Article ID: 1000 - 5641(2003)01 - 0021 - 04

The Derivation Algebra of $gl_\infty(\mathbf{C})$

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Abstract: In this paper, it is proved that the derivation algebra of the Lie algebra of infinite matrices with finite nonzero entries is isomorphic to the quotient of the Lie algebra of infinite matrices with finite nonzero entries in each row and in each column nodulo its center. It is also proved that this quotient is a complete Lie algebra.

Key words: derivation; derivation algebra; Lie algebra
CLC number: O152 **Document code**: A

The infinite matrix Lie algebra $gl_\infty(\mathbf{C})$ has been discussed by many authors. Let's recall the definition:

$$gl_\infty(\mathbf{C}) = \langle E_{i,j} \mid i, j \in \mathbf{Z} \rangle,$$

which is a Lie algebra over the complex number field $\mathbf{C}$, with the bracket given by

$$[E_{i,j},\ E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{l,i} E_{k,j}.$$

In this paper, we will determine the derivation algebra Der $(gl_\infty(\mathbf{C}))$, and prove that Der $(gl_\infty(\mathbf{C}))$ is complete.

At first, we will give some properties of derivations of $gl_\infty(\mathbf{C})$. Given any $D \in \mathrm{Der}(gl_\infty(\mathbf{C}))$, assume that

$$D(E_{i,j}) = \sum_{s,t \in \mathbf{Z}} C_{s,t}^{i,j} E_{s,t}$$

for all $i, j \in \mathbf{Z}$, where $C_{s,t}^{i,j} \in \mathbf{C}$.

Since $D \in \mathrm{Der}(gl_\infty(\mathbf{C}))$, we have

$$[D(E_{i,j}), E_{k,l}] + [E_{i,j}, D(E_{k,l})] = D([E_{i,j},\ E_{k,l}]),$$

that is

$$\sum_{s,t \in \mathbf{Z}} (\delta_{j,k} C_{s,t}^{i,l} - \delta_{l,i} C_{s,t}^{k,j}) E_{s,t} = \sum_{s \in \mathbf{Z}} C_{s,k}^{i,j} E_{s,l} - \sum_{s \in \mathbf{Z}} C_{l,s}^{i,j} E_{k,s} + \sum_{s \in \mathbf{Z}} C_{j,s}^{k,l} E_{i,s} - \sum_{s \in \mathbf{Z}} C_{s,i}^{k,l} E_{s,j}.$$

If $i \neq k,\ l$ and $j \neq k, l$, then

$$\sum_{s \in \mathbf{Z}} C_{s,k}^{i,j} E_{s,l} - \sum_{s \in \mathbf{Z}} C_{l,s}^{i,j} E_{k,s} + \sum_{s \in \mathbf{Z}} C_{j,s}^{k,l} E_{i,s} - \sum_{s \in \mathbf{Z}} C_{s,i}^{k,l} E_{s,j} = 0,$$

and the coefficient of $E_{k,l}$ is $C_{k,k}^{i,j} - C_{l,l}^{i,j} = 0$.

收稿日期:2001 - 04
基金项目:国家自然科学基金(10271047),教育部博士点基金和上海市重点学科建设项目
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Consider the coefficient of $E_{s,i}$, we have

$$\sum_{s\in \mathbf{Z}}(\delta_{j,k}C_{s,i}^{i,l} - \delta_{l,i}C_{s,i}^{k,j})E_{s,i} = \sum_{s\in \mathbf{Z}}\delta_{l,i}C_{s,k}^{i,j}E_{s,i} - C_{l,i}^{i,j}E_{k,i} + C_{j,i}^{k,l}E_{i,i} - \sum_{s\in \mathbf{Z}}\delta_{i,j}C_{s,i}^{k,l}E_{s,i},$$

that is

$$\sum_{s\in \mathbf{Z}}(\delta_{j,k}C_{s,i}^{i,l} - \delta_{l,i}C_{s,i}^{k,j} - \delta_{l,i}C_{s,k}^{i,j} + \delta_{i,j}C_{s,i}^{k,l})E_{s,i} = - C_{l,i}^{i,j}E_{k,i} + C_{j,i}^{k,l}E_{i,i}.$$

When $j = k$; $i\neq j, l$, $\sum_{s\in \mathbf{Z}}C_{s,i}^{i,l}E_{s,i} = - C_{l,i}^{i,j}E_{j,i} + C_{j,i}^{j,l}E_{i.i}$, then comparing the coefficients of $E_{i,i}$ gives $C_{i,i}^{i,l} = C_{j,i}^{j,l}$, here l can be j. And by the same way we can get that $C_{i,i}^{l,i} = C_{i,j}^{l,j}$ for all $i\neq j$, l.

If $i = j$, $j\neq k$, l, then comparing the coefficient of $E_{k,i}$, we have $-C_{l,k}^{i,i} - C_{k,i}^{k,l} = 0$. Put $k = l$, then $C_{l,i}^{i,i} + C_{l,i}^{l,l} = 0$.

When $i\neq \{j, k, l\}$ and $j\neq k$, $C_{l,i}^{i,j} = C_{j,i}^{k,l} = 0$, here k or j can be l.

If $i = l$ and $i\neq j$, k; $j\neq k$, then

$$\sum_{s\in \mathbf{Z}}(- C_{s,i}^{k,j} - C_{s,k}^{i,i})E_{s,i} = - C_{i,i}^{i,j}E_{k,i} + C_{j,i}^{k,i}E_{i,i}.$$

When $s = k$, $C_{k,i}^{k,j} + C_{k,k}^{i,j} = C_{i,i}^{i,j}$, by $C_{i,i}^{i,l} = C_{j,i}^{j,l}$, hence $C_{k,k}^{i,j} = 0$, and by the same way, $C_{i,j}^{k,k} = 0$.

Let $i = l$ and $i\neq j, k$, then the coefficient of $E_{k,j}$,

$$\delta_{k,j}C_{k,j}^{i,i} - C_{k,j}^{k,j} = - C_{i,j}^{i,j} - C_{k,i}^{k,i},$$

so $C_{i,j}^{i,j} = C_{k,j}^{k,j} - C_{k,i}^{k,i}$, $C_{i,j}^{i,j} = - C_{j,i}^{j,i}$. Let k be j, then $C_{j,j}^{j,j} = 0$.

From above discussion, we have the properties of D:

Proposition 1 For all $i, j, s, t\in \mathbf{Z}$, we have

(1) $C_{s,t}^{i,j} = 0$, if $i\neq s$ and $j\neq t$;

(2) $C_{i,j}^{i,j} = C_{0,j}^{0,j} - C_{0,i}^{0,i}$;

(3) $C_{i,i}^{i,l} = C_{j,i}^{j,l}$, if $l\neq i$;

(4) $C_{i,i}^{l,i} = C_{i,j}^{l,j}$, if $l\neq i$;

(5) $- C_{i,l}^{l,l} = C_{l,l}^{l,i}$.

So, we can say that the action of D is

$$D(E_{i,j}) = \sum_{s\in \mathbf{Z},\, s\neq i} C_{s,j}^{i,j}E_{s,j} + \sum_{t\in \mathbf{Z},\, t\neq j} C_{i,t}^{i,j}E_{i,t} + C_{i,j}^{i,j}E_{i,j},$$

and obviously, thre are only finite number of $s\in \mathbf{Z}$ such that $C_{s,j}^{i,j}\neq 0$; and only finite number of $t\in \mathbf{Z}$ such that $C_{i,t}^{i,j}\neq 0$.

Now we will define an algebra. Suppose $A = \sum_{i,j\in \mathbf{Z}}a_{i,j}E_{i,j}$, $a_{i,j}\in \mathbf{C}$. *If for any* i$\in \mathbf{Z}$, there are only finite number of $j\in \mathbf{Z}$ such that $a_{i,j}\neq 0$, and for any $j\in \mathbf{Z}$, there are only finite number of $i\in \mathbf{Z}$ such that $a_{i,j}\neq 0$, we say that A is locally finite. Let $ml(\mathbf{C}) = \{A = \sum_{i,j\in \mathbf{Z}}a_{i,j}E_{i,j} \mid a_{i,j}\in \mathbf{C},\ A \text{ is locally finite}\}$. We can define multiplication and addition on $ml(\mathbf{C})$ as following:

$$\sum_{i,j\in \mathbf{Z}} a_{i,j}E_{i,j} * \sum_{k,l\in \mathbf{Z}} b_{k,l}E_{k,l} = \sum_{i,l\in \mathbf{Z}} \Big(\sum_{j\in \mathbf{Z}} a_{i,j}b_{j,l}\Big)E_{i,l},$$

$$\sum_{i,j\in\mathbf{Z}} a_{i,j}E_{i,j} + \sum_{k,l\in\mathbf{Z}} b_{k,l}E_{k,l} = \sum_{i,j\in\mathbf{Z}} (a_{i,j} + b_{i,j})E_{i,j}.$$

It is easy to check that $ml(\mathbf{C})$ is an associative algebra. Usually, we write $A * B$ as AB. Naturally, it is a Lie algebra with the bracket $[A,B] = AB - BA$ for all $A, B \in ml(\mathbf{C})$. Obviously, $gl_\infty(\mathbf{C})$ is its associative subalgebra and Lie subalgebra. It is easy to check that $gl_\infty(\mathbf{C})$ is also a Lie ideal of $ml(\mathbf{C})$. Thus there exists a Lie homomorphism Φ from $ml(\mathbf{C})$ to Der $(gl_\infty(\mathbf{C}))$ defined as following:

$$\Phi(A) = \mathrm{ad}\, A|_{gl_\infty(\mathbf{C})}.$$

Suppose D is a derivation of $gl_\infty(\mathbf{C})$ defined as above. Let

$$A = \Big(\sum_{\substack{s,t\in\mathbf{Z}\\ s\neq t}} C_{s,0}^{t,0} E_{s,t}\Big)$$

By the properties of D, A is an element of $ml(\mathbf{C})$. So for any $E_{i,i} \in gl_\infty(\mathbf{C})$,

$$(D - \Phi(A))(E_{i,j}) = 0,$$

that is

$$(D - \Phi(A))(H) = 0, \tag{1}$$

where $H = \langle E_{i,i} \mid i \in \mathbf{Z}\rangle$. And such A is unique, otherwise we can assume that $A' = A + B$ is scuh one, and B is not 0, then

$$(D - \Phi(A) - \Phi(B))(H) = -\Phi(B)(H) = [H,B] = 0.$$

It is easy to check that this is impossible. If we set

$$A = \Big(\sum_{\substack{s,t\in\mathbf{Z}\\ s\neq t}} C_{s,0}^{t,0} E_{s,t}\Big) + \Big(\sum_{s\in\mathbf{Z}} C_{s,0}^{s,0} E_{s,s}\Big),$$

then by the properties of $C_{s,t}^{i,j}$ we get

$$(D - \Phi(A))(gl_\infty(\mathbf{C})) = 0. \tag{2}$$

From (1) and (2) we know Φ is an epimorphism and the kernel is contained in

$$H^* =: \Big\{\sum_{i\in\mathbf{Z}} a_i E_{ii} \mid a_i \in \mathbf{C}\Big\}.$$

Define $g_m = \{\sum_{i,j=m,m+1} a_{i,j}E_{i,j} \mid a_{i,j} \in \mathbf{C}\}$. If $A = \sum_{i\in\mathbf{Z}} a_i E_{ii}$ is in the kernel, then A commutes with each g_m, but $\Phi(A)(g_m) = [a_m E_{m,m} + a_{m+1}E_{m+1,m+1}, g_m] = 0$, so $a_m = a_{m+1}$, and this equality holds for each $m \in \mathbf{Z}$. Therefore $A = a_0 I$, and Ker $(\Phi) = \mathbf{C}I$. Denote $sml(\mathbf{C}) := ml(\mathbf{C})/\mathbf{C}I$, which is the quotient Lie algebra of $ml(\mathbf{C})$ modulo its center.

Theorem 2 The derivation algebra Der $(gl_\infty(\mathbf{C}))$ is isomorphic to Lie algebra $sml(\mathbf{C})$.

In fact, $gl_\infty(\mathbf{C})$ can be regarded as a subalgebra of $sml(\mathbf{C})$.

In the rest we will show that Der $(gl_\infty(\mathbf{C}))$ is complete. If D' is a derivation of $sml(\mathbf{C})$, for any $A = \sum_{i,j\in\mathbf{Z}} a_{i,j}E_{i,j} + \mathbf{C}I \in sml(\mathbf{C})$ and $E_{k,l} \in gl_\infty(\mathbf{C})$, by the bracket of $sml(\mathbf{C})$,

$$\begin{aligned}[][D'(A), E_{k,l} + \mathbf{C}I] &= D'([A, E_{k,l} + \mathbf{C}I]) - [A, D'(E_{k,l} + \mathbf{C}I)] + \mathbf{C}I \\ &= D'\Big(\sum_{i\in\mathbf{Z}} a_{i,k}E_{i,l} - \sum_{j\in\mathbf{Z}} a_{l,j}E_{k,j} + \mathbf{C}I\Big) \\ &\quad - \sum_{i,j\in\mathbf{Z}} a_{i,j}[E_{i,j}, D'(E_{k,l} + I)] + \mathbf{C}I\end{aligned}$$

$$= \sum_{i,j\in\mathbf{Z}} a_{i,j}(D'([E_{i,j}, E_{k,l}] + \mathbf{C}I) - [E_{i,j}, D'(E_{k,l} + \mathbf{C}I)] + \mathbf{C}I$$
$$= \sum_{i,j\in\mathbf{Z}} a_{i,j}[D'(E_{i,j} + \mathbf{C}I), E_{k,l} + \mathbf{C}I] + \mathbf{C}I$$
$$= [\sum_{i,j\in\mathbf{Z}} a_{i,j} D'(E_{i,j} + \mathbf{C}I), E_{k,l} + \mathbf{C}I] + \mathbf{C}I.$$

So $D'(A) - \sum_{i,j\in\mathbf{Z}} a_{i,j} D'(E_{i,j} + \mathbf{C}I) \mid_{(gl_\infty(\mathbf{C})+\mathbf{C}I)/\mathbf{C}I} = 0$. In the proof of the properites of D, there is no limit about s, t of $C_{s,t}^{i,j}$, so we can show that the action of D' on $E_{k,l} + \mathbf{C}I$ is:

$$D'(E_{k,l} + \mathbf{C}I) = \sum_{s\in\mathbf{Z},\, s\neq k} C'^{k,l}_{s,l} E_{s,l} + \sum_{t\in\mathbf{Z},\, t\neq l} C'^{k,l}_{k,t} E_{k,t} + \mathbf{C}I.$$

Since $D'(E_{k,l} + \mathbf{C}I) \in sml(\mathbf{C})$, there are only finite number of $s, t \in \mathbf{Z}$ such that $C_{s,l}^{k,l} \neq 0$, $C_{k,t}^{k,l} \neq 0$, that is $D'(E_{k,l}) \in (gl_\infty(\mathbf{C}) + \mathbf{C}I)/\mathbf{C}I$, and $(gl_\infty(\mathbf{C}) + gl_\infty(\mathbf{C})I)/\mathbf{C}I$ is isomorphic to $gl_\infty(\mathbf{C})$. Thus we can regard D' as a derivation of $gl_\infty(\mathbf{C})$, then there exists unique $B \in sml(\mathbf{C})$ such that $(D' - \mathrm{ad}B)\ (gl_\infty(\mathbf{C})) = 0$. Of course $\mathrm{ad}B \in \mathrm{Der}(sml(\mathbf{C}))$, then

$$(D' - \mathrm{ad}B)(A) = \sum_{i,j\in\mathbf{Z}} a_{i,j}(D' - \mathrm{ad}B)(E_{i,j}) + \mathbf{C}I = \mathbf{C}I,$$

that is $D' = \mathrm{ad}B$ as derivation of $sml(\mathbf{C})$, so $\mathrm{Der}\ (sml(\mathbf{C})) = \mathrm{ad}(sml(C))$. It is clear that $C(sml(\mathbf{C})) = 0$, thus we have proved the following

Theorem 3 Der $(gl_\infty(\mathbf{C}))$ is a complete Lie algebra.

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gl_∞(C)的导子代数

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摘要： 该文证明了只有有限个非零元的无限矩阵构成的李代数的导子代数同构于每行每列都有限个非零元的无限矩阵构成的李代数模去其中心所成的商。同时证明这个商代数是完备李代数。

关键词： 导子； 导子代数； 李代数

中图分类号： O152 **文献标识码**:A