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Graded Groups and Lie Superalgebras

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Abstract: Let Γ be an abelian group. In this paper, the concept Γ-graded group is introduced. Since a Lie superalgebra can be associated with a $\boldsymbol{Z}_2$-graded group, properties of $\boldsymbol{Z}_2$-graded groups are discussed and $\boldsymbol{Z}_2$-graded structures of some special classes of groups are determined. Finally, structure of Lie superalgebra associated with dihedral group is discussed.

Key words: graded group; Lie superalgebra; $\boldsymbol{Z}_2$-graded group
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Let G be a group, $F[G]$ the group algebra of G over a field F. Then $F[G]$ is an associative algebra over F. We can define a Lie operation on $F[G]$ via $[x, y] := xy - yx$ then with this operation $F[G]$ becomes a Lie algebra. Usually, one denotes it by $F[G]^-$. On the other hand, we know that over a superalgebra $\mathscr{A}$ a Lie superalgebra structure can be defined via $[x, y] := xy - (-1)^{\deg(x)\deg(y)} yx$. This Lie superalgebra is called the Lie superalgebra associated with the superalgebra $\mathscr{A}$ (cf. [4]) and is denoted as $\mathscr{A}^L$. Then given a group G, can the group algebra $F[G]$ be defined into a (non - trival) superalgebra? If $F[G]$ is a superalgebra, $F[G]$ can become into a Lie superalgebra. But unfortunately, this case doesn't always happen. In this paper, we introduce the concept "Γ-gradation" over a group G, where Γ is an abelian group. It can be shown that if G is a $\boldsymbol{Z}_2$-graded group then $F[G]$ can be defined into a Lie superalgebra.

1 Definition and Properties of Graded Groups

Definition 1.1 Let G be a group, Γ an abelian group. A Γ-gradation of G is a family $(G_\alpha)_{\alpha\in\Gamma}$ of subsets such that

(a) $G_\alpha \neq \varnothing$ for any $\alpha \in \Gamma$.

(b) $G = \bigcup_{\alpha\in\Gamma} G_\alpha$, and $G_\alpha \cap G_\beta = \varnothing$, if $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$.

(c) $G_\alpha G_\beta \subset G_{\alpha+\beta}$, $\forall_\alpha, \beta \in \Gamma$.

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In this case G is called a Γ-graded group.

Example 1 Let G be an abelian group, then G is a G-graded group via a natural way (i.e., put $G_\alpha=\{\alpha\}$, for any $\alpha\in G$).

Example 2 Let G be any subgroup of the permutation group S_n on n elements ($n\geqslant 2$) and $G\not\subset A_n$. Put $G_{\bar{0}}=\{\sigma\mid\sigma$ is an even permutation in $G\}$,

$G_{\bar{1}}=\{\sigma\mid\sigma$ is an odd permutation in $G\}$.

Obviously, G is $\boldsymbol{Z}_2$-graded.

Example 3 If G is a $\boldsymbol{Z}_2$-graded group, $\{G_i\}_{i\in\boldsymbol{Z}}$ is its gradation. Define

$$G_{\bar{0}}=\bigcup_{i\in\boldsymbol{Z}}G_{2i} \text{ and } G_{\bar{1}}=\bigcup_{i\in\boldsymbol{Z}}G_{2i+1}.$$

Then $\{G_\alpha\}_{\alpha\in\boldsymbol{Z}_2}$ is a $\boldsymbol{Z}_2$-gradation of G, which is called the $\boldsymbol{Z}_2$-gradation induced by the $\boldsymbol{Z}$-gradation.

Definition 1.1′ Let G be a group, Γ an abelian group, G is called Γ-graded if there is a map deg: $G\to\Gamma$ such that

(a′) deg is surjective.

(b′) deg is a homomorphism of groups.

If x is an element of G such that deg $(x)=\alpha$, x is called an element of degree α. deg is called *a* Γ-graded mapping (or degree mapping) of G.

Theorem 1.1 Definition 1.1 and 1.1′ are equivalent.

Proof Let G be a group, Γ an abelian group, $\{G_\alpha\}_{\alpha\in\Gamma}$ a Γ-gradation under the Definition 1.1. We define graded mapping deg from G to Γ as following: for $x\in G_\alpha$, deg $(x)=\alpha$. By virtue of Definiton 1.1 (b), this is well-defined. We also want to prove that deg is a homomorphism of groups. For any $x\in G_\alpha$, $y\in G_\beta$, by condition (c), $xy\in G_{\alpha+\beta}$. deg $(x)=\alpha$, deg $(y)=\beta$ and deg $(xy)=\alpha+\beta$, so deg $(xy)=\deg(x)+\deg(y)$. Therefore, deg is a group homomorphism and condition (b′) is satisfied. According to condition (a), condition (a′) holds. Thus, deg is a Γ-graded mapping.

On the other hand, if G is a Γ-graded group under Defintion 1.1′ and deg is the graded mapping. For any $\alpha\in\Gamma$, set $G_\alpha=\deg^{-1}(\{\alpha\})$ (which is called the α component of G). Since deg is a map from G to Γ, condition (b) is clearly true. And because deg is a homomorphism of groups, condition (c) holds. Finally, deg is surjective implies that condition (a) is satisfied.

Example 4 Let G_i be Γ-graded groups and $\deg_i$ be the graded mappings of G_i ($i=1,2,\cdots,n$). Define a map deg grom the product group $G:=G_1\times G_2\times\cdots\times G_n$ to Γ as following

$$\deg(x_1,x_2,\cdots,x_n)=\deg_1(x_1)+\deg_2(x_2)+\cdots+\deg_n(x_n),(x_1,\cdots,x_n)\in G.$$

Then deg is a Γ-graded mapping.

The situation where $\Gamma=\{0\}$ is trival. So in the following discussion, we always assume that $\Gamma\neq\{0\}$.

Remark 1.1 Definition 1.1 (a) is equivalent to Definition 1.1 (a′). deg G is a subgroup of Γ so G is a deg (G)-graded. That is, condition (a) is not essential.

Proposition 1.1 Let G be a Γ-graded group, deg a Γ-graded mapping, then we have

(i) $\deg(x^{-1}) = -\deg(x), \forall x \in G$.

(ii) For $x \in G$ and $\alpha \in \Gamma$, if $o(x)$ and $o(\alpha)$ are their orders in groups respectively, then $o(\deg(x)) \mid o(x)$.

(iii) G_0 is a normal subgroup of G and $G_0 \supset G^{(1)}$, where $G^{(1)}$ is the commutator subgroup of G.

Proof By virtue of Theorem 1.1, Definition 1.1 and 1.1′ are equivalent, so deg is a homomorphism of groups from G to Γ. Thus, (i) is clearly true and $G_0 = \ker(\deg)$, which implies that G_0 is a normal subgroup of G and $G/\ker(\deg) \cong \Gamma$, that is, G/G_0 is abelian and $G_0 \supset G^{(1)}$, (iii) is correct.

Let $o(x) = n$, where x is an element of G, then $x^n = e$. But $\deg(e) = 0$, so we have $n \deg(x) = \deg(x^n) = \deg(e) = 0$, which means $o(\deg(x)) \mid o(x)$, (ii) is proved.

Corollary 1.1 If for a finite group G there exists a $\boldsymbol{Z}_2$-gradation then the number of elements of G with even order $\geqslant |G|/2$.

Proof According to the above theorem, if G has a $\boldsymbol{Z}_2$-gradation, then all elements in $G_{\bar{1}}$ have even orders. Thus, the number of elements with even order $\geqslant |G_{\bar{1}}| = |G|/2$.

Theorem 1.2 Let G be a group, Γ and abelian group. G has a Γ-gradation if and only if there exists a normal subgroup H of G such that $G/H \cong \Gamma$.

Corollary 1.2 Let G be a finite group. G has a $\boldsymbol{Z}_2$-gradation if and only if there is a normal subgroup of G such that its index in G is 2. Particularly, if G has a $\boldsymbol{Z}_2$-gradation, then $|G|$ is even.

Theorem 1.3 Let G be a finite simple group with order great than 2, then G does not possess any $\boldsymbol{Z}_2$-gradation.

Theorem 1.4 Any abelian group G of even order possesses a subgroup with index 2.

Proof Let G be an abelian group of even order. Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the order of G, where $p_1, \cdots, p_s$ are different primes, $p_1 = 2$, $\alpha_i \in \boldsymbol{N}$ for any i. By virtue of Sylow's theorems ([1](I),1.13), there exists a subgroup H_1, such that $|H_1| = 2^{\alpha_1 - 1}$. And if $s > 1$, then for every $i > 1$ there exists a p_i-Sylow's subgroup H_i of G. Put $H = H_1 \cdots H_s$, then H is the desired subgroup.

Corollary 1.4 Abelian finite group possesses a $\boldsymbol{Z}_2$-gradation if and only if it is of even order.

Theorem 1.5 Leg G be a finite group. Then G possesses a $\boldsymbol{Z}_2$-gradation if and only if the index of the commutator subgroup in G is even.

Proof Assume that G possesse a $\boldsymbol{Z}_2$-gradation, by Proposition 1.1(iii), the commutator subgroup $G^{(1)} \subset G_{\bar{0}}$, and $G_{\bar{0}}$ is a normal subgroup. The fact that $[G : G_{\bar{0}}] = 2$ and the third isomophism theorem show that

$$G/G^{(1)}/G_{\bar{0}}/G^{(1)} \cong G/G_{\bar{0}}.$$

Hence, $[G:G^{(1)}] = |G/G^{(1)}| = |G/G_{\bar{0}}| \cdot |G_{\bar{0}}/G^{(1)}| = 2 \cdot |G_{\bar{0}}/G^{(1)}|$, the necessity is proved.

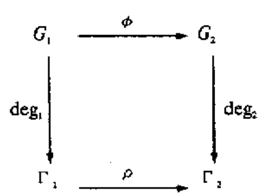
Suppose that $[G:G^{(1)}]$ is even. Then $\bar{G} = G/G^{(1)}$ is an abelian group with even order. By theorem 1.4, $\bar{G}$ has a subgroup $\bar{H}$ of index 2. Let π be the canonical homomorphism from G to $\bar{G}$. Then $H := \pi^{-1}(\bar{H}) \supset G^{(1)}$ is a subgroup of G, so it is a normal subgroup of G. By isomorphism theorems we know $G/H \cong \bar{G}/\bar{H}$. Therefore $|G/H| = |\bar{G}/\bar{H}| = 2$, which implies that $[G:H] = 2$. By corollary 1.2, G possesses a $\boldsymbol{Z}_2$-gradation such that H is its zero component.

Corollary 1.5 The number of $\boldsymbol{Z}_2$-gradations of G is equal to the number of subgroups of G which contain $G^{(1)}$ and have index 2 in G.

Example 5 Let $G = \{e, a, b, ab\}$ be an ableian group such that $a^2 = b^2 = e$. Then there are three subgroups with index 2 in G: $H_1 = \{e, a\}$, $H_2 = \{e, b\}$ and $H_3 = \{e, ab\}$. Hence, by Corollary 1.5, G possesses three $\boldsymbol{Z}_2$-gradations. Their graded mappings are defined by $\deg_1(a) = \bar{0}$, $\deg_1(b) = \bar{1}$, $\deg_2(a) = \bar{1}$, $\deg_2(b) = \bar{0}$, $\deg_3(a) = \bar{1}$ and $\deg_3(b) = \bar{1}$, respectively.

In fact, it is easy to see that above three $\boldsymbol{Z}_2$-gradations are essentialy same. In order to speak it clear, we introduce the following definition:

Definition 1.2 Let G_i be Γ_i-graded groups and $\deg_i$ be Γ_i-graded mappings, $i = 1, 2$. If there are two isomorphisms of groups $\rho: \Gamma_1 \to \Gamma_2$ and $\varphi: G_1 \to G_2$ such that following diagram is commutative then these two graded groups are called to be equivalent and these two gradations are called equivalent too. Denote it by $(G_1, \Gamma_1) \sim (G_2, \Gamma_2)$. In particular, Let G be a group and $\deg_1$ and $\deg_2$ are two $\boldsymbol{Z}_2$-graded mappings on G. Then the two $\boldsymbol{Z}_2$-gradations are equivalent if and only if there exists an automorphism φ of G such that $\varphi(\ker(\deg_1)) = \ker(\deg_2)$, that is, $\varphi|_{\ker(\deg_1)}$ is an isomorphism from $\ker(\deg_1)$ to $\ker(\deg_2)$.



If there is another graded group (G_3, Γ_3) such that $(G_2, \Gamma_2) \sim (G_3, \Gamma_3)$, then $(G_1, \Gamma_1) \sim (G_3, \Gamma_3)$. That is, $\sim$ is an equivalent relation between graded groups.

Example 6 Let $G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$ be an abelian group, that is $G = (a) \times (b)$ is a 2-group of type [2,1] (cf. [2]). Then G possesses three subgroups of index 2: $H_1 = (a)$, $H_2 = (ba)$ and $H_3 = (a^2, b)$. Therefore G has three $\boldsymbol{Z}_2$-gradations. Since H_3 is isomorphic neither to H_1 nor to H_2, the third gradation is not equivalent to the first two. Define an automorphism φ of G by $\varphi(a) = ba$ and $\varphi(b) = b$. We obtain $\varphi(H_1) = H_2$. Therefore, the first two gradations are equivalent. Thus there are only two $\boldsymbol{Z}_2$-gradations which are not equivalent.

2 Z_2-Gradations of some Special Kinds of Groups

In this section, we discuss the existence of $\boldsymbol{Z}_2$-gradations of symmetric groups, alternating groups, dihedral groups and elementary 2-groups. But firstly, let 's discuss cyclic groups.

Theorem 2.1 Let G is a cyclic group. Then we have

(i) If G is infinite, G possesses a unique $\boldsymbol{Z}_2$-gradation.

(ii) If G is of odd order, G does not possess any $\boldsymbol{Z}_2$-gradation.

(iii) If G is of even order, G possesses a unique $\boldsymbol{Z}_2$-gradation.

Theorem 2.2 Symmetric group on n letters S_n ($n \geqslant 2$) has a unique $\boldsymbol{Z}_2$-gradation.

Proof By example 2 we know there is a $\boldsymbol{Z}_2$-gradation on S_n. But the alternating subgroup A_n has index 2 in S_n. Now Corollary 1.5 concludes the result.

Theorem 2.3 Alternating group A_n ($n \geqslant 2$) does not possess any $\boldsymbol{Z}_2$-gradation.

Proof If $n \geqslant 5$, thanks to the Galois 'theorem ([1](Ⅰ),4.6) A_n is a simple group. By theorem 1.3 we obtain the result.

When $n = 2$ or 3 the result is clearly true.

If $n = 4$, there are only 4 elements in A_4 which have even order: (1), (12)(34), (13)(24) and (14)(23). But $4 < 6 = |A_4|/2$, by Corollary 1.1 there isn 't any $\boldsymbol{Z}_2$-gradation on A_4.

Theorem 2.4 Let $G = \boldsymbol{Z}_2^n$ be elementary 2-group. Then G possesses $2^n - 1$ $\boldsymbol{Z}_2$-gradations which are all equivalent.

Proof Consider G as an n dimensional $\boldsymbol{Z}_2$-vector space. H is a subgroup of G with index 2 in G if and only if H is an $n-1$ dimensional subspace of G. Let $\{e_1, e_2, \cdots, e_n\}$ is a basis of G. Then H is $1-1$ corresponding to subspace

$$\{x_1e_1 + x_2e_2 + \cdots x_ne_n \in G \mid k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0\},$$

where $k_1, k_2, \cdots, k_n \in \boldsymbol{Z}_2$, and are not all zeros. Since there are $2^n - 1$ such kinds of $(k_1, k_2, \cdots, k_n)$, G possesses $2^n - 1$ subgroups of index 2. Hence, G has $2^n - 1$ $\boldsymbol{Z}_2$-gradations.

Evidently, any two $n-1$ dimensional subspaces of G are isomorphic. And any linear independent subset of G with $n-1$ elements can be expanded into a basis of G. Therefore, the above isomorphism between two $n-1$ dimensional subspaces can be expanded into an automorphism of G. Hence, any two $\boldsymbol{Z}_2$-gradations of G are equivalent.

Theorem 2.5 Let $G = D_m$ be a dihedral group. Then G has a unique $\boldsymbol{Z}_2$-gradation if m is odd and 3 $\boldsymbol{Z}_2$-gradations if m is even, but in this case, G has only 2 $\boldsymbol{Z}_2$-gradations which are not equivalent.

Proof Let $G = (r, s)$, where $r^m = s^2 = 1$, $srs = r^{-1}$. So $G = \{r^i, r^is \mid 0 \leqslant i < m\}$. $G^{(1)} = (r)$ if m is odd, but $H := (r)$ is a cyclic subgroup of order m. So $[G : H] = 2$ and by Corollary 1.5, We obtain the desired result.

When $m = 2q$ is even, set $H_1 = (r)$, $H_2 = (r^2, s)$ and $H_3 = (r^2, rs)$. Then they are subgroups of G and have index 2 in G. Clearly, $G^{(1)}=(r^2)$. let H be any subgroup of G with index 2. Then H contains $G^{(1)}$. Put $\overline{H}=\pi(H)$, where π is the canonical homomorphism from G to $\overline{G}:=G/G^{(1)}$. Then $|\overline{H}|=2$. Because $\overline{G}=\{\overline{1},\overline{r},\overline{s},\overline{rs}\}$ has three subgroups of order 2, that is $\overline{H}_1=\{\overline{1},\overline{r}\}$, $\overline{H}_2=\{\overline{1},\overline{s}\}$ and $\overline{H}_3=\{\overline{1},\overline{rs}\}$. Thus, $H_i=\pi^{-1}\overline{H}_i$, $i=1,2,3$, are all subgroups of G with index 2 in G. Define an automorphism φ of G by $\varphi(r)=r$ and $\varphi(s)=rs$ we have $\varphi(H_2)=H_3$. As H_1 is a cyclic subgroup but H_2 and H_3 are not, the gradations defined by H_2 and H_3 are not equivalent to one defined by H_1.

3 Lie Superalgebra Associated with A Z_2-graded Group

In this section, F always denotes a field with characteristic zero or $p>2$.

Definition 3.1 Let G be a $\boldsymbol{Z}_2$-graded group and $A=F[G]$ the group algebra over F. Then A becomes an associative superalyebra by means of the definition

$$A_\alpha:=FG_\alpha,\ \text{for all } \alpha\in\boldsymbol{Z}_2.$$

We denote this superalgebra by $\mathscr{A}$. Let $\mathscr{A}^L$ be the Lie superalgebra associated with the associative superalyebra $\mathscr{A}$. $\mathscr{A}^L$ is called the Lie superalgebra associated with the $\boldsymbol{Z}_2$-graded group G.

It is well-known that if G is an abelian group and $A=F[G]$ is the group algebra of G, then Lie algebra A^- is commutative. But for Lie superalgebras, we have

Proposition 3.1 let G be a finite group, F an algebraically closed field such that char $F\nmid|G|$. Then the Lie algebra A^- associated with the group algebra $A=F[G]$ is isomorphic to a direct sum of a finite number of general linear Lie algebras.

Proof Since char $F\nmid|G|$ the group algebra A is semi-simple (cf. [1](Ⅱ)) and

$$A=F[G]=A_1\oplus A_2\oplus\cdots\oplus A_s,$$

where $A_i\cong M_{n_i}(\Delta_i)$, Δ_i is a division algebra over F. The assumption that F is algebraically closed implies that the only finite dimensional division algebra over F is F itself. So $A_i\cong M_{n_i}(F)$. Thus

$$A^-\cong M_{n_1}(F)^-\oplus\cdots\oplus M_{n_s}(F)^-=\mathfrak{gl}_{n_1}(F)\oplus\cdots\oplus\mathfrak{gl}_{n_s}F.$$

Corollary 3.1 Let G be a finite group. Then the Lie algebra A^- associated with the group algebra $A=\mathbf{C}[G]$ is solvable if and only if G is an abelian group.

Proof According to the above proposition, $A^-\cong\mathfrak{gl}_{n_1}\mathbf{C}\oplus\cdots\oplus\mathfrak{gl}_{n_s}\mathbf{C}$ and A^- is solvable if and only if every summand $\mathfrak{gl}_{n_i}(\mathbf{C})$ of A^- is solvable (cf. [5]). Thus A^- is solvable if and only if $n_1=n_2=\cdots=n_s=1$, i.e., A is commutative, which is equivalent to that G is an abelian group.

From above discussion we know the structure of Lie algebra A^- of the group algebra $A=\mathbf{C}[G]$ of finite group G is simple. In contrast, structure of Lie superalgebra $\mathscr{A}^L$ over $\mathbf{C}$ associated with a $\boldsymbol{Z}_2$-graded finite group G is more complicated.

Theorem 3.1 Let G is a $\boldsymbol{Z}_2$-graded finite group with $G_{\bar{0}}=H$. $\mathscr{A}^L$ the Lie superalgebra over $\mathbf{C}$ associated with G. Then $\mathscr{A}^L$ is solvable if and only if H is an abelian subgroup of index 2 in G.

Proof Thanks to [4,3.2], Lie superalgebra $\mathscr{A}^L$ is solvable if and only if its Lie algebra $\mathscr{A}_{\bar{0}}^L$ is solvable. But $\mathscr{A}_{\bar{0}}^L$ is isomorphic to the Lie algebra $\mathbf{C}[H]^-$. Bt using Corollary 3.1 we obtain the result.

Let G be a group. If H is a group which is isomorphic to G then the group algebra $F[G]$ is isomorphic to $F[G]$, which induces an isomorphism of Lie algebras from $F[G]^-$ to $F[H]^-$. For Lie superalgebras we have

Proposition 3.2 Let G and H be two $\boldsymbol{Z}_2$-graded groups, $\mathscr{A}^L$ and $\mathscr{B}^L$ the Lie superalgebras associated with G and H respectively. If G is equivalent to H then $\mathscr{A}^L$ is isomorphic to $\mathscr{B}^L$.

For a group G, it may possesses $\boldsymbol{Z}_2$-gradation or not. If it possesses $\boldsymbol{Z}_2$-gradation it may have one or many ones which are not equivalent to each other. Are the Lie superalgebras associated with these inequivalent $\boldsymbol{Z}_2$-gradations isomorphic to each other? The answer is very dependent on these gradations and on the base field F. Let's have some examples.

Example 7 Let G be the group as in example 6. We know there are two inequivalent $\boldsymbol{Z}_2$-gradations on G determined by $H_1=(a)$ and $H_3=(a^2, b)$ respectively. Let $\mathscr{A}$ and $\mathscr{B}$ be the associative superalgebras of $\mathbf{C}[G]$ with respect to these two $\boldsymbol{Z}_2$-gradations respectively, i. e., $\mathscr{A}_{\bar{0}}=\text{span}\ \{e, a, a^2, a^3\}$ and $\mathscr{B}_{\bar{0}}=\text{span}\ \{e, a^2, b, ba^2\}$. We define a linear map φ from $\mathscr{A}$ to $\mathscr{B}$ such that

$$e\to e,\ a\to\frac{1+i}{2}ba^2+\frac{1-i}{2}b,\ a^2\to a^2,\ a^3\to\frac{1-i}{2}ba^2+\frac{1+i}{2}b,$$

$$ba\to ba,\ b\to\frac{1+i}{2}a+\frac{1-i}{2}a^3,\ ba^3\to ba^3,\ ba^2\to\frac{1-i}{2}a+\frac{1+i}{2}a^3.$$

Then φ is an isomorphism of superalgebras. Thus φ is also an isomorphism of Lie superalgebra from $\mathscr{A}^L$ to $\mathscr{B}^L$.

This example shows that if G_1 and G_2 are two $\boldsymbol{Z}_2$-graded groups which are not equivalent, then their Lie superalgebras may be isomorphic.

Example 8 Put $F=\mathbf{C}$. Let $G=(a)\times(b)\times(c)$ be an elementary 2-group, i.e., $G=\{e, a, b, c, ab, ac, bc, abc\}$. Let $H=\{e, d, \cdots, d^7\}=(d)$ be a cyclic group of order 8. Put $G_{\bar{0}}=(a)\times(b)$ and $H_{\bar{0}}=(d^2)$, then G and H are $\boldsymbol{Z}_2$-graded. Let $\mathscr{A}$ and $\mathscr{B}$ be the superalgebras of $F[G]$ and $F[H]$ defined by these two $\boldsymbol{Z}_2$-gradations respectively. Define a linear map $\varphi:\mathscr{A}^L\to\mathscr{B}^L$ such that

$$\varepsilon\to d^2,\ a\to d^6,\ b\to\frac{1-i}{2}e+\frac{1+i}{2}d^4,\ ab\to\frac{1+i}{2}e+\frac{1-i}{2}d^4,$$

$$c\to d,\ ac\to d^5,\ bc\to\frac{1+i}{2}d^3+\frac{1-i}{2}d^7,\ abc\to\frac{1-i}{2}d^3+\frac{1+i}{2}d^7.$$

Then φ is an isomorphism of Lie superalgebras. But if we change the base field F to be $\boldsymbol{Z}_3$ the Lie superalgebra $\mathscr{A}^L$ and $\mathscr{B}^L$ are not isomorphic. The proof is based on the following consideration. In $\mathscr{A}^L$ we have

$$\langle c, c\rangle = \langle ac, ac\rangle = \langle bc, bc\rangle = \langle abc, abc\rangle = 2\varepsilon,$$

which is equivalent to the fact that the subspace spanned by elements x in $\mathscr{A}_{\bar{1}}^L$ such that $x^2 = \varepsilon$ is $\mathscr{A}_{\bar{1}}^L$ itself. But by computation we know for any $0 \neq y_0 \in \mathscr{B}_{\bar{0}}^L$, dimension of the subspace generated by elements y in $\mathscr{B}_{\bar{1}}^L$ such thay $y^2 = y_0$ is less or equal to $3 < 4 = \dim \mathscr{A}_{\bar{1}}^L$. Therefore as Lie superalgebras over $\boldsymbol{Z}_3$, $\mathscr{A}^L$ is not isomorphic to $\mathscr{B}^L$.

Finally, we discuss structures of Lie superalgebras associated with dihedral groups.

Theorem 3.2 Let the base field F be $\mathbf{C}$. Let $G = D_m = \langle r, s \mid r^m = s^2 = e, srs = r^{-1}\rangle$ be a dihedral group $(m > 2)$. Then if m is odd the Lie superalgebra $\mathscr{A}^L$ of D_m is solvable. If m is even and $\mathscr{A}^L$ and $\mathscr{B}^L$ are Lie superalgebras of D_m with respect to $\boldsymbol{Z}_2$-gradations determined by $H_1 = (r)$ and $H_2 = (r^2, s)$ respectively. Then $\mathscr{A}^L$ is isomorphic to $\mathscr{B}^L$ if $m = 4$, $\mathscr{A}^L$ is not isomorphic to $\mathscr{B}^L$ if $m > 4$. In all cases $\mathscr{A}^L$ is solvable, but $\mathscr{B}^L$ is not solvable if $m > 4$.

Proof According to Theorem 2.5, D_m has a unique $\boldsymbol{Z}_2$-gradation determined by H_1 if m is odd and has two inequivalent $\boldsymbol{Z}_2$-gradations if m is even. These two gradations are one given by H_1 and the other one given by H_2. Since H_1 is a cyclic group, by Theorem 3.1 $\mathscr{A}^L$ is solvable. When m is even and $m > 4$ the subgroup H_2 of D_m is not abelian. Thus Lie algebras $\mathscr{A}_{\bar{0}}^L$ and $\mathscr{B}_{\bar{0}}^L$ are not isomorphic. So $\mathscr{A}^L$ is not isomorphic to $\mathscr{B}^L$. Also from Theorem 3.1 we know $\mathscr{B}^L$ is not a solvale Lie superalgebra. For $G = D_4$ we can define a linear map φ from $\mathscr{A}$ to $\mathscr{B}$ by

$$e \to e,\ r \to \frac{1-i}{2} r^2 s + \frac{1+i}{2} s,\ r^2 \to r^2,\ r^3 \to \frac{1+i}{2} r^2 s + \frac{1-i}{2} s$$

$$rs \to rs,\ s \to \frac{1+i}{2} r + \frac{1-i}{2} r^3,\ r^3 s \to r^3 s,\ r^2 s \to \frac{1-i}{2} r + \frac{1+i}{2} r^3.$$

Then it is easy to check that φ is an isomorphism of superalgebras. Therefore, Lie superalgebra $\mathscr{A}^L$ and $\mathscr{B}^L$ are isomorphic.

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摘要 :设 Γ 是交换群。在该文中,作者引入了 Γ-阶化群的概念。因为一个 Z_2-阶化群可以对应一个李超代数,所以对 Z_2-阶化群的性质进行了讨论。并对一些特殊类型的群确定了它们的 Z_2-阶化结构。最后,讨论了与二面体群相伴的李超代数的结构。

关键词 :阶化群; 李超代数; Z_2-阶化群
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