Applications of stability of vector bundles and algebraic invariant theory

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1. Stability of Vector Bundles

X: **projective complex manifold**

$$X \subset \mathbb{P}^N, \qquad \dim_{\mathbb{C}} X = n$$

E: **holomorphic vector bundle over** X

$$\operatorname{rank} E = r$$

H: **hyperplane section on** X

$$\deg E = c_1(E) H^{n-1} \in \mathbb{Z}.$$

<u>Definition:</u> $\mathcal{F} \subset E$ **is a subsheaf of** E**.**

- E **is <u>semistable</u> if for any** $\mathcal{F}$**,**

$$\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}} \leq \frac{\deg E}{\operatorname{rank} E}$$

- E **is <u>stable</u> if for any** $\mathcal{F}$ **with rank** $F < r$**,**

$$\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}} < \frac{\deg E}{\operatorname{rank} E}$$

2. Bogomolov's Inequality

1. **If** X **is a surface,** E **is semistable, then**

$$c_1^2(E) \leq \frac{2r}{r-1} c_2(E)$$

2. **If** $c_1^2(E) > \dfrac{2r}{r-1} c_2(E)$, **then** $\exists\ \mathcal{F} \subset E$,

$$\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}} > \frac{\deg E}{r}, \qquad (\operatorname{rk} \mathcal{F} < r).$$

3. $(r = 2)$ **If** $c_1^2(E) > 4c_2(E)$, **then** $\exists\ L \subset E$,

$$\deg L > \frac{1}{2} \deg E, \qquad (L \text{ **is invertible**.})$$

3. Applications of Rank 2 Case

1. **Miyaoka-Yau's Inequality (1978)**
2. **Vanishing Theorem (Mumford 1982)**
3. **Bombieri's Theorem (Reider 1987)**
4. **Fujita's Conjecture (Reider 1987)**

5. **Cayley-Bacharach Theorem (2000)**
6. **Effective Serre Theorem (2003)**
7. **Effective Matsusaka Theorem (2003)**
8. **Riemann-Roch Problem (2003)**
9. **Effective Artin Theorem (2003)**
10. **Effective Postulation (2004)**
11. **Existence of Curves with Prescribed Singularities (2004)**
12. **Seshadri's constants (2004)**

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X: **smooth projective complex surface**

$$X \subset \mathbb{P}^n$$

C: **algebraic curve on** X **defined by**

$$f(x, y) = 0.$$

Linear equivalence: $C_1 \equiv C_2 \iff$

$$\frac{f_1}{f_2} \text{ is a meromorphic function on } X$$

Linear system: $|D| = \{\ C \mid C \equiv D\ \}$

$$H^0(D) = \left\{ f \ \middle|\ f = 0 \text{ or } C_f \in |D| \right\}$$

$$\dim |D| = \dim_{\mathbb{C}} H^0(D) - 1 = h^0(D) - 1$$

Linear subsystem:

$$|D - p_1 - \cdots - p_k| = \left\{ C \in |D| \ \middle|\ p_1, \cdots, p_k \in C \right\}$$

If $\{f_0, f_1, \cdots, f_N\}$ **is a base of** $H^0(D)$**, then**

$$\Phi_D : X - - \to \mathbb{P}^N,$$
$$p \longmapsto [f_0(p), \cdots, f_N(p)]$$

- Φ_D **is well defined**
  $\Longleftrightarrow$ **base point free**
  $\Longleftrightarrow$ 0-**very ample**
  $\Longleftrightarrow \forall\ p,\ \exists\ f \in H^0(D),\ f(p) \neq 0$
  $\Longleftrightarrow \forall\ p,\ \exists\ C \in |D|,\ p \notin C$

$$\dim |D - p| = \dim |D| - 1,\ \forall\ p \in X$$

- Φ_D **is a holomorphic embedding**
  $\Longleftrightarrow$ **very ample**
  $\Longleftrightarrow$ 1-**very ample**
  $\Longleftrightarrow \forall\ p,\ q \in X$ **(not necessarily distinct)**

$$\dim |D - p - q| = \dim |D| - 2,\ \forall\ p,\ q \in X$$

Definition I: $|D|$ **or** D **is**

- (-1)-**very ample** $\iff H^1(D) = 0$.

- k-**very ample** $\iff$ **any** $k+1$ **points give** $k+1$ **indep. conditions**
  $\iff \forall\, p_1, \cdots, p_{k+1}$

$$\dim|D - p_1 - \cdots - p_{k+1}| = \dim|D| - (k+1)$$

Definition II:

- H **is ample** $\iff \exists\, n >> 0$, nH **is very ample**.
  $\iff H^2 > 0,\ HC > 0,\ \ \forall\, C$

- A **is nef** $\iff AC \geq 0,\ \ \forall\, C$
  $\implies A^2 \geq 0$.

- A **is big** $\iff A^2 > 0$.

4. Problems on Effectivity

Problems solved in surface case:

1. **Fujita's Conjecture (1987):**
   If $n \geq k + \dim X + 1$**, then** $|nH + K_X|$ **is** k**-very ample**

2. **Effective Matsusaka Theorem (1993):**
   If $n \geq M(X, H)$**, then** $|nH|$ **is very ample.**

   (But the optimal bounds have not been found even for surfaces)

3. **Conjecture (1980's):** K_X **nef and big.**
   If $n \geq \dim X + 2$**, then** $|nK_X|$ **is free.**

Problems NOT solved in surface case:

4. **Effective Postulation ($<$ 1949):**
   X **is a surface. If** $n \geq P(X, H)$**, then**
   $$\dim |nH| = \chi(\mathcal{O}_X) - 1 + \frac{1}{2} nH(nH - K_X).$$
   Namely $H^1(nH) = H^2(nH) = 0$.
5. **Riemann-Roch Problem:** $\dim |nD| = ?$
6. **Effective Serre Theorem:** $|nH + L|$
7. **Effective Artin Theorem:** $|nA|$
8. **Effective Zariski Theorem:** $|nA + L|$
9. **Existence of Curves with Prescribed Singularities:**
   $$|nH|_T = |nH - S_1 p_1 - \cdots - S_k p_k| \subset |nH|$$

<u>**Main Problem**</u>: **When is the linear system**

$$|nA + L|$$

k-**very ample?**

5. Main Theorem

$$\mathcal{B}(A,L) := \frac{((K_X - L)A + 2)^2}{4A^2} - \frac{(K_X - L)^2}{4}$$

$\beta(A,L) := [\mathcal{B}(A,L)]$ **(integral part)**

Theorem 1. $k \geq 0$. **Assume that either**

(1) $n \geq \beta(A,L) + k + 1$, **or**

(2) $k = 0$, $A^2 = 1$, $nA + L \sim A + K_X$.

If $|nA + L|$ **is not** $(k-1)$**-very ample, then there is a curve** $D \neq 0$ **such that**

$$\begin{cases} LD - D^2 - K_X D \leq k, \\ DA = 0. \end{cases} \qquad (*)$$

$$\alpha(A,L) = \begin{cases} +\infty, & A \textbf{ ample,} \\ \min_D \{ LD - K_X D - D^2 \}, & \textbf{otherwise.} \end{cases}$$

$$\alpha = \alpha(A, L), \quad \beta = \beta(A, L).$$

Theorem 1: **Assume that**

$$\alpha \geq 1, \qquad n \geq \beta + 1.$$

Then $|nA + L|$ **is**

$$\min\{\, \alpha - 2,\ n - \beta - 2 \,\}$$

-very ample.

- **Fujita's Condition**: $|nA + K_X|$.

$$\alpha(A, K_X) = \min\{-D^2\} \geq 1.$$

- **Artin's Condition**: $|nA|$. $\alpha(A, 0) \leq 2$.

$$\alpha(A, 0) = 2 \iff D^2 + K_X D \leq -2, \quad (\forall\ D).$$

- **Ramanujian-Laufer's Condition**:

$$LD \geq K_X D, \quad (\forall\ D) \quad \Longrightarrow \quad \alpha(A, L) \geq 1.$$

6. Effective Serre Theorem

$\beta = \beta(H, L)$.

Effective Serre's Theorem.

- $|nH+L|$ **is** $(n-\beta-2)$**-very ample if** $n \geq \beta+1$.
- $|nH+L|$ **is** k**-very ample if** $n \geq \beta+k+2$.

Corollary 2. (Effective Serre's Theorem)

- $H^2(nH+L) = 0$, **if** $n \geq \beta + 1$
- $H^1(nH+L) = 0$, **if** $n \geq \beta + 1$
- $|nH+L|$ **is free, if** $n \geq \beta + 2$
- $|nH+L|$ **is very ample, if** $n \geq \beta + 3$

History : **No bounds are found for surfaces**

7. Effective Postulation

Corollary 3. (Effective Postulation)

If $n \geq \beta(H, L) + 1$**, then**

$$\dim |nH+L| = \chi(\mathcal{O}_X) - 1 + \frac{1}{2}(nH+L)(nH+L-K_X).$$

Problem ($<$ **1949)**: **Find** k **such that if** $n \geq k$

$$\dim |nH| = \chi(\mathcal{O}_X) - 1 + \frac{1}{2}nH(nH - K_X). \quad (*)$$

[1] **J. G. Semple, L. Roth: Introduction to Algebraic Geometry, 1949. (Page 444)**

Classical Criterion: $C \in |H|$**. If**

$$H^1(\mathcal{O}_X(kH)) = H^1(\mathcal{O}_C(kH)) = 0,$$

then $(*)$ **holds for any** $n \geq k$**.**

8. Effective Matsusaka Theorem

Corollary 4. (Effective Matsusaka THM)

$|nH|$ **is very ample if** $n \geq \beta(H, 0) + 3$, **i.e.**,

$$n > \frac{(K_X H + 2)^2}{4H^2} - \frac{K_X^2}{4} + 2$$

- **Curve:** $n \geq (2g(X) + 1)/\deg H$
- **Surface: (1) Fernandez del Busto, 96**
  (2) Beltrametti-Sommese, 98

$$n > \frac{(4H^2 + HK_X + 1)^2}{2H^2} + \frac{3}{2} \qquad (1)$$

$$n > \frac{(2H^2 + HK_X + 1)^2}{2H^2} + \frac{7}{2} \qquad (2)$$

Problem (Ein, 95): Find optimal bound.

- **Y. T. Siu, 1993, 2002**: $d = \dim X$

$$n \geq N(d, H^d, K_X H^{d-1}).$$

9. Fujita's Conjecture

Fujita's Conjecture: $\dim X = d$,

(1) $|nH + K_X|$ **is base point free if** $n \geq d+1$**.**
(2) $|nH + K_X|$ **is very ample if** $n \geq d+2$**.**
(3) $|nH + K_X|$ **is** $(n-d-1)$**-very ample.**

- $d = 1$: (1), (2), (3) **Riemann-Roch Thm**
- $d = 2$: (1), (2) **Reider 88**

  (3) **Beltrametti-Sommese 91**
- $d = 3$: (1) **Ein-Lazarsfeld, 1993**
  $d = 4$: (1) **Kawamata, 1997**
- (1) **Angehrn-Siu 95**: $n \geq \frac{1}{2}(d^2 + d + 2)$

  (1) **Heier 02:** $n > \left(e + \frac{1}{2}\right) d^{4/3} + \frac{1}{2} d^{2/3} + 1$

Example 1: **(**$L = K_X$**)** $\mathcal{B}(H, K_X) = \dfrac{1}{H^2} \leq 1$

$$\beta(H, K_X) = \begin{cases} 0, & \textbf{if} \;\; H^2 > 1, \\ 1, & \textbf{if} \;\; H^2 = 1. \end{cases}$$

Fujita's Conjecture: **If** H **is ample, then**

- $|nH + K_X|$ **is** $(n-3)$**-very ample.**
- $|nH + K_X|$ **is** $(n-2)$**-very ample if** $H^2 > 1$.

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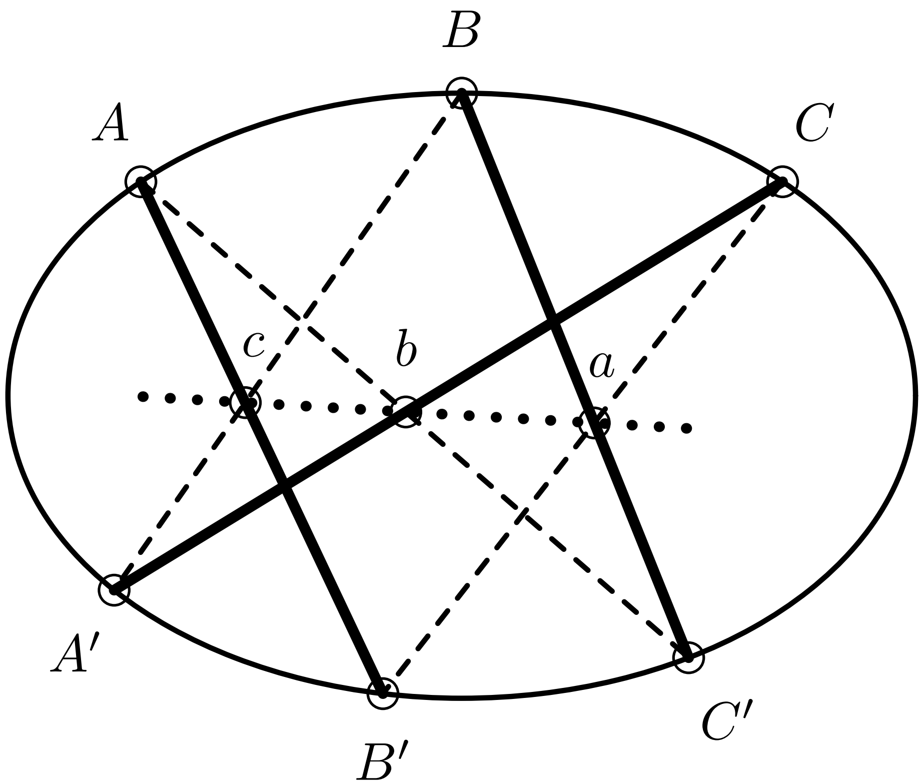
10. Cayley-Bacharach Theorem

Pascal Theorem:



C_1: **Three fat lines**
C_2: **Three dotted lines**
C_3: **Conic + line passing through a and b**

$$C_1 \cap C_2 = \{ A,\ B,\ C,\ A',\ B',\ C',\ a,\ b,\ c \}$$

<u>**Chasles Theorem:**</u> C_1, C_2, C_3 **cubic curves.**

$$C_1 \cap C_2 = \{ p_1,\ p_2,\ \cdots,\ p_9 \}.$$

If C_3 passes through $p_1, \cdots, p_8$, then it passes through p_9.

<u>**Cayley-Bacharach Theorem:**</u> $(\ell = 3)$

$$\deg C_1 = m,\ \deg C_2 = n,\ \deg C_3 = m + n - 3.$$

$$C_1 \cap C_2 = \{ p_1,\ p_2, \cdots, p_{mn} \}$$

If C_3 passes through $mn - 1$ points, then it passes through the last one.

<u>**Theorem (2000)**</u>: $(\ell \geq 3)$. C_1, $C_2 \subset X$. $\#(C_1 \cap C_2) < +\infty$.

$$C_3 \in |C_1 + C_2 - \ell H|$$

If C_3 passes through $\#(C_1 \cap C_2) - (\ell - 2)$ points, then it passes through all of them.

$\dim X \geq 2$:

C-B Problem = Fujita's Conjecture

11. Effective Riemann-Roch Problem

Riemann-Roch Problem: Find the function

$$n \longmapsto \dim |nD| = h^0(nD) - 1$$

In particular, for $n \gg 0$.

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1. **Castelnuovo 1897: Assume** $|D|$ **is free,** $\dim |D| \geq 2$**. If** $n >> 0$**, then**

   $$\dim |nD| = \chi(\mathcal{O}_X) - 1 + \frac{1}{2} nD\,(nD - K_X) + s$$

   where $s = h^1(nD)$ **is a constant.**

2. **Zariski 1962 (Cutkosky-Srinivas 1993): There is a quadratic polynomial** $P(n)$,

   $$h^0(nD) = P(n) + s(n),$$

   where $s(n)$ **is periodic when** $n \gg 0$.

12. Linear Systems in Classical Sense

S: **a curve singularity**

$m_1, \cdots, m_r$: **multiplicity sequence of** S

$$\mathcal{B}(S) := \sum_{i=1}^{r} \frac{(m_i+1)^2}{4},$$

$$\ell(S) := \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2}.$$

$T = p_1^{S_1} \cdots p_k^{S_k}$

$$|C|_T = |C - S_1 p_1 - \cdots - S_k p_k|.$$

Theorem: **If** $n >> 0$**, then**

1. $\dim |nH + L|_T = \dim |nH + L| - \sum_{i=1}^{k} \ell(S_i)$

2. **Generic curves in** $|nH + L|_T$ **are irreducible, and have exactly curve singularity at** p_i **of type** S_i**.**

<u>Problem</u>: What is the effective bound?

(Cayley, Noether, Hilbert, Zariski, ...)

<u>Our Solutions:</u>

$$1) \quad n > \mathcal{B}(H, L) + \sum_{i=1}^{k} \mathcal{B}(S_i)$$

$$2) \quad n > \mathcal{B}(H, L) + \sum_{i=1}^{k} \mathcal{B}(S_i) + 1$$

<u>Severi Variety:</u> $V = V_{|nH+L|}(S_1, \cdots, S_k)$

2) $\implies \dim V = \dim |nH + L| - \sum_{i=1}^{k} c(S_i)$,

$c(S_i) =$ **# of conditions given by** S_i

$\implies$ V **is a Zariski open set of a variety**

$$\overline{V} = \overline{V}_{|nH+L|}(S_1, \cdots, S_k)$$

which is a $\mathbb{P}^N$-fibration.

13. About Vector Bundle Method

dim $X = d \geq 2$.

Theorem (Tan 2000). (1), (2) equivalent

(1) $|K_X + L|$ **is** $(k-1)$**-very ample. (Any** k**-points give** k **independent conditions)**

(2) $Z = F_1 \cap \cdots \cap F_d = \{ p_1, \cdots, p_m \}$.

$$F_0 \in |F_1 + \cdots + F_d - L|.$$

If F_0 **passes through** $m-k$ **points of** Z**, then** F **passes through all.**

Translate it to a problem on vector bundles

$$0 \longrightarrow \mathcal{F} \longrightarrow \oplus_{i=0}^{d} \mathcal{O}_X(-F_i) \xrightarrow{f} \mathcal{O}_X$$
$$(x_0, \cdots, x_d) \mapsto f_0 x_0 + \cdots + f_d x_d$$

$d = 2$: $\mathcal{F}$ **is a rank** 2 **vector bundle.**

$d \geq 3$: $\mathcal{F}$ **is a rank** d **reflexive sheaf, with at most a finite number of singular points.**

Problem: Find numerical conditions such that $\mathcal{F}$ **admits an invertible subsheaf** $\mathcal{L} \subset \mathcal{F}$ **satisfying**

$$\deg \mathcal{L} > \frac{\deg \mathcal{F}}{d}$$

In surface case, i.e., rank two vector bundle case, Bogomolov's inequality $c_1^2 > 4c_2$ **gives us a solution.**

14. Invariant Theory of Binary Forms

(= Theory of Rank 2 Vector Bundles)

Binary Form:

$$f(x_1, x_2) =$$

$$a_0 x_1^n + \binom{n}{1} a_1 x_1^{n-1} x_2 + \binom{n}{2} a_2 x_1^{n-2} x_2 + \cdots + a_n x_2^n$$

Linear Transformation $\varphi \in G = GL(2, k)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \qquad (*)$$

New form: $f(c_{11}\bar{x}_1 + c_{12}\bar{x}_2, c_{21}\bar{x}_1 + c_{22}\bar{x}_2)$:

$$\bar{f}(\bar{x}_1, \bar{x}_2) =$$

$$\bar{a}_0 \bar{x}_1^n + \binom{n}{1} \bar{a}_1 \bar{x}_1^{n-1} \bar{x}_2 + \binom{n}{2} \bar{a}_2 \bar{x}_1^{n-2} \bar{x}_2 + \cdots + \bar{a}_n \bar{x}_2^n$$

$$\begin{cases} \bar{a}_0 = \bar{a}_0(a_0, a_1, \cdots, a_n), \\ \bar{a}_1 = \bar{a}_1(a_0, a_1, \cdots, a_n), \quad \text{(Polynomials)} \\ \quad \vdots \\ \bar{a}_n = \bar{a}_n(a_0, a_1, \cdots, a_n). \end{cases} \tag{**}$$

Covariant: Homogeneous Polynomial
(in a and x)

$$I(a_0, a_1, \cdots, a_n; x_1, x_2) =$$

$$C_0 x_1^m + \binom{m}{1} C_1 x_1^{m-1} x_2 + \cdots + C_m x_2^m$$

satisfying

$$I(\bar{a}_0, \cdots, \bar{a}_n; \bar{x}_1, \bar{x}_2) = (\det \varphi)^p I(a_0, \cdots, a_n; x_1, x_2).$$

Invariant: A covariant with $m = 0$.

Nullform $f \iff$ for any invariant I, $I(f) = 0$.

$\iff f = \ell^h f_1$, $(h > n/2)$. (Hilbert, 1897).

15. Stability of Vector Bundles

E: Rank two vector bundle over X.

$x_1,\ x_2$: Local coordinates (or base).

L: Line bundle over X.

Binary Form: A section f of $S^nE \otimes L$.

Covariant: An algebraic map

$$\begin{aligned} I:\ S^nE \otimes L &\to S^mE \otimes (\det E)^p \otimes L^g \\ f &\mapsto I(f) \end{aligned}$$

Invariant: An algebraic map

$$I:\ S^nE \otimes L \to (\det E)^p \otimes L^g = \Big((\det E)^{n/2} \otimes L\Big)^g$$

Nullforms: = Unstable Points

16. Proof of Bogomolov's Inequality

$$c_1^2 > 4c_2 \implies L \subset E, \quad \deg L > \frac{1}{2} \deg E.$$

Idea of Proof: $E_n := S^n E \left(-\frac{n}{2} \det E\right)$.

Step 1: By R-R Theorem, for any D

$$h^0(E_n(-D)) = \frac{n^3}{12}\left(c_1^2 - 4c_2\right) + O(n^2).$$

Step 2: n is a minimal positive integer such that there is a D satisfying

$$H^0(E_n(-D)) \neq 0, \quad DH > 0.$$

Step 3: Any form $0 \neq f \in H^0(E_n(-D))$ is a nullform. $\forall$ invariant I,

$$\begin{array}{rccc} I: & H^0(E_n(-D)) & \longrightarrow & H^0(-gD) = 0, \\ & f & \mapsto & I(f) = 0 \end{array}$$

$\Longrightarrow f$ nullform

$\Longrightarrow f = \ell^h f_1 \neq 0, \ \left(\frac{n}{2} < h \leq n\right)$ **(Hilbert)**

$$\Longrightarrow \begin{cases} 0 \neq \ell \in H^0(E(-L)), & \Longrightarrow L \subset E \\ 0 \neq f_1 \in H^0(E_{n-h}(-D_1))), & \Longrightarrow \end{cases}$$

$$D_1 = D + h(-L + c_1/2).$$

By the minimality of n, $D_1 H \leq 0$, thus

$$DH + h(-LH + c_1 H/2) \leq 0$$

and

$$\deg L = LH > \frac{1}{2} c_1 H = \frac{1}{2} \deg E.$$

□

(1) Riemann-Roch Theorem

(2) Hilbert's factorization of nullform

(3) We use $\mathrm{Sym}^n E$ **and** $\det E$

17. Possible Generalization

H. Weyl:

(1) All geometric facts are expressed by the vanishing of invariants.

(2) All invariants are invariants of tensors.

E: Rank d vector bundle

$$E^{\otimes n} = S^n E \oplus \Lambda^n E \oplus E^{T_1} \oplus E^{T_2} \oplus \cdots .$$

Usual stability comes from $S^n E$ and $\Lambda^m E$.

What kind of stability can we get from E^T?

[1] H. Weyl: The Classical Groups. Their invariants and representations. 1939

[2] G.-C. Rota: Two turning points in invariant theory, The Math. Intelligencer, **15** (1999), no.1, 20–27.