau_i+\tau_{i'}}=[x^{k\varepsilon_n+\tau_i}, x^{\varepsilon_n+\tau_{i'}}]\in Y$.

If $k\equiv0(2)$, then $(k+1)\varepsilon_n\leqslant\tau_n$ and $k+1\equiv1(2)$. Hence

$$x^{(k+1)\varepsilon_n+\tau_i+(\tau_{i'}-\varepsilon_i)}=[x^{(k+1)\varepsilon_n+\tau_i+\tau_{i'}}, x^{\varepsilon_i}]\in Y, \tag{iii}$$

Then $[x^{(k+1)\varepsilon_n+\tau_{i'}}, x^{\tau_i}]=\mu_{i'}x^{k\varepsilon_n+\tau_i+\tau_{i'}}+x^{(k+1)\varepsilon_n+(\tau_i-\varepsilon_i)+(\tau_{i'}-\varepsilon_{i'})}\in Y$, $[x^{(k+1)\varepsilon_n+\tau_i+(\tau_{i'}-\varepsilon_{i'})}, x^{\varepsilon_{i'}}]=\mu_{i}x^{k\varepsilon_n+\tau_i+\tau_{i'}}+x^{(k+1)\varepsilon_n+(\tau_i-\varepsilon_i)+(\tau_{i'}-\varepsilon_{i'})}\in Y$. We add the right sides of above two identites, then $x^{k\varepsilon_n+\tau_i+\tau_{i'}}\in Y$.

(5) $x^{\varepsilon_i+\varepsilon_{i'}+\varepsilon_{j'}}\in Y$, $i\neq j$, $i, j\leqslant2r$.

If $\mu_j\neq0$, then $x^{\varepsilon_i+\varepsilon_{i'}+\varepsilon_{j'}}=\frac{1}{\mu_j}[x^{\varepsilon_n+\varepsilon_i+\varepsilon_{i'}}, x^{\varepsilon_{j'}}]\in Y$. If $\mu_j=0$, then $x^{\varepsilon_j+\varepsilon_{j'}+\varepsilon_{i'}}=[x^{\varepsilon_n-\varepsilon_{i'}+\varepsilon_{j'}}, x^{\varepsilon_j}]+x^{\varepsilon_n+\varepsilon_{i'}}\in Y$. Hence $x^{\varepsilon_i-\varepsilon_{i'}+\varepsilon_{j'}}=[x^{\varepsilon_j+\varepsilon_{j'}+\varepsilon_{i'}}, x^{\varepsilon_i+\varepsilon_{j}}]\in Y$.

(6) Let $\sigma=x^{\varepsilon_n+\tau_i+\tau_{i'}+\tau_j+(\tau_j-\varepsilon_{j'})}$, $\delta=x^{\varepsilon_n+\tau_i+\tau_{i'}+(\tau_j-\varepsilon_j)+\tau_{j'}}$. Then $\sigma, \delta\in Y$.

By (ii) and (5), $x^{\varepsilon_n+\tau_j+\varepsilon_{i'}}=[x^{\varepsilon_n+\tau_j}, x^{\varepsilon_j+\varepsilon_{j'}+\varepsilon_{i'}}]\in Y$. If $\mu_{j'}\neq0$, by (iii), $x^{\varepsilon_n+\tau_i+\tau_{i'}+\tau_j}=\frac{1}{\mu_{j'}}[x^{\varepsilon_n+\tau_i+(\tau_{i'}-\varepsilon_{i'})}, x^{\varepsilon_n+\tau_j+\varepsilon_{i'}}]\in Y$. By (3), $\sigma=[x^{\varepsilon_n+\tau_i+\tau_{i'}+\tau_j}, x^{\varepsilon_j+\tau_{j'}}]\in Y$. $\delta=[[\sigma, x^{\varepsilon_{i'}+\varepsilon_{j'}}], x^{\varepsilon_i+\varepsilon_j}]\in Y$.

If $\mu_j\neq0$, symmetrically, we can get $x^{\varepsilon_n+\tau_i+\tau_{i'}+\tau_{j'}}\in Y$ and $\delta, \sigma\in Y$.

(7) Let $\eta_{ij}(k)=x^{k\varepsilon_n-\tau_i+\tau_{i'}+\tau_j-\tau_{j'}}$. Then $\eta_{ij}(1), \eta_{ij}(0)\in Y$.

Since $r\geqslant3$, there is l such that $1\leqslant l\leqslant n$ and $l\in\{i, j, i', j'\}$. If $\mu_l\neq0$, by (6), $\eta_{ij}(1)x^{\varepsilon_l}=\frac{1}{\mu_l}[\delta, x^{\varepsilon_n+\varepsilon_j+\varepsilon_l}]\in Y$. Hence $\eta_{ij}(1)=[\eta_{ij}(1)x^{\varepsilon_l}, x^{\varepsilon_{l'}}]-\mu_l[[\eta_{ij}(1)x^{\varepsilon_l}, x^{\varepsilon_n+\varepsilon_l}], 1]\in Y$.

If $\mu_l=0$, then $\mu_{l'}\neq0$. Symmetrically, we can get $\eta_{ij}(1)\in Y$. If $\mu_{l'}\neq0$, then $\eta_{ij}(0)x^{\varepsilon_l}=\frac{1}{\mu_{l'}}[\eta_{ij}(1), x^{\varepsilon_l}]\in Y$. Hence $\eta_{ij}(0)=[\eta_{ij}(0)x^{\varepsilon_l}, x^{\varepsilon_{l'}}]\in Y$. If $\mu_l\neq0$, symmetrically, we have $\eta_{ij}(0)\in Y$.

(8) Let $Q_h(k)=x^{k\varepsilon_n+\tau_1+\tau_{1'}+\cdots+\tau_h+\tau_{h'}}$. Then $Q_h(k)\in Y$.

We use induction on h. If $h\equiv1(2)$, by (4), $Q_h(k)=\frac{1}{h}[x^{\varepsilon_n+\tau_h+\tau_{h'}}, Q_{h-1}(k)]\in Y$. Suppose $h\equiv0(2)$, then $h+1\leqslant r$. If $k\varepsilon_n<\tau_n$, by (7), $Q_{h+1}(k)=[Q_{h-1}(k+1), \eta_{h\,h+1}(0)]\in Y$. If $k\varepsilon_n=\tau_n$, by (7), $Q_{h+1}(k)=[Q_{h-1}(k), \eta_{h\,h+1}(1)]Y$.

Using following identites

$$[Q_h(k)x^{t\varepsilon_{h+1}+l\varepsilon_{(h+1)'}}, x^{\varepsilon_{(h+1)'}}]=Q_h(k-1)y_1+Q_h(k)x^{(t-1)\varepsilon_{h+1}+l_{(h+1)'}}$$

$$[Q_h(k)x^{t\varepsilon_{h+1}+l\varepsilon_{(h+1)'}}, x^{\varepsilon_{h+1}}]=Q_h(k-1)y_2-Q_h(k)x^{t\varepsilon_{h+1}+(l-1)\varepsilon_{(h+1)'}}, \qquad \text{(iv)}$$

where

$$y_1=(l+1)\mu_{h+1}x^{t\varepsilon_{h+1}+(l+1)\varepsilon_{(h+1)'}},$$

$$y_2=(t+1)\mu_{(h+1)'}x^{(t+1)\varepsilon_{h+1}+l\varepsilon_{(h+1)'}},$$

using induction on $d=(2^{m_{h+1}}-1)+(2^{m_{(h+1)'}}-1)-(t+l)$, we have $Q_h(k)\in Y$.

(9) $x^a\in Y$, $0\leqslant a\leqslant\tau$.

By (8), $Q_r(a_n)\in Y$. Using the identites (iv) and induction on

$$d=\sum_{i=1}^{r}((2^{m_i}-1)+(2^{m_{i'}}-1))-\sum_{i=1}^{r}(a_i+a_{i'}),$$

we can prove that $x^a\in Y$.

Theorem 7. *Let $r\equiv 1(2)$ and $r>1$. Then* $\operatorname{Der} K(n, \mu_j, \boldsymbol{m})=adK(n, \mu_j, \boldsymbol{m})\oplus M$, *where* $M=\langle D_i^{2^{k_i}} \mid i=1, \cdots, n, 1\leqslant k_i\leqslant m_i-1\rangle$.

Proof $\forall D\in \operatorname{Der} K(n, \mu_j, \boldsymbol{m})$, by (i) and (iii) in the proof of Theorem 4.1 of paper [2] (now $G=0$ in [2]), we know that there exists $g\in K(n, \mu_j, \boldsymbol{m})$ such that $D^{(3)}=D$-adg satisfies $D^{(3)}(x^{\varepsilon_n})=0$, $D^{(3)}(x^{\varepsilon_i})=D^{(3)}(x^{\varepsilon_i+\varepsilon_{i'}})=0$, $i=1, \cdots, 2r$.

We affirm that $D^{(3)}(x^{\varepsilon_i+\varepsilon_j})=0$, $1\leqslant i, j\leqslant 2r$, $j\neq i, i'$. In fact, applying $D^{(3)}$ to the identities $[x^{\varepsilon_i+\varepsilon_j}, 1]=0$ and

$$[x^{\varepsilon_i+\varepsilon_j}, x^{\varepsilon_l}]=\begin{cases}0, & \text{if } l\neq i', j',\\ x^{\varepsilon_i} \text{ or } x^{\varepsilon_j}, & \text{if } s=j' \text{ or } i',\end{cases}$$

by Lemma 4.2 of [2], we have $D^{(3)}(x^{\varepsilon_i+\varepsilon_j})=\alpha 1$, $\alpha\in F$. Applying $D^{(3)}$ to the indentity $[x^{\varepsilon_i+\varepsilon_j}, x^{\varepsilon_j+\varepsilon_{j'}}]=x^{\varepsilon_i+\varepsilon_j}$, we have $[\alpha 1, x^{\varepsilon_j+\varepsilon_{j'}}]=\alpha 1$. Then $\alpha 1=0$ and $D^{(3)}(x^{\varepsilon_i+\varepsilon_j})=0$.

Since $[x^{\varepsilon_n+\varepsilon_i}, 1]=x^{\varepsilon_i}$, $[x^{\varepsilon_n+\varepsilon_i}, x^{\varepsilon_j}]=\mu_{j'}(1+\delta_{ij})x^{\varepsilon_i+\varepsilon_j}+\delta_{ij'}x^{\varepsilon_n}$, applying $D^{(3)}$, by Lemma 4.2 of [2], we have $D^{(3)}(x^{\varepsilon_n+\varepsilon_i})=\alpha 1$, $\alpha\in F$. Applying $D^{(3)}$ to the identity $[x^{\varepsilon_n+\varepsilon_i}, x^{\varepsilon_i+\varepsilon_{i'}}]=x^{\varepsilon_n+\varepsilon_i}$, we have $D^{(3)}(x^{\varepsilon_n+\varepsilon_i})=0$.

Using Theorem 6, imitating the proof of part (iv) of Theorem 4.1 in [2], we have $D\in \operatorname{ad}K(n, \mu_j, \boldsymbol{m})\oplus M$.

Similarly, using Theorem 6, we can prove

Theorem 8. *Let $r\equiv 0(2)$ and $r\neq 0$. Then* $\operatorname{Der} K(n, \mu_j,\boldsymbol{m})=ad\ K(n, \mu_j, \boldsymbol{m})\oplus adx^{\tau}|_{K(n, \mu_j, \boldsymbol{m})}\oplus M$.

Using Theorem 3 and Theorem 8, imitating the proof of Theorem 2.3 of [1], we have

Theorem 9. *Let $r\equiv 0(2)$ and $r>0$. Then*

(I) $I(\operatorname{Der} K(n, \mu_j, \boldsymbol{m}))=n$. (II) $\forall D\in \operatorname{Der} K(n, \mu_j, \boldsymbol{m})$, $I(D)=n$ *if and only if* $0\neq D\in\langle \operatorname{ad} x^{\tau}\rangle$.

By Theorem 9, $\langle \operatorname{ad} x^{\tau}\rangle$ is an invariant subspace of $\operatorname{Der} K(n, \mu_j, \boldsymbol{m})$. Let $R'=\langle x^a \mid \deg x^a\geqslant 2, a\neq\tau\rangle$.

Corollary 3. *Let $r\equiv 0(2)$ and $r>0$. Then R' is an invariant subalgebra of*

$K(n, \mu_j, \boldsymbol{m})$.

Proof Let σ be an automorphism of $K(n, \mu_j, \boldsymbol{m})$. Then $D \mapsto \sigma D \sigma^{-1}$, $\forall D \in$ Der $K(n, \mu_j, \boldsymbol{m})$, is an automorphism of Der $K(n, \mu_j, \boldsymbol{m})$. Hence $\sigma\langle \text{ad } x^\tau \rangle \sigma^{-1} = \langle \text{ad } x^\tau \rangle$. Since $R' = \{y \in K(n, \mu_j, \boldsymbol{m}) \mid \langle \text{ad } x^\tau \rangle (y) = 0\}$, $\langle \text{ad } x^\tau \rangle (\sigma R') = \sigma \langle \text{ad} x^\tau \rangle \sigma^{-1} (\sigma R') = 0$. Therefore $\sigma(R') \subset R'$ and R' is an invariant subalgebra.

When $\tau \equiv 0(2)$ and $\tau > 0$, we also have the filtrations

$$K(n, \mu_j, \boldsymbol{m}) = L'_{-2} \supset L'_{-1} \supset \cdots \supset L'_s = 0, \tag{2.1}$$

$$K(n, \mu_j, \boldsymbol{m}) = \overline{L}'_{-1} \supset \overline{L}'_0 \supset \cdots \supset \overline{L}'_r = 0, \tag{2.2}$$

where $L'_{-1} = V \oplus R'$, $L'_0 = R'$, $L'_i = \{x \in L'_{i-1} \mid [x, L'_{-1}] \subset L'_{i-1}\}$, $i \geq 1$; $\overline{L}'_0 = R'$, $\overline{L}'_i = \{x \in \overline{L}'_{i-1} \mid [x, \overline{L}'_{-1}] \subset L'_{i-1}\}$, $i \geq 1$.

Thus the results of Throrem 4 and Theorem 5 hold for $\tau \equiv 0(2)$ and $\tau > 1$.

Referances

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