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Based Rings and Representations of Affine Hecke Algebras

Purpose: Given a positive answer to a question of G. Lusztig on classification of irreducible Representations of an Affine Hecke Algebras (1989, Astérisque 171-172, 73-84)

Why Hecke Algebras?

$\mathbb{F}_q$ Finite field of q elements

$G = GL_n(\mathbb{F}_q)$ $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset G$

{Complex Representations of G with non zero points fixed by B}

$\simeq$ {Complex Representations of the Hecke algebra of the symmetric group S_n} parameter $= q$

S_n a Weyl group

For general finite groups of Lie type, we need Hecke algebras of Weyl groups.

~~of Lie ty~~

F p-adic field, residue field $= \mathbb{F}_q$

$\mathcal{G}$ reductive algebraic group over F

{Complex admissible representations of $\mathcal{G}$ with non-zero points fixed by an Iwahori subgroup I}

A. Borel (1976) Inv. Math.

$\longleftrightarrow$ {Complex representations of the corresponding affine Hecke algebra H_q with parameter q}

The affine Hecke algebra H_q can be defined for aribitary non-zero complex number q.

Question: classification of irreducible representations of H_q.

H_q G the corresponding reductive algebraic group over $\mathbb{C}$

$\operatorname{Irr} H_q$ isomorphism classes of irreducible representations of H_q

Conjecture (Langlands).

$$\operatorname{Irr} H_q \longleftrightarrow \{\text{semisimple classes of } G\}$$

Refinement (Deligne)

$$\operatorname{Irr} H_q \longleftrightarrow \left\{ (s,u) \,\middle|\, \begin{array}{l} s \in G \text{ semisimple} \\ u \in G \text{ unipotent} \\ sus^{-1} = u^q \end{array} \right\} \Big/ G$$

$$(g(s,u) = (gsg^{-1}, gug^{-1}))$$

This is true for affine type A, q power of a prime (Bernstein and Zelevinski, 1976)

Further Refinement (Lusztig, 1983)

$$A(s,u) = Z_G(s) \cap Z_G(u) \Big/ (Z_G(s) \cap Z_G(u))^\circ$$

(a finite group) acts on the cohomology $H^\bullet(\mathcal{B}_u^s) = \bigoplus H^i(\mathcal{B}_u^s)$

$\mathcal{B}$ flag variety $= \{$all Borel subgroups of $G\}$

$$\mathcal{B}_u^s = \{B \in \mathcal{B} \mid s, u \in B\}$$

Deligne - Langlands - Lusztig

(*) $\mathrm{Irr}\, H_q \longleftrightarrow \{(s, u, \rho)\}/G$

$s \in G$ semisimple
$u \in G$ unipotent
ρ irreducible representation of $A(s,u)$ that appears in $H^\bullet(\mathcal{B}_u^s)$

Kazhdan & Lusztig (1987. Invent Math.)

the final version of the conjecture (*) is true when q is not a root of 1.

Main point of KL's approach: H_q can be realized through the equivariant K-group $K^{G \times \mathbb{C}^*}(Z)$

Z. Steinberg triple

$$= \{ (u, B, B') \mid u \in G \text{ unipotent}, B, B' \in \mathcal{B}, \; u \in B \cap B' \}$$

Question. What happens when q is a root of 1?

W_0 the Weyl group of G

$l : W_0 \to \mathbb{N}$. length function

Lusztig (1989) If $\sum_{w \in W_0} q^{l(w)} \neq 0$, the classification (*) remains true.

Xi (2004). It is.

How to see this?

Based ring

Based Ring

Kazhdan - Lusztig Theory

Coxeter group (W, S)

W generated by S and relations

$s^2 = 1 \quad \forall s \in S$

$(st)^{m_{st}} = 1 \quad \forall s \neq t$ in S.

$(m_{st} \in \{2, 3, 4, 5, \dots, \infty\})$

Example. S_n.

$S = \{(12), (23), (34), \dots, (n-1, n)\}$

Hecke algebra $\overbrace{}^{\mathcal{H}}$ of (W, S) with parameter v

$\mathcal{A} = \mathbb{Z}[v^{1/2}, v^{-1/2}]$ $v^{1/2}$ ~~indeterminant~~ indeterminate

$\mathcal{H}$: free $\mathcal{A}$-module with a basis $\{T_w \mid w \in W\}$.

Multiplication is defined by relations

$$(T_s - q)(T_s + 1) = 0 \quad \forall s \in S$$

$$T_w T_u = T_{wu} \text{ if } l(wu) = l(w) + l(u)$$

($l : W \to \mathbb{N}$. length function).

Kazhdan - Lusztig basis

$\bar{\cdot} \quad \mathcal{H} \longrightarrow \mathcal{H}$ ring isomorphism

$$v^{1/2} \to v^{-1/2}$$

$$T_w \to T_{w^{-1}}^{-1}$$

Kazhdan & Lusztig (1979. Invent. Math.)

For each $w \in W$, $\exists ! \; C_w \in \mathcal{H}$ such that

(1) $C_w = \overline{C_w}$

(2) $C_w = v^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w} T_y$

$P_{y,w}$ are polynomials in v with degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$.

($y \leq w$. Bruhat order)

$P_{y,w}$ are the famous KL polynomials.

function $a : W \to \mathbb{N} \cup \{\infty\}$

For $z \in W$, set

$$a(z) = \min\{ i \in \mathbb{N} \mid v^{i/2} h_{x,y,z} \in \mathbb{Z}[v^{1/2}] \quad \forall\, x, y \in W \}.$$

($a(z)$ can be ∞ in principal, but no such z is found)

where $h_{x,y,z}$ is defined by

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z , \quad h_{x,y,z} \in \mathbb{Z}[v^{1/2}, v^{-1/2}]$$

This defines a function

$$a : W \to \mathbb{N} \cup \{\infty\}$$

Lusztig (1985). The function a is bounded if W is an affine Weyl group.

More precisely, let W_0 be the corresponding Weyl group of W, w_0 is the longest element of W_0, then ~~a(x)~~ $a(w) \leq a(w_0) \quad \forall\, w \in W$.
$= l(w_0)$

From now on, we consider such W that the function a is bounded.

Define $\gamma_{x,y,z}$ by

$$h_{x,y,z} = \gamma_{x,y,z}\, q^{-a(z)/2} + \text{higher degree terms}$$

Based ring

Let $J_{\mathbb{Z}}$ be the free $\mathbb{Z}$-module with basis elements t_x, $x \in W$. Define

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_z$$

It turns out that this defines an associative ring structure of $J_{\mathbb{Z}}$ ~~We called this ring.~~ This ring is called the based ring of W. (Lusztig 1987).

Let $J_{\mathcal{A}} = J_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{A}$.

Define $\varphi : \mathcal{H} \longrightarrow J_{\mathcal{A}}$ by

$$\varphi(C_w) = \sum_{\substack{d \in \mathcal{D} \\ a(d) = a(u)}} h_{w,d,u} t_u$$

where $\mathcal{D}$ is the set of distinguished involutions.

$\mathcal{D} = \{ d \in W \mid l(d) - a(d) - 2 \deg P_{e,d} = 0 \}$

e is the neutral element of W.

Then φ is an $\mathcal{A}$-algebra homomorphism

Let $J = J_{\mathbb{Z}} \otimes \mathbb{C}$

Specializing $v^{1/2}$ to a square root $q^{1/2}$ of q

Then we get a $\mathbb{C}$-algebra homomorphism

$$\varphi_q : H_q \longrightarrow J_{\mathbb{C}}$$

Thus any J-module E has an H_q-module structure by means of φ_q. Denote this H_q-module structure by E_{φ_q}

A decomposition of J

Lusztig: If $\gamma_{x,y,z} \neq 0$, then $a(x) = a(y) = a(z)$

Let J_i be the subspace of J spanned by all t_w with $a(w) = i$.

Then J_i is a two-sided ideal of J and

$$J = \bigoplus J_i .$$

So each J_i is also a $\mathbb{C}$-algebra

For each irreducible J-module E, there exists a unique i such that $J_i E \neq 0$. Then

Set $a(E) = i$

Let M be an H_q-module, define $a(M)$ by the following condition

$$C_w M = 0 \quad \text{if} \quad a(w) > a(M)$$

$$C_w M \neq 0 \quad \text{for some } w \text{ with } a(w) = a(M).$$

Then $a(M)$ is well defined.

From now on, W is an affine Weyl group.

Lusztig (1987): If M is an irreducible H_q-module, then $\exists$ irreducible J-module E such that

(i) $a(E) = a(M)$

(ii) M is a quotient of E_{φ_q}.

Lusztig. For any irreducible constituent M of E_{φ_q} (E an irreducible J-module), we have ~~$a(M) \leq a(E_{\varphi_q})$~~, $a(M) \leq a(E)$.

Xi (2004). (i) Let E be an irreducible J-module, then E_{φ_q} has at most one irreducible constituent M such that $a(M) = a(E)$

(ii) If E_{φ_q} has an irreducible constituent M such that $a(M) = a(E)$, then M must be a quotient of E_{φ_q}. Moreover, ~~the quotient is~~ M is the unique quotient of E_{φ_q}.

(iii) Let E, E' be irreducible J-modules such that $a(E) = a(E')$. Assume that ~~both~~ E_{φ_q} and E'_{φ_q} have irreducible constituents M and M' such that $a(M) = a(M') = a(E) = a(E')$ then

$$E \simeq E' \iff M \simeq M'$$

Remark. (i), (ii), (iii) are true for any field k such that $q^{1/2} \in k$. (Then H_q is replaced by $\mathcal{H} \otimes_A k$. J is replaced by $J_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$)

Xi. (2004)

If $\sum_{w \in W_0} q^{l(w)} \neq 0$, then ~~each~~ for irreducible J-module E, E_q has a unique irreducible ~~Consitutex~~ constituent M such that $a(M) = a(E)$.

The map

$$E \longrightarrow M$$

defines a bijection

$$\mathrm{Irr}\, J = \{\text{isomorphism classes of irreducible } J\text{-modules}\}$$

$$\downarrow$$

$$\mathrm{Irr}\, H_q$$

Lusztig: 1987

$$\mathrm{Irr}\, J \longleftrightarrow \{(s, u, \rho) \mid s \in G \text{ semisimple},\ u \in G \text{ unipotent},\ \rho \text{ irreducible } A(s,u)\text{-module appearing in } H^{\cdot}(\mathcal{B}_u^s)\}/G.$$

$sus^{-1} = u^q$

Conclusion.

If $\sum_{w \in W_0} q^{l(w)} \neq 0$, then

$$\text{Irr } H_q \longleftrightarrow \{(s, u, \rho)\}/G.$$

i.e. Deligne - Langlands - Lusztig Conjecture remains true in this case.

Remark 1. If $\sum_{w \in W_0} q^{l(w)} = 0$. this Conjecture needs modification.

~~That is.~~ In fact, $\exists$ irreducible J-module E with $a(E) = a(w_0)$. such that $C_w E \varphi_q \neq 0 \Rightarrow a(w) < a(w_0)$

Remark 2. 1994. I. Gronojwski (a student of Luoliz) ? claimed he proved the conclusion and announced that the detailed proof would appear soon. But ~~unit~~ until now, no evidence shows that he will ~~pub~~ publish a proof.

~~Question. What~~ Is ~~DLL conjecture true for any field k (assume q)~~

Question. Is DLL Conjecture true for any algebrically closed field k ?

<u>Suggestion</u> Should be.

Evidence. rank 2 cases. ($\widehat{A_2}$, $\widehat{B_2}$, $\widehat{G_2}$)
type $\widehat{A}$