Left cells in the affine Weyl groups $\widetilde{E}_6$

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Abstract. The aim of the present paper is to describe all the left cells L with $a(L) \leqslant 12$ in the affine Weyl groups $\widetilde{E}_6$. We find a representative set of those left cells which occurs as the vertex sets of the corresponding left cell graphs by applying Shi's algorithm.

Let W be a Coxeter group with S its distinguished generator set. In [**8**], Kazhdan and Lusztig introduced the concept of left, right and two-sided cells in W in order to construct representations of W and the associated Hecke algebra $\mathcal{H}$. Later Lusztig raised a theme for the description of all the cells in the affine Weyl groups W_a (see [**10**]). Lusztig defined a function $a : W \to \mathbf{N} \cup \{\infty\}$, which is upper-bounded and is constant on any two-sided cell of $W = W_a$ (see [**12**]). Since then, the cells (in particular, the left cells) of W_a have been studied extensively by many people. So far, the left cells L of W_a in the following cases have been described explicitly:

(i) $W_a = \widetilde{A}_n$, $n \geqslant 1$ (see [**17**]);

(ii) The rank of W_a is $\leqslant 4$ (see [**12**], [**1**], [**7**], [**6**], [**23**], [**24**], [**25**], [**31**]);

(iii) $a(L)$ is either $\frac{1}{2}|\Phi|$ or $\leqslant 4$, where Φ is the root system of the Weyl group associated to W_a (see [**19**], [**20**], [**9**], [**16**], [**3**], [**4**], [**5**], [**29**]);

(iv) L contains a fully-commutative element of W_a (see [**26**], [**27**]).

Now consider an irreducible affine Weyl group W_a. For any $k \in \mathbf{N}$, let $W_{(k)} = \{w \in W_a \mid a(w) = k\}$. Then $W_{(k)}$ is a union of some two-sided cells of W_a. In the present paper, we shall describe all the left cells in the set $W_{(i)}$ with $i \leqslant 12$ for the affine Weyl group $W_a = \widetilde{E}_6$.

The main tool in describing the left cells is Algorithm 3.6, which was constructed in [**22**] and improved in [**25**] by Shi. We apply it to find a representative set $E(\Omega)$ for all the

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left cells (or an *l.c.r. set* for brevity) of W_a in a two-sided cell Ω. $E(\Omega)$ is given in terms of left cell graphs $\mathcal{M}_L(x)$, $x \in P(\Omega)$, for a certain subset $P(\Omega)$ of Ω as follows.

(i) $P(\Omega) \subseteq E(\Omega)$;

(ii) There exists a bijective map $\psi : E(\Omega) \to \cup_{x\in P(\Omega)} M_L(x)$ ($M_L(x)$ is the vertex set of $\mathcal{M}_L(x)$) such that for any $y \in E(\Omega)$, $\psi(y)$ is the left cell of W_a containing y, and that there exists a unique $x \in P(\Omega)$ and a path $L_0 = \psi(x), L_1, ..., L_r = \psi(y)$ in $\mathcal{M}_L(x)$, where $\{\psi^{-1}(L_{i-1}), \psi^{-1}(L_i)\}$ is a string for any $1 \leqslant i \leqslant r$ (see 2.1).

The main technical difficulties for doing this is in applying Processes **B** and **C** since the jointed relation x —— y and the value $a(z)$ for $x, y, z \in W_a$ are hard to be determined in general. To avoid this obstruction, we find many (right) primitive pairs in a quite ingenious way.

By expressing the elements of the group $\widetilde{E}_6$ in their alcove forms, we use the computer programme GAP to execute Algorithm 3.6. However, finding various primitive pairs is a flexible and ingenious task, which has to be done by hands. Since the rank of $\widetilde{E}_6$ is higher (compare with the rank 4 cases in [**23**], [**24**], [**25**]), we only work out all the left cells L of $\widetilde{E}_6$ with $a(L) \leqslant 12$ so far.

Unlike what we did in the lower rank cases, $\widetilde{E}_6$ is the first group we have dealt with so far for which we have to apply process **C** to enlarge the set P in order to get an l.c.r. set of $\widetilde{E}_6$ in some two-sided cell Ω (e.g., when $\Omega = W_{(9)}$, see 4.11).

The contents of the paper are organized as follows. Sections 1–3 are served as preliminaries, we collect some concept, terms and known results there. We introduce Kazhdan–Lusztig cells and affine Weyl groups in Section 1, star operations, primitive pairs and generalized τ-invariants in Section 2, and an algorithm for finding an l.c.r. set in a two-sided cell in Section 3. Then in Sections 4, we concentrate our attention on the affine Weyl groups $W_a = \widetilde{E}_6$. we find an l.c.r. set for any two-sided cell Ω of W_a with $a(\Omega) \leqslant 12$ in terms of left cell graphs.

1. Preliminaries

1.1. Let **N** (resp., **Z**, **R**, **C**) be the set of all the non-negative integers (resp., integers, real numbers, complex numbers). Let W be a Coxeter group with S its distinguished generator set. Denote by $\leqslant$ (resp., l) the Bruhat-Chevalley order (resp., the length function) on W. Let $\mathcal{A} = \mathbf{Z}[u, u^{-1}]$ be the ring of all the Laurent polynomials in an indeterminate u with integer coefficients. The Hecke algebra $\mathcal{H}$ of W over $\mathcal{A}$ has two $\mathcal{A}$-bases $\{T_x\}_{x\in W}$ and $\{C_w\}_{w\in W}$ satisfying the following relations

$$\begin{cases} T_wT_{w'} = T_{ww'}, & \text{if } l(ww') = l(w) + l(w'), \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S, \end{cases}$$

and

$$C_w = \sum_{y \leqslant w} u^{l(w)-l(y)} P_{y,w}(u^{-2}) T_y,$$

where $P_{y,w} \in \mathbf{Z}[u]$ satisfies that $P_{w,w} = 1$, $\quad P_{y,w} = 0$ if $y \not\leqslant w$ and $\deg P_{y,w} \leqslant (1/2)(l(w) - l(y) - 1)$ if $y < w$. The $P_{y,w}$'s are called *Kazhdan-Lusztig polynomials* (see [**8**]).

1.2. For $y, w \in W$ with $l(y) \leqslant l(w)$, denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(l(w)-l(y)-1)}$ in $P_{y,w}$. The elements y and w are called *jointed*, written y —— w, if $\mu(y, w) \neq 0$. To any $x \in W$, we associate two subsets of S:

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

1.3. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on W defined as in [**8**], and let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation on W determined by $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$). The corresponding equivalence classes of W are called *left* (resp., *right*, *two-sided*) *cells* of W. $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) induces a partial order on the set of left (resp., right, two-sided) cells of W.

1.4. Define $h_{x,y,z} \in \mathcal{A}$ by

$$C_xC_y = \sum_z h_{x,y,z} C_z$$

for any $x, y, z \in W$. In [**12**], Lusztig defined a function $a : W \to \mathbf{N} \cup \{\infty\}$ by setting

$$a(z) = \min\{k \in \mathbf{N} \mid u^k h_{x,y,z} \in \mathbf{Z}[u], \ \forall x, y \in W\} \quad \text{for any } z \in W$$

with the convention that $a(z) = \infty$ if the minimum on the RHS of the above equation does not exist.

1.5. An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, reductive algebraic group over $\mathbf{C}$. Fix a maximal torus T of G, let X be the character group of T and let $\Phi \subset X$ be the root system of G with $\Delta = \{\alpha_1, ..., \alpha_l\}$ a choice of simple root system. Then $E = X \otimes_Z \mathbf{R}$ is a euclidean space with an inner product$\langle\ ,\ \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i, $1 \leqslant i \leqslant l$. Denote by N the group of all the translations $T_\lambda : x \mapsto x + \lambda$ on E with λ ranging over X. Then the semidirect product $W_a = N \rtimes W_0$ of W_0 with N is an *affine Weyl group*. Let K be the type dual to the type of G. Then the type of W_a is $\widetilde{K}$. In the case where no danger of confusion causes, W_a is denoted simply by its type $\widetilde{K}$. Let $w \mapsto \bar{w}$ be the canonical homomorphism from W_a to $W_0 \cong W_a/N$.

The following properties of the function a on (W_a, S) were proved by Lusztig:

(1) $x \underset{LR}{\leqslant} y \Longrightarrow a(x) \geqslant a(y)$. In particular, $x \underset{LR}{\sim} y \Longrightarrow a(x) = a(y)$. So we may define the value $a(\Gamma)$ for a left (resp., right, two-sided) cell Γ of W_a to be the common value $a(x)$ of all the $x \in \Gamma$ (see [**12**]).

(2) $a(x) = a(y)$ and $x \underset{L}{\leqslant} y$ (resp., $x \underset{R}{\leqslant} y) \Longrightarrow x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) (see [**13**]).

(3) Let $\delta(z) = \deg P_{e,z}$ for $z \in W_a$, where e is the identity of the group W_a. Define

$$\mathcal{D} = \{w \in W_a \mid l(w) = 2\delta(w) + a(w)\}$$

Then $\mathcal{D}$ is a finite set of involutions. Each left (resp., right) cell of W_a contains a unique element of $\mathcal{D}$ (called a *distinguished involution* of W_a)(see [**13**]).

(4) For any $I \subset S$, let w_I be the longest element in the subgroup W_I of W_a generated by I (note that W_I is always finite). Then $w_I \in \mathcal{D}$ and $a(w_I) = l(w_I)$ (see [**12**]).

Let $W_{(i)} = \{w \in W_a \mid a(w) = i\}$ for any $i \in \mathbf{N}$. Then the set $W_{(i)}$ is a union of some two-sided cells of W_a by (1).

(5) If $W_{(i)}$ contains an element of the form w_I for some $I \subset S$, then the set $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W_a by (1)–(2).

Call $s \in S$ *special* if the group $W_{S\setminus\{s\}}$ has the maximum possible order among all the standard parabolic subgroups of W_a of the form W_I, $I \subset S$. For $s \in S$, let

$$Y_s = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s\}\}.$$

Then Lusztig and Xi proved the following

(6) Let $s \in S$ be special. Then $\Omega \cap Y_s$ is non-empty and forms a single left cell of W_a for any two-sided cell Ω of W_a (see [**15**]).

Lusztig also proved the following

1.6. Theorem. (see [**14**]) *Let an algebraic group G and an affine Weyl group W_a be as in 1.5. Then there exists a bijective map $\mathbf{u} \mapsto c(\mathbf{u})$ from the set $\mathcal{U}(G)$ of unipotent conjugacy classes in G to the set* $\mathrm{Cell}(W)$ *of two-sided cells in W which satisfies the equation $a(c(\mathbf{u})) = \dim \mathcal{B}_u$, where u is any element in $\mathbf{u}$, and $\dim \mathcal{B}_u$ is the dimension of the variety of all the Borel subgroups of G containing u.*

1.7. Keep the notation in 1.5. Let $-\alpha_0$ be the highest short root in Φ. Denote $s_0 = s_{\alpha_0} T_{-\alpha_0}$, where s_{α_0} is the reflection in E with respect to α_0. Then $S = S_0 \cup \{s_0\}$ forms a Coxeter generator set of W_a.

The alcove form of an element $w \in W_a$ is, by definition, a Φ-tuple $(k(w,\alpha))_{\alpha\in\Phi}$ over $\mathbf{Z}$ determined by the following conditions.

(a) $k(w,-\alpha) = -k(w,\alpha)$ for any $\alpha \in \Phi$;

(b) $k(e,\alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity element of W_a;

(c) If $w' = ws_i$ $(0 \leqslant i \leqslant l)$, then

$$k(w',\alpha) = k(w,(\alpha)\overline{s}_i) + \varepsilon(\alpha,i)$$

with

$$\varepsilon(\alpha,i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

where $\overline{s}_i = s_i$ if $1 \leqslant i \leqslant l$, and $\overline{s}_0 = s_{\alpha_0}$.

By condition (a), we can also denote the alcove form of $w \in W_a$ by a Φ^+-tuple $(k(w,\alpha))_{\alpha\in\Phi^+}$, where Φ^+ is the positive root system of Φ containing Δ.

Condition (c) defines a set of operators $\{s_i \mid 0 \leqslant i \leqslant l\}$ on the alcove forms of elements w of W_a:

$$s_i : \quad (k(w;\alpha))_{\alpha\in\Phi} \longmapsto (k(w;(\alpha)\overline{s}_i) + \varepsilon(\alpha,i))_{\alpha\in\Phi}.$$

These operators could be described graphically (see [**22**]).

For $w, w' \in W_a$, w' is called a *left extension* of w if $l(w') = l(w) + l(w'w^{-1})$.

Then the following results were shown by Shi:

1.8. Proposition. (see [**18**]) *Let $w \in W_a$.*

(1) *$l(w) = \Sigma_{\alpha\in\Phi^+}|k(w,\alpha)|$, where the notation $|x|$ stands for the absolute value of $x \in \mathbf{Z}$;*

(2) *$\mathcal{R}(w) = \{s_i \mid k(w,\alpha_i) < 0\}$;*

(3) *w' is a left extension of w if and only if the inequalities $k(w',\alpha)k(w,\alpha) \geqslant 0$ and $|k(w',\alpha)| \geqslant |k(w,\alpha)|$ hold for any $\alpha \in \Phi^+$.*

2. Graphs, strings and generalized τ-invariants

In the present section, we assume that (W_a, S) is an irreducible affine Weyl group of simply-laced type, that is, the order $o(st)$ of the product st is not greater than 3 for any $s, t \in S$, or equivalently, W_a is of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.

2.1. Given $s \neq t$ in S with $o(st) = 3$, a set of the form $\{ys, yst\}$ is called a (right) $\{s,t\}$-*string* (or a *string* in short), if $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

We have the following result.

2.2. Proposition. (see [**22**]) *For* $s, t \in S$ *with* $o(st) = 3$*, let* $\{x_1, x_2\}$ *and* $\{y_1, y_2\}$ *be two* $\{s,t\}$*-strings. Then*

(1) $x_1 \text{ --- } y_1 \Longleftrightarrow x_2 \text{ --- } y_2$;

(2) $x_1 \underset{L}{\sim} y_1 \Longleftrightarrow x_2 \underset{L}{\sim} y_2$.

2.3. x is obtained from w by a (right) $\{s,t\}$-*star operation* (or a *star operation* in short), if $\{x, w\}$ is an $\{s,t\}$-string. Note that the resulting element x for an $\{s,t\}$-star operation on w is always unique whenever it exists.

Two elements $x, y \in W_a$ form a (right) *primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, \ldots, x_r$ and $y_0 = y, y_1, \ldots, y_r$ in W_a such that the following conditions are satisfied:

(a) For every $1 \leqslant i \leqslant r$, there exist some $s_i, t_i \in S$ with $o(s_i t_i) = 3$ such that both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are $\{s_i, t_i\}$-strings.

(b) $x_i \text{ --- } y_i$ for some (and then for all, under the condition (a)) $0 \leqslant i \leqslant r$ (see [**8**]).

(c) Either $\mathcal{R}(x) \not\subseteq \mathcal{R}(y)$ and $\mathcal{R}(y_r) \not\subseteq \mathcal{R}(x_r)$, or $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $\mathcal{R}(x_r) \not\subseteq \mathcal{R}(y_r)$ hold.

2.4. Proposition. (see [**22**]) $x \underset{R}{\sim} y$ *if* $\{x, y\}$ *is a primitive pair.*

In order to describe the left cells of W_a, we need introduce the concept of a left cell graph.

2.5. By a graph $\mathcal{M}$, we mean a set M of vertices together with a set of edges, where each edge is a two-element subset of M, and each vertex is labelled by some subset of S. A graph is *finite* if it contains finite number of vertices, and is *infinite* if otherwise.

By a path $\mathcal{P}$ in a graph $\mathcal{M}$, we mean a sequence of vertices $z_0, z_1, \ldots, z_r$ in M with some $r > 0$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathcal{M}$ for any $1 \leqslant i \leqslant r$. In this case, we say that the length of $\mathcal{P}$ is r.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two graphs with the vertex sets M and M' respectively. They are called *isomorphic*, written $\mathcal{M} \cong \mathcal{M}'$, if there exists a bijective map η from M to M' satisfying the following conditions.

(a) The label of $\eta(x)$ is the same as that of x for any $x \in M$.

(b) For $x, y \in M$, $\{x, y\}$ is an edge of $\mathcal{M}$ if and only if $\{\eta(x), \eta(y)\}$ is an edge of $\mathcal{M}'$.

This is an equivalence relation on graphs.

2.6. For any $x \in W_a$, denote by $M(x)$ the set of all the elements $y \in W_a$ such that there is a sequence of elements $x = x_0, x_1, \dots, x_r = y$ in W_a with some $r \geqslant 0$, where $\{x_{i-1}, x_i\}$ is a string for every $1 \leqslant i \leqslant r$.

Define a *graph* $\mathcal{M}(x)$ associated to an element $x \in W_a$ as follows. Its vertex set is $M(x)$; its edge set consists of all the two-element subsets in $M(x)$ each of which forms a string; each vertex $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$.

A *left cell graph* associated to an element $x \in W_a$, written $\mathcal{M}_L(x)$, is by definition a graph, whose vertex set $M_L(x)$ consists of all the left cells Γ of W_a with $\Gamma \cap M(x) \neq \emptyset$; two vertices $\Gamma, \Gamma' \in M_L(x)$ are jointed by an edge, if there are two elements $y \in M(x) \cap \Gamma$ and $y' \in M(x) \cap \Gamma'$ with $\{y, y'\}$ an edge of $\mathcal{M}(x)$; each vertex Γ of $\mathcal{M}_L(x)$ is labelled by the set $\mathcal{R}(\Gamma)$ (see 1.5 (1)).

Clearly, for any $x \in W$, both graphs $\mathcal{M}(x)$ and $\mathcal{M}_L(x)$ are connected.

2.7. Two elements $x, x' \in W_a$ are said to have the same *generalized τ-invariants*, if for any path $z_0 = x, z_1, \dots, z_r$ in $\mathcal{M}(x)$, there is a path $z'_0 = x', z'_1, \dots, z'_r$ in $\mathcal{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every $0 \leqslant i \leqslant r$, and if the same condition holds when the roles of x and x' are interchanged.

Then we have the following known result.

2.8. Proposition. (see [**22**] [**30**]) (a) *If $x \underset{L}{\sim} y$ in W_a then x, y have the same generalized τ-invariants.*

(b) *If $x \underset{L}{\sim} y$ in W_a, then the left cell graphs $\mathcal{M}_L(x)$ and $\mathcal{M}_L(y)$ are isomorphic.*

3. An algorithm for finding an l.c.r. set of W_a in a two-sided cell

3.1. A subset $K \subset W_a$ is called a *representative set for the left cells* (or an *l.c.r set* for brevity) of W_a (resp., of W_a in a two-sided cell Ω), if $|K \cap \Gamma| = 1$ for any left cell Γ of W_a (resp., of W_a in Ω), where the notation $|X|$ stands for the cardinality of a set X.

Obviously, the set $\mathcal{D}$ (see. 1.5 (3)) is an l.c.r. set of W_a. But it is not easy to find the whole set $\mathcal{D}$ of W_a explicitly in general since it may involve the complicated computation of Kazhdan-Lusztig polynomials.

We shall obtain an l.c.r. set $E(\Omega)$ of W_a in a two-sided cell Ω in terms of left cell graphs $\mathcal{M}_L(x)$, $x \in P(\Omega)$, for a certain subset $P(\Omega)$ of Ω as follows.

(i) $P(\Omega) \subseteq E(\Omega)$;

(ii) There exists a bijective map $\psi : E(\Omega) \to \cup_{x \in P(\Omega)} M_L(x)$ ($M_L(x)$ is the vertex set of $\mathcal{M}_L(x)$) such that for any $y \in E(\Omega)$, $\psi(y)$ is the left cell of W_a containing y, and that there exists a unique $x \in P(\Omega)$ and a path $L_0 = \psi(x), L_1, ..., L_r = \psi(y)$ in $\mathcal{M}_L(x)$, where $\{\psi^{-1}(L_{i-1}), \psi^{-1}(L_i)\}$ is a string for any $1 \leqslant i \leqslant r$.

Note that such an l.c.r. set $E(\Omega)$ of W_a in Ω can be easily obtained from the set $P(\Omega)$ and the corresponding left cell graphs $\mathcal{M}_L(x)$, $x \in P(\Omega)$. However, $E(\Omega)$ is not uniquely determined by these data in general. It is so if and only if the set $\cup_{x\in P(\Omega)} M(x)$ is distinguished (see 3.4).

Shi constructed an algorithm for finding an l.c.r. set of W_a in a two-sided cell, which is based on the following

3.2. Theorem. (see [**22**]) *Let Ω be a two-sided cell of W_a. Then a non-empty subset $E \subset \Omega$ is an l.c.r. set of W_a in Ω, if E satisfies the following conditions:*

(1) *$x \underset{L}{\not\sim} y$ for any $x \neq y$ in E;*

(2) *For any $y \in W_a$, if there exists some $x \in E$ satisfying that y —— x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$, then there exists some $z \in E$ with $y \underset{L}{\sim} z$.*

3.3. We know that the relations y —— x and $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ hold if and only if one of the following cases occurs:

(1) $\{x, y\}$ is a string;

(2) $y = x \cdot s$ for some $s \in S$ with $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$, where by the notation $a = b \cdot c$ $(a, b, c \in W_a)$, we mean $a = bc$ and $l(a) = l(b) + l(c)$;

(3) $y < x$, y —— x and $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$.

3.4. A subset $P \subset W_a$ is called *distinguished* if $P \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in P.

For a non-empty subset in a two-sided cell Ω of W_a, consider the following processes (see [**22**]).

(**A**) Find a distinguished subset Q of the largest possible cardinality from the set $\bigcup_{x\in P} M(x)$.

(**B**) Let $B_x = \{y \in W_a \mid y = x \cdot s \notin M(x) \text{ for some } s \in S \text{ with } a(y) = a(x)\}$ for any $x \in P$. Find a distinguished subset Q of the largest possible cardinality from the set $B = P \cup (\cup_{x\in P} B_x)$.

(**C**) Let $C_x = \{y \in W_a \mid y < x, y \text{ —— } x, \mathcal{R}(y) \supsetneqq \mathcal{R}(x), a(y) = a(x)\}$ for any $x \in P$. Find a distinguished subset Q of the largest possible cardinality from the set $C = P \cup (\cup_{x\in P} C_x)$.

3.5. A subset P of W_a is **A**-*saturated* (resp., **B**-*saturated,* **C**-*saturated*), if the Process **A** (resp., **B**, **C**) on P cannot produce any element z with $z \underset{L}{\not\sim} x$ for all $x \in P$.

Clearly, a set of the form $\bigcup_{x\in K} M(x)$ for any $K \subseteq W_a$ is always **A**-saturated.

It follows from Theorem 3.2 that an l.c.r. set of W_a in a two-sided cell Ω is exactly a distinguished subset of Ω which is **ABC**-saturated simultaneously. In order to get such a subset, we apply the following algorithm.

3.6. Algorithm. (see [**22**]).

(1) Find a non-empty subset P of Ω (It is usual to take P distinguished and consisting of elements of the form w_I, $I \subset S$, whenever it is possible);

(2) Perform Processes **A**, **B** and **C** alternately on P until the resulting distinguished set cannot be further enlarged by any of these processes.

4. The left cells of the affine Weyl group $\widetilde{E}_6$

In this section, we shall explicitly describe all the left cells of the affine Weyl group $W_a = \widetilde{E}_6$ in all the two-sided cells Ω with $a(\Omega) \leqslant 12$. We shall find an l.c.r. set in terms of left cell graphs for each of such two-sided cells by applying Algorithm 3.6 (in the way explained in 3.1). This will be achieved by expressing the elements of W_a in their alcove forms and then in virtue of the computer programme GAP. The work is hard in applying Process **B** and is even harder in applying Process **C** since it is not easy to determine the joint relations and the a-values for the related elements in general. To avoid these difficult points, we find various primitive pairs in an ingenious way.

4.1. The Coxeter graph of the group $\widetilde{E}_6$ is as follows.

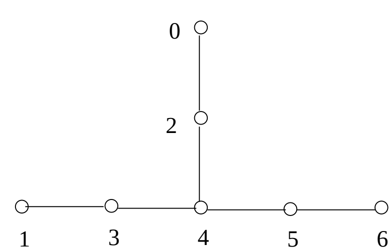


Figure 1.

Recall the notation $W_{(i)}$ for $i \geqslant 0$ in 1.5. By Theorem 1.6, $W_{(i)}$ is a single two-sided cell of $\widetilde{E}_6$ if $i \in \{0, 1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20, 25, 36\}$, and is a union of two two-sided cells of $\widetilde{E}_6$ if $i \in \{4, 6\}$ (see [**2**, Chapter 13]).

For $i \in \mathbf{N}$, denote by $P(i)$ the set of all the elements in $W_{(i)}$ of the form w_I for some $I \subset S$.

For the sake of simplifying the notation, denote by $\mathbf{i}$ (bold-faced) the reflection s_i corresponding to the vertex in Figure 1.

4.2. For any $i \neq j$ in $\{1, 0, 6\}$, let ψ_{ij} be the unique automorphism of $\widetilde{E}_6$ which stabilizes the set S and transposes $\mathbf{i}$ and $\mathbf{j}$. For example, we have

$$(\psi_{10}(\mathbf{0}), \psi_{10}(\mathbf{1}), \psi_{10}(\mathbf{2}), \psi_{10}(\mathbf{3}), \psi_{10}(\mathbf{4}), \psi_{10}(\mathbf{5}), \psi_{10}(\mathbf{6})) = (\mathbf{1}, \mathbf{0}, \mathbf{3}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}).$$

Then ψ_{ij} preserves the value of the function a and the joint relation on elements, i.e., for any $x, y, z \in \widetilde{E}_6$, we have $a(\psi(z)) = a(z)$, and x — y if and only if $\psi_{ij}(x)$ — $\psi_{ij}(y)$. Hence ψ_{ij} stabilizes the set $W_{(k)}$ for any $k \geqslant 0$.

4.3. The two-sided cell $W_{(0)}$ consists of a single element: the identity element of the group $\widetilde{E}_6$. The two-sided cell $W_{(1)}$ consists of all the non-identity elements of $\widetilde{E}_6$ each of which has a unique reduced expression. The set $E(W_{(1)}) = S$ forms an l.c.r. set of $W_{(1)}$ (see [**10**]). The left cell graph of $W_{(1)}$ is isomorphic to Fig. A (see Appendix).

4.4. For the two-sided cell $W_{(2)}$, we have

$$P(2) = \{\mathbf{14},\ \mathbf{15},\ \mathbf{16},\ \mathbf{12},\ \mathbf{10},\ \mathbf{32},\ \mathbf{30},\ \mathbf{35},\ \mathbf{36},\ \mathbf{40},\ \mathbf{46},\ \mathbf{25},\ \mathbf{26},\ \mathbf{05},\ \mathbf{06}\}.$$

The graph $\mathcal{M}(\mathbf{12})$ is infinite. Take a connected subgraph $\mathcal{M}'(\mathbf{12})$ from $\mathcal{M}(\mathbf{12})$ as in Fig. B with the vertex labelled by $\boxed{\mathbf{12}}$ being the element $\mathbf{12}$ (see Appendix). Then its vertex set $M'(\mathbf{12})$ is distinguished by Proposition 2.9, and is also **ABC**-saturated. So

$$E(W_{(2)}) = M'(\mathbf{12})$$

forms an l.c.r. set of $W_{(2)}$ by Theorem 3.2 (in the subsequent discussion, we shall frequently apply Proposition 2.9 and Theorem 3.2 but without mentioning them explicitly).

4.5. For the two-sided cell $W_{(3)}$, we have

$$P(3) = \{\mathbf{131},\ \mathbf{343},\ \mathbf{424},\ \mathbf{454},\ \mathbf{202},\ \mathbf{565},\ \mathbf{146},\ \mathbf{140},\ \mathbf{150},\ \mathbf{152},\ \mathbf{162},\ \mathbf{160},\ \mathbf{352},\ \mathbf{350},\ \mathbf{362},\ \mathbf{360},\ \mathbf{460}\}.$$

Consider the graph $\mathcal{M}(\mathbf{131})$ (see Fig. $C1$ in Appendix). Its vertex set $M(\mathbf{131})$ is distinguished, and is also **A**-saturated, but not **B**-saturated. Take $x = \mathbf{131420} \in M(\mathbf{131})$ and $x' = x \cdot \mathbf{5}$. We see from Fig. 2 (a) that $\{x, x'\}$ forms a primitive pair. Hence $x' \in W_{(3)}$ by Proposition 2.2 and 1.5 (1) (in the subsequent discussion, we shall frequently apply Proposition 2.2 and 1.5 (1) to primitive pairs but without mentioning them explicitly). The graph $\mathcal{M}(x')$ is isomorphic to the graph $\mathcal{M}(\mathbf{140})$ (see Fig. $C2$ in Appendix). By 1.5 (5) and

Proposition 2.3, the sets $M(x')$ and $M(\mathbf{140})$ represent the same set of left cells in $W_{(3)}$ since both contain a vertex labelled by $\boxed{\mathbf{140}}$. The set

$$E(W_{(3)}) = M(\mathbf{131}) \cup M(\mathbf{140})$$

is distinguished and also **ABC**-saturated. Thus it forms an l.c.r. set of $W_{(3)}$.

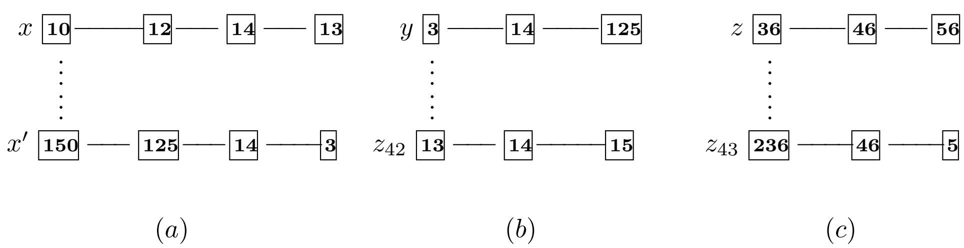


Figure 2.

4.6. There are two two-sided cells in $W_{(4)}$ (see 4.1). We have

$$P(4) = \{\, w_{132}, w_{130}, w_{135}, w_{136}, w_{340}, w_{346}, w_{450}, w_{451}, w_{241}, w_{246}, \\ w_{021}, w_{023}, w_{025}, w_{026}, w_{562}, w_{560}, w_{561}, w_{563}, w_{1046} \,\}.$$

where the notation $w_{ij\cdots k}$ stands for the longest element in the subgroup of $\widetilde{E}_6$ generated by $\mathbf{i}, \mathbf{j}, ..., \mathbf{k}$. The graph $\mathcal{M}(\mathbf{1312})$ is infinite. Take a subgraph $\mathcal{M}'(\mathbf{1312})$ in $\mathcal{M}(\mathbf{1312})$ as in Fig. D with the vertex labelled by $\boxed{\mathbf{1312}}$ being the element $\mathbf{1312}$ (see Appendix). Then its vertex set $\mathcal{M}'(\mathbf{1312})$ is distinguished, and also **ABC**-saturated. Let $W^1_{(4)}$ be the two-sided cell of $\widetilde{E}_6$ containing the element $\mathbf{1312}$. Then the set

$$E(W^1_{(4)}) = M'(\mathbf{1312})$$

forms an l.c.r. set of $W^1_{(4)}$.

Since there is no vertex in $\mathcal{M}'(\mathbf{1312})$ with the label $\boxed{\mathbf{1046}}$, we have $z_{41} = \mathbf{1046} \notin W^1_{(4)}$. Let $W^2_{(4)}$ be the two-sided cell of $\widetilde{E}_6$ containing the element z_{41}. The graph $\mathcal{M}(z_{41})$ is as in Fig. E (see Appendix), whose vertex set $M(z_{41})$ is distinguished and also **A**-saturated, but not **B**-saturated. Let $y = z_{41} \cdot \mathbf{5243} \in M(z_{41})$ and $z_{42} = y \cdot \mathbf{1}$. Thus we see from Fig. 2 (b) that $\{y, z_{42}\}$ forms a primitive pair. Hence $z_{42} \in W^2_{(4)}$. The graph $\mathcal{M}(z_{42})$ is as in Fig. $C1$ (see Appendix). But the set $M(z_{41}) \cup M(z_{42})$ is still not **B**-saturated. Let $z = z_{42} \cdot \mathbf{4156} \in M(z_{42})$ and $z_{43} = z \cdot \mathbf{2}$. Then we see from Fig. 2 (c) that $\{z, z_{43}\}$ forms a primitive pair. So $z_{43} \in W^2_{(4)}$. The graph $\mathcal{M}(z_{43})$ is as in Fig. $C2$ (see Appendix). The set

$$E(W^2_{(4)}) = \bigcup_{i=1}^{3} M(z_{4i})$$

is distinguished and also **ABC**-saturated. Thus it forms an l.c.r. set of $W^2_{(4)}$.

The results in 4.6 were obtained in our previous paper [**29**].

4.7. The set $W_{(5)}$ forms a single two-sided cell of W by 4.1. We have

$$P(5) = \{ w_{1325}, w_{1326}, w_{1305}, w_{1306}, w_{5612}, w_{5610}, w_{5623}, w_{5630}, \\ w_{0216}, w_{0215}, w_{0236}, w_{0235}, w_{3406}, w_{2416}, w_{4510} \}.$$

The graph $\mathcal{M}(\mathbf{13125})$ is infinite. Take a subgraph $\mathcal{M}'(\mathbf{13125})$ of $\mathcal{M}(\mathbf{13125})$ as in Fig. F (see Appendix). Its vertex set

$$E(W_{(5)}) = M'(\mathbf{13125})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(5)}$.

4.8. There are two two-sided cells in $W_{(6)}$ by 4.1. We have

$$P(6) = \{ w_{134}, w_{456}, w_{024}, w_{234}, w_{245}, w_{345}, w_{1356}, w_{1302}, w_{0256} \}.$$

The graph $\mathcal{M}(z_{61})$ with $z_{61} = \mathbf{143143}$ is infinite, Take a subgraph $\mathcal{M}'(z_{61})$ of $\mathcal{M}(z_{61})$ as in Fig.s G1–2 (see Appendix). Its vertex set

$$E(W^1_{(6)}) = M'(z_{61})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of the two-sided cell $W^1_{(6)}$ containing z_{61}.

Since no vertex in $\mathcal{M}'(z_{61})$ is labelled by $\boxed{\mathbf{1356}}$, $\boxed{\mathbf{1320}}$ or $\boxed{\mathbf{0256}}$, the elements $z_{62} = \mathbf{131565}$, $z_{63} = \mathbf{131202}$, $z_{64} = \mathbf{202565}$ in $P(6)$ are in the two-sided cell $W^2_{(6)} = W_{(6)} \setminus W^1_{(6)}$. The graphs $\mathcal{M}(z_{62})$, $\mathcal{M}(z_{63})$, $\mathcal{M}(z_{64})$ are as in Fig.s H, $\psi_{06}(H)$, $\psi_{01}(H)$, respectively (see Appendix), where the graph $\psi_{ij}(H)$ is obtained from H by applying the automorphism ψ_{ij} of $\widetilde{E}_6$ (see 4.2). Take $x = z_{62} \cdot \mathbf{425434120456243}$ and $y = z_{62} \cdot \mathbf{425434120456245}$ in $M(z_{62})$ and let $z = x \cdot \mathbf{1}$, $w = y \cdot \mathbf{6}$.

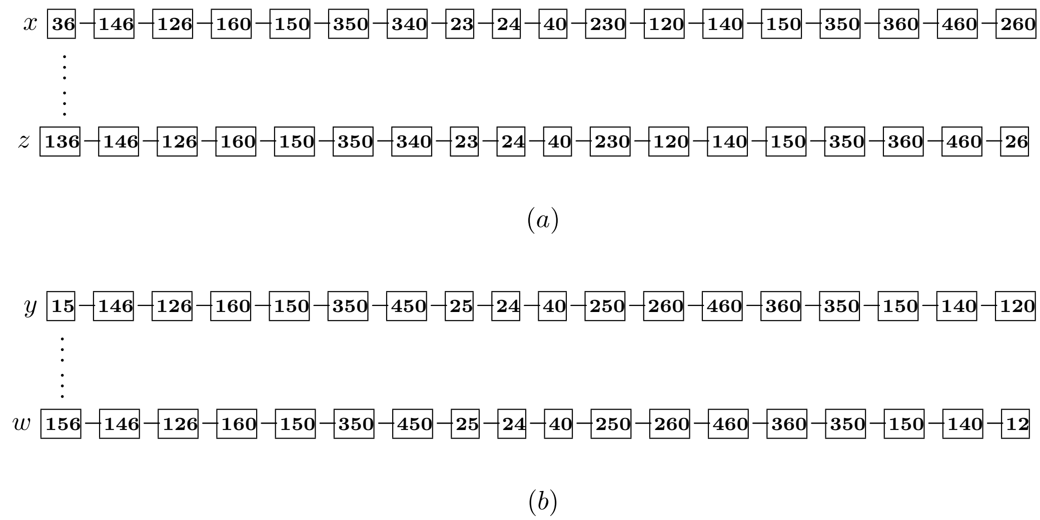


Figure 3.

We see from Fig. 3 (a)–(b) that both $\{x, z\}$ and $\{y, w\}$ are primitive pairs. We have $\mathcal{M}(z) \cong \mathcal{M}(z_{63})$ (resp., $\mathcal{M}(w) \cong \mathcal{M}(z_{64})$) since $a(z) = a(z_{63}) = 6$ (resp., $a(w) = a(z_{64}) = 6$) and both graphs contain a vertex of the label $\boxed{\mathbf{0256}}$ (resp., $\boxed{\mathbf{1302}}$). The set

$$E(W^2_{(6)}) = \bigcup_{i=2}^{4} M(z_{6i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W^2_{(6)}$.

By 4.1, we see that $W_{(i)}$ is a single two-sided cell of $\widetilde{E}_6$ for any $i > 6$ with $W_{(i)} \neq \emptyset$.

4.9. Consider the two-sided cell $W_{(7)}$ of W. We have

$$P(7) = \{ w_{1346}, w_{1340}, w_{0246}, w_{0241}, w_{4561}, w_{4560}, w_{2346}, w_{2451}, \\ w_{3450}, w_{13562}, w_{13560}, w_{13025}, w_{13026}, w_{02561}, w_{02563} \}.$$

The graph $\mathcal{M}(z_{71})$ with $z_{71} = \mathbf{1431436}$ is infinite. Take a subgraph $\mathcal{M}'(z_{71})$ of $\mathcal{M}(z_{71})$ as in Fig. *I* (see Appendix). Then its vertex set $M'(z_{71})$ is distinguished. Let $x = z_{71} \cdot \mathbf{5464} \in M'(z_{71})$ and $x' = x \cdot \mathbf{2}$. We see from Fig. 4 (a) that $\{x, x'\}$ forms a primitive pair. Take $y = \mathbf{143143020}$ in $M(\mathbf{1431430})$ and $y' = y \cdot \mathbf{3}$. Also, take $z = \mathbf{465465020}$ in $M(\mathbf{4654650})$ and

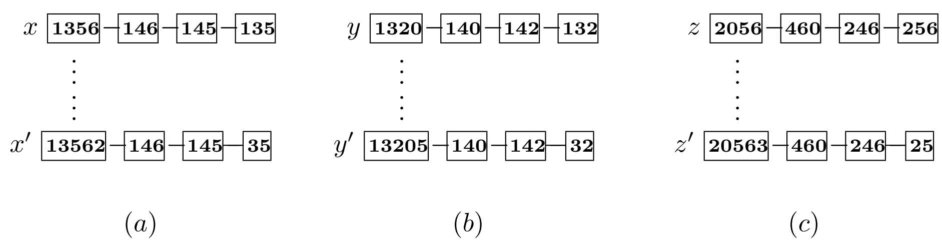


Figure 4.

$z' = z \cdot \mathbf{3}$. Then by Fig. 4 (b)–(c), both $\{y, y'\}$ and $\{z, z'\}$ are primitive pairs. Since they all contain an element with the label $\boxed{\mathbf{1340}}$, the sets $M(z_{71})$, $M(\mathbf{1431430})$, $M(\mathbf{4654650})$ represent the same set of left cells of $\widetilde{E}_6$ by 1.5 (5), i.e., for any left cell L of $\widetilde{E}_6$, the intersections $M(z_{71}) \cap L$, $M(\mathbf{1431430}) \cap L$, $M(\mathbf{4654650}) \cap L$ are either all empty or all non-empty. Let $z_{72} = \mathbf{2025653}$, $z_{73} = \mathbf{1312025}$ and $z_{74} = \mathbf{1315652}$. The graphs $\mathcal{M}(z_{72})$, $\mathcal{M}(z_{73})$, $\mathcal{M}(z_{74})$ are as in Fig.s $\psi_{01}(J)$, $\psi_{06}(J)$, J, respectively (see Appendix). Then the graph $\mathcal{M}(z')$ (resp., $\mathcal{M}(y')$, $\mathcal{M}(x')$) is isomorphic to $\mathcal{M}(z_{72})$ (resp., $\mathcal{M}(z_{73})$, $\mathcal{M}(z_{74})$) since both contain a vertex with the label $\boxed{\mathbf{20563}}$ (resp., $\boxed{\mathbf{13205}}$, $\boxed{\mathbf{13562}}$). The set

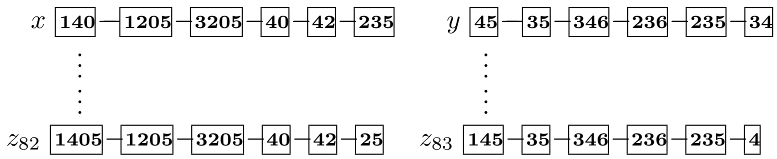
$$E(W_{(7)}) = M'(z_{71}) \cup \left(\bigcup_{i=2}^{4} M(z_{7i}) \right)$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(7)}$.

4.10. Next consider the two-sided cell $W_{(8)}$. We have

$$P(8) = \{w_{13406}, w_{45601}, w_{02416}\}.$$

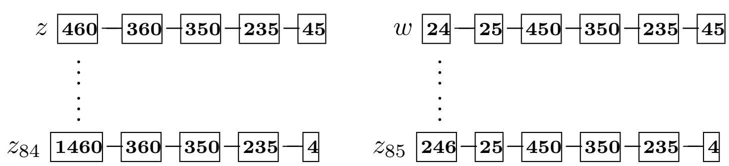
The graph $\mathcal{M}(z_{81})$ with $z_{81} = \mathbf{14314360}$ is infinite. Take a subgraph $\mathcal{M}'(z_{81})$ of $\mathcal{M}(z_{81})$ as in Fig. $K1$ with the vertex labelled by $\boxed{\mathbf{13460}}$ being the element z_{81} (see Appendix). Let $x = z_{81} \cdot \mathbf{52420} \in M'(z_{81})$ and $z_{82} = x \cdot \mathbf{5}$. We see from Fig. 5 (a) that $\{x, z_{82}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{82})$ is as in Fig. I (see Appendix). Take $y = z_{82} \cdot \mathbf{065345} \in M(z_{82})$ and $z_{83} = y \cdot \mathbf{1}$. We see from Fig. 5 (b) that $\{y, z_{83}\}$ forms a primitive pair. Take $z = z_{82} \cdot \mathbf{23456}$



(a) (b)

Figure 5.

and $w = z_{82} \cdot \mathbf{065234614520425142}$ in $M(z_{82})$. Let $z_{84} = z \cdot \mathbf{1}$ and $z_{85} = w \cdot \mathbf{6}$. We see from Fig. 6 (a)–(b) that both $\{z, z_{84}\}$ and $\{w, z_{85}\}$ are primitive pairs. The graphs $\mathcal{M}(z_{83})$,



(a) (b)

Figure 6.

$\mathcal{M}(z_{84})$, $\mathcal{M}(z_{85})$ are as in Fig.s J, $\psi_{06}(J)$, $\psi_{01}(J)$, respectively (see Appendix). Then the set

$$E(W_{(8)}) = M'(z_{81}) \cup \left(\bigcup_{i=2}^{5} M(z_{8i}) \right)$$

is distinguished and also **ABC**-saturated, thus it forms an l.c.r. set of $W_{(8)}$.

4.11. Consider the two-sided cell $W_{(9)}$. We have

$$P(9) = \{w_{130256}\}.$$

The graph $\mathcal{M}(z_{91})$ with $z_{91} = \mathbf{131202565}$ is as in Fig. $K2$ (see Appendix). Take $x = z_{91} \cdot \mathbf{42534} \in M(z_{91})$ and $z_{92} = x \cdot \mathbf{3}$. Then we see from Fig. 7 (a) that $\{x, z_{92}\}$ forms a

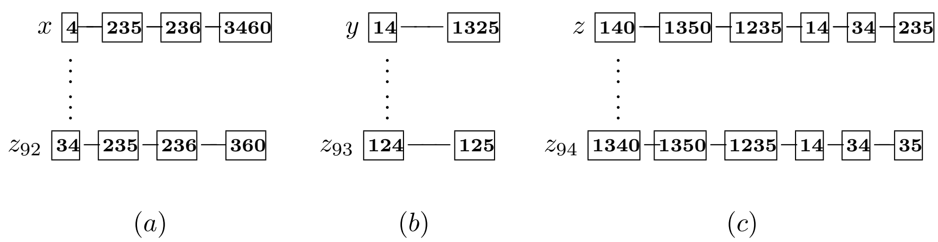


Figure 7.

primitive pair. The graph $\mathcal{M}(z_{92})$ is as in Fig. $K1$ (see Appendix). Let $y = z_{92} \cdot \mathbf{1} \in M(z_{92})$ and $z_{93} = y \cdot \mathbf{2}$. We see from Fig, 7 (b) that $\{y, z_{93}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{93})$ is as in Fig. $K3$ (see Appendix). Take $z = z_{92} \cdot \mathbf{1404} \in M(z_{92})$ and $z_{94} = z \cdot \mathbf{3}$. We see from Fig. 7 (c) that $\{z, z_{94}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{94})$ is as in Fig. I (see Appendix). The set $M(z_{91}) \cup M(z_{92}) \cup M(z_{93}) \cup M(z_{94})$ is distinguished and also **AB**-saturated, but is not **C**-saturated. Take

$$w = z_{91} \cdot \mathbf{4235423654130245\dot{5}620}, \qquad u = \mathbf{561302454324130245632450243\dot{3}1},$$

$$z_{95} = z_{91} \cdot \mathbf{423542365413024 \cdot 620}, \qquad z_{96} = \mathbf{5613024543241302456324502 4 \cdot 1},$$

$$v = z_{91} \cdot \mathbf{42345645324 1\dot{3}0245},$$

$$z_{97} = z_{91} \cdot \mathbf{423456453241 \cdot 0245}.$$

in $M(z_{94})$. The graphs $\mathcal{M}(z_{95})$, $\mathcal{M}(z_{96})$, $\mathcal{M}(z_{97})$ are as in Fig.s J, $\psi_{06}(J)$, $\psi_{01}(J)$, respectively (see Appendix). By Fig. 8 (a)–(c), $\{w, z_{95}\}$, $\{u, z_{96}\}$ and $\{v, z_{97}\}$) are all primitive pairs. The set

$$E(W_{(9)}) = \bigcup_{i=1}^{7} M(z_{9i})$$

is distinguished and also **ABC**-saturated, so it forms an l.c.r. set of $W_{(9)}$.

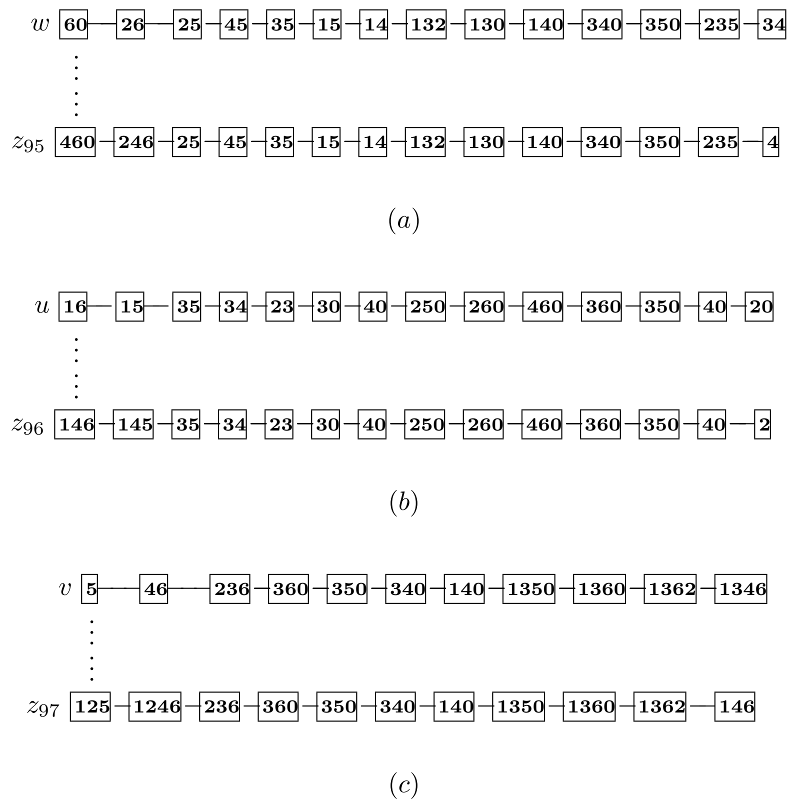


Figure 8

4.12. In the two-sided cell $W_{(10)}$, we have

$$P(10) = \{w_{1345}, w_{1342}, w_{0243}, w_{0245}, w_{3456}, w_{2456}\}.$$

The graph $\mathcal{M}(z_{10,1})$ with $z_{10,1} = \mathbf{1342134131}$ is as in the Fig.s $L1$–10 (see [**28**]). Its vertex set $M(z_{10,1})$ is not **B**-saturated. Take

$x = z_{10,1} \cdot \mathbf{054652543}$, $y = z_{10,1} \cdot \mathbf{05465342403541324534}$,

$z = z_{10,1} \cdot \mathbf{05465342405253413542}$, $w = z_{10,1} \cdot \mathbf{543204146412454624234132463}$,

$u = z_{10,1} \cdot \mathbf{02456452434153520451424}$

in $M(z_{10,1})$. Let $z_{10,2} = x \cdot \mathbf{0}$, $z_{10,3} = y \cdot \mathbf{0}$, $z_{10,4} = z \cdot \mathbf{4}$, $z_{10,5} = w \cdot \mathbf{0}$, $z_{10,6} = u \cdot \mathbf{6}$.

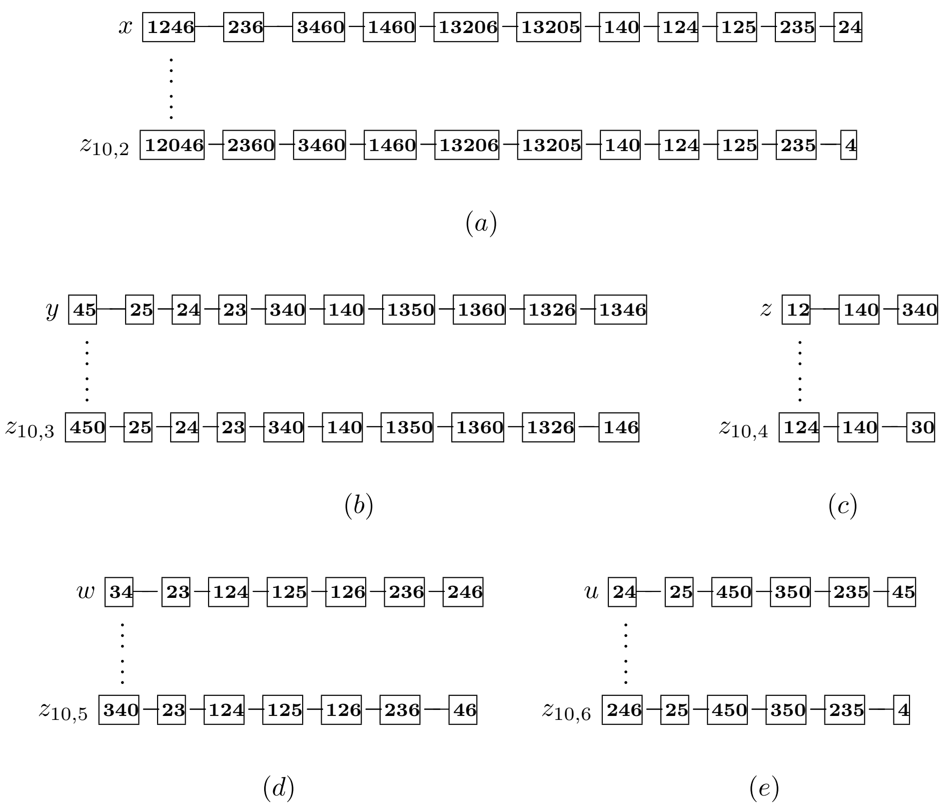


Figure 9.

The graphs $\mathcal{M}(z_{10,2})$, $\mathcal{M}(z_{10,3})$, $\mathcal{M}(z_{10,4})$, $\mathcal{M}(z_{10,5})$ and $\mathcal{M}(z_{10,6})$) are as in Fig.s $K1$, I, J, $\psi_{06}(J)$, $\psi_{01}(J)$, respectively (see Appendix). From the graphs in Fig. 9 (a)-(e), we see that $\{x, z_{10,2}\}$, $\{y, z_{10,3}\}$, $\{z, z_{10,4}\}$, $\{w, z_{10,5}\}$ and $\{u, z_{10,6}\}$ are all primitive pairs. The set

$$E(W_{(10)}) = \bigcup_{i=1}^{6} M(z_{10,i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(10)}$.

4.13. In the two-sided cell $W_{(11)}$, we have

$$P(11) = \{w_{13450}, w_{13426}, w_{02436}, w_{02451}, w_{34560}, w_{24561}\}.$$

The graph $\mathcal{M}(z_{11,1})$ with $z_{11,1} = \mathbf{13421341316}$ is as in Fig.s $M1$–13 (see [**28**]). Its vertex set $M(z_{11,1})$ is not **B**-saturated. Take

$x = z_{11,1} \cdot \mathbf{02543}$, $w = z_{11,1} \cdot \mathbf{0245342340463543451342403235465345 0}$,

$y = z_{11,1} \cdot \mathbf{543241464021454264234162434 53}$, $z = z_{11,1} \cdot \mathbf{54324146254204254342132546521 45}$

in $M(z_{11,1})$. Let $z_{11,2} = x \cdot \mathbf{6}$, $z_{11,3} = y \cdot \mathbf{0}$, $z_{11,4} = z \cdot \mathbf{0}$, $z_{11,5} = w \cdot \mathbf{1}$. From Fig. 10 (a)–(d),

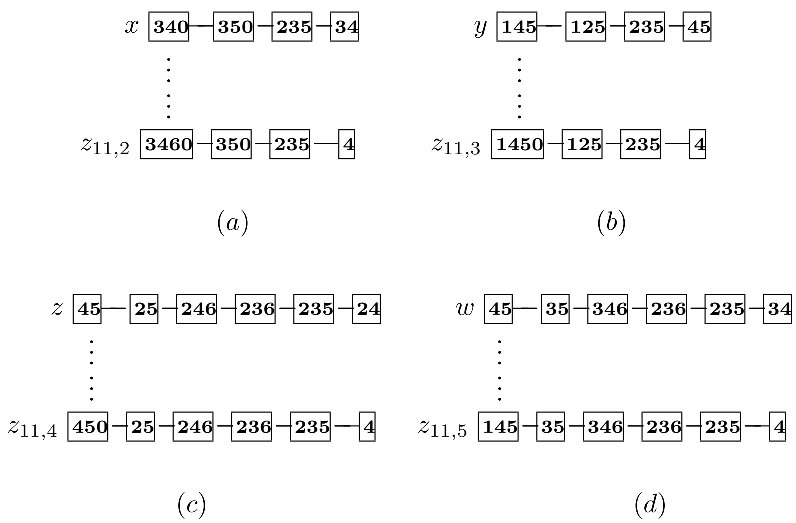


Figure 10.

we see that $\{x, z_{11,2}\}$, $\{y, z_{11,3}\}$, $\{z, z_{11,4}\}$ and $\{w, z_{11,5}\}$ are all primitive pairs. The graphs $\mathcal{M}(z_{11,2})$, $\mathcal{M}(z_{11,3})$, $\mathcal{M}(z_{11,4})$, $\mathcal{M}(z_{11,5})$ are as in Fig.s $K2$, $\psi_{06}(J)$, $\psi_{01}(J)$, J, respectively (see Appendix). We may check that the set

$$E(W_{(11)}) = \bigcup_{i=1}^{5} M(z_{11,i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(11)}$.

4.14. In the two-sided cell $W_{(12)}$, we have

$$P(12) = \{w_{2345}\}.$$

Let w_0 be the longest element in the Weyl group E_6. Then by [**2**]Chapter 13, we have $z_{12,1} = w_0 w_{1356} \in W_{(12)}$. The graph $\mathcal{M}(z_{12,1})$ is as in Fig. N (see Appendix). Take $x = z_{12,1} \cdot \mathbf{452340}$, $y = z_{12,1} \cdot \mathbf{04352}$ and $z = y \cdot \mathbf{534}$ in $M(z_{12,1})$ and let $z_{12,2} = x \cdot \mathbf{2}$, $z_{12,3} = y \cdot \mathbf{4}$, $z_{12,4} = z \cdot \mathbf{2}$. We see from Fig. 11 (a)-(c) that $\{x, z_{12,2}\}$, $\{y, z_{12,3}\}$ and

$\{z, z_{12,4}\}$ are all primitive pairs. The graph $\mathcal{M}(z_{12,2})$ (resp., $\mathcal{M}(z_{12,3})$, $\mathcal{M}(z_{12,4})$) is as in Fig. K2 (resp., K1, K3) (see Appendix). Take $w = z_{12,4} \cdot \mathbf{3} \in M(z_{12,4})$, $z_{12,5} = w \cdot \mathbf{5}$; $v = z_{12,5} \cdot \mathbf{6452435436045321302} \in M(z_{12,5})$, $z_{12,6} = v \cdot \mathbf{0}$; $u = z_{12,6} \cdot \mathbf{546424320542} \in M(z_{12,6})$, $z_{12,7} = u \cdot \mathbf{0}$; $t = z_{12,7} \cdot \mathbf{24142340424624}$, $s = z_{12,7} \cdot \mathbf{24642540424214}$ in $M(z_{12,7})$ $z_{12,8} = t \cdot \mathbf{2}$, $z_{12,9} = t \cdot \mathbf{3}$, $z_{12,10} = s \cdot \mathbf{5}$. Then by Fig. 11 (d)-(i), we see that $\{w, z_{12,5}\}$, $\{v, z_{12,6}\}$, $\{u, z_{12,7}\}$, $\{t, z_{12,8}\}$, $\{t, z_{12,9}\}$ and $\{s, z_{12,10}\}$ are all primitive pairs. The graphs $\mathcal{M}(z_{12,5})$, $\mathcal{M}(z_{12,6})$, $\mathcal{M}(z_{12,7})$, $\mathcal{M}(z_{12,8})$, $\mathcal{M}(z_{12,9})$, $\mathcal{M}(z_{12,10})$ are Fig.s I, $K1$, I, J, $\psi_{01}(J)$, $\psi_{06}(J)$, respectively (see Appendix). One can check that the set

$$E(W_{(12)} = \bigcup_{i=1}^{10} M(z_{12,i})$$

is distinguished and also **ABC**-saturated, so it forms an l.c.r. set of $W_{(12)}$.

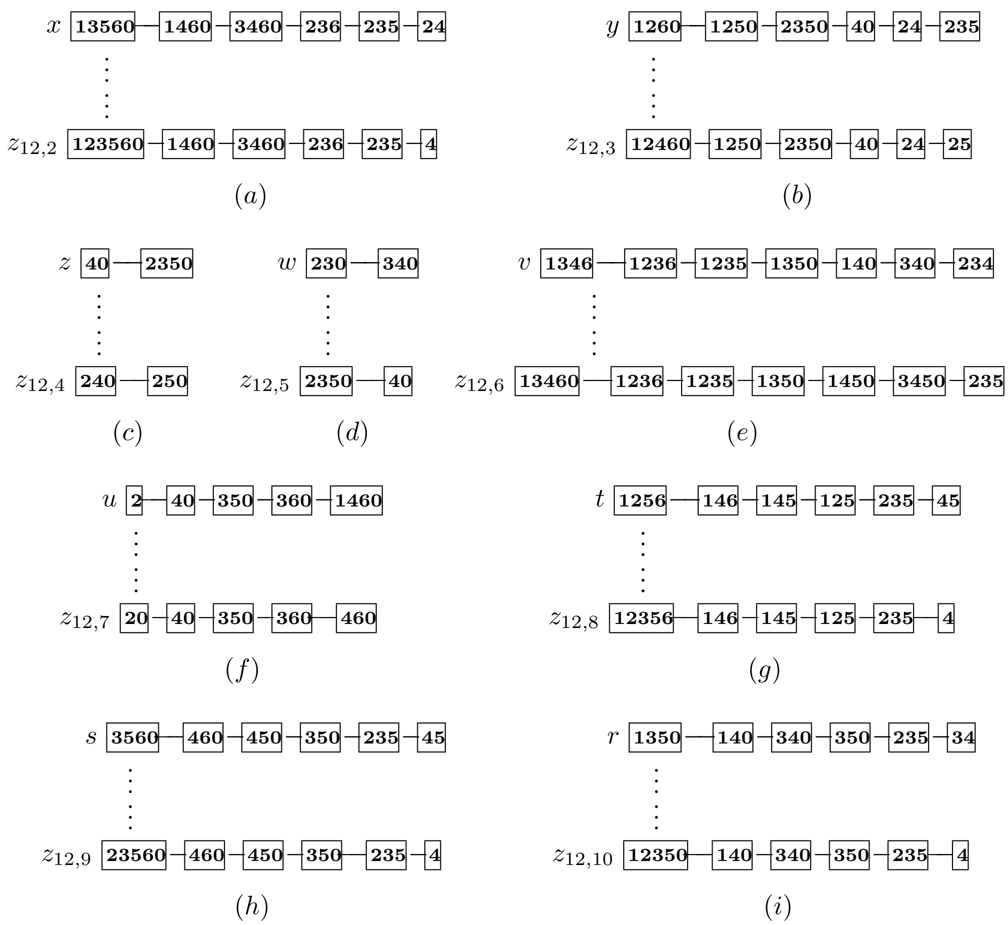


Figure 11

4.15. Keep the notation in 1.5–1.6 with G the reductive algebraic group of type E_6. For a unipotent class $\mathbf{u}$ of G, let $c(\mathbf{u})$ be the corresponding two-sided cell of $\widetilde{E}_6$. Denote by $n(c(\mathbf{u}))$ the number of left cells of $\widetilde{E}_6$ contained in $c(\mathbf{u})$ and by $E(c(\mathbf{u}))$ an l.c.r. set of $\widetilde{E}_6$ in $c(\mathbf{u})$. Then the above results can be summarized into the following table.

Unipotent class of G	$c(\mathbf{u})$	$n(c(\mathbf{u}))$	$E(c(\mathbf{u}))$
E_6	$W_{(0)}$	1	$\{e\}$
$E_6(a_1)$	$W_{(1)}$	7	S
D_5	$W_{(2)}$	27	$M'(\mathbf{12})$
$E_6(a_3)$	$W_{(3)}$	57	$M(\mathbf{131}) \cup M(\mathbf{140})$
A_5	$W^1_{(4)}$	162	$M'(\mathbf{1312})$
$D_5(a_1)$	$W^2_{(4)}$	72	$\cup_{i=1}^3 M(z_{4i})$
$A_4 + A_1$	$W_{(5)}$	216	$M'(\mathbf{13125})$
A_4	$W^1_{(6)}$	432	$M'(z_{61})$
D_4	$W^2_{(6)}$	270	$\cup_{i=2}^4 M(z_{6i})$
$D_4(a_1)$	$W_{(7)}$	540	$M'(z_{71}) \cup (\cup_{i=2}^4 M(z_{7i}))$
$A_3 + A_1$	$W_{(8)}$	675	$M'(z_{81}) \cup (\cup_{i=2}^5 M(z_{8i}))$
$2A_2 + A_1$	$W_{(9)}$	720	$\cup_{i=1}^7 M(z_{91})$
A_3	$W_{(10)}$	1890	$\cup_{i=1}^6 M(z_{10,i})$
$A_2 + 2A_1$	$W_{(11)}$	2160	$\cup_{i=1}^5 M(z_{11,i})$
$2A_2$	$W_{(12)}$	1325	$\cup_{i=1}^{10} M(z_{12,i})$

Table 1

where the unipotent conjugacy classes of G are parameterized as in [**2**, Chapter 13].

For any graph $\mathcal{M}$, denote by $n(M)$ the number of vertices in $\mathcal{M}$. The figures with a top-left $*$ are large, hence they are not displayed in the paper, which can be found in the web-side of the first-named author (see [**28**]).

Graphs $\mathcal{M}$	Left-cell graphs isomorphic to $\mathcal{M}$	$n(M)$
$\boxed{\emptyset}$	$\mathcal{M}_L(e)$	1
A	$\mathcal{M}_L(\mathbf{1})$	7
B	$\mathcal{M}_L(\mathbf{12})$	27
$C1$	$\mathcal{M}_L(\mathbf{131})$, $\mathcal{M}_L(z_{42})$	21
$C2$	$\mathcal{M}_L(\mathbf{140})$, $\mathcal{M}_L(z_{43})$	36
D	$\mathcal{M}_L(\mathbf{1312})$	162
E	$\mathcal{M}_L(z_{41})$	15
F	$\mathcal{M}_L(\mathbf{13125})$	216
$G1$–2	$\mathcal{M}_L(z_{61})$	432
H	$\mathcal{M}_L(z_{62})$	90
$\psi_{06}(H)$	$\mathcal{M}_L(z_{63})$	90
$\psi_{01}(H)$	$\mathcal{M}_L(z_{64})$	90
I	$\mathcal{M}_L(z_{71})$, $\mathcal{M}_L(z_{82})$, $\mathcal{M}_L(z_{94})$, $\mathcal{M}_L(z_{10,3})$, $\mathcal{M}_L(z_{12,5})$, $\mathcal{M}_L(z_{12,7})$	300
$\psi_{06}(J)$	$\mathcal{M}_L(z_{73})$, $\mathcal{M}_L(z_{84})$, $\mathcal{M}_L(z_{96})$, $\mathcal{M}_L(z_{10,5})$, $\mathcal{M}_L(z_{11,3})$, $\mathcal{M}_L(z_{12,10})$	80
$\psi_{01}(J)$	$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{85})$, $\mathcal{M}_L(z_{97})$, $\mathcal{M}_L(z_{10,6})$, $\mathcal{M}_L(z_{11,4})$, $\mathcal{M}_L(z_{12,9})$	80
J	$\mathcal{M}_L(z_{74})$, $\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{10,4})$, $\mathcal{M}_L(z_{11,5})$, $\mathcal{M}_L(z_{12,8})$	80
$K1$	$\mathcal{M}_L(z_{81})$, $\mathcal{M}_L(z_{92})$, $\mathcal{M}_L(z_{10,2})$, $\mathcal{M}_L(z_{12,3})$, $\mathcal{M}_L(z_{12,6})$	135
$K2$	$\mathcal{M}_L(z_{91})$, $\mathcal{M}_L(z_{11,2})$,$\mathcal{M}_L(z_{12,2})$	10
$K3$	$\mathcal{M}_L(z_{93})$, $\mathcal{M}_L(z_{12,4})$	35
*$L1$–10	$\mathcal{M}_L(z_{10,1})$	1215
*$M1$–13	$\mathcal{M}_L(z_{11,1})$	1910
N	$\mathcal{M}_L(z_{12,1})$	170

Table 2

In Tables 1–2, we have $\mathcal{M}_L(x) \cong \mathcal{M}(x)$ for any $x \notin F := \{\mathbf{12}, \mathbf{1312}, \mathbf{13125}, z_{61}, z_{71}, z_{81}\}$; while for any $x \in F$, the graph $\mathcal{M}(x)$ is infinite. By 4.1, 4.5 and 4.7, we see that the automorphism ψ_{ij} of $\widetilde{E}_6$ (see 4.2) stabilizes each two-sided cell Ω of $\widetilde{E}_6$. So ψ_{ij} gives rise to a permutation on the left cells of $\widetilde{E}_6$ in Ω and further a permutation on the left-cell graphs in Ω. More precisely, ψ_{ij} stabilizes each of the above left-cell graphs of $\widetilde{E}_6$ except for the ones in the following table, where ψ_{ij} transposes two members in each pair.

Transposed by ψ_{10}	Transposed by ψ_{16}	Transposed by ψ_{06}
$\mathcal{M}_L(z_{62})$, $\mathcal{M}_L(z_{64})$	$\mathcal{M}_L(z_{63})$, $\mathcal{M}_L(z_{64})$	$\mathcal{M}_L(z_{62})$, $\mathcal{M}_L(z_{63})$
$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{74})$	$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{73})$	$\mathcal{M}_L(z_{73})$, $\mathcal{M}_L(z_{74})$
$\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{85})$	$\mathcal{M}_L(z_{84})$, $\mathcal{M}_L(z_{85})$	$\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{84})$
$\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{97})$	$\mathcal{M}_L(z_{96})$, $\mathcal{M}_L(z_{97})$	$\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{96})$
$\mathcal{M}_L(z_{10,3})$, $\mathcal{M}_L(z_{10,4})$	$\mathcal{M}_L(z_{10,4})$, $\mathcal{M}_L(z_{10,5})$	$\mathcal{M}_L(z_{10,3})$, $\mathcal{M}_L(z_{10,5})$
$\mathcal{M}_L(z_{11,3})$, $\mathcal{M}_L(z_{11,4})$	$\mathcal{M}_L(z_{11,3})$, $\mathcal{M}_L(z_{11,5})$	$\mathcal{M}_L(z_{11,4})$, $\mathcal{M}_L(z_{11,5})$

Table 3

In the following table, we list the position for each $L(z_{ij})$ as a vertex of the left-cell graph $\mathcal{M}_L(z_{ij})$, where $L(z_{ij})$ is the left cell of $\widetilde{E}_6$ containing the element z_{ij}. In the most cases, such a position is determined uniquely by the label of the vertex $L(z_{ij})$; in the case where there exist some other vertices of $\mathcal{M}_L(z_{ij})$ sharing the same label as $L(z_{ij})$, we need indicate the label of some adjacent vertex in addition, to distinguish $L(z_{ij})$ from the others.

z_{ij}	Position of $L(z_{ij})$ in $\mathcal{M}_L(z_{ij})$	z_{ij}	Position of $L(z_{ij})$ in $\mathcal{M}_L(z_{ij})$
z_{61}	$\boxed{\mathbf{134}}$	z_{62}	$\boxed{\mathbf{1356}}$
z_{63}	$\boxed{\mathbf{1230}}$	z_{64}	$\boxed{\mathbf{2560}}$
z_{71}	$\boxed{\mathbf{1346}}$	z_{72}	$\boxed{\mathbf{23560}}$
z_{73}	$\boxed{\mathbf{12350}}$	z_{74}	$\boxed{\mathbf{12356}}$
z_{81}	$\boxed{\mathbf{13460}}$	z_{82}	$\boxed{\mathbf{1450}}$, adjacent to $\boxed{\mathbf{1250}}$
z_{83}	$\boxed{\mathbf{145}}$	z_{84}	$\boxed{\mathbf{1460}}$, adjacent to $\boxed{\mathbf{360}}$
z_{85}	$\boxed{\mathbf{246}}$	z_{91}	$\boxed{\mathbf{123560}}$
z_{92}	$\boxed{\mathbf{34}}$, adjacent to $\boxed{\mathbf{235}}$	z_{93}	$\boxed{\mathbf{124}}$
z_{94}	$\boxed{\mathbf{1340}}$	z_{95}	$\boxed{\mathbf{460}}$, adjacent to $\boxed{\mathbf{246}}$
z_{96}	$\boxed{\mathbf{146}}$, adjacent to $\boxed{\mathbf{145}}$	z_{97}	$\boxed{\mathbf{125}}$, adjacent to $\boxed{\mathbf{1246}}$
$z_{10,1}$	$\boxed{\mathbf{1234}}$	$z_{10,2}$	$\boxed{\mathbf{12460}}$
$z_{10.3}$	$\boxed{\mathbf{450}}$, distance 2 to $\boxed{\mathbf{560}}$	$z_{10,4}$	$\boxed{\mathbf{124}}$
$z_{10,5}$	$\boxed{\mathbf{340}}$	$z_{10,6}$	$\boxed{\mathbf{246}}$
$z_{11,1}$	$\boxed{\mathbf{12346}}$	$z_{11,2}$	$\boxed{\mathbf{3460}}$
$z_{11,3}$	$\boxed{\mathbf{1450}}$	$z_{11,4}$	$\boxed{\mathbf{450}}$
$z_{11,5}$	$\boxed{\mathbf{145}}$	$z_{12,1}$	$\boxed{\mathbf{2345}}$
$z_{12,2}$	$\boxed{\mathbf{123560}}$	$z_{12,3}$	$\boxed{\mathbf{12460}}$
$z_{12,4}$	$\boxed{\mathbf{240}}$	$z_{12,5}$	$\boxed{\mathbf{2350}}$, distance 2 to $\boxed{\mathbf{460}}$, $\boxed{\mathbf{140}}$
$z_{12,6}$		$z_{12,7}$	
$z_{12,8}$	$\boxed{\mathbf{12356}}$	$z_{12,9}$	$\boxed{\mathbf{23560}}$
$z_{12,10}$	$\boxed{\mathbf{12350}}$		

Table 4

where by " $\boxed{\mathbf{450}}$, distance 2 to $\boxed{\mathbf{560}}$ ", we mean that the vertex $L(z_{10,3})$ of $\mathcal{M}_L(z_{10,3})$ is labelled by $\boxed{\mathbf{450}}$ and that there is a path of length 2 connecting $L(z_{10,3})$ and a vertex labelled by $\boxed{\mathbf{560}}$ (see 2.5). Also, by " $\boxed{\mathbf{2350}}$, distance 2 to $\boxed{\mathbf{460}}$, $\boxed{\mathbf{140}}$ ", we mean that the vertex $L(z_{12,5})$ of $\mathcal{M}_L(z_{12,5})$ is labelled by $\boxed{\mathbf{2350}}$ and that there are two paths of length 2, one connecting $L(z_{12,5})$ and a vertex labelled by $\boxed{\mathbf{460}}$, and the other connecting $L(z_{12,5})$ and a vertex labelled by $\boxed{\mathbf{140}}$.

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5. Appendix

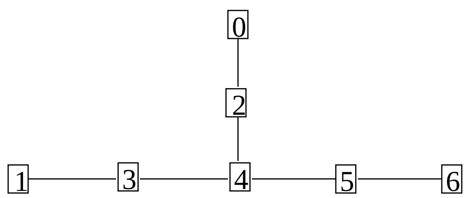


Fig. A

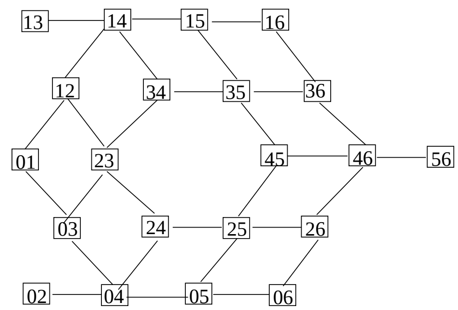


Fig. $C1$

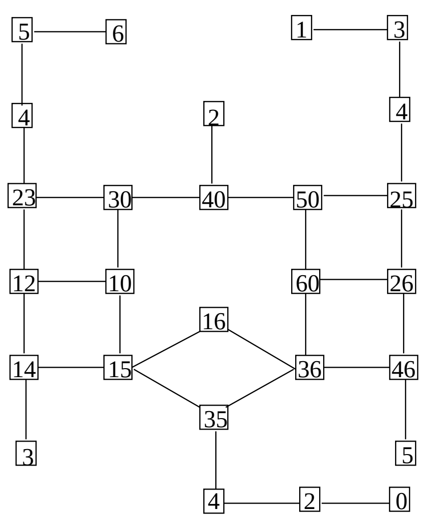


Fig. B

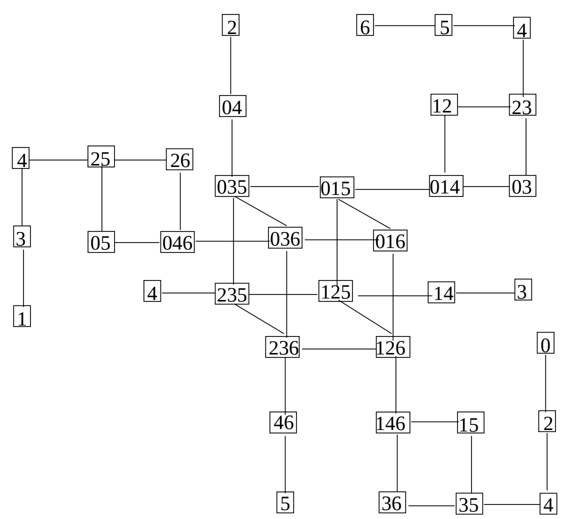
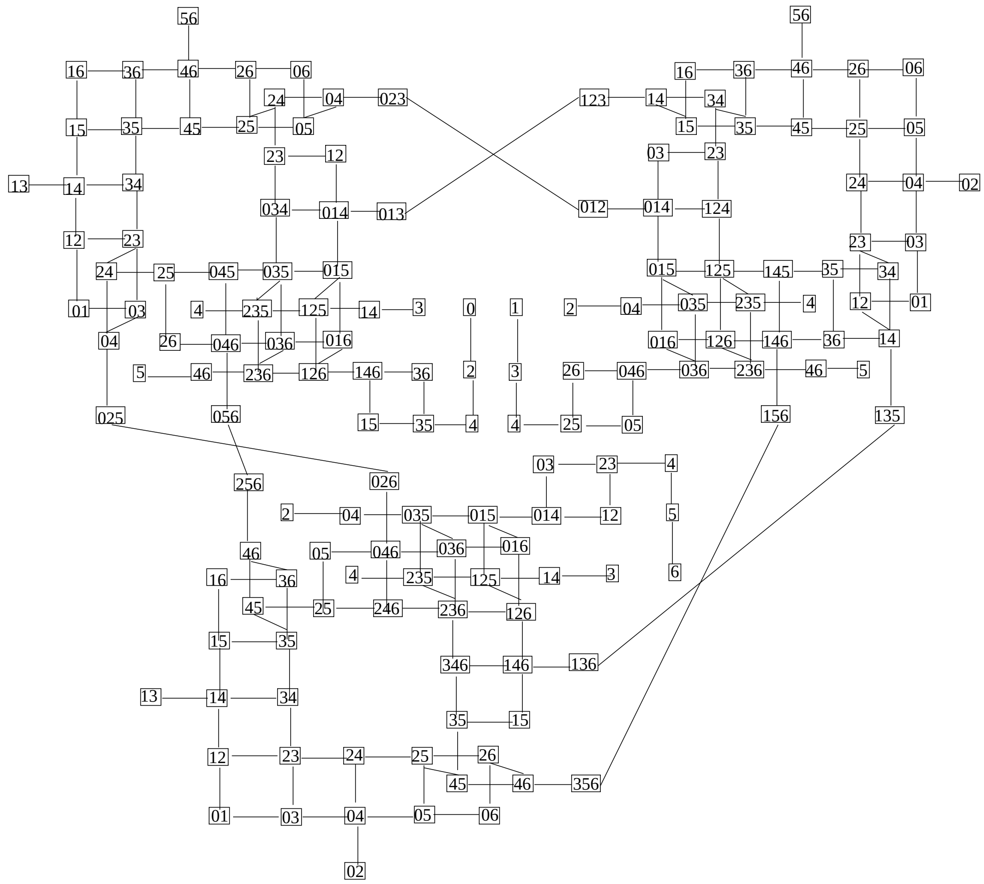


Fig. $C2$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012 014 124
12 23
23 03
24 25 045 035 015
015 125 145 35 34
01 03 4 235 125 14 3
0 1 2 04 035 235 4
12 01
04 26 046 036 016
016 126 146 36 14
5 46 236 126 146 36
2 3 26 046 036 236 46 5
025 056 15 35 4
4 25 05 156 135
256 026
03 23 4
2 04 035 015 014 12 5
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02

Fig. *D*

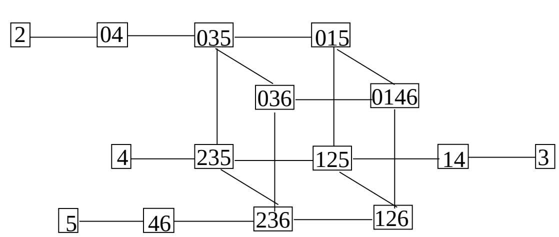
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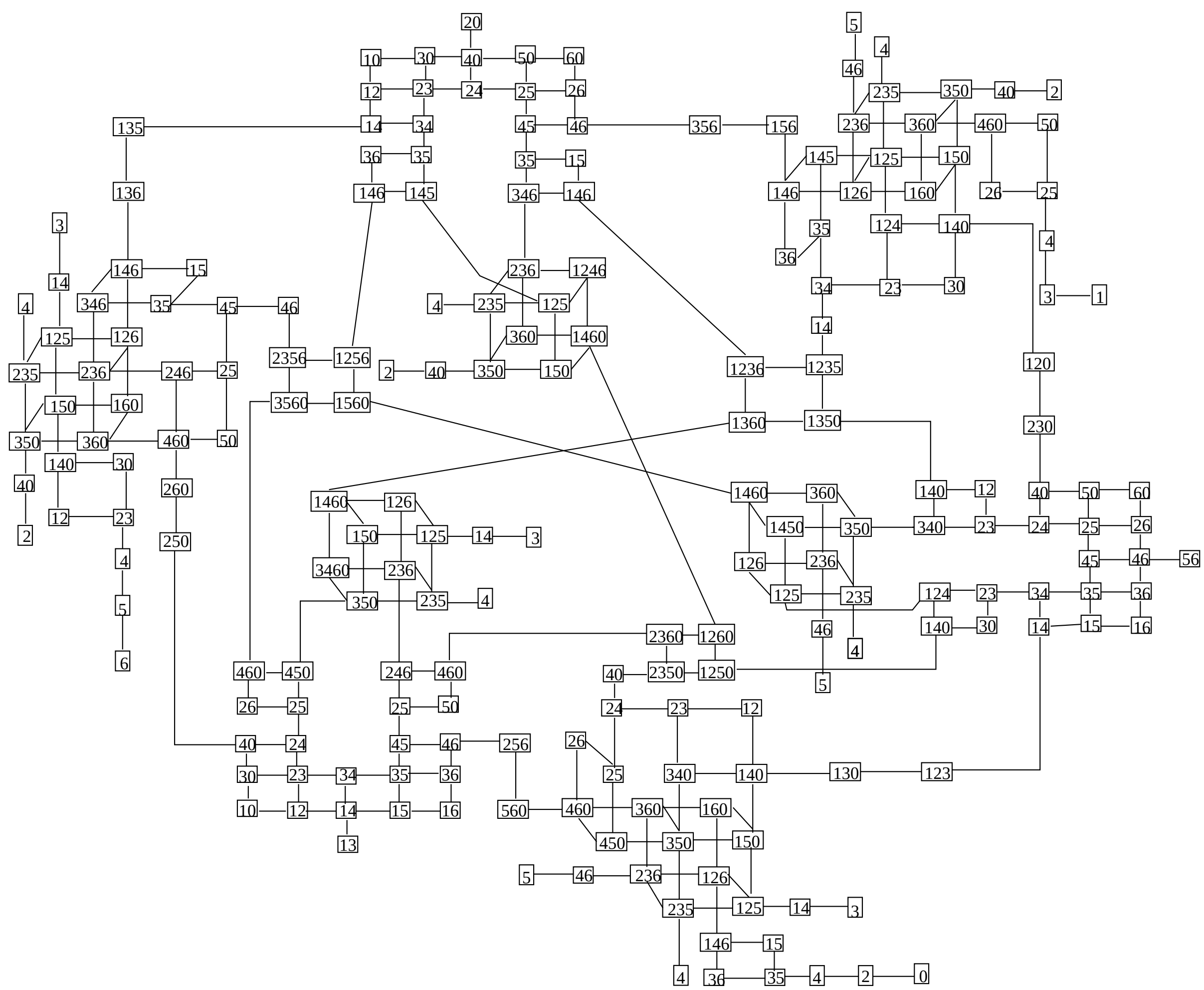
Fig. *E*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146 15 236 1246
14
34 23 30
3 1
4 346 35 45 46 4 235 125
14
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1360 1350
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40 24 45 46 256
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Fig. F

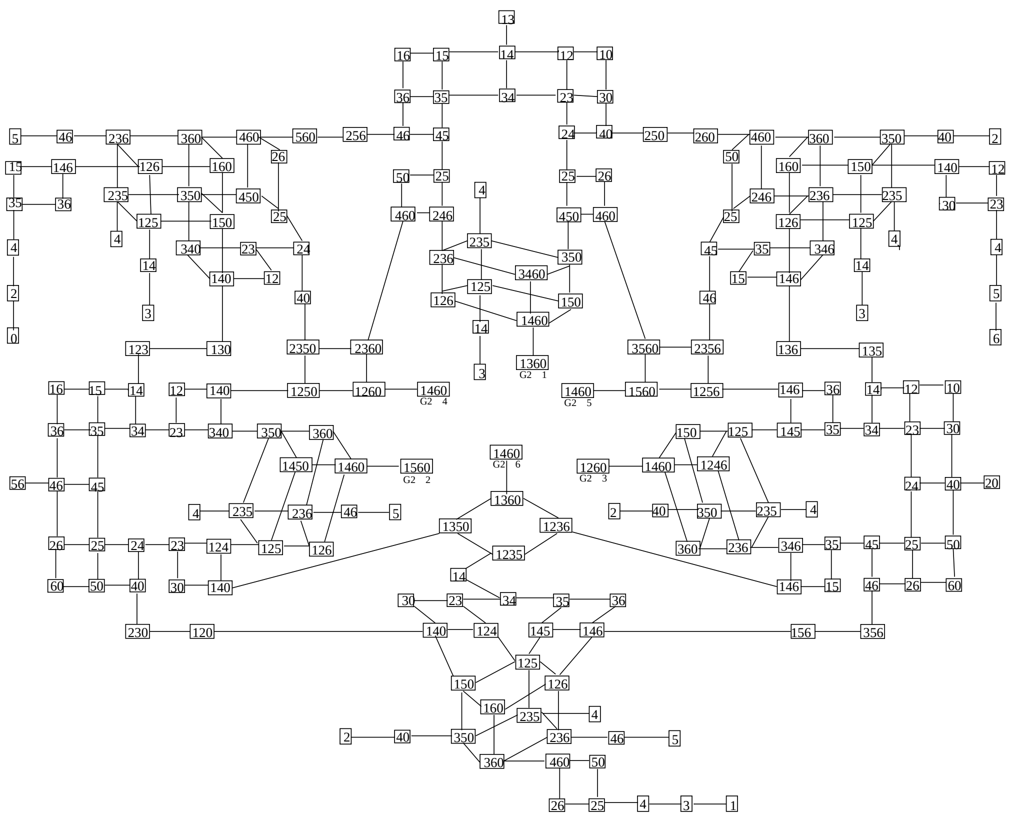
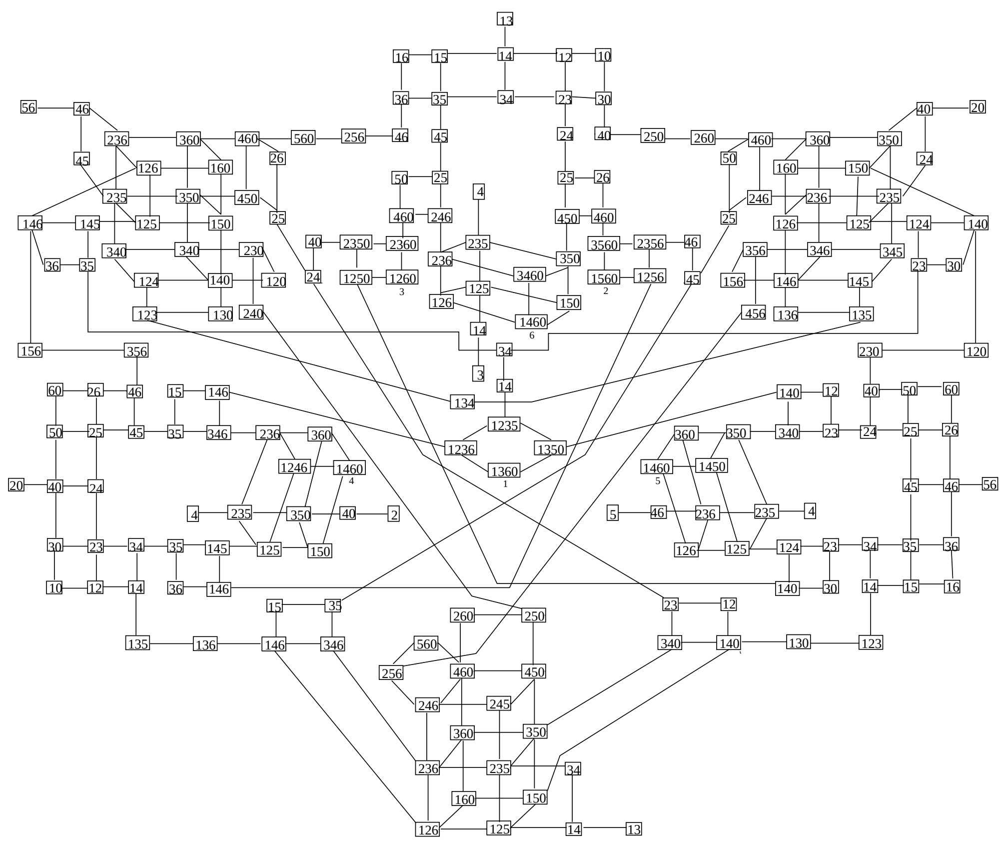


Fig. G1

where by a node with an underneath label " G2 i ", we mean that the node is in Fig. G2 with the underneath label i. For example, **1460** (G2 1) is the node **1460** (1) in Fig. G2.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126 125 14 13

Fig. G2

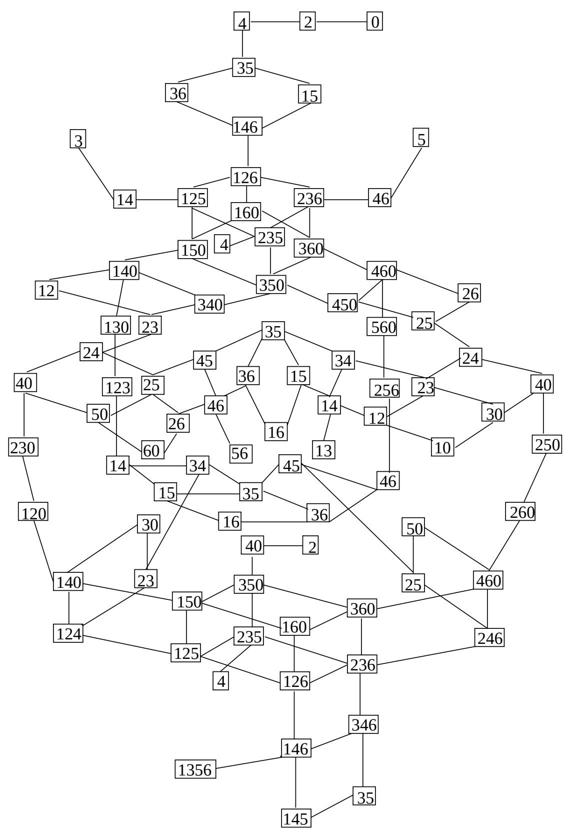
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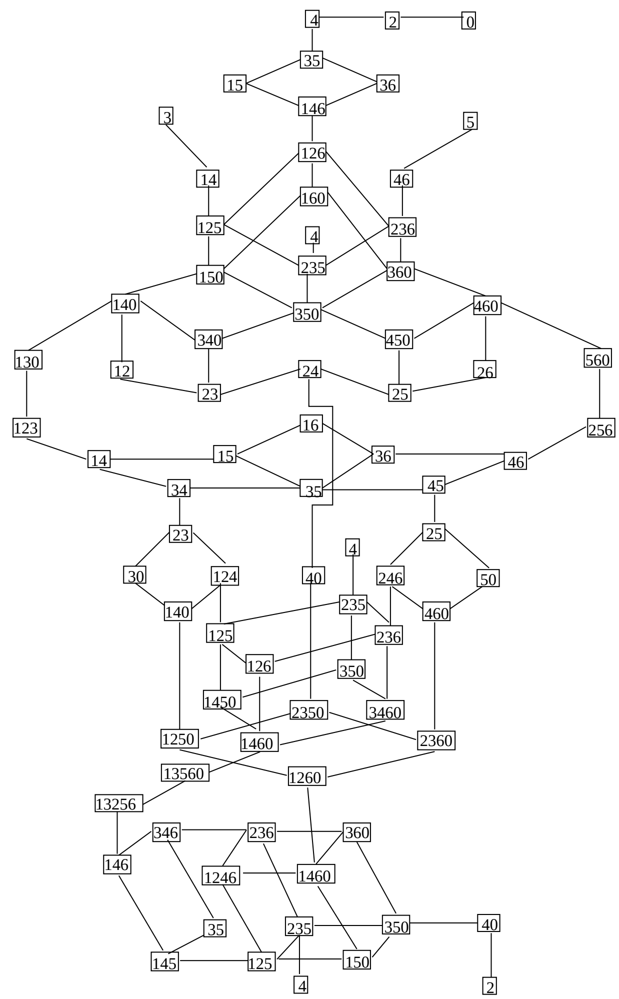
Fig. *H*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13560
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Fig. *J*

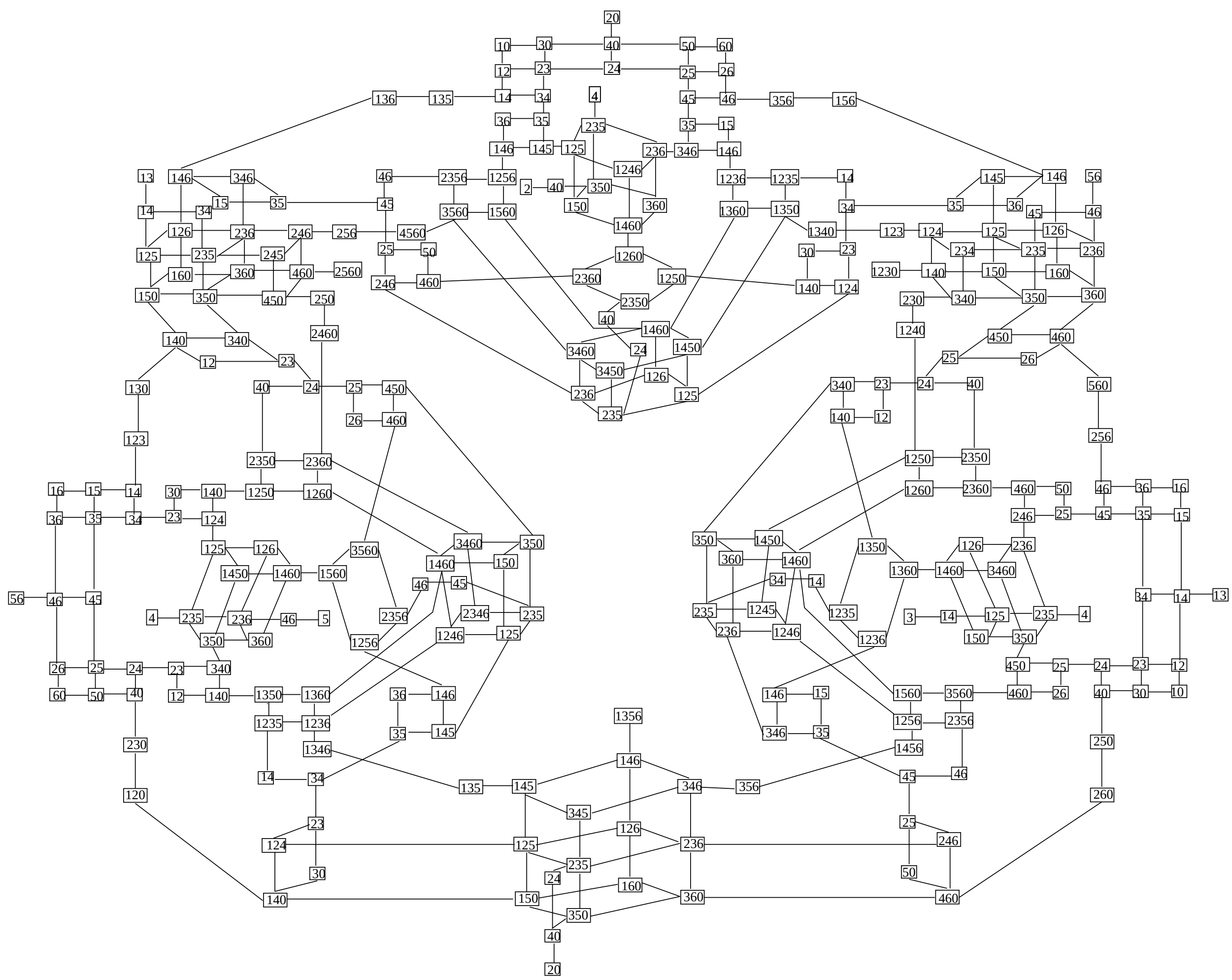


Fig. I

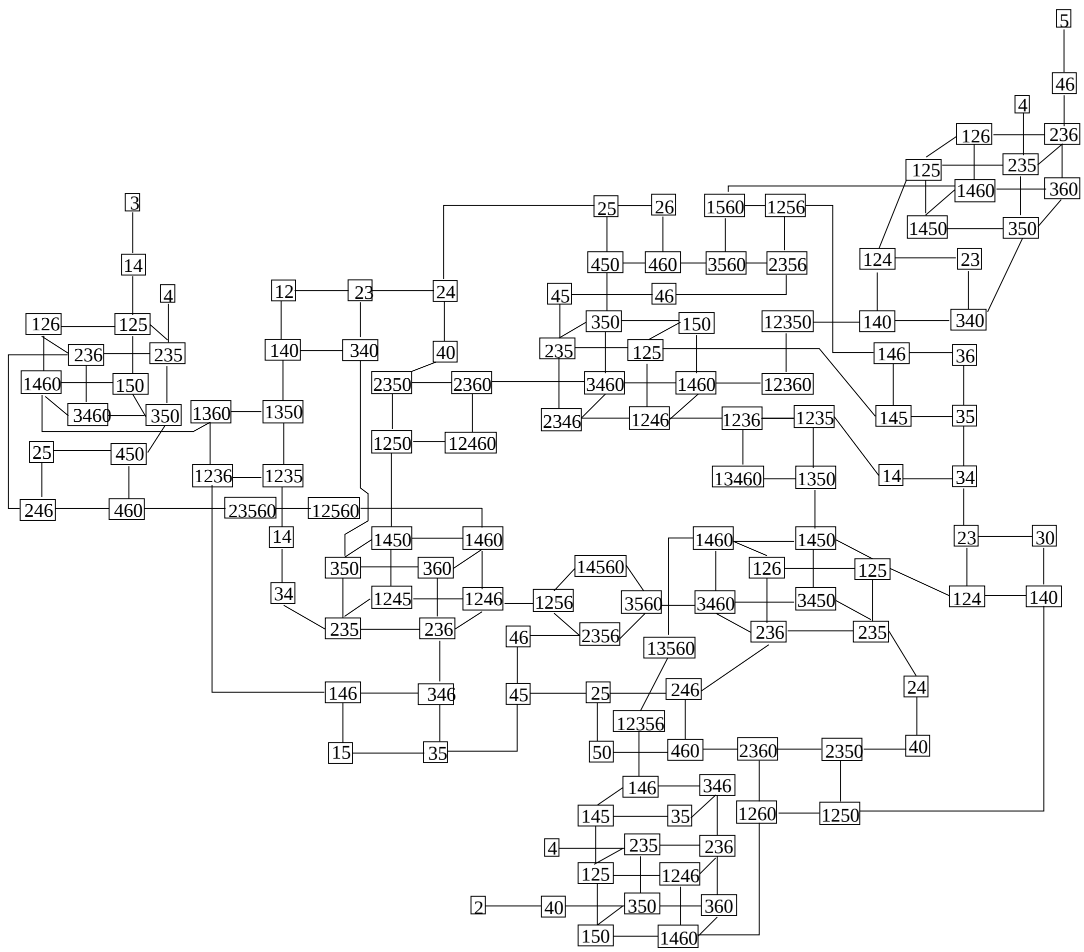


Fig. $K1$

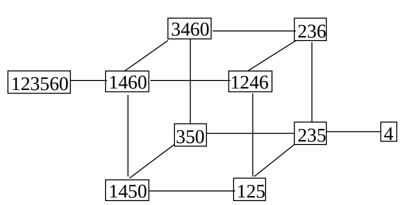


Fig. $K2$

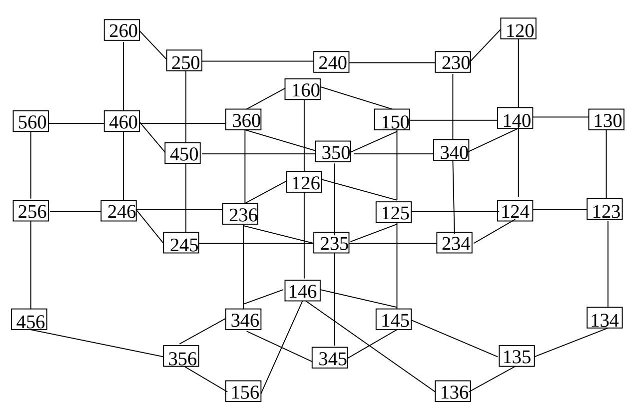


Fig. K3

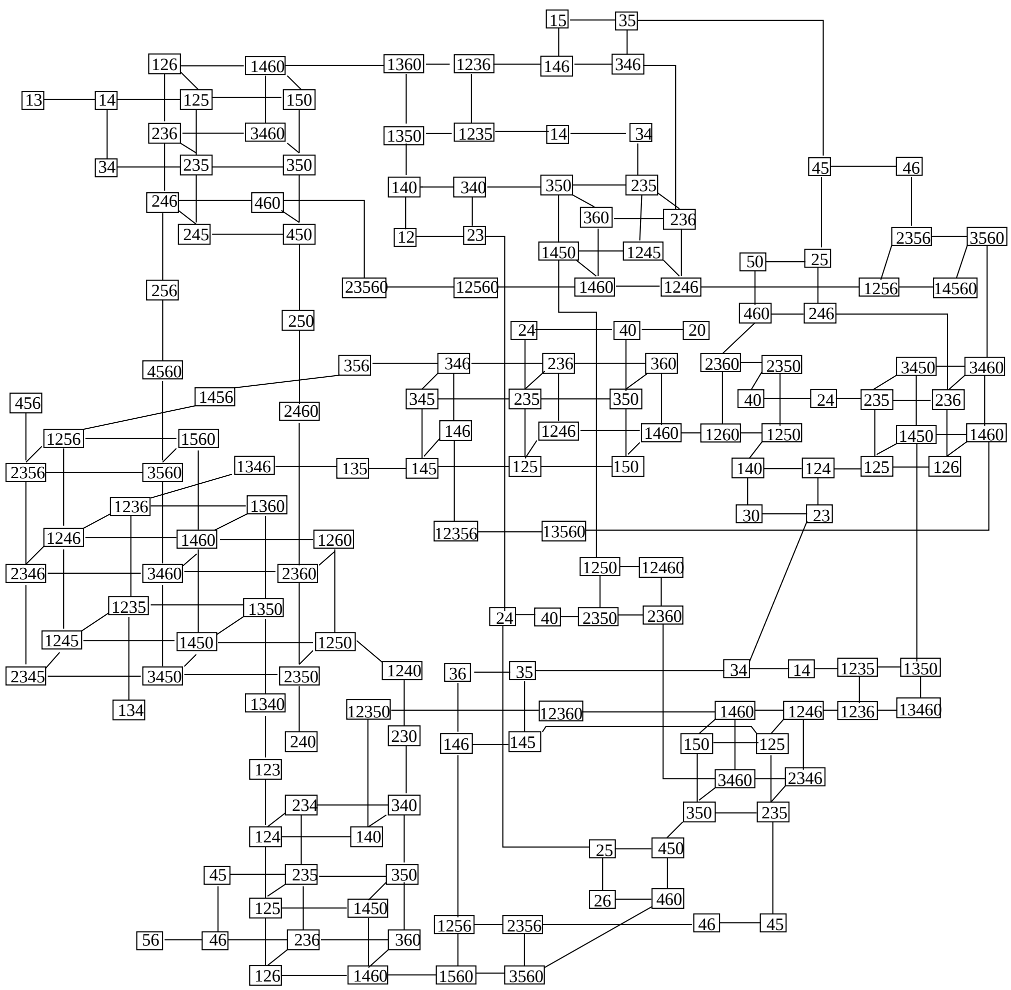


Fig. N

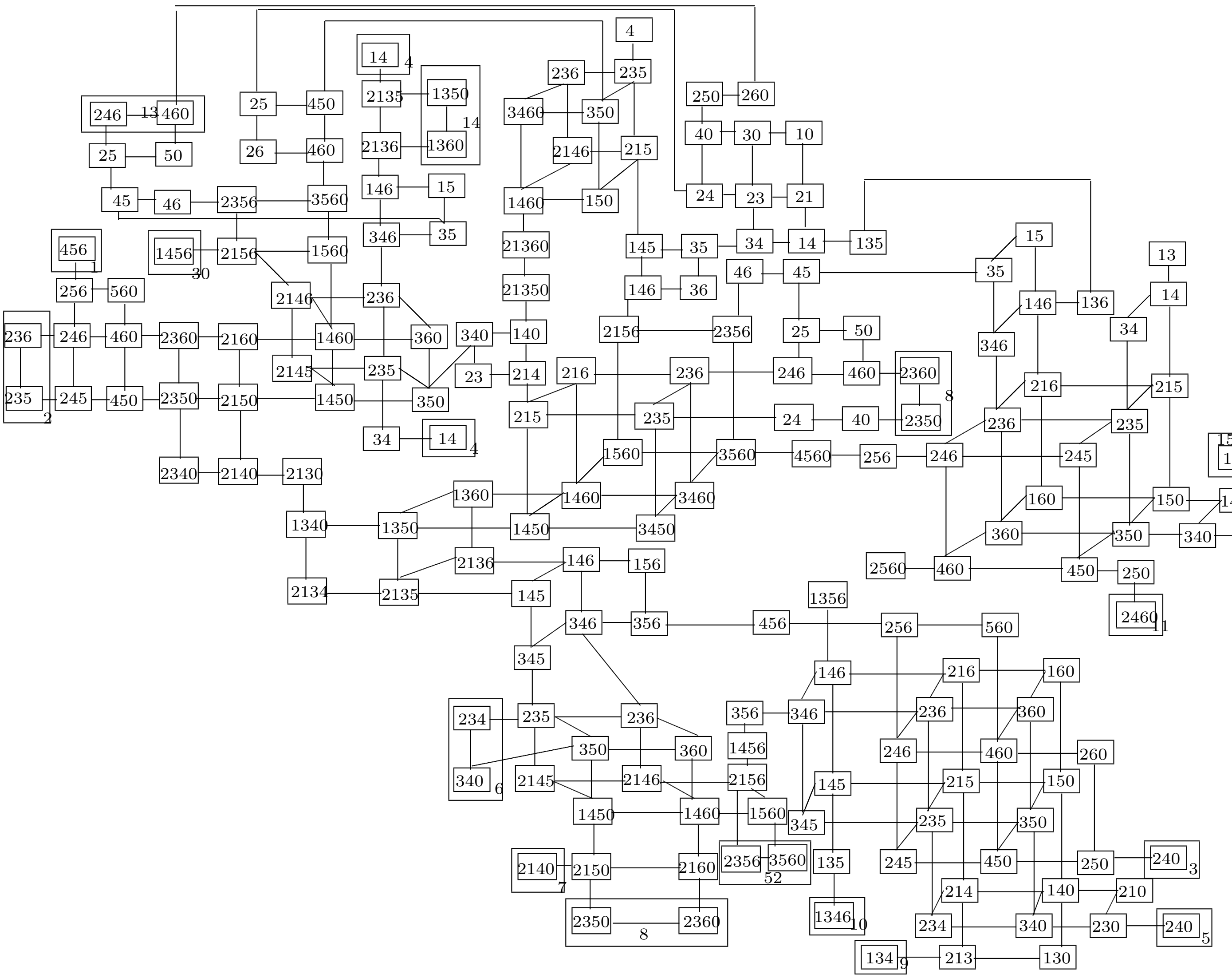


Fig.19(1)

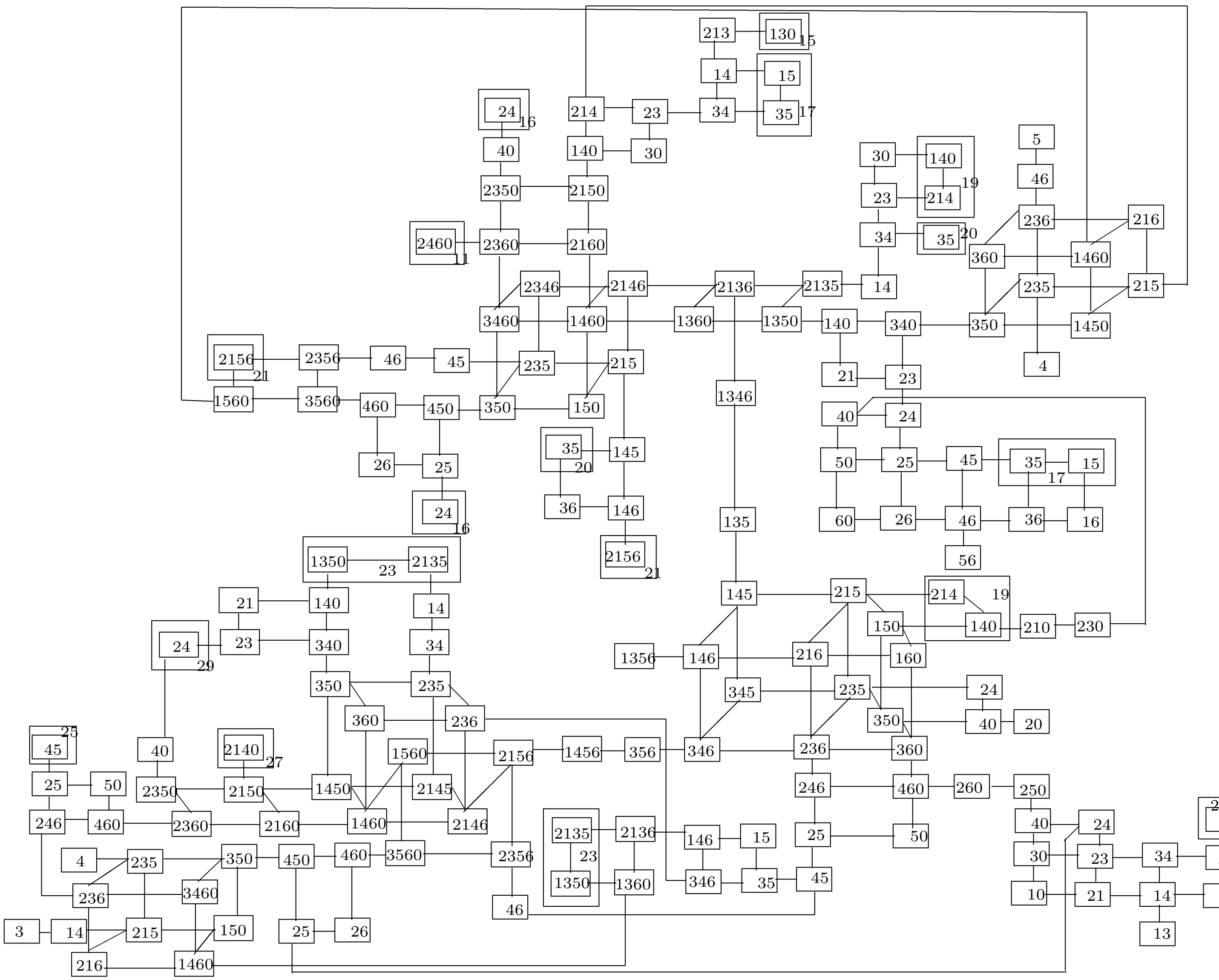
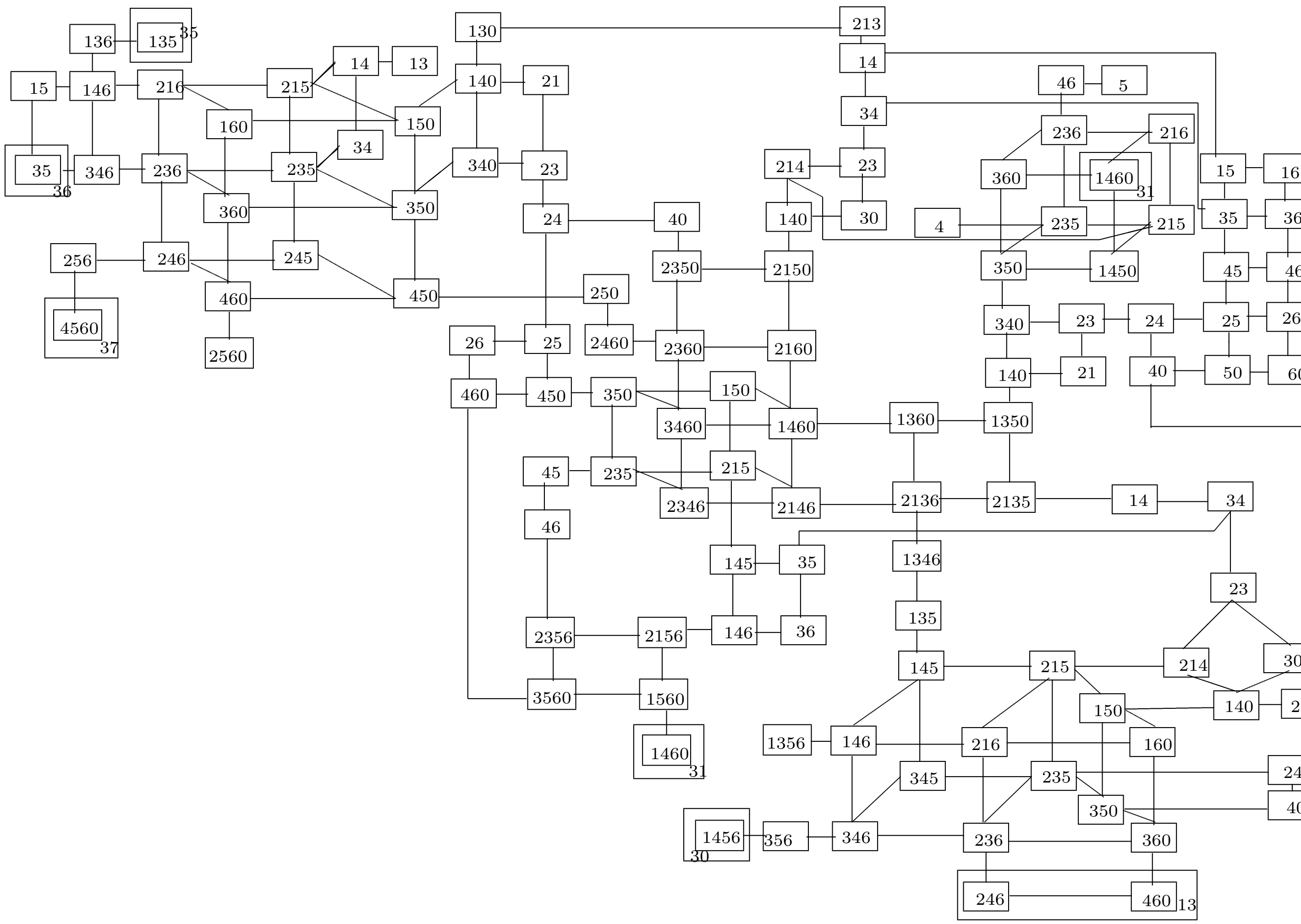


Fig.19(2)



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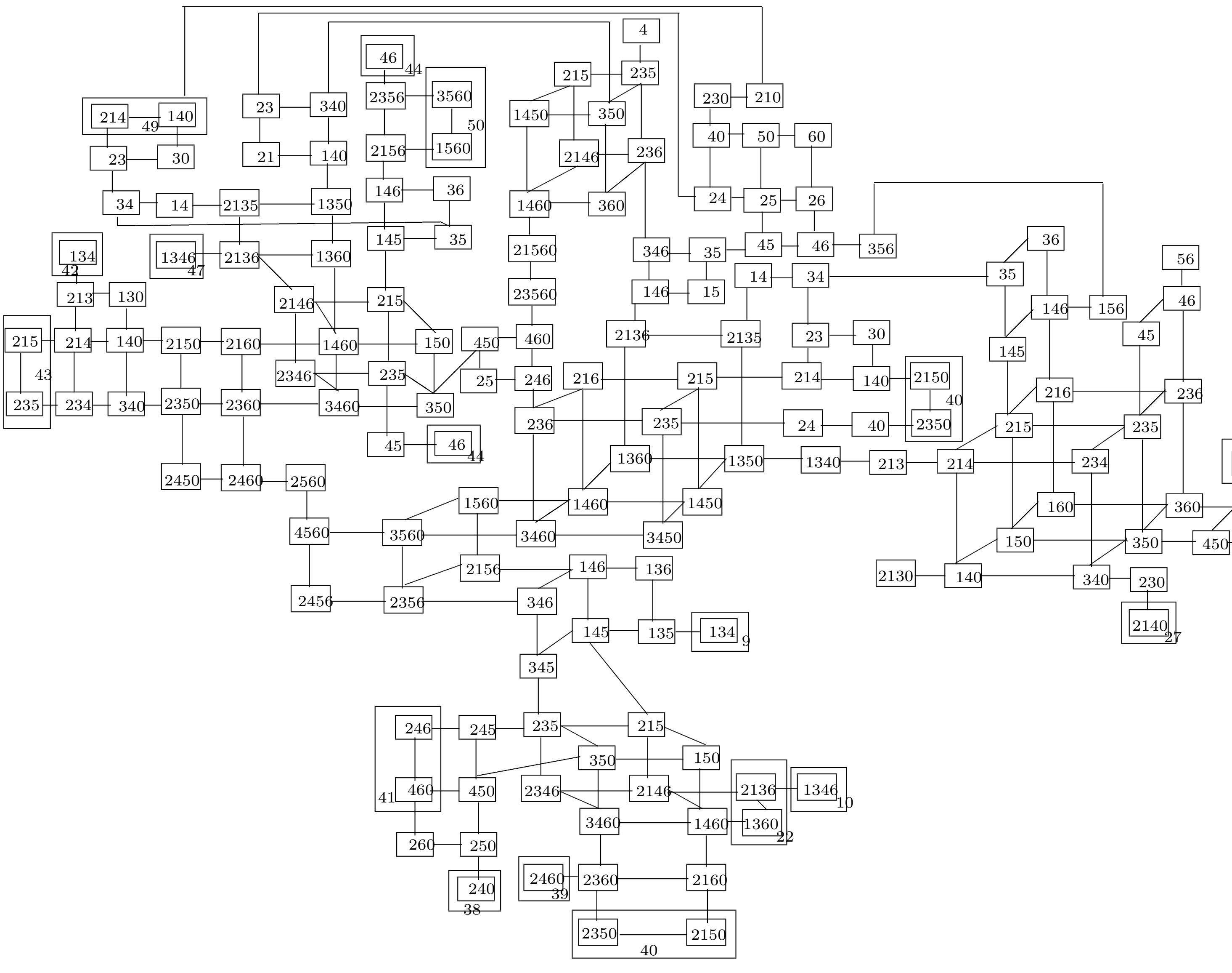

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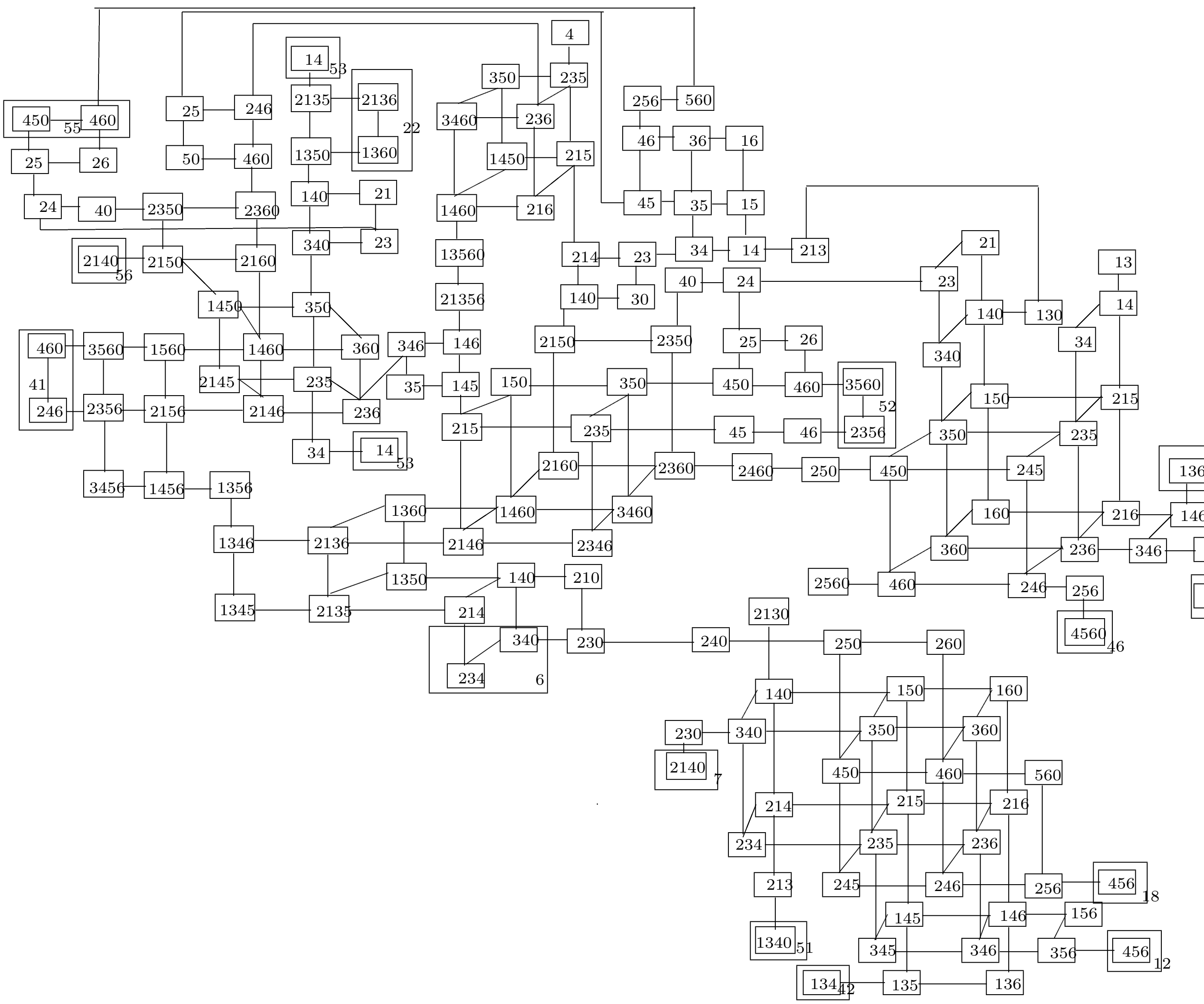

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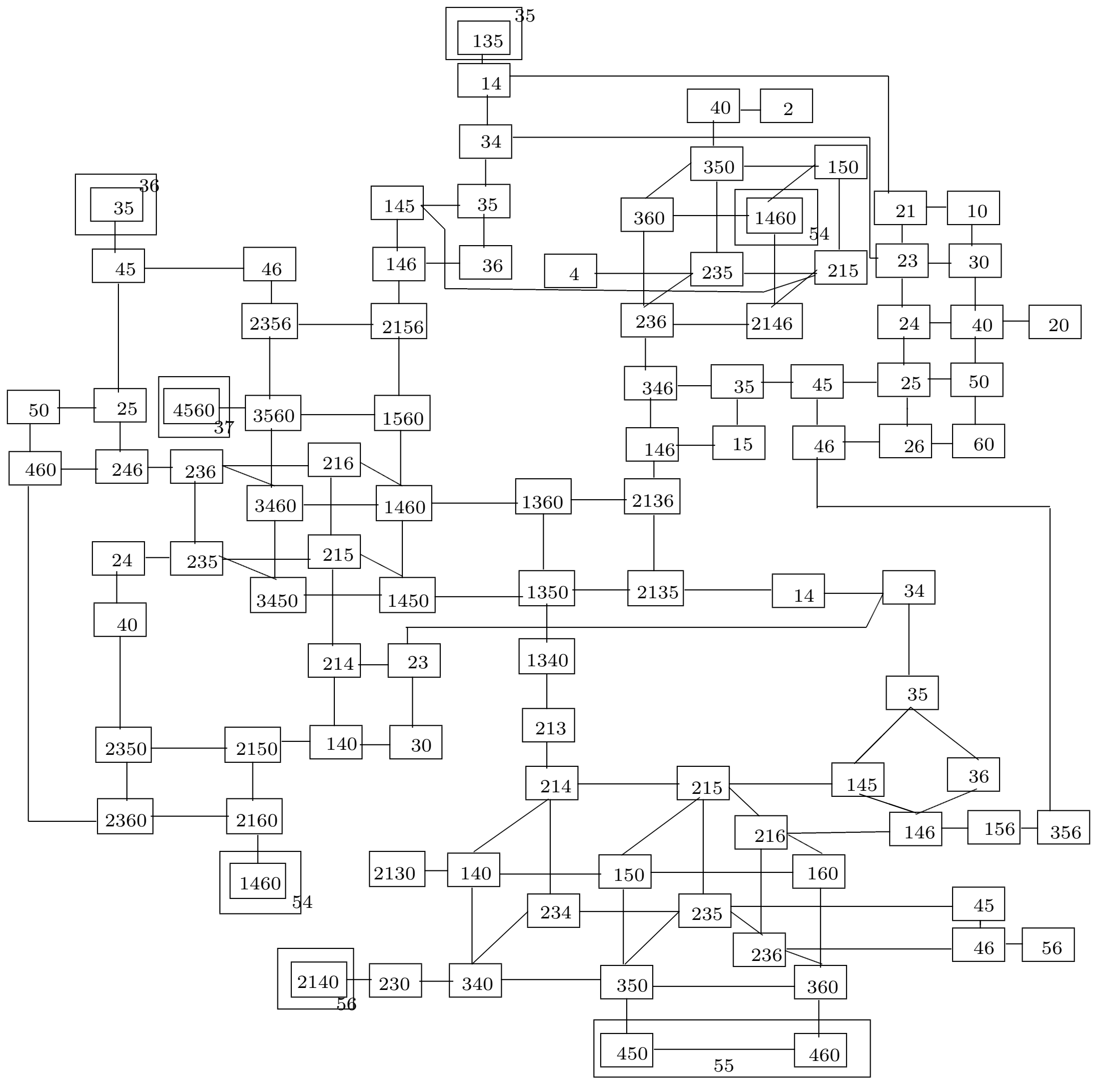
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Fig.19(6)

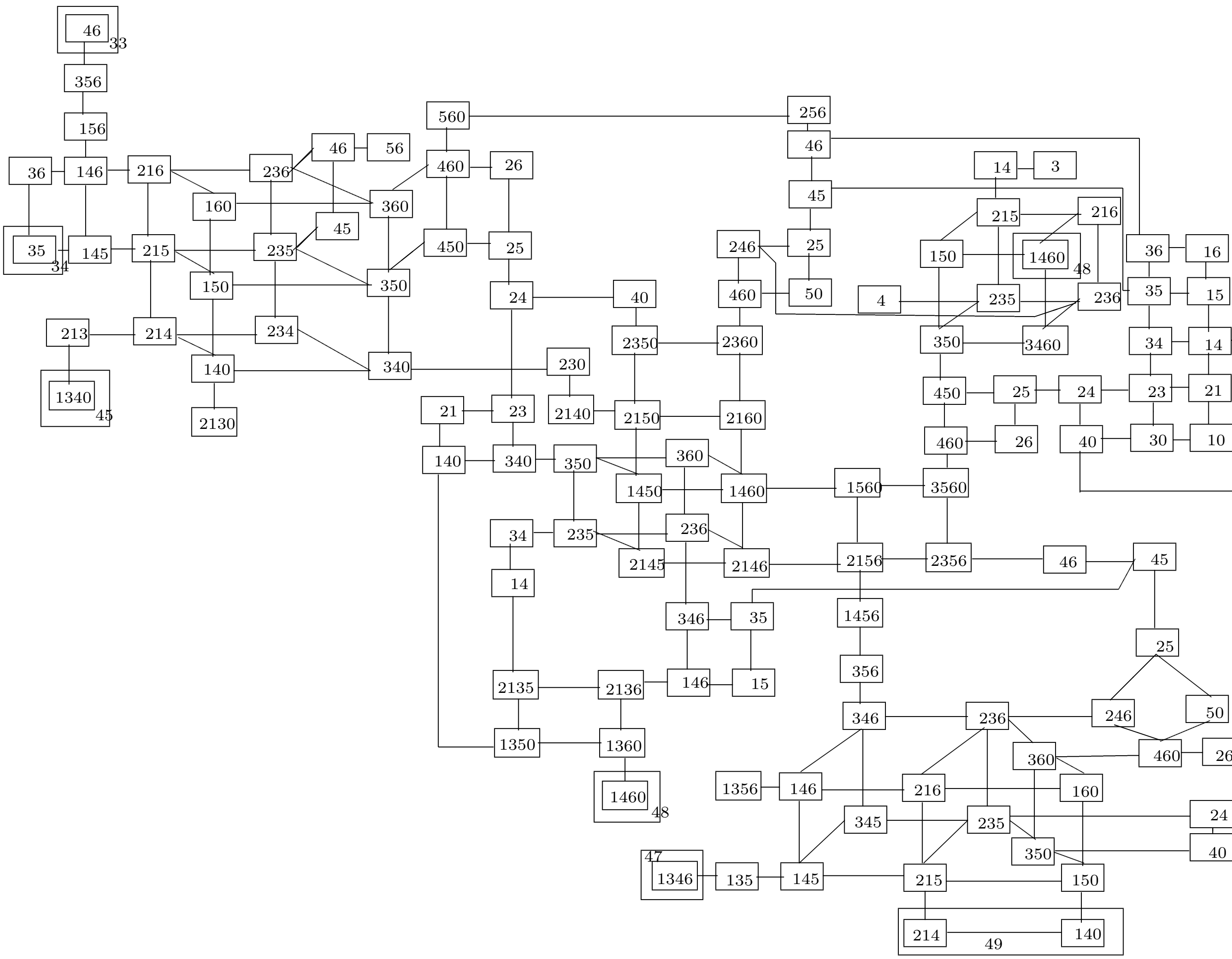
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Fig.19(7)

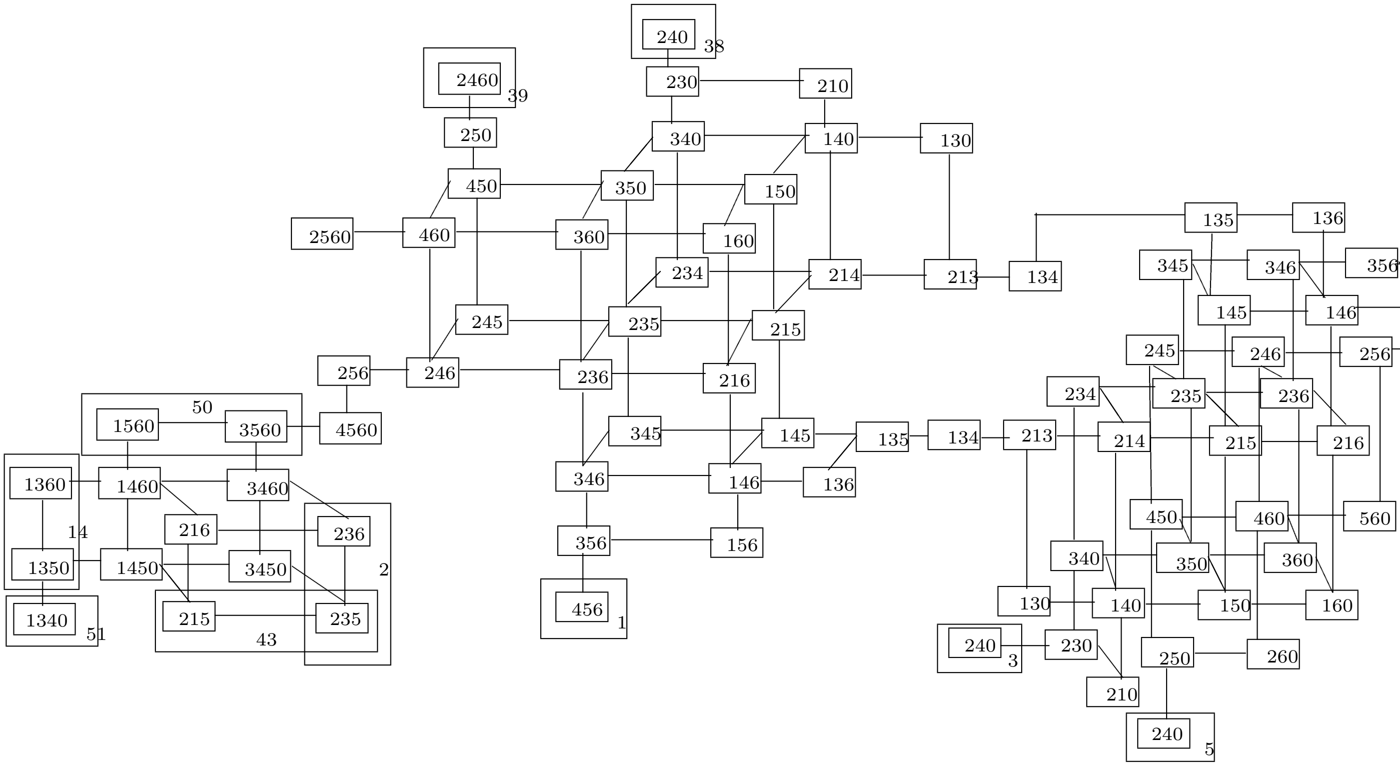
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Fig.19(8)

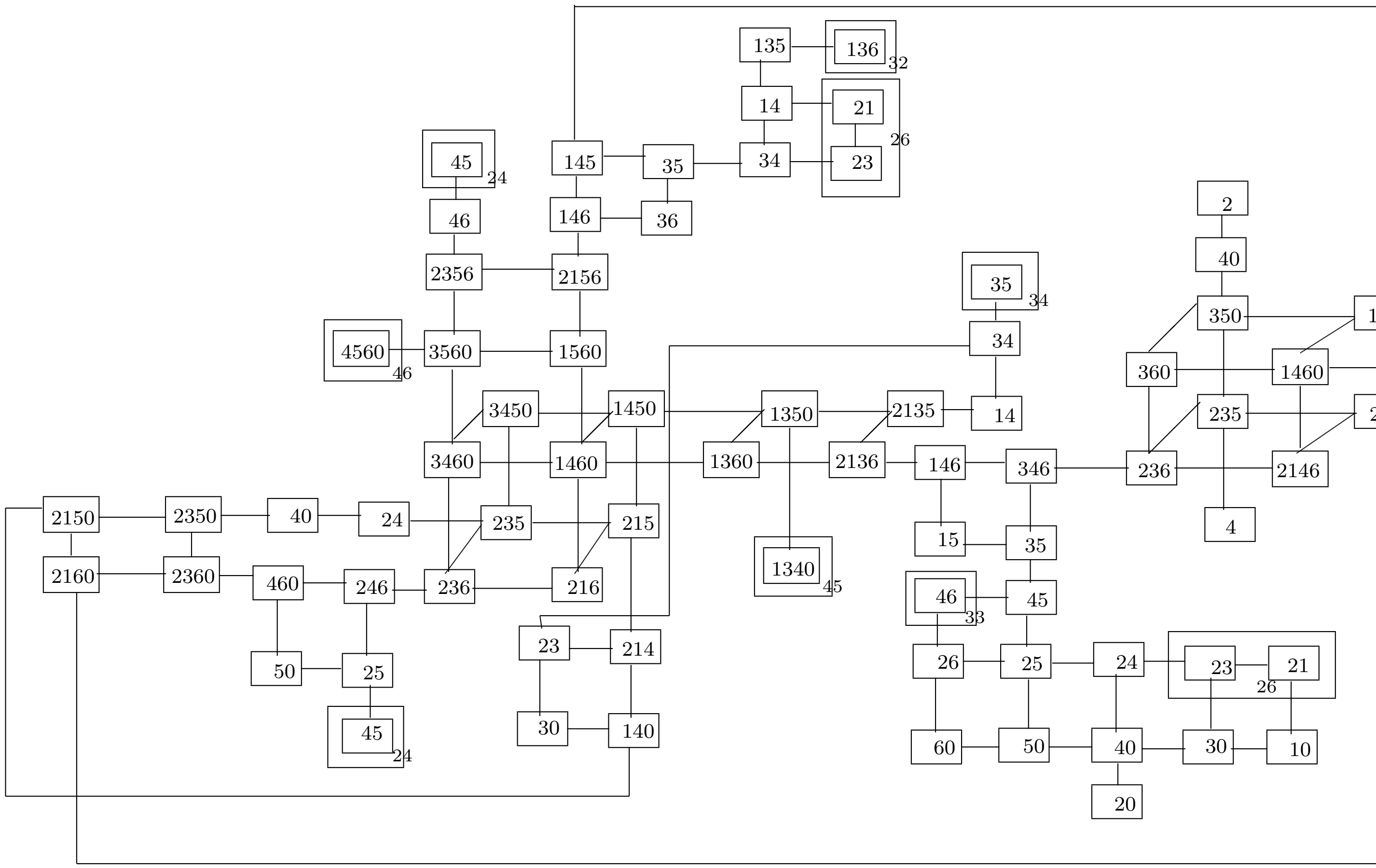
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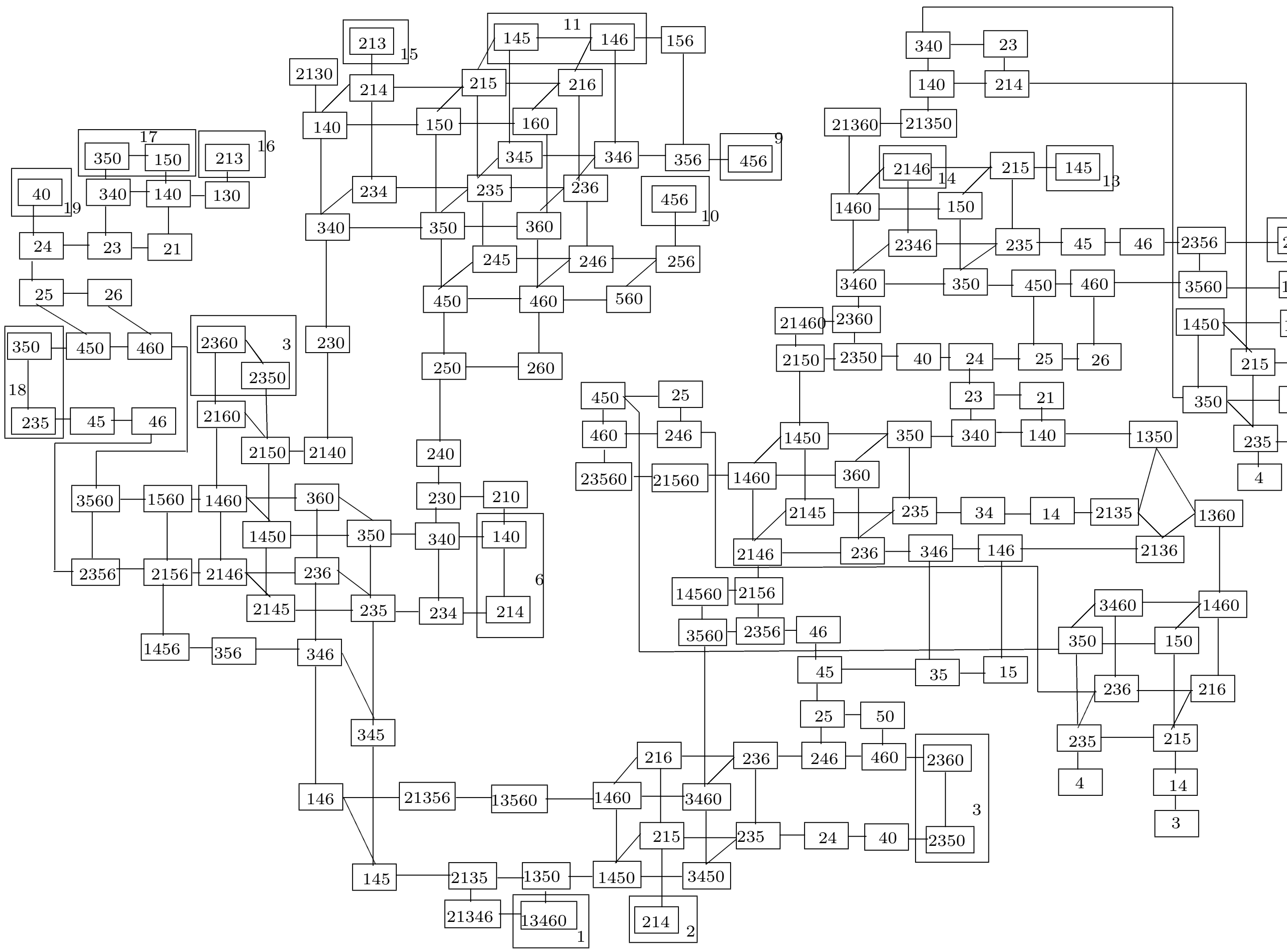
Fig.19(9)



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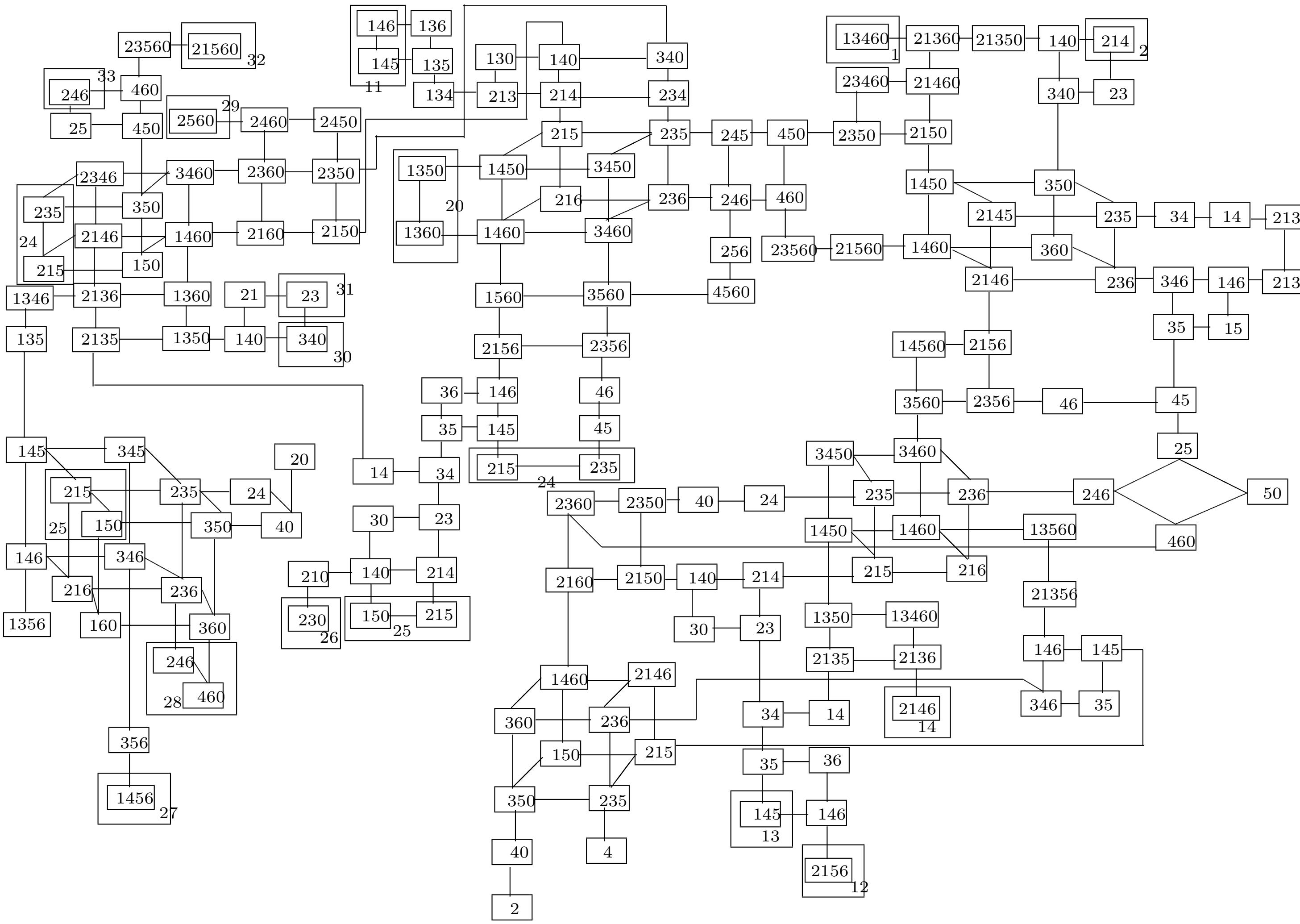
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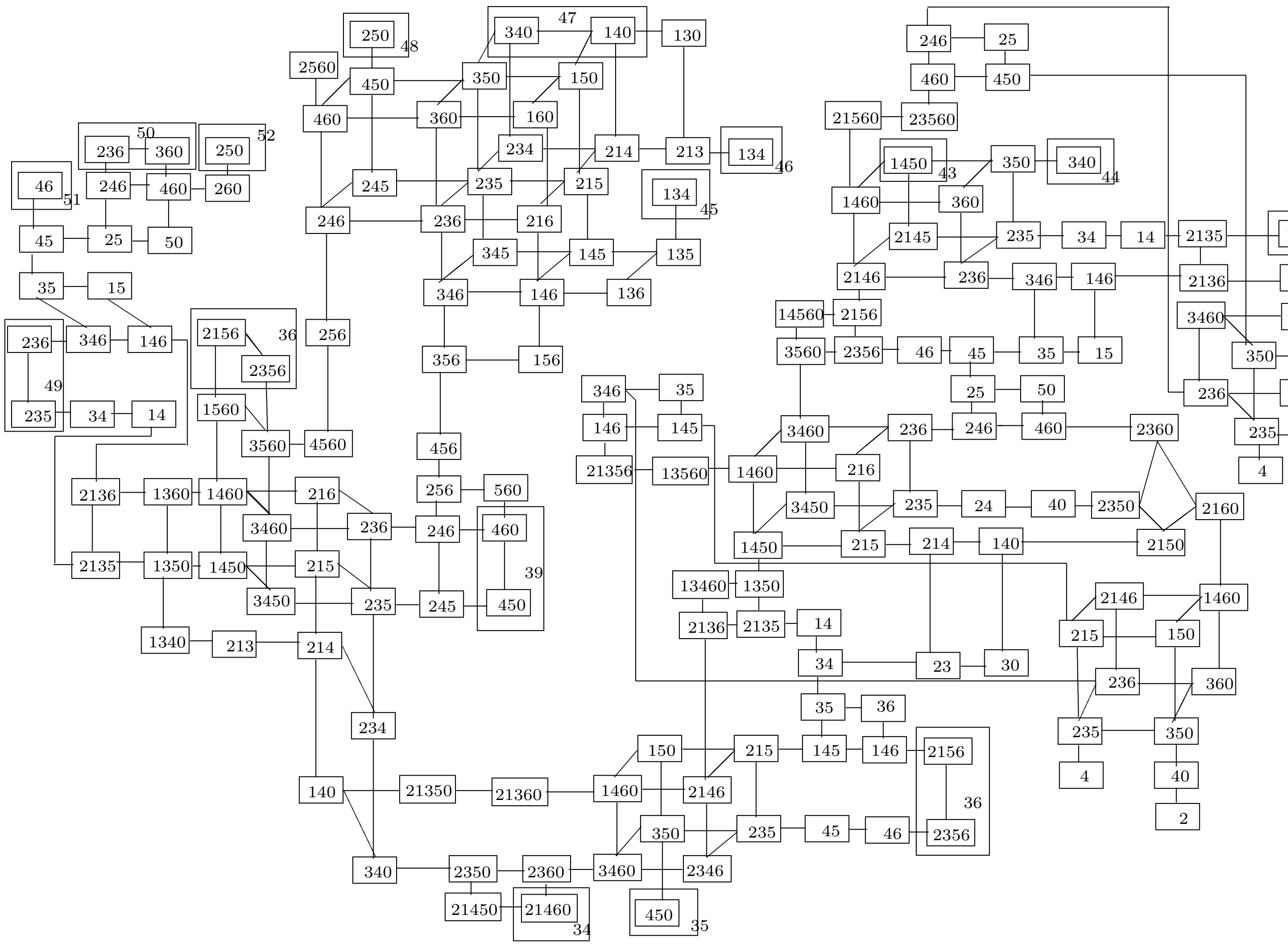
Fig.20(2)



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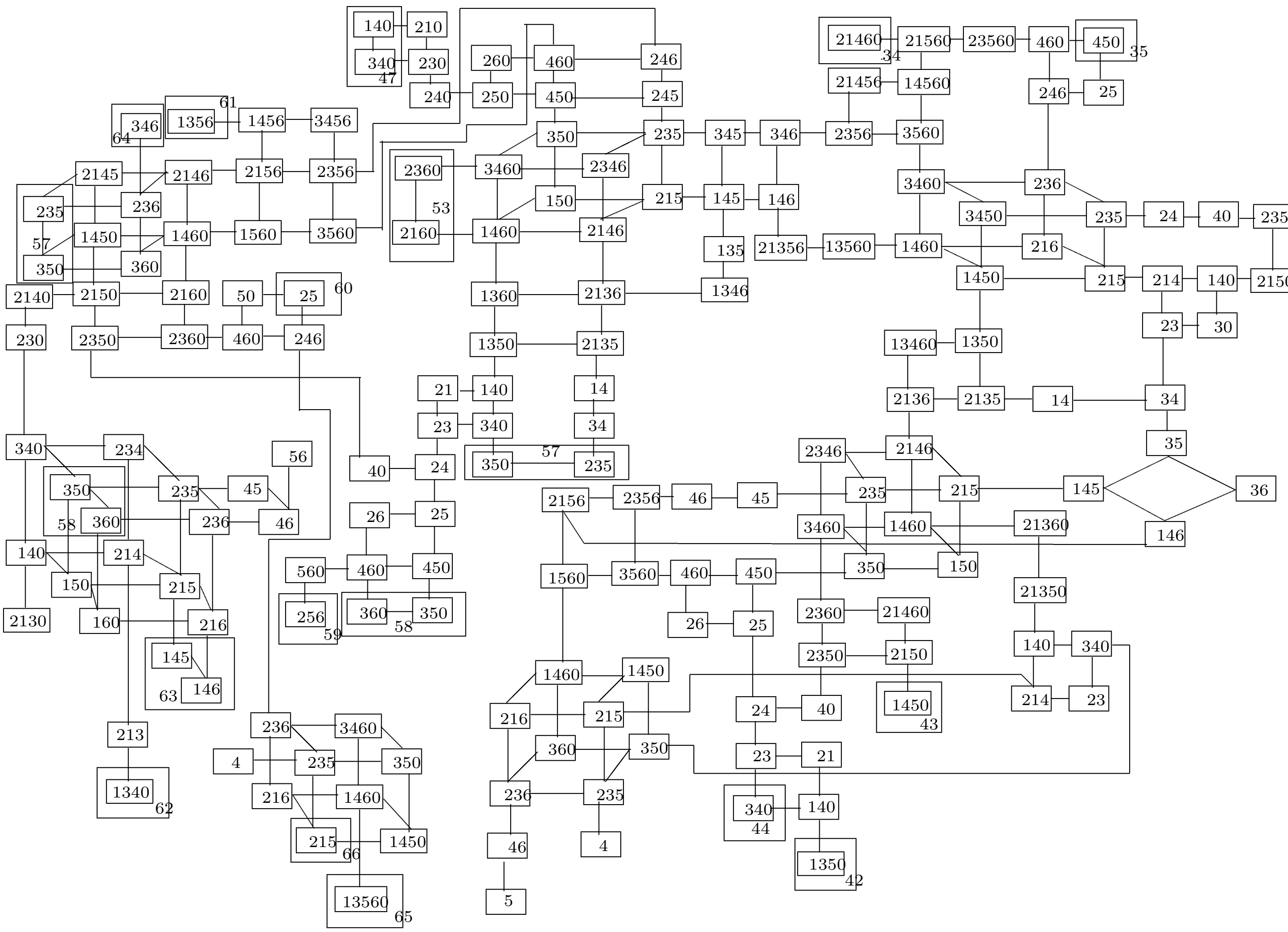
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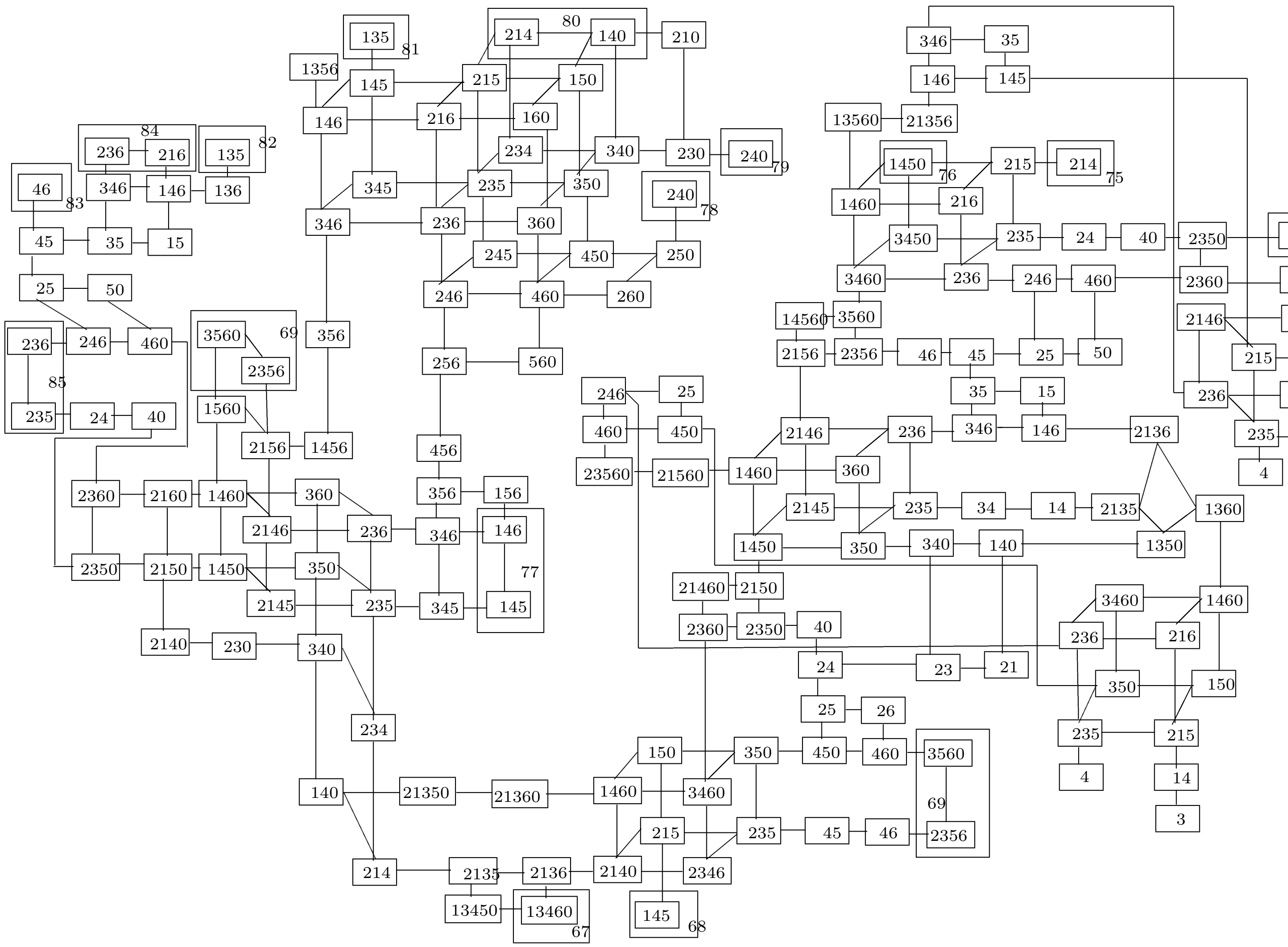
Fig.20(4)



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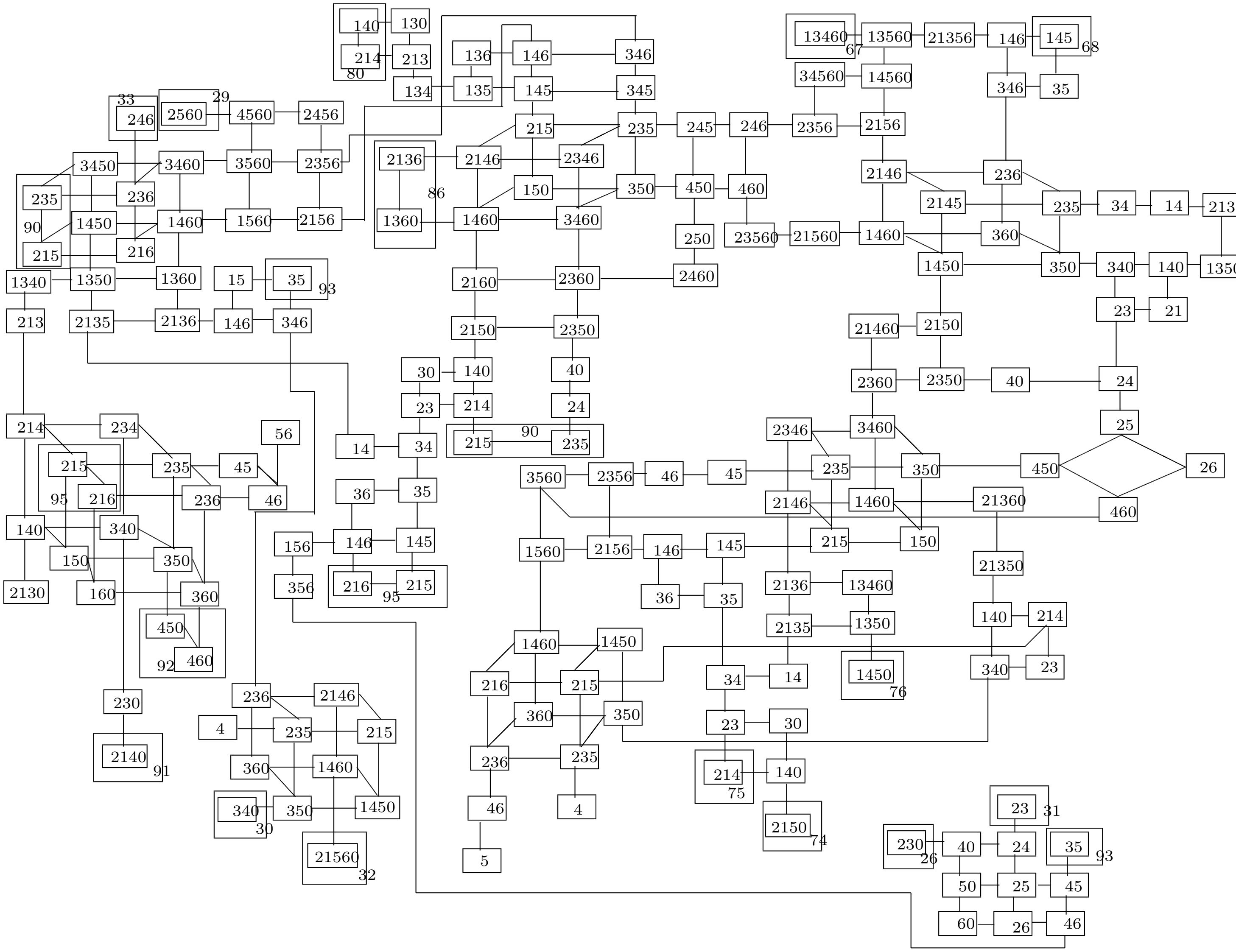

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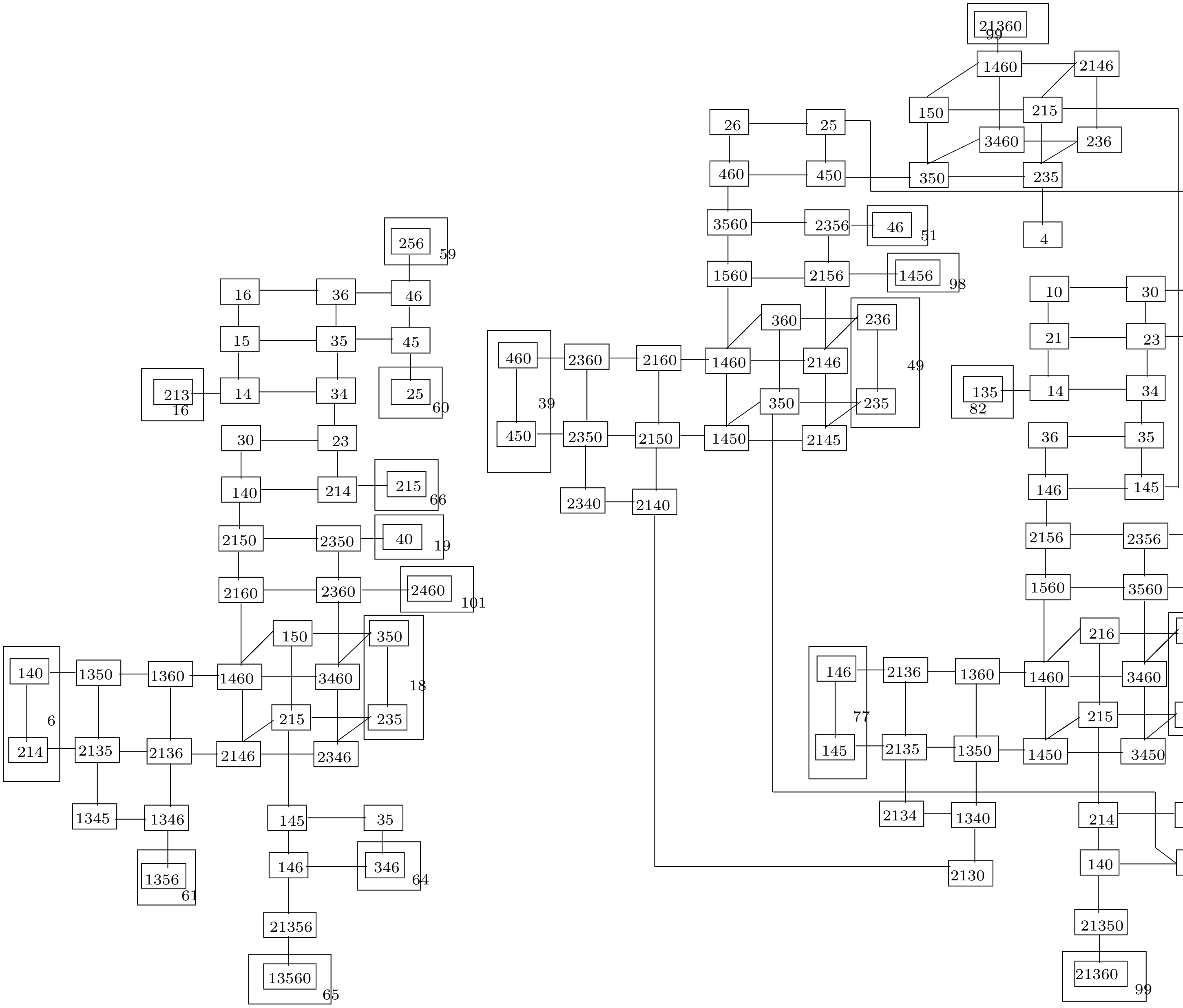
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Fig.20(7)

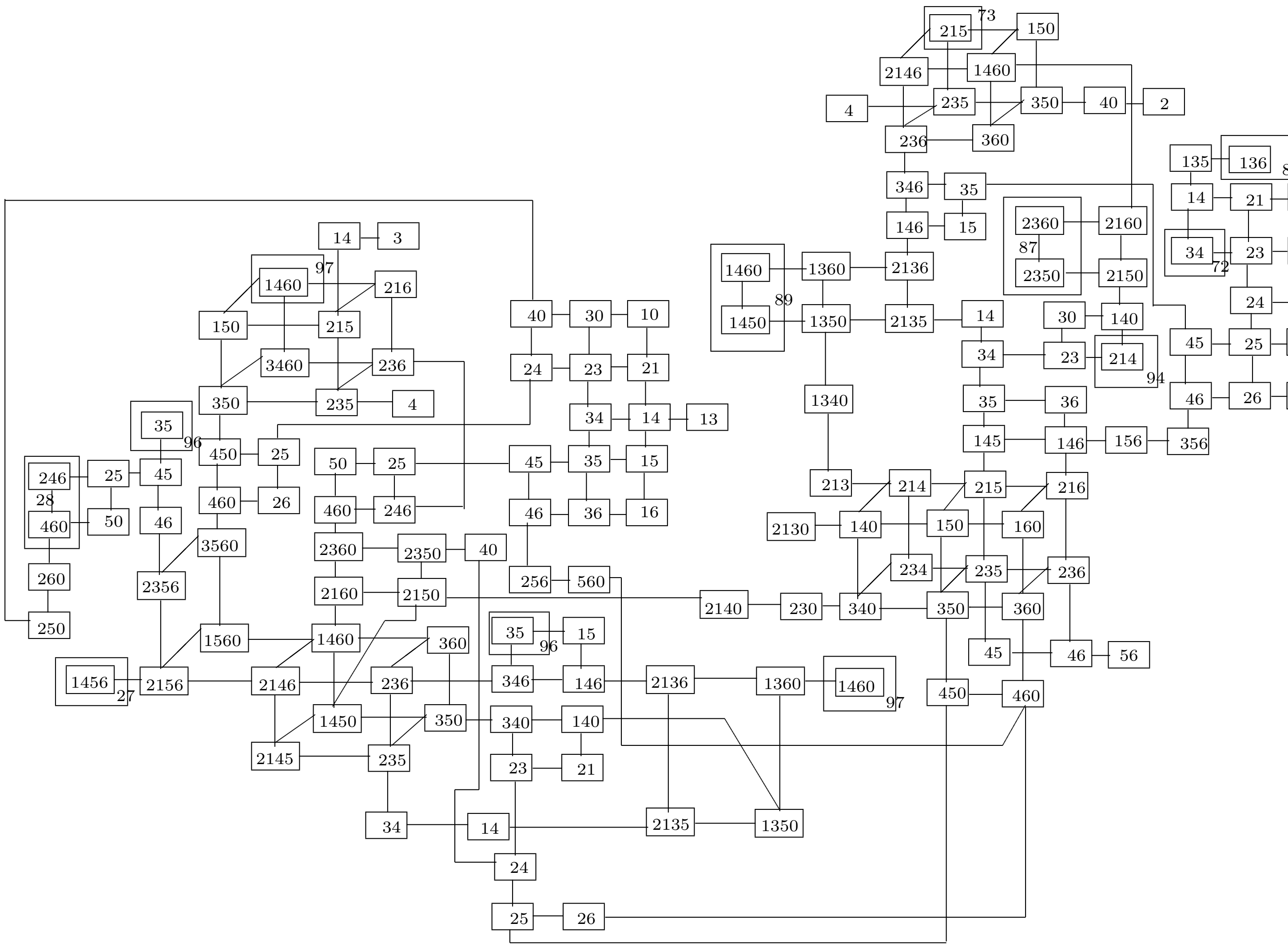
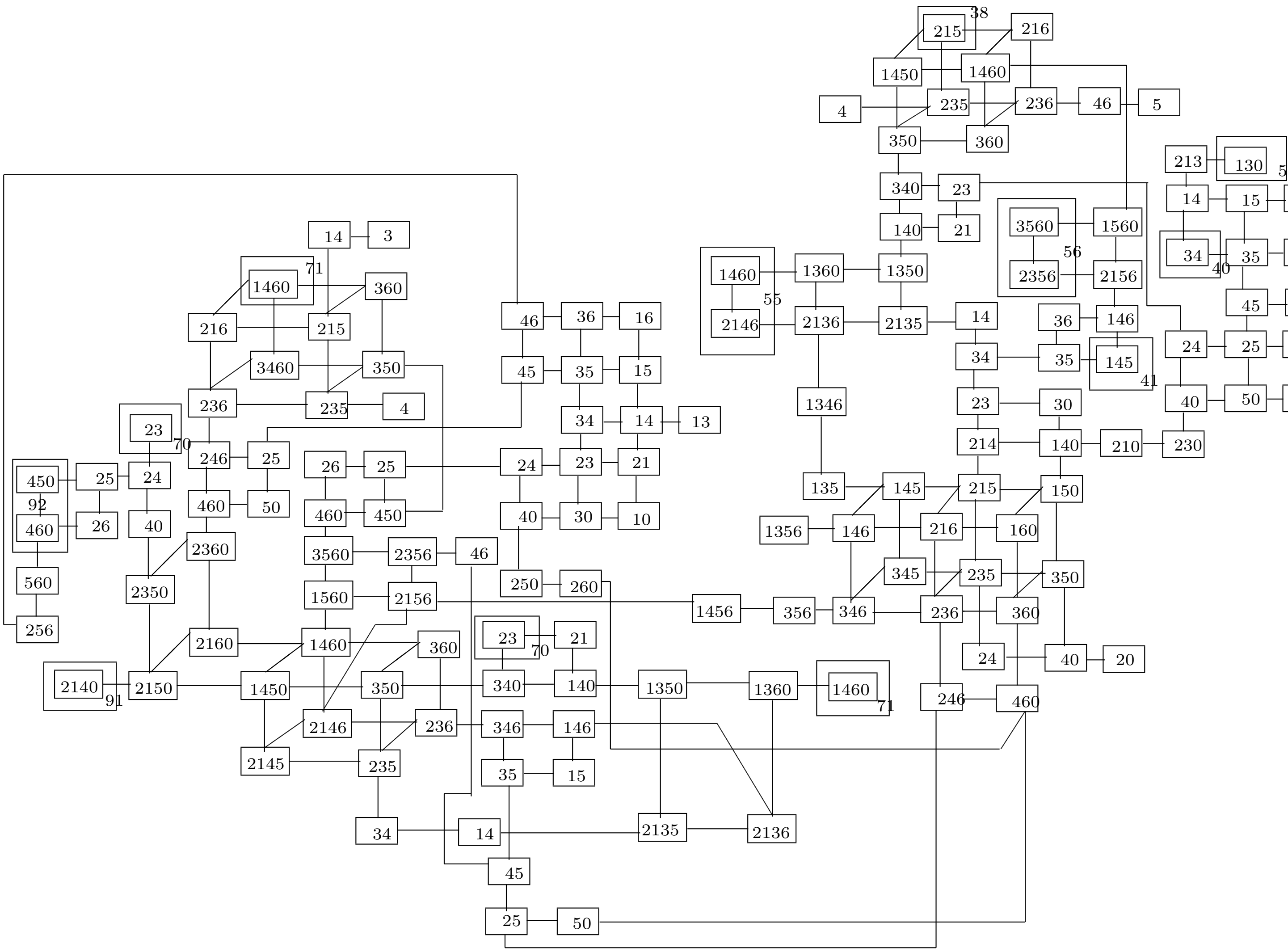


Fig.20(8)



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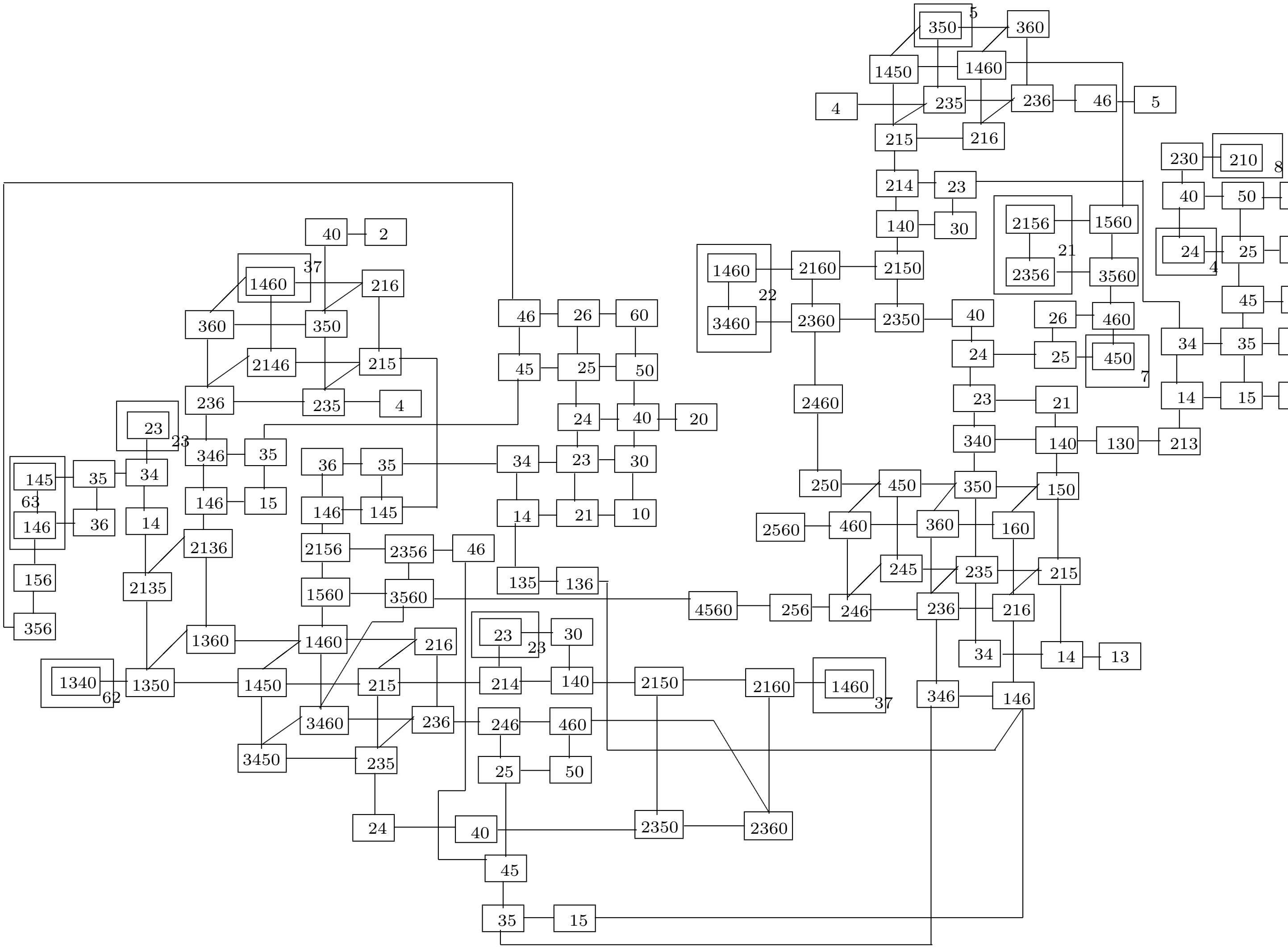
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Fig.20(10)

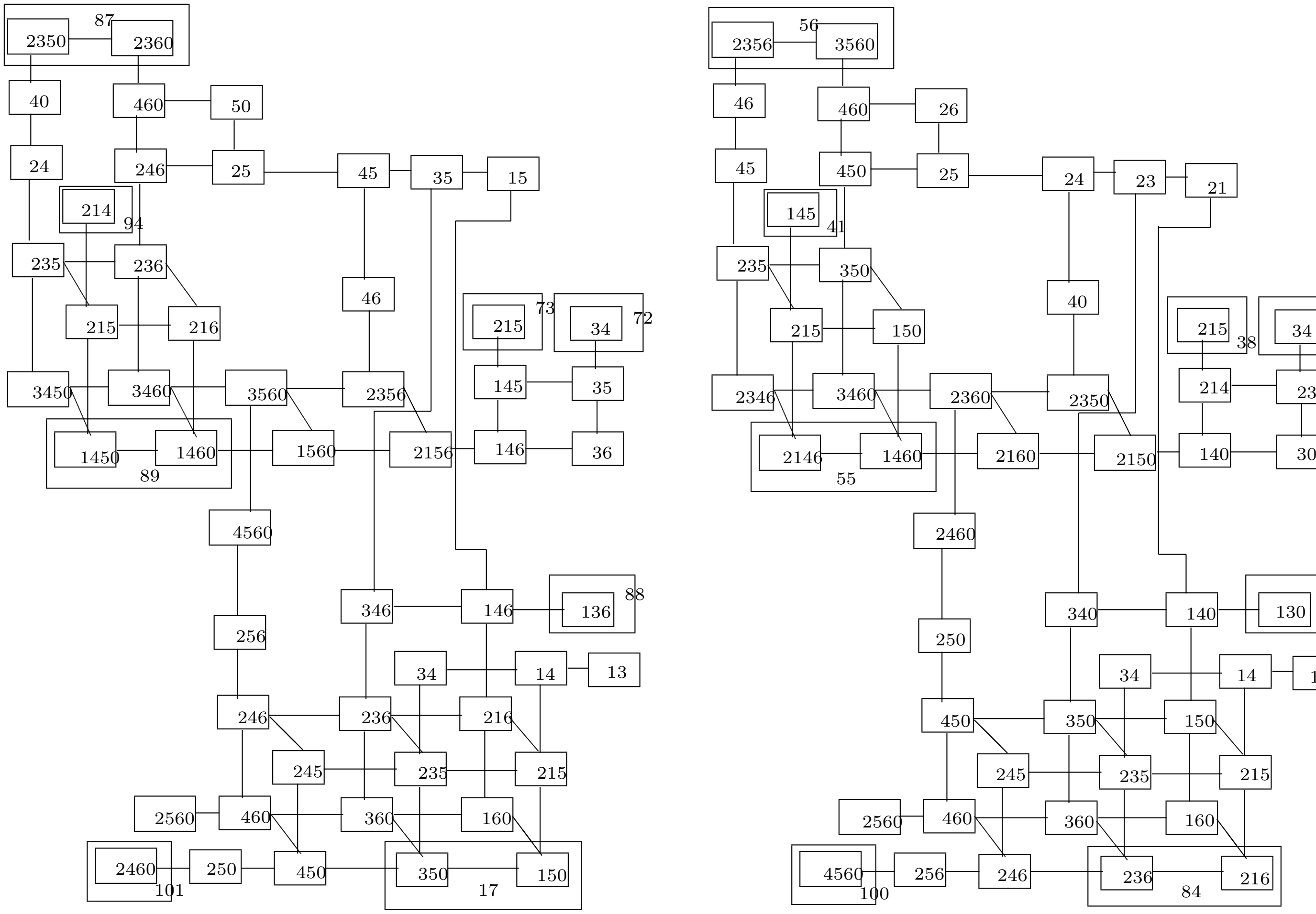
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Fig.20(11)

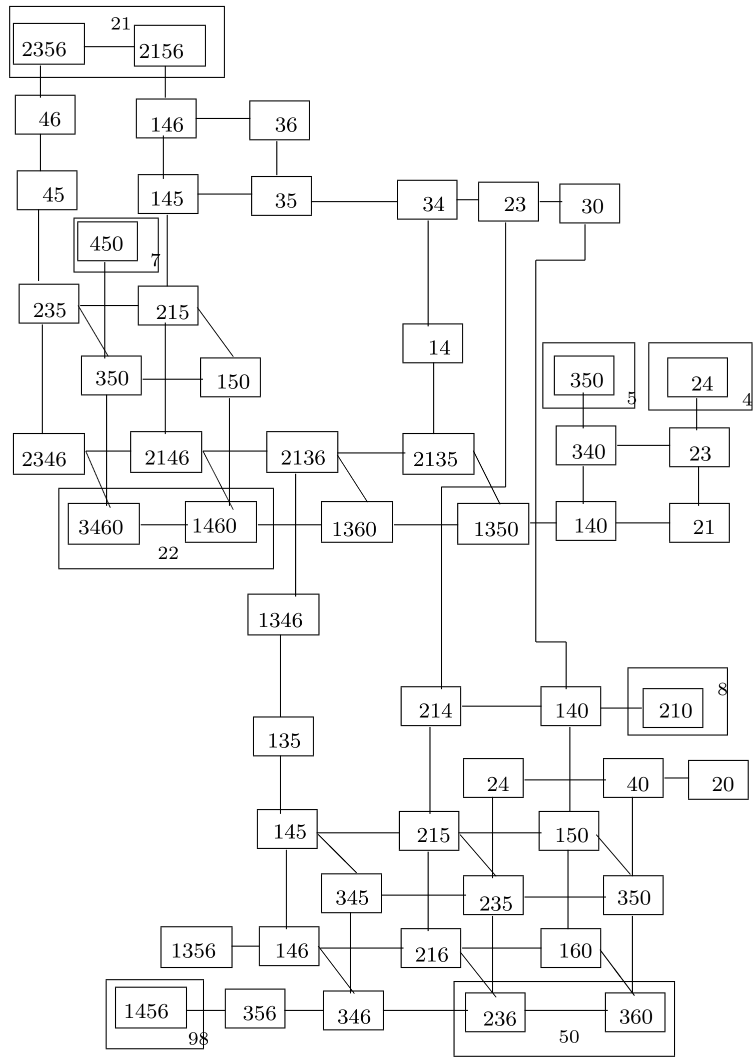
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Fig.20(12)

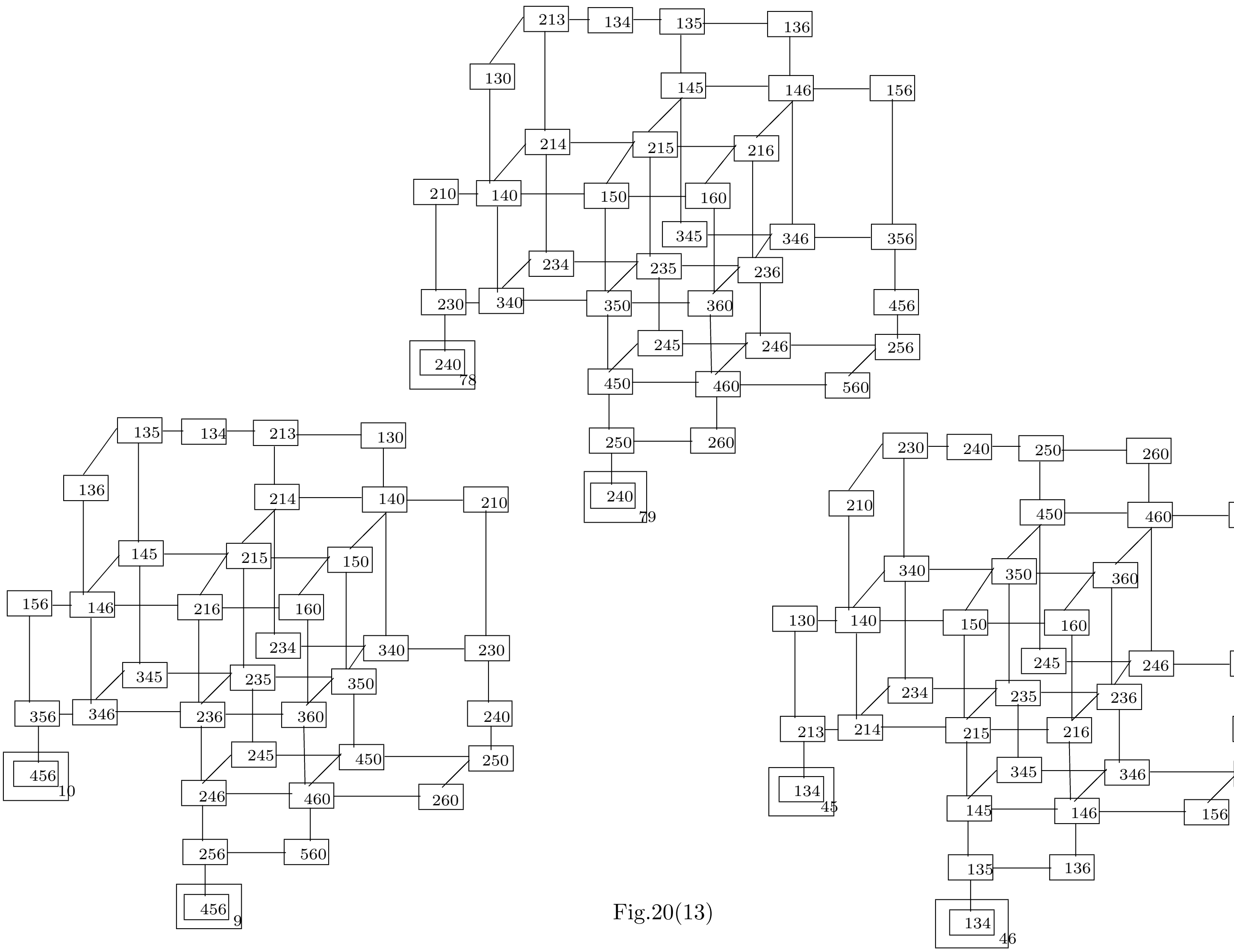

Fig.20(13)

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E-mail address: jyshi@math.ecnu.edu.cn; xyzhang71@yahoo.com.cn