Classification of simple amenable C^*-algebras

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References

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★. A C^*-algebra is a complete normed algebra over $\mathbb{C}$ with an involution $(*)$ for which

$$\|a^*\| = \|a\| \quad \text{and} \quad \|a^*a\| = \|a\|^2.$$

★. Every C^*-algebra is closed and adjoint closed subalgebra of $B(H)$, where $B(H)$ is the C^*-algebra of all bounded operators on a Hilbert space H.

★. Examples: $\mathbb{C}$, M_n, $C_0(X)$, where X is a locally compact Hausdorff space, $\mathcal{K}$, the algebra of all compact operators on l^2.

★. Gelfand: Every (unital) commutative C^*-algebra is isomorphic (as C^*-algebra) to $C(X)$ for some compact Hausdorff space X.

★. C^*-algebra are viewed as non-commutative topology.

★. Minimal dynamical systems.

Let X be a compact metric space and $\alpha : X \to X$ be a minimal homeomorphism. There is an α-invariant normalized Borel measure μ. Consider the Hilbert space $H = L^2(X, \mu)$ and homomorphism $\pi : C(X) \to B(H)$ defined by

$$\pi(g)(f) = gf \text{ for } f \in L^2(X, \mu)$$

for all $g \in C(X)$. Define

$$U(f) = f \circ \alpha^{-1} \; f \in L^2(X, \mu).$$

Then U gives a homomorphism from $\mathbb{Z}$ into the unitary group of $B(H)$. The C^*-algebra generated by $\pi(C(X)$ and α is denoted by $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and is called the crossed product of $C(X)$ by $\mathbb{Z}$ via α.

In this special case A_α is a unital separable simple C^*-algebra. (no proper ideal).

★. Irrational rotations.

Let S^1 be the unit circle and θ be an irrational number. Define $\alpha : S^1 \to S^1$ by $\alpha(e^{2\pi i t}) = e^{2\pi i(t+\theta)}$. This is an irrational rotation.

One can show that A_α is the universal C^*-algebra generated by two uniatries u and v with relation:

$$uv = e^{2\pi i\theta} vu.$$

(non-commutative torus).

★. Question: when two C^*-algebras are isomorphic?

★.

Two unital commutative C^*-algebras $A = C(X)$ and $B = C(Y)$ are isomorphic if and only if X and Y are homeomorphic.

★. We are NOT going to classify commutative C^*-algebras.

★. We consider simple separable amenable C^*-algebras with lower rank (for this talk zero rank).

★. AF-algebras can be classified by (scaled) dimension groups (G. A. Elliott—1978).

AF==== approximately finite dimensional. $C(X)$ is AF if and only if $\dim X = 0$.

★. A C^*-algebra has real rank zero if the set of invertible self-adjoint elements is dense in $A_{s.a.}$.

Every AF-algebra has real rank zero.

$C(X)$ has real rank zero if and only if $\dim X = 0$.

Every Von-Neumann algebras has real rank zero.

★. AH-algebras:

$A = \lim_{n\to\infty}(A_n, \phi_n)$, where

$$A_n = P(C(X_n) \otimes F_n)P = P(C(X, F_n))P,$$

where X_n is a finite CW-complex, P is a projection in $C(X, F_n)$.

A is said to be $A\mathbb{T}$-algebra if each X_n can be taken as the unit circle.

★. Theorem (Elliott-Gong) (On the classification of C^*-algebras of real rank zero. II. Ann. of Math. **144** (1996), 497–610.)

Let A and B be two unital AH-algebras with no dimension growth and with real rank zero. Then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A))$$
$$\cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

★. Moreover, for any countable abelian group G_1 and any countable weakly unperforated (partial) ordered group G_0 with order unit $u \in G_0$ with the Riesz interpolation, there is a unital simple AH-algebra with no dimension growth and with real rank zero such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (G_0, (G_0)_+, u, G_1).$$

Theorem (Elliott and Evans)
The structure of the irrational rotation C^*-algebra, Ann. of Math. **138** (1993), 477–501.)

Let θ be an irrational number and $\alpha : S^1 \to S^1$ is defined by $\alpha(z) = e^{2\pi i\theta} z$. Then $C(X) \rtimes_\alpha \mathbb{Z}$ is a unital simple $A\mathbb{T}$-algebra with real rank zero.

★. All purely infinite simple C^*-algebras have real rank zero. (S. Zhang)

★. Tracial (topological) rank was introduced in 1998. If A has tracial rank zero, it will written $TR(A) = 0$.

★. Every unital simple C^*-algebra with $TR(A) = 0$ is quasidiagonal, has real rank zero, stable rank and weakly unperforated K_0.

★. Every unital simple AH-algebra with no dimension growth and with real rank zero has tracial rank zero.

★. $\mathcal{N}$: the so-called Bootstrap class of C^*-algebras.

It contains most interesting separable C^*-algebras. It contains all commutative C^*-algebras, type I C^*-algebras, closed under inductive limit, quotient, ideal, tensor product with AF-algebras, crossed products by $\mathbb{Z}$,....

We are only interested in C^*-algebras in $\mathcal{N}$.

★. **Theorem A** (L—)

Let A and B be two unital separable simple C^*-algebras in $\mathcal{N}$. Suppose that $TR(A) = TR(B) = 0$. Then

$$A \cong B$$

if and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A))$$
$$\cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

★ Consider

$$\alpha : K_*(A) \longrightarrow K_*(B).$$

We hope to establish homomorphisms $\phi : A \to B$ so that $[\phi] = \alpha$.

One can settle for "approximately multiplicative maps".

★ A sequence of positive linear maps $\phi_n : A \to B$ is said to be asymptotically multiplicative if

$$\lim_{n\to\infty} \|\phi_n(a)\phi_n(b) - \phi_n(ab)\| = 0$$

for all a, $b \in A$.

In general, asymptoticcally multiplicative completely positive linear maps can not be "close" to any homomorphisms

★ (–2001) "Existence Theorem"

For unital separable amenable simple C^*-algebras A and B with $TR(A) = TR(B) = 0$, given any

$$\alpha : (K_0(A), K_0(A)_+, [1_A], K_1(A))$$
$$\to K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

There existence a (sequence) of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\}$: $A \to B$ such that "locally" $\{\phi_n\}$ gives α.

★ (A non-commutaive diagram)

$$\begin{array}{ccccc} A & \stackrel{\text{id}}{\to} & A & \stackrel{\text{id}}{\to} & A \\ \downarrow \phi_1 & \nearrow \psi_1 & \downarrow \phi_2 & & \\ B & \stackrel{\text{id}}{\to} & B & \stackrel{\text{id}}{\to} & B \end{array}$$

★ Question:

Given two maps from $L_1, L_2 :\to B$.

1) When are they the "same"?

2) When are they unitarily equivalent?

★ Consider a spacial case:

Let X be a compact metric space and let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Suppose $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. When they are "equivalent?"

★. **Theorem B** (L—)

Let A be a unital simple C^*-algebra with tracial rank zero and X be a compact metric space. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent if and only if

$$[h_1] = [h_2] \text{ in } KL(C(X), A) \quad \text{and} \quad \tau(h_1(f)) = \tau(h_2(f))$$

for every $f \in C(X)$ and every trace τ of A.

★. Approximately unitarily equivalent:

There exists a sequence of unitaries $u_n \in A$ such that

$$\lim_{n\to\infty} \|u_n^* h_1(a) u_n - h_2(a)\| = 0$$

for alll $f \in C(X)$.

★. Recall the BDF-theory.

Let A be the Calkin algebra. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then h_1 and h_2 are unitarily equivalent if and only if

$$[h_1] = [h_2] \text{ in } KK(C(X), A).$$

★. If $K_*(C(X))$ is torsion free, in Theorem A, condition about KL can be replaced by

$$(h_1)_{*i} = (h_2)_{*i}, \quad i = 0, 1,$$

where $(h_j)_{*i} : K_i(C(X)) \to K_i(A)$ $(i = 0, 1)$ is the induced homomorphism on K_i.

★. If A has a trace τ and h_1 and h_2 are approximately unitarily equivalent, then

$$\tau(h_1(f)) = \tau(h_2(f))$$

for all $f \in C(X)$.

★. Calkin algebra is purely infinite and simple—no trace.

★. **Theorem C** Let A and B be two unital separable simple C^*-algebras with $TR(A) = TR(B) = 0$. Suppose that $\{\phi_n\}, \{\psi_n\} : A \to B$ are two sequence of completely positive linear maps which are asymptotically multiplicative such that

$$[\{\phi_n\}] = [\{\psi_n\}] \text{ in } KL(A, B)$$

. Then there exists a sequnece of unitaries $\{u_k\} \subset B$ such that

$$\lim_{k\to\infty} \|u_k^* \phi_{n_k}(a) u_k - \psi_{n_k}(a)\| = 0$$

for all $a \in A$.

★ If $K_*(A)$ is torsion free,

$$KL(A, B) = Hom(K_*(A), K_*(B)).$$

By using the "existence theorem" and the "uniqueness theorem" one can construct an approximate intertwining:

★

$$\begin{array}{ccccc} A & \stackrel{\text{id}}{\to} & A & \stackrel{\text{id}}{\to} & A \\ \downarrow_{\phi_1} & \nearrow_{\text{ad}\, u_1 \circ \psi_1} & \downarrow_{\text{ad}\, v_2 \circ \phi_2} & & \\ B & \stackrel{\text{id}}{\to} & B & \stackrel{\text{id}}{\to} & B \end{array}$$

Minimal dynamical systems

★. Let X be a compact metric space and α be a homeomorphism on X. Set $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ and $j_\alpha : C(X) \to A_\alpha$ the obvious embedding map.

★. Let X be a compact metric space and $\alpha, \beta : X \to X$ be minimal homeomorphisms. We say α and β are conjugate if there exists homeomorphism $\sigma : X \to X$ such that

$$\sigma^{-1} \circ \beta \circ \sigma = \alpha.$$

We say α and β are flip conjugate if either α and β^{-1} (or α^{-1} and β or α and β) are conjugate.

★. Theorem T (J. Tomiyama)

Let X be a compact metric space and $\alpha, \beta : X \to X$ be homeomorphisms. Suppose that (X, α) and (X, β) are topologically transitive. Then α and β are flip conjugate if and only if there is an isomorphism $\phi : C(X) \rtimes_\alpha \mathbb{Z} \to C(X) \rtimes_\beta \mathbb{Z}$ such that $\phi \circ j_\alpha = j_\beta \circ \chi$ for some isomorphism $\chi : C(X) \to C(X)$.

It should be noted that all minimal dynamical systems are transitive.

★. Definition

Let (X, α) and (X, β) be two topological transitive systems. (X, α) and (X, β) are C^**-strongly approximately flip conjugate* if there exists an $\phi : A_\alpha \to A_\beta$, a sequence of unitaries $u_n \in C(X) \times_\alpha \mathbb{Z}$ and an isomorphism $\chi : C(X) \to C(X)$ such that

$$\lim_{n \to \infty} \|\mathrm{ad}\, u_n \circ \phi \circ j_\alpha(f) - j_\beta \circ \chi(f)\| = 0 \text{ for } f \in C(X).$$

★. In Theorem T, let $\theta = [\phi]$ in $KK(A_\alpha, A_\beta)$. Let $\Gamma(\theta)$ be the induced element in $Hom(K_*(A_\alpha), K_*(A_\beta))$ which preserves the order and the unit. Then one has

$$[j_\alpha] \times \theta = [j_\beta \circ \chi]$$

★ Let A be a stably finite C^*-algebra and $T(A)$ be the space of tracial states on A. There is a positive homomorphism $\rho_A : K_0(A) \to Aff(T(A))$, where $Aff(T(A))$ is the set of all real affine continuous functions on $T(A)$.

Suppose that $TR(A_\alpha) = TR(A_\beta) = 0$. Then $\rho_{A_\alpha}(K_0(A_\alpha))$ and $\rho_{A_\beta}(K_0(A_\beta))$ are dense in $Aff(T(A_\alpha))$ and $Aff(T(A_\beta))$ respectively. Thus $\Gamma(\theta)$ induces an order and unit preserving affine isomorphism $\theta_\rho : Aff(T(A_\alpha)) \to Aff(T(A_\beta))$. For each $a \in A_{s.a.}$, one defines an element $\hat{a} \in Aff(T(A_\alpha))$ by $\hat{a}(\tau) = \tau(a)$. In particular, each element in $j_\alpha(C(X)_{s.a})$ gives an element in $Aff(T(A_\alpha))$. Therefore, in terms of K-theory and KK-theory, one has the following: If α and β are flip conjugate, then there is an isomorphism

$\chi : C(X) \to C(X)$ such that

$$[j_\alpha] \times \theta = [j_\beta \circ \chi] \text{ in } KK(C(X), A_\beta) \quad \text{and}$$

$$\theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi.$$

★. **Theorem D** (L—2004)

Let (X, α) and (X, β) be two minimal dynamical systems such that A_α and A_β have tracial rank zero. Then α and β are C^*-strongly approximately flip conjugate if and only if the following hold: There is an sequence of isomorphism $\chi_n : C(X) \to C(X)$ and $\theta \in KL(A_\alpha, A_\beta)$ such that $\Gamma(\theta)$ gives an isomorphism from
$(K_0(A_\alpha), K_0(A_\alpha)_+, [1], K_1(A_\alpha))$ to
$(K_0(A_\beta), K_0(A_\beta)_+, [1], K_1(A_\beta))$,

$$[j_\alpha] \times \theta = [j_\beta \circ \chi_n] \text{ in } KL(C(X), A_\beta) \text{ for all } n \quad \text{and}$$

$$\lim_{n\to\infty} \|\rho_{A_\beta} \circ j_\beta \circ \chi_n(f) - \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha(f)\| = 0$$

for all $f \in C(X)$.

★. **Cor C**. Let X be a compact metric space with torsion free K-theory. Let (X, α) and (X, β) be two minimal dynamical systems such that $TR(A_\alpha) = TR(A_\beta) = 0$. Suppose that there is a unit preserving order isomorphism

- (i) $\gamma : (K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta))$,
- (ii) $[j_\alpha] \times \theta = [j_\beta \circ \chi]$ in $KL(C(X), A_\beta)$ and
- (iii) $\gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi$

for some isomorphism $\chi : C(X) \to C(X)$. Then (X, α) and (X, β) are C^*-strongly approximately flip conjugate.

★. The Cantor set.

In the case when X is the Cantor set, $K_0(C(X)) = C(X, \mathbb{Z})$. It follows that, if there is $\theta : K_i(A_\alpha) \to K_i(A_\beta)$ that is an order and unit preserving isomorphism, then there exists $\chi : C(X) \to C(X)$ such that

$$\theta \circ (j_\alpha)_{*0} = (j_\beta \circ \chi)_{*0}.$$

Moreover, it implies that

$$\theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi.$$

In other words, in the case that X is the Cantor set condition (ii) and (iii) is automatic.

★. Approximate conjugacy.

Two dynamical systems (X, α) and (X, β) are said to be weakly approximately conjugate if there are $\sigma_n, \gamma_n : X \to X$ such that

$$\lim_{n\to\infty} \|f(\sigma_n^{-1} \circ \beta \circ \sigma_n) - f(\alpha)\| = 0 \quad \text{and}$$
$$\lim_{n\to\infty} \|f(\gamma_n^{-1} \circ \alpha \circ \gamma_n) - f(\beta)\| = 0$$

for all $f \in C(X)$.

★. This is too weak since there is no consistency among σ_n and γ_n.

Suppose

$$\lim_{n\to\infty} \|f(\sigma_n \circ \alpha \circ \sigma_n^{-1}) - f(\beta)\| = 0$$

for all $f \in C(X)$. Then there exists a sequence of completely positive linear maps $\psi_n : B \to A$ such that

$$\lim_{n\to\infty} \|\psi_n(ab) - \psi(a)\psi(b)\| = 0$$

for all $a, b \in B$ and

$$\lim_{n\to\infty} \|\psi_n(f) - f \circ \sigma_n\| = 0$$

for all $f \in C(X)$ and

$$\lim_{n\to\infty} \psi_n(u_\beta) = u_\alpha,$$

where u_α and u_β denote the implementing unitaries in $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta \mathbb{Z}$.

Let (X, α) and (X, β) be dynamical systems on compact metrizable spaces X and Y. Suppose that a sequence of homeomorphisms $\sigma_n : X \to X$ satisfies $\sigma_n \alpha \sigma_n^{-1} \to \beta$.

Let $\{\psi_n\}$ be the asymptotic morphism arising from σ_n.

★. We say that the sequence $\{\sigma_n\}$ induces an order and unit preserving homomorphism $H_* : K_*(C(X) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z})$ between K-groups, if for every projection $p \in M_\infty(C(X) \rtimes_\beta \mathbb{Z})$ and every unitary $u \in M_\infty(C(X) \rtimes_\beta \mathbb{Z})$, there exists $N \in \mathbb{N}$ such that

$$[\psi_n(p)] = H_*([p]) \in K_0(C(X) \rtimes_\alpha \mathbb{Z}) \quad \text{and}$$

$$[\psi_n(u)] = H_*([u]) \in K_1(C(X) \rtimes_\beta \mathbb{Z})$$

for every $n \geq N$.

★. (For torsion free case). We say that (X, α) and (X, β) are approximately K-conjugate, if there exist homeomorphisms $\sigma_n : X \to X$, $\gamma_n : X \to X$ and a (unit preserving) order isomorphism $H_* : K_*(C(X) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z})$ between K-groups such that

$$\lim_{n\to\infty} \|f(\sigma_n \alpha \sigma_n^{-1}) - f(\beta)\| = 0 \quad \text{and}$$

$$\lim_{n\to\infty} \|f(\gamma_n \beta \gamma_n^{-1}) - f(\alpha)\| = 0$$

for all $f \in C(X)$ and the associated asymptotic morphisms $\{\psi_n\} : B \to A$ and $\{\phi_n\} : A \to B$ induce the isomorphisms H_* and H_*^{-1}.

Theorem E (with H. Matui)

Let X be the Cantor set and α and β be minimal homeomorphisms. Then the following are equivalent:

(i) α and β are C^*-strongly approximately flip conjugate,

(ii) α and β are approximately K-conjugate,

(iii) A_α and A_β are isomorphic,

(iv) $(K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}]) \cong (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}])$,

By a theorem of Giordano, Putnam and Skau, the above also equivalent to

(v) (X, α) and (X, β) are strong orbit equivalent.