LEFT CELLS WITH a-VALUE 4 IN THE AFFINE WEYL GROUPS $\widetilde{E}_i$ $(i = 6, 7, 8)$

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Abstract. The aim of the present paper is to describe all the left cells of a-value 4 in the affine Weyl groups $\widetilde{E}_i$ $(i = 6, 7, 8)$. We find a representative set of those left cells which occur as the vertex set of the corresponding left cell graphs by applying Shi's algorithm. Then we find all the distinguished involutions in those left cells. We show that those left cells are left-connected, verifying a conjecture of Lusztig in our case.

§0. Introduction.

Let W be a Coxeter group with S its distinguished generator set. In [9], Kazhdan and Lusztig introduced the concept of left, right and two-sided cells in W in order to construct representations of W and the associated Hecke algebra $\mathcal{H}$. When W is either a Weyl group or an affine Weyl group, Lusztig further introduced the function $a : W \to \mathbb{N} \cup \{\infty\}$ which is constant on any two-sided cell of W, and then Lusztig introduced a special kind of elements, called distinguished involutions in W, and proved that each left cell of W contains a unique distinguished involution (see [15]). Distinguished involutions play an important role in the representation theory of W and $\mathcal{H}$. Thus this yields a big project to describe all the left cells of a Coxeter group W, and to find all the distinguished involutions of W when W is either a Weyl group or an affine Weyl group.

The left cells L in the affine Weyl groups W have been described explicitly in the following cases:

(i) $W = \widetilde{A}_n$, $n \geqslant 1$ (see [19], [13]);

(ii) The rank of W is $\leqslant 4$ (see [2], [7], [8], [14], [26], [27], [28], [33]);

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(iii) $a(L)$ is either $\frac{1}{2}|\Phi|$ or $\leqslant 3$, where Φ is the root system of the Weyl group associated to W (see [21], [22], [10], [11], [18]);

(iv) $W \in \{\widetilde{B}_l, \widetilde{C}_m, \widetilde{D}_n \mid l \geqslant 3, m \geqslant 2, n \geqslant 4\}$ and $a(L) = 4$ (see [4], [5], [6]);

(v) L contains a fully-commutative element of W (see [29], [30]).

In the cases where either (iv)–(v) or $a(L) = \frac{1}{2}|\Phi|$, all the distinguished involutions contained in those left cells L have been described.

From now on, we always assume that W is an irreducible affine Weyl group unless otherwise specified. For any $k \in \mathbb{N}$, let $W_{(k)} = \{w \in W \mid a(w) = k\}$. Then $W_{(k)}$ is a union of some two-sided cells of W. In the present paper, we shall describe all the left cells and find all the distinguished involutions in the set $W_{(4)}$ for $W = \widetilde{E}_i$, $i = 6, 7, 8$.

The main tool in describing the left cells is Algorithm 3.4, which was constructed by Shi in [25]. We apply it to find a representative set for all the left cells (or an *l.c.r. set* for brevity) in a two-sided cell of W and construct the corresponding left cell graphs. Then we find all the distinguished involutions contained in these left cells by virtue of these left cell graphs and some results in [24], [31]. The distinguished involutions will be displayed as the vertex set in certain graphs, the latter are closely related to the corresponding left cell graphs.

A subset K of W is *left-connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, ..., x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ for $1 \leqslant i \leqslant r$. Lusztig conjectured in [1] that if W is an affine Weyl group then any left cell L of W is left-connected. The conjecture is supported by all the existing data. In $W = \widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, define $E(4) = \{w \in W_{(4)} \mid a(sw) < a(w),\ \forall\ s \in \mathcal{L}(w)\}$ and $E_{\min}(4) = \{w \in W_{(4)} \mid \ell(w) \leqslant \ell(y)\ \ \forall\ y \in W \text{ with } y \underset{L}{\sim} w\}$. Then by Proposition 6.2 (a special case of the result in [31, Proposition 2.6]), any element z of $E(4)$ has the expression $z = w_J \cdot x$ for some $x \in W$ and $J \subset S$ with $\ell(w_J) = 4$. We show that any distinguished involution d of W in $W_{(4)}$ has the form $d = x^{-1} \cdot w_J \cdot x$ for any $z = w_J \cdot x \in E(4)$ with $z \underset{L}{\sim} d$ and $J = \mathcal{L}(z)$ (see Theorem 6.6). The crucial step in showing this is Lemma 6.4, where we get the equality $E(4) = E_{\min}(4)$ by practically finding out the sets $E(4)$ and then by observing that the condition $x \underset{L}{\sim} y$ in $E(4)$ implies $\ell(x) = \ell(y)$ (see 6.3). Theorem 6.6 and Lemma 6.4 are used to prove the left-connectedness of any left cell L of W in $W_{(4)}$ (see Theorem 6.12), verifying the conjecture of Lusztig in our case.

The contents of the paper are organized as follows. Sections 1–4 are served as preliminaries, we collect some concepts, terms and known results there. We introduce Kazhdan–Lusztig cells in Section 1, star operations, primitive pairs and generalized τ-invariants in Section 2, an algorithm for finding an l.c.r. set in a two-sided cell in Section 3, and alcove form of an element in Section 4. Then in Sections 5–6, we concentrate our attention on the affine Weyl groups $W = \widetilde{E}_i$, $i = 6, 7, 8$. In Section 5, we find an l.c.r. set of any two-sided cell of W in the set $W_{(4)}$ and construct all the corresponding left cell graphs. In Section 6, we find all the distinguished involutions of W in $W_{(4)}$ and showed that all the left cells of W in $W_{(4)}$ are left-connected under the assumption of Proposition 6.2, the latter is shown in Section 7. In Appendix, we draw some graphs, from which we can get all the left cell graphs and all the distinguished involutions of $\widetilde{E}_6$ in $W_{(4)}$; we also list the elements of $\widetilde{E}_6$ in $E_{\min}(4)$.

§1. Cells.

1.1. Let W be a Coxeter group with S its distinguished generator set. Let $\leq$ be the Bruhat order on W: the notation $y \leqslant w$ in W means that there exist some reduced forms $w = s_1 s_2 \ldots s_l$ and $y = s_{i_1} s_{i_2} \ldots s_{i_t}$ with $s_i \in S$ such that $i_1, i_2, \ldots, i_t$ is a subsequence of $1, 2, \ldots, l$. For $w \in W$, we denote by $\ell(w)$ the length of w.

1.2. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of all Laurent polynomials in an indeterminate u with integer coefficients. The Hecke algebra $\mathcal{H}$ of W over $\mathcal{A}$ has two sets of $\mathcal{A}$-bases $\{T_x \mid x \in W\}$ and $\{C_w \mid w \in W\}$ which satisfy the relation

$$\text{(1.2.1)} \qquad \begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'); \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S, \end{cases}$$

and

$$\text{(1.2.2)} \qquad C_w = \sum_{y \leqslant w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y,$$

where $P_{y,w} \in \mathbb{Z}[u]$ satisfies that $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \not\leqslant w$ and $\deg P_{y,w} \leqslant (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$. The $P_{y,w}$'s are called Kazhdan-Lusztig polynomials (see [9]).

1.3. For $y,w \in W$ with $\ell(y) < \ell(w)$, we denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. Written y—w, if $\mu(y, w) \neq 0$. To any $x \in W$, we associate two subsets of S:

$$\mathcal{L}(x) = \{s \in S | sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S | xs < x\}.$$

The following relations hold: for any $x \in W$ and $s \in S$,

$$C_s C_x = \begin{cases} (u^{-1} + u) C_x, & s \in \mathcal{L}(x); \\ \sum_{\substack{y — x \\ sy < y}} \mu(x, y) C_y, & s \notin \mathcal{L}(x); \end{cases} \tag{1.3.1}$$

and

$$C_x C_s = \begin{cases} (u^{-1} + u) C_x, & s \in \mathcal{R}(x); \\ \sum_{\substack{y — x \\ ys < y}} \mu(x, y) C_y, & s \notin \mathcal{R}(x); \end{cases} \tag{1.3.2}$$

where each of the summations on the RHS of (1.3.1) and (1.3.2) contains finite terms. Moreover, $\{C_s | s \in S\}$ forms a generator set of the algebra $\mathcal{H}$ over $\mathcal{A}$.

1.4. For any $x, y, z \in W$, set $h_{x,y,z} \in \mathcal{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z.$$

Following Lusztig in [12, Chapter 5], denote $x \underset{L}{\leqslant} y$ (resp. $x \underset{R}{\leqslant} y$), if there exists some $w \in W$ with $h_{w,y,x} \neq 0$. Denote $x \underset{LR}{\leqslant} y$, if there exists some $w \in W$ with $x \underset{L}{\leqslant} w \underset{R}{\leqslant} y$ (or equivalently, if there exists some $w' \in W$ with $x \underset{R}{\leqslant} w' \underset{L}{\leqslant} y$). Write $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$, $x \underset{LR}{\sim} y$), if the relation $x \underset{L}{\leqslant} y \underset{L}{\leqslant} x$ (resp. $x \underset{R}{\leqslant} y \underset{R}{\leqslant} x$, $x \underset{LR}{\leqslant} y \underset{LR}{\leqslant} x$) holds. These are equivalence relations on W, and the equivalence classes of W with respect to $\underset{L}{\sim}$ (resp. $\underset{R}{\sim}$, $\underset{LR}{\sim}$) are called the left (resp. right, two-sided) cells of W. The preorder $\underset{L}{\leqslant}$ (resp. $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) on elements of W induces a partial order on the left (resp., right, two-sided) cells of W.

1.5. From now on, we always assume that W is an irreducible affine Weyl group unless otherwise specified. It is well known that for $x, y, z \in W$, $h_{x,y,z}$ has non-negative coefficients as a Laurent polynomial in u and that there exists some $N \in \mathbb{N}$ with $u^N h_{x,y,z} \in \mathbb{Z}[u]$ for any $x, y, z \in W$ (see [14]). So we can define a function $a : W \to \mathbb{N}$ by

$$a(z) = \min\{k \in \mathbb{N} \mid u^k h_{x,y,z} \in \mathbb{Z}[u], \ \forall\ x, y \in W\} \qquad \text{for } z \in W. \tag{1.5.1}$$

The following are some known properties of the a-function:

(1) If $x \underset{LR}{\leqslant} y$ then $a(x) \geqslant a(y)$. In particular, $x \underset{LR}{\sim} y$ implies $a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ on a left (resp. right, two-sided) cell Γ of W to be $a(x)$ for any $x \in \Gamma$ (see [14]).

(2) $a(w_J) = \ell(w_J)$ for any $J \subseteq S$ with W_J finite, where W_J is the subgroup of W generated by J and w_J is the longest element in W_J.

(3) For $x, y, w \in W$, we use the notation $w = x \cdot y$ to mean that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$, call w a *left* (resp., *right*) *extension* of y (resp., x). In this case, we have $w \underset{L}{\leqslant} y$, $w \underset{R}{\leqslant} x$ and $a(w) \geqslant a(x), a(y)$.

(4) If $a(x) = a(y)$ and $x \underset{L}{\leqslant} y$ (resp. $x \underset{R}{\leqslant} y$) then $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$) (see [15]).

(5) Let $\delta(z) = \deg P_{e,z}$ for $z \in W$, where e is the identity of the group W. Then the inequality

$$\ell(z) - 2\delta(z) - a(z) \geqslant 0 \tag{1.5.2}$$

holds for any $z \in W$. For $i \in \mathbb{N}$, define

$$\mathcal{D}_i = \{w \in W | \ell(w) - 2\delta(w) - a(w) = i\} \tag{1.5.3}$$

Then Lusztig proved in [15] that $\mathcal{D}_0$ is a finite set of involutions (called *distinguished involution* by Lusztig in [15]) and that each left (resp. right) cell of W contains a unique element of $\mathcal{D}_0$. For any $x \in W$, we have $h_{x^{-1},x,d} \neq 0$ for $d \in \mathcal{D}_0$ with $d \underset{L}{\sim} x$.

Let $W_{(i)} = \{w \in W | a(w) = i\}$ for any nonnegative integer i. Then by (1), $W_{(i)}$ is a union of some two-sided cells of W.

(6) If $W_{(i)}$ contains an element of the form w_I for some $I \subseteq S$, then $\{w \in W_{(i)} | \mathcal{R}(w) = I\}$ forms a single left cell of W.

1.6. We say that $s \in S$ is *special* if the subgroup of W generated by $S \setminus \{s\}$ has maximally possible order. For $s \in S$, let

$$Y_s = \{w \in W | \mathcal{R}(w) \subseteqq \{s\}\}.$$

Then the following result is due to Lusztig and Xi.

Theorem. (see [17]) *Let $s \in S$ be special. Then for any two-sided cell Ω of W, $\Omega \cap Y_s$ consists of exactly one left cell.*

1.7. Let W be an irreducible affine Weyl group of type $\widetilde{X}$. Let G be the connected reductive algebraic group over $\mathbb{C}$ of type $X^\vee$,where $X^\vee$ is the dual of X. Then the following result is due to Lusztig.

Theorem. (see [16, Theorem 4.8]) *There exists a bijection* $\mathbf{u} \mapsto c(\mathbf{u})$ *from the set* $\mathfrak{U}(G)$ *of unipotent conjugacy classes in* G *to the set* Cell*(W) of two-sided cells in* W *satisfying* $a(c(\mathbf{u})) = \dim \mathcal{B}_u$*, where* u *is any element in* $\mathbf{u}$*, and* $\dim \mathcal{B}_u$ *is the dimension of the variety* $\mathcal{B}_u$ *of Borel subgroups of* G *containing* u*.*

1.8. According to Theorem 1.7 and [3, Chapter 13], we see that the number of two-sided cells of W in the set $W_{(4)}$ is 2 if $W = \widetilde{E}_i$, $i = 6, 7, 8$.

§2. Star operations, primitive pairs and generalized τ-invariants.

The three concepts in the title are useful tools in finding an l.c.r. set of W and in the description of left cells of W.

2.1. Now assume that (W, S) is an irreducible affine Weyl group of simply-laced type. That is, for any $s \neq t$ in S, the order $o(st)$ of the product st is not greater than 3, or equivalently, W is of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.

Given $s \neq t$ in S with $o(st) = 3$, a set of the form $\{ys, yst\}$ is called a *right* $\{s, t\}$*-string* (or just called a *right string*), if $\mathcal{R}(y) \cap \{s, t\} = \emptyset$.

An element x is obtained from w by a *right* $\{s, t\}$*-star operation* (or a *right star operation* for brevity), if x, w are two neighboring terms in a right $\{s, t\}$-string. Note that the resulting element x of a right $\{s, t\}$-star operation on w is always unique.

Two elements $x, y \in W$ form a *right primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, \ldots, x_r$ and $y_0 = y, y_1, \ldots, y_r$ in W such that the following conditions are satisfied.

(a) For each $1 \leqslant i \leqslant r$, there exist some $s_i, t_i \in S$ with $o(s_i t_i) = 3$ such that both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are right $\{s_i, t_i\}$-strings.

(b) x_i —— y_i for some (and then for all) $0 \leqslant i \leqslant r$.

(c) Either $\mathcal{R}(x) \not\subseteq \mathcal{R}(y)$ and $\mathcal{R}(y_r) \not\subseteq \mathcal{R}(x_r)$, or $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $\mathcal{R}(x_r) \not\subseteq \mathcal{R}(y_r)$ hold.

Similarly, we can define a *left* $\{s, t\}$*-string*, a *left* $\{s, t\}$*-star operation* on an element, and a *left primitive pair*.

Note that elements in a right (resp., left) string form a right (resp., left) primitive pair.

The following result is well known.

Lemma. (see [23, Section 3]) *If x, y is a right (resp., left) primitive pair, then $x \underset{R}{\sim} y$ (resp., $x \underset{L}{\sim} y$).*

2.2. For each element $x \in W$, we denote by $M(x)$ the set of all elements y such that there is a sequence of elements $x = x_0, x_1, \ldots, x_r = y$ in W with some $r \geqslant 0$, where $\{x_{i-1}, x_i\}$ is a right string for each $1 \leqslant i \leqslant r$.

2.3. By a *graph* $\mathcal{M}$, we mean that a set M of vertices together with a set of edges, where each edge is a two-element subset of M, and each vertex is labelled by some subset of S.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two graphs with the vertex sets M and M' respectively. They are *isomorphic*, written $\mathcal{M} \cong \mathcal{M}'$, if there exists a bijective map η from M to M' satisfying the following conditions.

(a) The label of x is the same as that of $\eta(x)$ for any $x \in M$.

(b) For $x, y \in M$, $\{x, y\}$ is an edge of $\mathcal{M}$ if and only if $\{\eta(x), \eta(y)\}$ is an edge of $\mathcal{M}'$.

This defines an equivalence relation on graphs.

2.4. We define a *graph* $\mathcal{M}(x)$ *associated to* an *element* $x \in W$ as follows. Its vertex set is $M(x)$ and its edge set consists of all the pairs $\{y, z\}$ in $M(x)$ which form a right string. Each vertex $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$. Clearly, for any $x \in W$, the graph $\mathcal{M}(x)$ is always connected.

A left cell graph associated to an element $x \in W$, written $\mathcal{M}_L(x)$, is by definition a graph, whose vertex set $M_L(x)$ consists of all the left cells Γ of W with $\Gamma \cap M(x) \neq \emptyset$. Two vertices $\Gamma, \Gamma' \in M_L(x)$ are joined by an edge, if there are two elements $y \in M(x) \cap \Gamma$ and $y' \in M(x) \cap \Gamma'$ with $\{y, y'\}$ an edge of $\mathcal{M}(x)$. Each vertex Γ of $\mathcal{M}_L(x)$ is labelled by the common label of the elements in $M(x) \cap \Gamma$. Clearly, the graph $\mathcal{M}_L(x)$ is always connected.

For example, there are three vertices labelled by $\boxed{\mathbf{350}}$ (meaning that the corresponding left cells L satisfy $\mathcal{R}(L) = \{\mathbf{3},\mathbf{5},\mathbf{0}\}$) in Figure A: Denote the one at the northeast by L. Then the vertex L is joined with four other vertices by edges, corresponding to the right $\{\mathbf{4},\mathbf{5}\}$-, $\{\mathbf{3},\mathbf{4}\}$-, $\{\mathbf{1},\mathbf{3}\}$-, $\{\mathbf{0},\mathbf{2}\}$- and $\{\mathbf{5},\mathbf{6}\}$-star operations, respectively. The edge joining with the vertex labelled by $\boxed{\mathbf{40}}$ corresponds to two star operations: $\{\mathbf{3},\mathbf{4}\}$- and $\{\mathbf{4},\mathbf{5}\}$-, while any of the other three edges corresponds to just one star operation.

2.5. By a *path* in the graph $\mathcal{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_r$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathcal{M}(x)$ for any $1 \leqslant i \leqslant r$. Two elements $x, x' \in W$ have the same *generalized* τ*-invariants*, if for any path $z_0 = x, z_1, ..., z_r$ in $M(x)$, there is a path $z'_0 = x', z'_1, ..., z'_r$ in $M(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for any $0 \leqslant i \leqslant r$, and if the same condition holds when the roles of x and x' are interchanged.

Then we have the following known result.

Proposition 2.6. *(a) Any $x, y \in W$ with $x \underset{L}{\sim} y$ have the same generalized τ-invariants.*

(b) For $x \underset{L}{\sim} y$ in W, the left cell graphs $\mathcal{M}_L(x)$ and $\mathcal{M}_L(y)$ are isomorphic.

Remark 2.7. 2.1–2.5 are due to Shi (see [25]), while 2.6 is due to Vogan (see [32]).

§3. An algorithm for finding an l.c.r. set in a two-sided cell.

A subset $K \subset W$ is called a *representative set of left cells* (or an *l.c.r. set* for brevity) in W (resp., in a two-sided cell Ω of W), if $|K \cap \Gamma| = 1$ for any left cell Γ in W (resp., in Ω), where the notation $|X|$ stands for the cardinality of a set X.

Obviously, the set $\mathcal{D}_0$ (see 1.5 (5)) is an l.c.r. set of W. In this section, we state an algorithm for finding an l.c.r. set (not necessarily contained in $\mathcal{D}_0$) in a two-sided cell of W, which was first introduced by Shi in [25].

The algorithm is based on the following

Theorem 3.1. (see [25]) *Let Ω be a two-sided cell of W. Then N, $\emptyset \neq N \subset \Omega$, is an l.c.r. set in Ω, if N satisfies the following conditions:*

(1) $x \underset{L}{\not\sim} y$ for any $x \neq y$ in N;

(2) Let $y \in W$. Suppose that there exists an element $x \in N$ such that $y — x, \mathcal{R}(y) \nsubseteq \mathcal{R}(x)$ and $a(y) = a(x)$. Then there exists some $z \in N$ with $y \underset{L}{\sim} z$.

3.2. A subset $P \subset W$ is said to be *distinguished* if $P \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in P. Assume that P is a subset of Ω. We introduce the following processes (see [25]).

(**A**) Find a largest possible subset Q from the set $\cup_{x\in P} M(x)$ with Q distinguished.

(**B**) For each $x \in P$, set

$$B_x = \{y \in W | y^{-1}x \in S, \mathcal{R}(y) \supsetneqq \mathcal{R}(x), a(y) = a(x)\}$$

and $B = P \cup (\cup_{x\in P} B_x)$. Take a largest possible subset Q from B with Q distinguished.

(**C**) For each $x \in P$, set

$$C_x = \{y \in W | y < x, y \text{—} x, \mathcal{R}(y) \supsetneqq \mathcal{R}(x), a(y) = a(x)\}$$

and define $C = P \cup (\cup_{x \in P} C_x)$. Take a largest possible subset Q from C with Q distinguished.

3.3. A subset P of W is **A**-*saturated* (resp., **B**-*saturated*, **C**-*saturated*), if Process (**A**) (resp., (**B**), (**C**)) on P cannot produce any element z satisfying $z \not\sim_L x$ for any $x \in P$.

Clearly, a set of the form $\cup_{x \in K} M(x)$ for any $K \subseteq W$ is always **A**-saturated.

It follows from Theorem 3.1 that an l.c.r. set of W in a two-sided cell Ω is exactly a distinguished subset of Ω which is **ABC**-saturated. In order to get such a subset, we need the following algorithm on a set $P \neq \emptyset$.

Algorithm 3.4. (see [25])

(1) Find a non-empty subset P of Ω (we take P to be distinguished to avoid unnecessary complication whenever possible);

(2) Perform Processes (**A**), (**B**) and (**C**) alternately on P until the resulting distinguished set cannot be further enlarged by any of these three processes.

In applying Algorithm 3.4 on a two-sided cell Ω, it is convenient to take the starting set P to consist of some elements of the form w_J, $J \subseteq S$, in Ω whenever such kind of elements are available.

§4. Alcove form of an element.

Alcove form is a useful expression of an element w in an affine Weyl group W from which one can easily read out the length $\ell(w)$ and the set $\mathcal{R}(w)$ (see Proposition 4.4). In our running Algorithm 3.4 in Sections 5 and 6, all the elements will be expressed in their alcove forms. Alcove form was originally introduced by Shi in [19], [20].

4.1. An affine Weyl group W has the following geometric realization. Let G be a connected, adjoint reductive algebraic group over $\mathbb{C}$. Fix a maximal torus T of G. Let X be the character group of T and let Φ be the root system in X with $\Delta = \{\alpha_1, ..., \alpha_l\}$, Φ^+ a choice of simple root system and the corresponding positive root system, respectively. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a euclidean space with an inner product $\langle\ ,\ \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections $s_i := s_{\alpha_i}$, $1 \leqslant i \leqslant l$. We denote by N the group of all translations T_λ $(\lambda \in X)$ on E: T_λ sends

x to $x+\lambda$. Then the semidirect product $W = N \rtimes W_0$ is called an affine Weyl group. Let K be the dual of the type of G. Then the type of W is $\widetilde{K}$. Sometimes we denote W simply by $\widetilde{K}$ when no danger of confusion. There is a canonical homomorphism from W to W_0: $w \mapsto \bar{w}$.

Let $-\alpha_0$ be the highest short root in Φ. We define $s_0 = s_{\alpha_0} T_{-\alpha_0}$, where s_{α_0} is the reflection corresponding to α_0. Then the generator set of W can be taken as $S = S_0 \cup \{s_0\}$.

4.2. The *alcove form* of an element $w \in W$ is, by definition, a Φ-tuple $(k(w,\alpha))_{\alpha\in\Phi}$ over $\mathbb{Z}$ subject to the following conditions.

(a) $k(w,-\alpha) = -k(w,\alpha)$ for any $\alpha \in \Phi$;

(b) $k(e,\alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity element of W;

(c) If $w' = ws_i$ $(0 \leqslant i \leqslant l)$, then

$$k(w',\alpha) = k(w,(\alpha)\bar{s_i}) + \varepsilon(\alpha,i)$$

with

$$\varepsilon(\alpha,i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

where $\bar{s_i} = s_i$ if $1 \leqslant i \leqslant l$, and $\bar{s_0} = s_{\alpha_0}$.

By condition (a), we can also denote the alcove form of $w \in W$ by a Φ^+-tuple $(k(w,\alpha))_{\alpha\in\Phi^+}$.

4.3. Condition 4.2.(c) defines a set of operators $\{s_i | 0 \leqslant i \leqslant l\}$ on the alcove forms of elements of W:

$$s_i: \quad (k_\alpha)_{\alpha\in\Phi} \mapsto (k_{(\alpha)\bar{s}_i} + \varepsilon(\alpha,i))_{\alpha\in\Phi}.$$

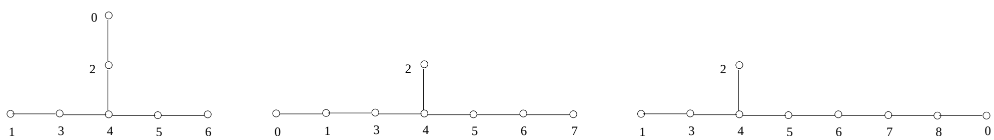
The following result of Shi describes the functions $\ell(w)$ and $\mathcal{R}(w)$ for any $w \in W$.

Proposition 4.4. (see [20]) *Let w be an element in an affine Weyl group W.*

(1) $\ell(w) = \Sigma_{\alpha\in\Phi^+} |k(w,\alpha)|$, where the notation $|x|$ stands for the absolute value of x;

(2) $\mathcal{R}(w) = \{s_i | k(w,\alpha_i) < 0\}$.

4.5. The Coxeter diagrams of the affine Weyl groups $\widetilde{E}_i$ $(i = 6,7,8)$ are as follows.



$\widetilde{E}_6$ $\widetilde{E}_7$ $\widetilde{E}_8$

Let us explain how to use alcove forms of elements in the study of left cells of an affine Weyl group W by taking $W = \widetilde{E}_6$ as an example. The alcove form of any $w \in \widetilde{E}_6$ consists of 36 integer entries indexed by positive roots of the root system $\Phi(E_6)$. We arrange them in a fixed order as in Figure 1. The notation $\begin{smallmatrix} & & b & & \\ a & c & d & e & f \end{smallmatrix}$ stands for the root $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6$, where $\alpha_1, ..., \alpha_6$ are simple roots in $\Phi(E_6)$ whose labels coincide with those in the Coxeter graph $\Gamma(E_6)$. By Proposition 4.4, the set $\mathcal{R}(w)$ can be easily read out from the alcove form of w, for example, s_1 (resp., s_0) is in $\mathcal{R}(w)$ if and only if $a' < 0$ (resp., $u > 0$). Also, by 4.3, the relation between the alcove forms of w and ws is displayed graphically, where we take $s = s_1, s_0$ in Figure 1 as examples. w is in a right $\{s_1, s_3\}$- (resp., $\{s_0, s_2\}$-) string if and only if exactly

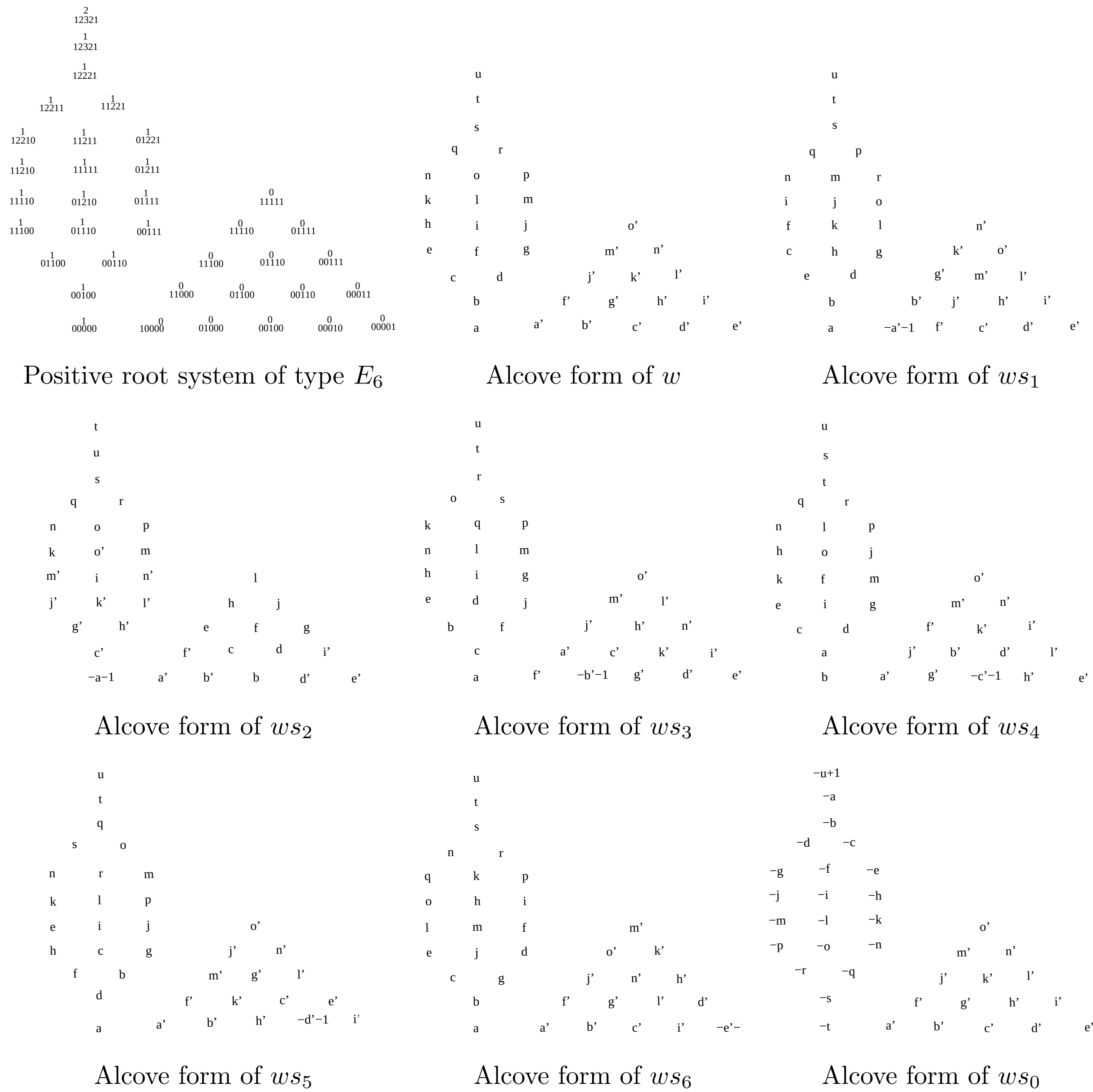


Figure 1.

one of the entries a', b' (resp., $a, -u$) is negative. In this case, ws_1, resp., ws_3, (resp., ws_0, resp., ws_2,) is obtained from w by a right $\{s_1, s_3\}$- (resp., $\{s_0, s_2\}$-) star operation if and only if the integers f', a', resp., b', f', (resp., $-u, -t$, resp., $a, -t$,) are either both negative or both not. Hence one can successively apply various right star operations on the alcove form of w to get the graph $\mathcal{M}(w)$, in particular, its vertex set $M(w)$. Such a process can be run by a computer programme in GAP. The results in 1.5 (1), (6), Theorem 1.6 and Proposition 2.6 will be useful tools in checking the relation $x \underset{L}{\sim} y$ or $x \underset{L}{\not\sim} y$ in $M(w)$. In applying Processes **B** and **C**, one need find various primitive pairs $\{x, y\}$. This can be done by chasing the trace on the graphs $\mathcal{M}(x)$ and $\mathcal{M}(y)$, the latter could be worked out in terms of alcove forms of elements again by 4.3 and Proposition 4.4. We refer the readers to any of [25], [26], [27], [28] for applying the above method in the groups $\widetilde{C}_4$, $\widetilde{D}_4$ and $\widetilde{F}_4$.

Similar for the case of $\widetilde{E}_7$, $\widetilde{E}_8$.

§5. An l.c.r. set in $W_{(4)}$.

5.1. In the present section, we concentrate our attention on the groups $W = \widetilde{E}_i$, $i = 6, 7, 8$. We find an l.c.r. set for any two-sided cell of W in $W_{(4)}$ and construct all the corresponding left cell graphs by running a computer programme in GAP in terms of alcove forms of elements. Owing to the limitation of the space, we only sketch the major steps with the most technical details omitted. Also, we only draw out the resulting left cell graphs and certain small graphs, the latter will be used to show the primitivity for certain pairs of elements, while many other graphs occurring in the intermediate steps (e.g. $\mathcal{M}(x)$, $\mathcal{M}'(x)$, etc) will often be omitted.

From now on, we use the boldfaced letter **i** to denote the simple reflection s_i corresponding to the vertex of the Coxeter diagram labelled by i (see 4.5).

5.2. For the group $W = \widetilde{E}_6$, there are two two-sided cells in $W_{(4)}$ (see 1.8). All the elements of the form w_J, $J \subseteq S$, in $W_{(4)}$ are as follows:

1312, **1310**, **1315**, **1316**, **3430**, **3436**, **4540**, **4541**, **4241**, **4246**,

2021, **2023**, **2025**, **2026**, **5652**, **5650**, **5651**, **5653**, **1046**.

Let $\Omega(6, 1)$ be the two-sided cell of $\widetilde{E}_6$ containing $x =$ **1312**. Take $P = \{x\}$ (here and later, the notation P is always used as a starting set in applying Algorithm 3.4 for finding an l.c.r. set of the related two-sided cell). The graph $\mathcal{M}(x)$ contains infinite number of vertices. Take a subgraph $\mathcal{M}'(x)$ of $\mathcal{M}(x)$ with the vertex labelled by $\boxed{\mathbf{123}}$ being the element **1312**, such that the restriction to $M'(x)$ of the natural map $\eta : M(x) \to M_L(x)$ with $\eta(y) = L_y$ for any $y \in M(x)$ is bijective. Here L_x is the left cell of W containing x, $M'(x)$ is the vertex set of $\mathcal{M}'(x)$, and

$M_L(x)$ is the vertex set of the left cell graph $\mathcal{M}_L(x)$, the latter is displayed as in Figure A but with all the edge labels forgotten (see 2.4 and Appendix). It can be checked that the set

$$M_1 := M'(x) \qquad (5.2.1)$$

is distinguished, and also **ABC**-saturated. Hence M_1 forms an l.c.r. set in $\Omega(6,1)$.

Since there is no element w in M_1 with $\mathcal{R}(w) = \{\mathbf{1},\mathbf{0},\mathbf{4},\mathbf{6}\}$, the element $y = \mathbf{1046}$ is not in $\Omega(6,1)$. Consider the two-sided cell $\Omega(6,2)$ containing y. Take $P = \{y\}$. The graph $\mathcal{M}(y)$ (or rather, the left cell graph $\mathcal{M}_L(y)$) is as in Figure B but with all the edge labels forgotten (see Appendix). The vertex set $M(y)$ of $\mathcal{M}(y)$ is distinguished and **A**-saturated, but is not **B**-saturated. Take $y_0 = y\mathbf{5243} \in M(y)$ and $z = y_0\mathbf{1}$. Consider the pair $\{y_0, z\}$. We have $\mathcal{R}(y_0) = \{\mathbf{3}\}$ and $\mathcal{R}(z) = \{\mathbf{1},\mathbf{3}\}$. Let $y_0' = y_0\mathbf{3}$, $y_0'' = y_0'\mathbf{4}$, $z' = z\mathbf{4}$ and $z'' = z'\mathbf{5}$. Then $y_0', y_0'' \in M(y)$ with $\mathcal{R}(y_0') = \{\mathbf{1},\mathbf{4}\}$ and $\mathcal{R}(y_0'') = \{\mathbf{1},\mathbf{2},\mathbf{5}\}$. Also, $z', z'' \in M(z)$ with $\mathcal{R}(z') = \{\mathbf{1},\mathbf{4}\}$ and $\mathcal{R}(z'') = \{\mathbf{1},\mathbf{5}\}$. This implies that $\{y_0, z\}$ forms a right primitive pair (see 2.1 and Figure 1 (a)).

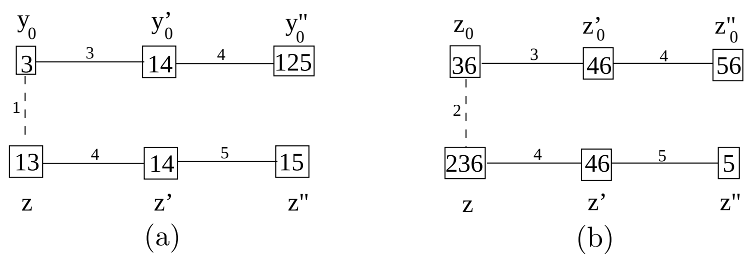


Figure 1

Hence $z \in \Omega(6,2)$ by Lemma 2.1. The graph $\mathcal{M}(z)$ (or rather, the left cell graph $\mathcal{M}_L(z)$) is as in Figure C but with all the edge labels forgotten (see Appendix). The set $M(y) \cup M(z)$ is distinguished, but is still not **B**-saturated. Consider the pair $\{z_0, z_1\}$ with $z_0 = z\mathbf{4156} \in M(z)$ and $z_1 = z_0\mathbf{2}$. We have $\mathcal{R}(z_0) = \{\mathbf{3},\mathbf{6}\}$ and $\mathcal{R}(z_1) = \{\mathbf{2},\mathbf{3},\mathbf{6}\}$. Let $z_0' = z_0\mathbf{3}$, $z_0'' = z_0'\mathbf{4}$, $z_1' = z_1\mathbf{4}$ and $z_1'' = z_1'\mathbf{5}$. Then $z_0', z_0'' \in M(z)$ with $\mathcal{R}(z_0') = \{\mathbf{4},\mathbf{6}\}$ and $\mathcal{R}(z_0'') = \{\mathbf{5},\mathbf{6}\}$. Also, $z_1', z_1'' \in M(z_1)$ with $\mathcal{R}(z_1') = \{\mathbf{4},\mathbf{6}\}$ and $\mathcal{R}(z_1'') = \{\mathbf{5}\}$. This implies that $\{z_0, z_1\}$ is a right primitive pair (see Figure 1 (b)). So $z_1 \in \Omega(6,2)$ by Lemma 2.1. The graph $\mathcal{M}(z_1)$ (or rather, the left cell graph $\mathcal{M}_L(z_1)$) is as in Figure D but with all the edge labels forgotten (see Appendix). It can be checked that the set $M(y) \cup M(z) \cup M(z_1)$ is distinguished and is also **ABC**-saturated. Thus

(5.2.2) $$M_2 := M(y) \cup M(z) \cup M(z_1)$$

forms an l.c.r. set of $\Omega(6,2)$.

Now we obtain the following

Theorem 5.3. *In the affine Weyl group $\widetilde{E}_6$, there are two two-sided cells $\Omega(6,1)$ and $\Omega(6,2)$ in the set $W_{(4)}$, where $\Omega(6,1)$ consists of 162 left cells with M_1 in (5.2.1) as an l.c.r. set, and $\Omega(6,2)$ consists of 72 left cells with M_2 in (5.2.2) as an l.c.r. set.*

5.4. For the affine Weyl group $\widetilde{E}_7$, there are two two-sided cells in $W_{(4)}$ (see 1.8). All the elements of the form w_J, $J \subseteq S$, in $W_{(4)}$ are as follows.

1014, **1012**, **1015**, **1016**, **1017**, **1312**, **1315**, **1316**, **1317**, **3430**, **3436**, **3437**,

4240, **4241**, **4246**, **4247**, **4540**, **4541**, **4547**, **5652**, **5653**, **5651**, **5650**, **6764**,

6762, **6763**, **6761**, **6760**, **0325**, **0326**, **0327**, **2357**, **1257**, **0257**, **0357**.

Let $\Omega(7,1)$ be the two-sided cell of $\widetilde{E}_7$ containing the element $t_1 = \mathbf{2350}$. Take $P = \{t_1\}$. The set $M(t_1)$ is distinguished and **A**-saturated, but is not **B**-saturated. Take $y_0 = t_1\mathbf{16745342654310}$ and $z_0 = y_0\mathbf{7}$. Then $y_0 \in M(t_1)$. The elements y_0, z_0 form a right primitive pair, as shown in Figure 2 (a), where the vertices on the top (resp., bottom) line are $y_0, y_1, ..., y_{12}$ (resp., $z_0, z_1, ..., z_{12}$) (from left to right). We see that z_{12} can be obtained from z_0 by the same sequence of right star operations as y_{12} from y_0 and that $\mathcal{R}(z_0) = \{\mathbf{7},\mathbf{0}\} \not\subseteq \{\mathbf{0}\} = \mathcal{R}(y_0)$ and $\mathcal{R}(y_{12}) = \{\mathbf{2},\mathbf{3},\mathbf{7},\mathbf{0}\} \not\subseteq \{\mathbf{2},\mathbf{3},\mathbf{7}\} = \mathcal{R}(z_{12})$.

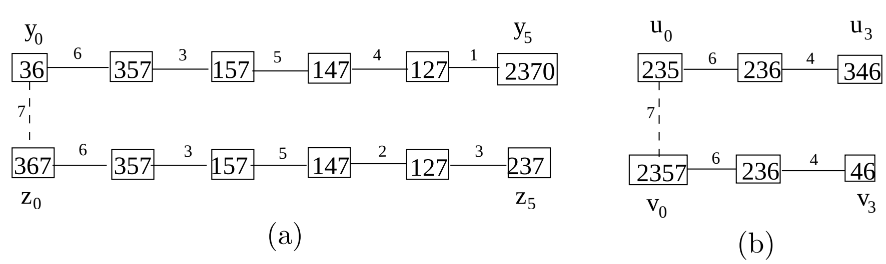


Figure 2

This implies $M(z_0) \subset \Omega(7,1)$.

Since there exists a vertex in $M(z_0)$ with label $\boxed{\mathbf{2357}}$, we have $t_2 := \mathbf{2357} \in \Omega(7,1)$. The set $M(t_1) \cup M(t_2)$ is distinguished and **A**-saturated, but is still not **B**-saturated. Take $u_0 = t_2\mathbf{56734275134} \in M(t_2)$ and $v_0 = u_0\mathbf{7}$. Then u_0, v_0 form a right primitive pair, as shown in Figure 2 (b).

We have $t_3 := \mathbf{1014} \in \Omega(7,1)$ since $M(v_0) \subset \Omega(7,1)$ and there exists a vertex in $\mathcal{M}(v_0)$ labelled by $\boxed{\mathbf{014}}$. The set

$$M_3 := M(t_1) \cup M(t_2) \cup M(t_3) \tag{5.4.1}$$

is distinguished and is also **ABC**-saturated. Hence it forms an l.c.r. set of $\Omega(7,1)$.

Since there is no element w in M_3 with $\mathcal{R}(w) = \{\mathbf{0},\mathbf{3},\mathbf{5},\mathbf{7}\}$, the element $t_4 := \mathbf{0357}$ is not in $\Omega(7,1)$. So $\Omega(7,2) := W_{(4)} \setminus \Omega(7,1)$ is the two-sided cell of $\widetilde{E}_7$ containing t_4. Take $P = \{t_4\}$. The set $M(t_4)$ is still not **B**-saturated. Consider the pair $\{t_5, t_6\}$ with $t_5 = t_4\mathbf{4625134} \in M(t_4)$ and $t_6 = t_5\mathbf{2}$. We have $\mathcal{R}(t_5) = \{\mathbf{4}\}$ and $\mathcal{R}(t_6) = \{\mathbf{2},\mathbf{4}\}$. Let $t_5' = t_5\mathbf{4}$ and $t_6' = t_6\mathbf{5}$. Then $t_5' \in M(t_4)$ with $\mathcal{R}(t_5') = \{\mathbf{2},\mathbf{3},\mathbf{5}\}$. Also, $t_6' \in M(t_6)$ with $\mathcal{R}(t_6') = \{\mathbf{2},\mathbf{5}\}$. This implies that $\{t_5, t_6\}$ forms a right primitive pair (see Figure 3).

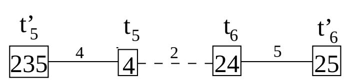


Figure 3

So $M(t_6) \subset \Omega(7,2)$. The set $M(t_4) \cup M(t_6)$ is still not **B**-saturated. Consider the pair $\{t_7, t_8\}$ with $t_7 = t_6\mathbf{3} \in M(t_6)$ and $t_8 = t_7\mathbf{5}$. We have $\mathcal{R}(t_7) = \{\mathbf{2},\mathbf{3}\}$ and $\mathcal{R}(t_8) = \{\mathbf{2},\mathbf{3},\mathbf{5}\}$. Let $t_7' = t_7\mathbf{2} \in M(t_6)$ and $t_8' = t_8\mathbf{4} \in M(t_8)$. Then $\mathcal{R}(t_7') = \{\mathbf{3},\mathbf{4}\}$ and $\mathcal{R}(t_8') = \{\mathbf{4}\}$. This implies that $\{t_7, t_8\}$ forms a right primitive pair. The set

$$M_4 := M(t_4) \cup M(t_6) \cup M(t_8) \tag{5.4.2}$$

is distinguished, and is also **ABC**-saturated. Hence it forms an l.c.r. set of $\Omega(7,2)$. Thus we get the following

Theorem 5.5. *In the affine Weyl group $\widetilde{E}_7$, there are two two-sided cells $\Omega(7,1)$ and $\Omega(7,2)$, where $\Omega(7,1)$ contains 232 left cells with M_3 in (5.4.1) as an l.c.r. set, and $\Omega(7,2)$ contains 126 left cells with M_4 in (5.4.2) as an l.c.r. set.*

Here and later, see website: http://www.math.ecnu.edu.cn/~jyshi for all the left cell graphs of the groups $\widetilde{E}_7$ and $\widetilde{E}_8$ in $W_{(4)}$.

5.6. For the affine Weyl group $\widetilde{E}_8$, there are also two two-sided cells in $W_{(4)}$ (see 1.8). All the elements of form w_J, $J \subseteq S$, in $W_{(4)}$ are as follows.

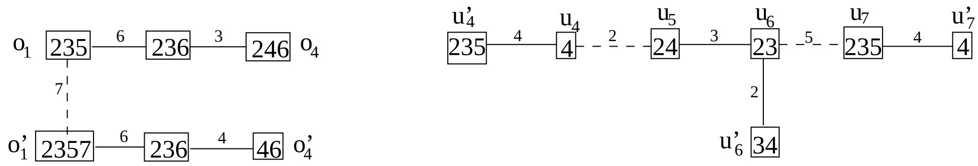
1312, **1315**, **1316**, **1317**, **1318**, **1310**, **3436**, **3437**, **3438**, **3430**, **4241**, **4246**, **4247**, **4248**, **4240**, **4541**, **4547**, **4548**, **4540**, **5652**, **5653**, **5651**, **5650**, **5658**, **6764**, **6762**, **6763**, **6761**, **6760**, **7871**, **7872**, **7873**, **7874**, **7875**, **8081**, **8083**, **8082**, **8084**, **8085**, **8086**, **1257**, **1258**, **1250**, **1468**, **1460**, **1268**, **1260**, **1270**, **1470**, **1570**, **2357**, **2358**, **2350**, **3570**, **2368**, **2360**, **3270**, **2570**.

Let $\Omega(8,1)$ be the two-sided cell of $\widetilde{E}_8$ containing $u_1 :=$ **1312**. Take $P = \{u_1\}$. Let $o_1 := u_1\mathbf{4534243124642514} \in M(u_1)$ and $o_1' = o_1\mathbf{7}$. Then $\{o_1, o_1'\}$ form a right primitive pair (see Figure 4 (a)). Let $u_2 =$ **1570**. Then $M(u_2) = M(o_1')$ since $M(o_1')$ has a vertex labelled by **1570**. Hence $M(u_2) \subset \Omega(8,1)$. The set

$$M_5 := M(u_1) \cup M(u_2) \qquad (5.6.1)$$

is distinguished, and also **ABC**-saturated. So it forms an l.c.r. set of $\Omega(8,1)$.

Since there is no element w in M_5 with $\mathcal{R}(w) = \{\mathbf{0},\mathbf{2},\mathbf{5},\mathbf{7}\}$, we see that $u_3 :=$ **0257** is not in $\Omega(8,1)$. So u_3 must be in the two-sided cell $\Omega(8,2) = W_{(4)} \setminus \Omega(8,1)$. Take $P = \{u_3\}$. The set $M(u_3)$ is not **B**-saturated. Take $u_4 := u_3\mathbf{6843514726534} \in M(u_3)$, $u_4' = u_4\mathbf{4}$, $u_5 = u_4\mathbf{2}$ and $u_6 = u_5\mathbf{3}$. Then $\mathcal{R}(u_4) = \{\mathbf{4}\}$, $\mathcal{R}(u_4') = \{\mathbf{2},\mathbf{3},\mathbf{5}\}$, $\mathcal{R}(u_5) = \{\mathbf{2},\mathbf{4}\}$ and $\mathcal{R}(u_6) = \{\mathbf{2},\mathbf{3}\}$. This implies that $\{u_4, u_5\}$ forms a right primitive pair (see Figure 4 (b)). So $u_5 \in \Omega(8,2)$. The set $M(u_3) \cup M(u_5)$ is still not **B**-saturated. Take $u_6' = u_6\mathbf{2}$, $u_7 = u_6\mathbf{5}$ and $u_7' = u_7\mathbf{4}$. Then $\mathcal{R}(u_6') = \{\mathbf{3},\mathbf{4}\}$, $\mathcal{R}(u_7) = \{\mathbf{2},\mathbf{3},\mathbf{5}\}$, $\mathcal{R}(u_7') = \{\mathbf{4}\}$. This implies that $\{u_6, u_7\}$ form a



(a) (b)

Figure 4

right primitive pair (see Figure 4 (b)). So $u_7 \in \Omega(8,2)$. The set

$$M_6 := M(u_3) \cup M(u_5) \cup M(u_7) \tag{5.6.2}$$

is distinguished, and is also **ABC**-saturated. Hence it forms an l.c.r. set of $\Omega(8,2)$. So we get

Theorem 5.7. *In the affine Weyl group $\widetilde{E}_8$, there are two two-sided cells $\Omega(8,1)$ and $\Omega(8,2)$, where $\Omega(8,1)$ contains 366 left cells with M_5 in (5.6.1) as an l.c.r. set, and $\Omega(8,2)$ contains 240 left cells with M_6 in (5.6.2) as an l.c.r. set.*

Remark 5.8. By an easy observation on all the left cell graphs of $\widetilde{E}_i$, $i=6,7,8$, in $W_{(4)}$, we see that each of the two-sided cells $\Omega(i,2)$, $i=6,7,8$, contains a unique element of the form w_J with $\ell(w_J)=4$. They are w_{0146} in $\Omega(6,2)$, w_{0357} in $\Omega(7,2)$ and w_{0257} in $\Omega(8,2)$.

§6. The distinguished involutions in $W_{(4)}$.

In this section, we describe the distinguished involutions of W in the set $W_{(4)}$ for $W=\widetilde{E}_i$, $i=6,7,8$ (see 1.5 and Theorem 6.6). As a consequence, we show that any left cell of W in $W_{(4)}$ is left-connected, verifying a conjecture of Lusztig in our case (see Corollary 6.11).

First we state two results. The first result is an easy consequence of [24, Proposition 5.12]:

Lemma 6.1. *Suppose that $s,t \in S$ satisfy $o(st)=3$ and that $x \in W$ is in $\mathcal{D}_0$ with $|\mathcal{L}(x) \cap \{s,t\}|=1$. If y is obtained from x by a right $\{s,t\}$-star operation followed by a left $\{s,t\}$-star operation, then y is also a distinguished involution of W.*

The second result is a special case of [31, Proposition 2.6]

Proposition 6.2. *Let W be an affine Weyl group of type $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Then any $w \in W_{(4)}$ has an expression of the form $w = x \cdot w_J \cdot y$ for some $x,y \in W$ and some $J \subseteq S$ with $\ell(w_J)=4$.*

We assume Proposition 6.2 in the present section and give its proof in Section 7.

6.3. Define $E(4)=\{w\in W_{(4)}\mid a(sw)<a(w),\ \forall\, s\in\mathcal{L}(w)\}$. For any left cell L of W, define two sets $E(L)=\{w\in L\mid a(sw)<a(w),\ \forall\, s\in\mathcal{L}(w)\}$ and $E_{\min}(L)=\{w\in L\mid \ell(x)\geqslant\ell(w),\ \forall\, x\in L\}$. By 1.5 (3)-(4), the relations $E_{\min}(L)\subseteq E(L)$ and $E(4)=\coprod_L E(L)$ always hold, where L in the last expression ranges over all the left cells of W in $W_{(4)}$.

By Proposition 6.2, we see that any $w\in E(4)$ has the form $w=w_J\cdot y$ for some $y\in W$ and some $J\subset S$ with $\ell(w_J)=4$. Consider the following process:

(i) Let X_0 be the set of all the elements of the form w_J for some $J\subset S$ with $\ell(w_J)=4$.

(ii) Let $k>0$. Suppose that the sets X_j have been defined for any $0\leqslant j<k$. Define $X_k=\{xs\mid x\in X_{k-1}, s\in S\setminus\mathcal{R}(x), xs\in E(4)\}$.

By practically applying the above process in each of the groups $\widetilde{E}_i$, $i=6,7,8$, we see that there exists some $m\in\mathbb{N}$ such that $X_j\neq\emptyset$ and $X_h=\emptyset$ for any $0\leqslant j\leqslant m<h$. This produces the set $E(4)=\cup_{i=0}^m X_i$ (as an example, we list the elements of $E(4)$ for $W=\widetilde{E}_6$ in Appendix). Note that checking the including relation $xs\in E(4)$ for $x\in X_{k-1}$ and $s\in S\setminus\mathcal{R}(x)$ needs some technical skills in determining the a-values. Once the set $E(4)$ is obtained, one can easily check the equality $E(4)=E_{\min}(4)$ by observing the fact that any $x\underset{L}{\sim}y$ in $E(4)$ satisfy $\ell(x)=\ell(y)$, the latter need make use of our left cell graphs and the results in 1.5 (1), (6), Theorem 1.6 and Proposition 2.6. So we have

Lemma 6.4. *Let W be one of the groups $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$.*

(1) There exists some $m\geqslant 0$ such that $|X_j|\neq\emptyset$ and $X_h=\emptyset$ for any $0\leqslant j\leqslant m<h$ and that $E(4)=\cup_{i=0}^m X_i$.

(2) $E(L)=E_{\min}(L)$ for any left cell L of W in $W_{(4)}$.

6.5. For any $x,y\in W$, write $C_xC_y=\sum_z h_{x,y,z}C_z$. Shi showed in [22, Proposition 2.3] that there exists a unique element (denoted by $\lambda(x,y)$) in W such that $h_{x,y,\lambda(x,y)}\neq 0$ and that any $z\in W$ with $h_{x,y,z}\neq 0$ satisfies the relation $z\leqslant\lambda(x,y)$. In [22, Proposition 2.3], Shi also described the element $\lambda(x,y)$ in either way below.

(i) Take any reduced expression $x=s_rs_{r-1}\cdots s_1$ of x with $s_i\in S$. Define a sequence of elements $x_0=y,x_1,...,x_r$ in W such that for $1\leqslant i\leqslant r$, $x_i=x_{i-1}$ if $s_ix_{i-1}<x_{i-1}$, and $x_i=s_ix_{i-1}$ if $s_ix_{i-1}>x_{i-1}$. Then $\lambda(x,y)=x_r$. In particular, $\lambda(x,y)$ is a left extension of y (see 1.5 (3)).

(ii) Take any reduced expression $y = t_1t_2\cdots t_u$ of y with $t_i \in S$. Define a sequence of elements $y_0 = x, y_1, ..., y_u$ in W such that for $1 \leqslant i \leqslant u$, $y_i = y_{i-1}$ if $y_{i-1}t_i < y_{i-1}$, and $y_i = y_{i-1}t_i$ if $y_{i-1}t_i > y_{i-1}$. Then $\lambda(x, y) = y_u$. In particular, $\lambda(x, y)$ is a right extension of x (see 1.5 (3)).

In particular, the element $\lambda(x^{-1}, x)$ is an involution for any $x \in W$.

By a result of Lusztig in [15, Theorem 1.10], we have

(6.5.1) $$d \leqslant \lambda(x^{-1}, x) \qquad \text{for any } x \in W \text{ and } d \in \mathcal{D}_0 \text{ with } x \underset{L}{\sim} d.$$

For an irreducible Weyl or affine Weyl group W, Shi conjectured that any distinguished involution of W has the form $d = \lambda(z^{-1}, z)$ for any $z \in E_{\min}(L_d)$, where L_d is the left cell of W containing d (see [22, Conjecture 8.10]). The following result supports the conjecture in our case.

Theorem 6.6. *Let W be as in Lemma 6.4. Then any distinguished involution of W in $W_{(4)}$ has the form $d = \lambda(z^{-1}, z) = {z'}^{-1} \cdot w_J \cdot z'$ for any $z = w_J \cdot z' \in E(L_d)$ with $J = \mathcal{L}(z)$ and some $z' \in W$.*

Proof. The proof follows the line of that in [31, Theorem A (3)].

Let d be a distinguished involution of W in $W_{(4)}$. By Lemma 6.4, we have an expression of the form $d = x \cdot w_J \cdot y$ for some $x, y \in W$ and $J \subseteq S$ with $\ell(w_J) = 4$. Choose such an expression with $\ell(y)$ smallest possible. Then $w_J \cdot y \underset{L}{\sim} d$ by 1.5 (2)–(4). We also have $\ell(x) \geqslant \ell(y)$ since both d and w_J are involutions. Hence

(6.6.1) $$\ell(d) \geqslant 2\ell(y) + 4.$$

We claim that the element $w_J \cdot y$ is in $E_{\min}(4)$. For, take any $z \in E_{\min}(4)$ with $z \underset{L}{\sim} d$. By Lemma 6.4, we have an expression $z = w_I \cdot z'$ for some $z' \in W$ and $I \subseteq S$ with $\ell(w_I) = 4$. By 6.5 (in particular, (6.5.1)), we have $\lambda(z^{-1}, z) = \lambda({z'}^{-1}, w_I \cdot z') \geqslant d$ and hence

(6.6.2) $$\ell(d) \leqslant \ell(\lambda(z^{-1}, z)) = \ell(\lambda({z'}^{-1}, w_I \cdot z')) \leqslant 2\ell(z') + \ell(w_I) = 2\ell(z') + 4$$

Since $w_J \cdot y \underset{L}{\sim} d \underset{L}{\sim} z$, this implies that

(6.6.3) $$2\ell(y) + 4 \geqslant 2\ell(z') + 4 \geqslant \ell(d) \geqslant 2\ell(y) + 4$$

by (6.6.1)–(6.6.2). So all the equalities in (6.6.3) should hold. Hence $w_J \cdot y$ is in the set $E_{\min}(4)$, as claimed. This further implies that $d = \lambda(z^{-1}, z) = {z'}^{-1} \cdot w_I \cdot z'$ for any $z = w_I \cdot z' \in E_{\min}(4)$, proving our assertion. $\square$

Remark 6.7. Owing to the limitation of the space, we only list the elements of $E_{\min}(4)$ for the group $W = \widetilde{E}_6$ (see Appendix). We are pleased to provide the similar data for the groups $W = \widetilde{E}_7, \widetilde{E}_8$ upon request. Here we would like to indicate the following fact: the set $E_{\min}(L)$ consists of

(i) a unique element if $L \subset \Omega(i, 2)$ for some $i = 6, 7, 8$;

(ii) at most 4 elements if $L \subset \Omega(i, 1)$ for some $i = 6, 7, 8$. e.g., for L being the vertex **126** at the northwest of Figure A, the set $E_{\min}(L)$ consists of four elements $z_1 =$**13104562**, $z_2 =$**34310562**, $z_3 =$**45431062** and $z_4 =$**56543102**. We have $d_L = \lambda(z_j^{-1}, z_j) =$ **2654·1310·4562** for any $1 \leqslant j \leqslant 4$.

6.8. By Lemma 6.1, we can find out all the distinguished involutions in the left cells occurring as vertices in any left cell graph $\mathcal{M}$ of W in $W_{(4)}$ via the following process.

(i) Find out the distinguished involution d_L in at least one of these left cells, say L.

(ii) For any other left cell L' occurring as a vertex in $\mathcal{M}$, let $L_0 = L, L_1, ..., L_r = L'$ be a sequence of vertices in $\mathcal{M}$ such that for $1 \leqslant i \leqslant r$, L_i can be obtained from L_{i-1} by a right $\{s_{h_i}, s_{k_i}\}$-star operation for some $s_{h_i}, s_{k_i} \in S$ with $o(s_{h_i} s_{k_i}) = 3$. Then by Lemma 6.1, the distinguished involution $d_{L'}$ in L' can be obtained from d_L by successively applying the left $\{s_{h_1}, s_{k_1}\}$-, $\{s_{h_2}, s_{k_2}\}$-, ..., $\{s_{h_r}, s_{k_r}\}$-star operations followed by the corresponding sequence of right star operations.

If $\mathcal{M}$ contains some vertices of labels $J \subset S$ with $\ell(w_J) = 4$ (e.g., when $\mathcal{M}$ is in one of Figures A, B), then we can take all such vertices as L in (i). We have $d_L = w_J$ with $J = \mathcal{R}(L)$. If we are not in the case, then by Theorem 6.6, the calculation in step (i) can be carried out as follows. Take some $x \in W$ with the left cell L_x occurring as a vertex in the graph $\mathcal{M}$ (e.g., x can be taken from the l.c.r. set of $\mathcal{M}$ in Section 5). Finding out a left retraction (say z) of x in $E(4)$, the distinguished involution d in L_x is equal to $\lambda(z^{-1}, z)$ by Theorem 6.6.

An alternative way for finding out all the distinguished involutions in $W_{(4)}$ is based on the computational results claimed in 6.3 and by applying Theorem 6.6.

The advantage for the first way to find out distinguished involutions is that the results can be displayed explicitly via graphs.

6.9. By Theorems 5.3, 5.5 and 5.7, there are 15 left cell graphs in total for the set $W_{(4)}$ of the groups $\widetilde{E}_i$, $i = 6, 7, 8$. We apply the method described in 6.8 to find out 15 distinguished involutions (one for each left cell graph) as follows.

$f_A = \mathbf{1312}$, $f_B = \mathbf{1460}$, $f_C = \mathbf{13425}f_B\mathbf{52431}$, $f_D = \mathbf{26154}f_C\mathbf{45162}$, $f_E = \mathbf{2350}$,

$f_F = \mathbf{2357}$, $f_G = \mathbf{1014}$, $f_H = \mathbf{3570}$, $f_I = \mathbf{24352146}f_H\mathbf{64125342}$, $f_J = \mathbf{53}f_I\mathbf{35}$,

$f_K = \mathbf{1570}$, $f_L = \mathbf{1312}$, $f_M = \mathbf{2570}$, $f_N = \mathbf{26154}f_M\mathbf{45162}$, $f_O = \mathbf{53}f_N\mathbf{35}$.

where $f_A, f_B, f_C, f_D \in \widetilde{E}_6$, $f_E, f_F, f_G, f_H, f_I, f_J \in \widetilde{E}_7$ and $f_K, f_L, f_M, f_N, f_O \in \widetilde{E}_8$.

6.10. Consider the graphs in Figures A–D of Appendix, which describe all the distinguished involutions in the left cells of $\widetilde{E}_6$ in $W_{(4)}$. For any of these graphs (say $\mathcal{M}$), we find out the distinguished involution d_L in at least one left cell L occurring as a vertex of $\mathcal{M}$ (e.g., the elements f_A, f_B, f_C, f_D in 6.9, which are in Figures A–D respectively). In particular, when $\mathcal{M}$ contains some vertices L with $\mathcal{R}(L) = J$ satisfying $\ell(w_J) = 4$, we have $d_L = w_J$. Then for any other vertex L' in $\mathcal{M}$, the distinguished involution $d_{L'}$ in L' can be found out in the following way. Let $L_0 = L, L_1, ..., L_r = L'$ be a sequence of vertices in $\mathcal{M}$ such that for $1 \leqslant i \leqslant r$, the vertices L_{i-1}, L_i are joined by an edge labelled with h_ik_i (note that when $h_i = k_i$ —— which happens in most of the cases —— we abbreviate h_ik_i to k_i in the graphs). Then we have $d_{L'} = \mathbf{h}_r\mathbf{h}_{r-1}\cdots\mathbf{h}_1 \cdot d_L \cdot \mathbf{k}_1\cdots\mathbf{k}_{r-1}\mathbf{k}_r$, where $\mathbf{h}_i, \mathbf{k}_i$ are the Coxeter generators of $\widetilde{E}_6$ corresponding to h_i, k_i. For example, the distinguished involution in the vertex $L = \boxed{\mathbf{045}}$ of Figure A is $d_L = \mathbf{0454}$, while the distinguished involution in the left-most vertex $L' = \boxed{\mathbf{13}}$ of Figure A is $d_{L'} = \mathbf{314342} \cdot d_L \cdot \mathbf{243413}$. It can be easily checked that $d_{L'}$ is independent of the choice of the vertex sequence from L to L'.

The corresponding graphs for the groups $\widetilde{E}_7$, $\widetilde{E}_8$ in $W_{(4)}$ can be found at website:

http://www.math.ecnu.edu.cn/~jyshi

6.11. A subset K of W is *left-connected* (resp., *right-connected*), if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, ..., x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ (resp., $x_{i-1}^{-1}x_i \in S$) for $1 \leqslant i \leqslant r$.

By Lemma 6.4 (1), we see that the subset $E(4)$ of $\widetilde{E}_i$ for any $i = 6, 7, 8$ is right-connected.

Lusztig conjectured in [1] that if W is an affine Weyl group then any left cell L of W is left-connected. The conjecture is supported by all the existing data. Now we can use Theorem 6.6 to prove the following

Theorem 6.12. *Let $W = \widetilde{E}_i$, $i = 6, 7, 8$. Then any left cell L of W in $W_{(4)}$ is left-connected.*

Proof. Following the line of the proof for [31, Theorem A (4)], we prove our result as follows. Let d_L be the distinguished involution of W in L. For any $w \in L$, there exist a sequence of elements $x_0 = w, x_1, ..., x_r = w'$ in L_d with some $w' \in E(L)$ such that $x_{i-1}x_i^{-1} \in S$ and $\ell(x_i) = \ell(x_{i-1}) - 1$ for $1 \leqslant i \leqslant r$. By Lemma 6.4, we have $w' \in E_{\min}(L_d)$. By Theorem 6.6, we have $d_L = \lambda({w'}^{-1}, w')$, which is a left extension of w' by 6.5 (i). So there exist a sequence of elements $y_0 = w', y_1, ..., y_t = d_L$ in L such that $y_{i-1}y_i^{-1} \in S$ and $\ell(y_i) = \ell(y_{i-1}) + 1$ for $1 \leqslant i \leqslant t$. This implies that L is left-connected. □

§7. Proof of Proposition 6.2.

In the present section, we assume that W is one of the affine Weyl groups $\widetilde{E}_i$, $i = 6, 7, 8$ and $w \in W_{(4)}$ with $J = \mathcal{L}(w)$. Let Γ be the Coxeter graph of W. We want to show Proposition 6.2. To do this, we need the following three lemmas.

Lemma 7.1. *If $J = \{s, r\}$ and $\ell(w_J) = 2$, then either $sw \underset{L}{\sim} w$ or $rw \underset{L}{\sim} w$ holds.*

Lemma 7.2. *If $J = \{s, r, u\}$ and $\ell(w_J) = 3$, then there exists some $v \in J$ with $vw \underset{L}{\sim} w$.*

Lemma 7.3. *If $J = \{s, t\}$ and $\ell(w_J) = 3$, then either $sw \underset{L}{\sim} w$ or $tw \underset{L}{\sim} w$ holds..*

Now we show Proposition 6.2 by assuming Lemmas 7.1–7.3.

7.4. *Proof of Proposition 6.2.* Write

$$w = w_J \cdot y \quad \text{ with } J = \mathcal{L}(w) \text{ and some } y \in W.$$

By the assumption of $w \in W_{(4)}$ and 1.5 (1)–(3), we have $1 \leqslant \ell(w_J) \leqslant 4$. If $\ell(w_J) = 4$, then we are done. If $\ell(w_J) = 1$, say $J = \{s\}$, then any $t \in \mathcal{L}(sw)$ must satisfy $st \neq ts$. Hence sw can be obtained from w by a left $\{s, t\}$-star operation and so $sw \underset{L}{\sim} w$ by Lemma 2.1. If $\ell(w_J) = 2$ (resp., $\ell(w_J) = 3$), then $vw \underset{L}{\sim} w$ for some $v \in J$ by Lemma 7.1 (resp., by Lemmas 7.2–7.3). Then Proposition 6.2 follows by applying induction on $\ell(w) \geqslant 4$. □

It remains to show Lemmas 7.1–7.3. We need the following result in proving Lemmas 7.1–7.3.

Lemma 7.5. *Let $w \in W_{(4)}$ be with $J = \mathcal{L}(w)$ and $\ell(w_J) < 4$. Set $I = \mathcal{L}(w_J w)$. Then $I \neq \emptyset$. Write $w = w_J \cdot w_I \cdot w_1$ for some $w_1 \in W$. Then $vw \underset{L}{\sim} w$ for some $v \in J$ if one of the following cases occurs:*

(1) There exist some element $t \in I$ such that s is the unique element in J satisfying $st \neq ts$ and that $sr = rs$ for any $r \in J$.

(2) $J = \{s, r\}$ and $t \neq u$ in I with $sr \neq rs$ and with r, t, u pairwise commutative (hence $st \neq ts$ and $su \neq us$).

(3) There exist some $t \in I$ and some $r \neq s$ in J with $sr = rs$, $st \neq ts$, $rt \neq tr$ and $r \in \mathcal{L}(w_1)$ such that t, r commutes with all the elements in $I \setminus \{t\}$.

(4) $J = \{s, r\}$ and $t \in I$ with $sr \neq rs$, $s \in \mathcal{L}(w_1)$, $st \neq ts$ and $rt = tr$ such that t, s commute with all the elements in $I \setminus \{t\}$.

(5) $J = \{s, r, u\}$, $I = \{t\}$ and $J \cap \mathcal{L}(w_1) \neq \emptyset$ with $|J| = 3$ and $\ell(w_J) = 3$.

Proof. This can be checked directly. □

7.6. *Proof of Lemma 7.1.* By the assumption of $a(w) > 2$, we can write $w = srt \cdot w_1$ for some $t \in \mathcal{L}(srw)$ and some $1 \neq w_1 \in W$ with Γ having a subgraph as in Figure 5 (a).

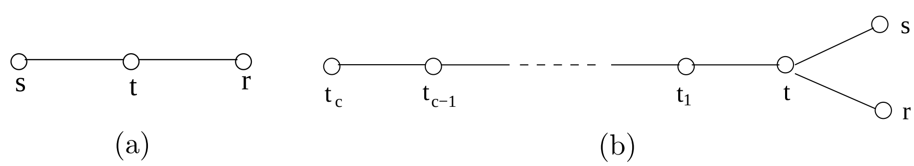


Figure 5

Clearly, $t \notin \mathcal{L}(w_1)$. We claim that $ut \neq tu$ for any $u \in \mathcal{L}(w_1)$. For otherwise, there would exist some $u \in \mathcal{L}(w_1)$ with $tu = ut$. Then $su \neq us$ and $ru \neq ur$ by the assumption of $\mathcal{L}(w) = \{s, r\}$. But then Γ has a circle with s, r, t, u its four vertices, contradicting the assumption on the type of W. Next we claim $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$. For otherwise, by the assumption on the type of W and by the first claim, t would be the branching node of Γ and $w_1 = t_1 t_2 \cdots t_c$ for some $c \in \mathbb{N}$ with Γ having a subgraph as in Figure 5 (b). But then we have $a(w) = 2$, contradicting the assumption of $a(w) = 4$. Now that $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$. We may assume $s \in \mathcal{L}(w_1)$ for the sake of definiteness. Then $rw \underset{L}{\sim} w$ by Lemma 7.5 (3). This proves our result. □

7.7. *Proof of Lemma 7.2.* Let $I = \mathcal{L}(w_J w)$. Then $I \cap J = \emptyset$ and $w = w_J \cdot w_I \cdot w_1$ for some

$w_1 \in W$. By Lemma 7.5 (1), we need only consider the case where there exist at least two $\alpha \neq \beta$ in J with $o(t\alpha) = o(t\beta) = 3$ for any $t \in I$. Thus by the assumption on the type of W, we have $|I| \leqslant 2$ and $tv = vt$ in the case of $I = \{t, v\}$.

(1) First assume $|I| = 1$. Say $I = \{t\}$. Then $w = srut \cdot w_1$ with $w_1 \neq 1$ by the assumption of $a(w) = 4$. By relabelling s, r, u if necessary, Γ has a subgraph displayed as in Figure 6 (a) or (b):

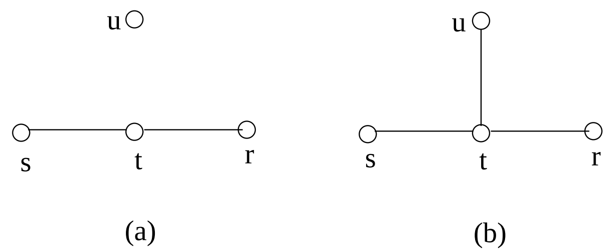


Figure 6.

We claim $J \cap \mathcal{L}(w_1) \neq \emptyset$. For otherwise, $J \cap \mathcal{L}(w_1) = \emptyset$. Then any $v \in \mathcal{L}(w_1)$ satisfies $tv \neq vt$ by the assumption $|I| = 1$. Hence by the assumption on the type of W, we have $w_1 = t_1 t_2 \cdots t_c$ and $u = t_k$ for some $1 < k \leqslant c$ with Γ having a subgraph as in Figure 5 (b) . But then we have $a(w) = 3$, contradicting our assumption of $a(w) = 4$. The claim is proved. So our result follows by Lemma 7.5 (5).

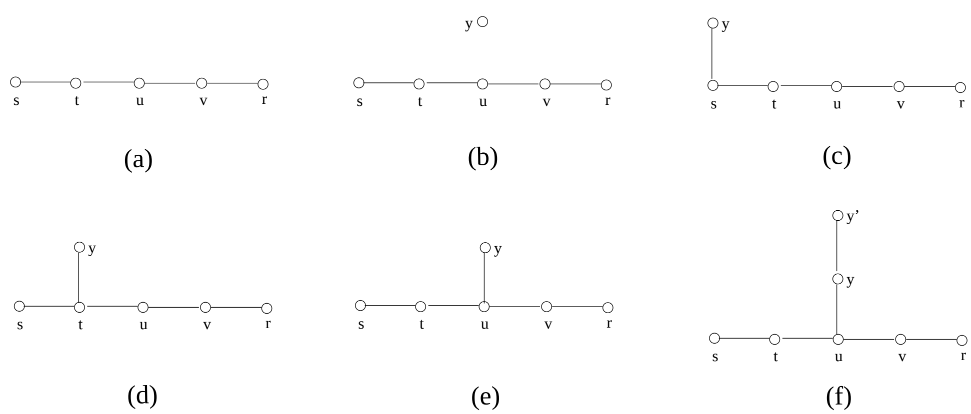


Figure 7.

(2) Next assume $|I| = 2$, say $I = \{t, v\}$. Then $tv = vt$. By Lemma 7.5 (1) and the assumption on the type of W, we need only consider the case where Γ has a subgraph as in Figure 7 (a).

Write $w = sru \cdot tv \cdot w_1$ for some $w_1 \in W$. Then $w_1 \neq 1$ by the assumption of $a(w) = 4$.

Any $x \in \mathcal{L}(w_1)$ satisfies either $tx \neq xt$ or $vx \neq xv$ by the assumption $|I| = 2$. By Lemma 7.5 (3), we need only consider the case where $\mathcal{L}(w_1) \cap \{s, r\} = \emptyset$. If $u \notin \mathcal{L}(w_1)$, then by relabelling s, t, u, v, r if necessary, we have $\mathcal{L}(w_1) = \{y\}$ with Γ having a subgraph as in Figure 7 (d) by the assumption on the type of W. Then $w_1 \in \{y_1, y_1y_2\}$ with Γ having a subgraph as one of those in Figure 8.

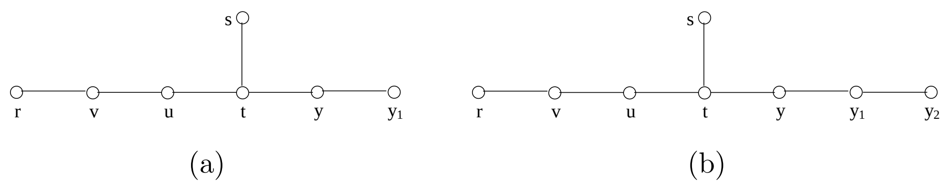


Figure 8.

which would imply $a(w) = 3$, contradicting the assumption of $a(w) = 4$.

It remains to consider the case of $u \in \mathcal{L}(w_1)$.

Then $w = sur \cdot tv \cdot u \cdot w_2$ for some $w_2 \in W$. Clearly, $w_2 \neq 1$ by the assumption of $a(w) = 4$. We also have $\mathcal{L}(w_2) \cap \{s, u, r\} = \emptyset$ by the assumption of $\mathcal{L}(w_1) \cap \{s, r\} = \emptyset$. If $t \in \mathcal{L}(w_2)$, then $w = sur \cdot tv \cdot ut \cdot w_3 = rv \cdot suvtu \cdot w_3$ for some $w_3 \in W$. Then rw can be obtained from w by a left $\{r, v\}$-star operation and hence $rw \underset{L}{\sim} w$. By symmetry, if $v \in \mathcal{L}(w_2)$ then we can show that $sw \underset{L}{\sim} w$.

Now assume $\mathcal{L}(w_2) \cap \{s, u, r, t, v\} = \emptyset$ and $y \in \mathcal{L}(w_2)$. By symmetry, we need only consider the case where Γ has one of the subgraphs in Figure 7 (b)–(e). In the case of Figure 7 (b), we have $w = y \cdot sur \cdot tv \cdot u \cdot w_3$ for some $w_3 \in W$. Hence $\{y, s, u, r\} \subseteq \mathcal{L}(w)$, contradicting the assumption $\mathcal{L}(w) = \{s, u, r\}$. In the case of Figure 7 (c), we have $w = sy \cdot rutvu \cdot w_3$ for some $w_3 \in W$. The element sw can be obtained from w by a left $\{s, y\}$-star operation. Hence $sw \underset{L}{\sim} w$.

By the assumption on the type of W, Γ can't simultaneously have a subgraph in Figure 7 (d) for some $y \in \mathcal{L}(w_2)$ and a subgraph in Figure 7 (e) for another $y \in \mathcal{L}(w_2)$.

Now assume that we are in the case of Figure 7 (d) for some $y \in \mathcal{L}(w_2)$ but not in the case of Figure 7 (b)–(c) for any $y \in \mathcal{L}(w_2)$. Then $w = sur \cdot tv \cdot u \cdot y \cdot w_3$ for some $1 \neq w_3 \in W$, where any $z \in \mathcal{L}(w_3)$ satisfies $zy \neq yz$. Then any $x \in \mathcal{L}(w_3)$ is either t or a node in the branch Γ_y of Γ with respect to t. In this case, if $t \not\leqslant w_3$, then by Lemma 7.5 and by the assumption on the type of W, we see that $w_3 \in \{y_1, y_1y_2\}$, with Γ having a subgraph as one of those in Figure 8. which would imply $a(w) = 3$, contradicting the assumption of $a(w) = 4$. If $t \leqslant w_3$, then we may write $w = sur \cdot tvuy \cdot z \cdot t \cdot w_4$ for some $z, w_4 \in W$ with $t \not\leqslant z$. We may take such an expression with $\ell(z)$ smallest possible. Then $\mathcal{R}(z) \subseteq \{s, u, y\}$. We also have $\mathcal{L}(yz) \subseteq \{y\}$ by the assumptions that all $x \in \mathcal{L}(z) \subseteq \mathcal{L}(w_3)$ satisfy $xy \neq yx$, that $t \not\leqslant z$ and that Γ has no subgraph in Figure 7 (b) for any $y \in \mathcal{L}(w_2)$. This implies that all $x \in S$ with $x \leqslant yz$ are the nodes in the branch Γ_y of Γ with respect to t, forcing $z = 1$ and hence $w = sur \cdot tvuyt \cdot w_4$ with $w_4 \neq 1$ by the assumption of $a(w) = 4$. By the assumption that all $x \in \mathcal{L}(w_3) = \mathcal{L}(tw_4)$ satisfy $xy \neq yx$, we have $\mathcal{L}(w_4) \subseteq \{u, s, y, y_1\}$, where $y_1 \in S$ satisfies $y_1 \neq t$ and $yy_1 \neq y_1 y$ whenever it exists. If $u \in \mathcal{L}(w_4)$, then $w = sur \cdot tv \cdot uytu \cdot w_5 = ysurvtyut \cdot w_5$ for some $w_5 \in W$, which implies $\mathcal{L}(w) \supseteq \{s, u, r, y\}$, contradicting the assumption of $\mathcal{L}(w) = \{s, u, r\}$. If $y \in \mathcal{L}(w_4)$, then $w = sur \cdot tv \cdot uyty \cdot w_5 = rvuvstuyt \cdot w_5$ for some $w_5 \in W$, hence rw can be obtained from w by a right $\{r, v\}$-star operation. If $\mathcal{L}(w_4) \subseteq \{s, y_1\}$, then one of the following cases must occur (note the assumption of $a(w) = 4$ and hence $w_4 \neq 1$ in particular):

(i) $w = sur \cdot tv \cdot uyt \cdot st \cdot w_5$ for some $w_5 \in W$;

(ii) $w = sur \cdot tv \cdot uyt \cdot y_1yt \cdot w_5$ for some $w_5 \in W$;

(iii) $w = sur \cdot tv \cdot uyt \cdot sy_1y_2y \cdot w_5$ for some $w_5 \in W$ (see Figure 8 (b) for y_2);

(iv) $w = sur \cdot tv \cdot uyt \cdot sy_1 \cdot w_5$ for some $w_5 \in W$ with $\mathcal{L}(w_5) \subseteq \{y\}$ (hence w is not in the case (iii)) and $\mathcal{L}(sy_1w_5) = \{s, y_1\}$ such that w is not in the case (ii) (hence $w \neq sur \cdot tv \cdot uyt \cdot sy_1yts \cdot w_5$ for any $w_5 \in W$).

Note that one may find one more possible case where $w = sur \cdot tv \cdot uyt \cdot y_1y_2yy_1 \cdot w_5$ for some $w_5 \in W$ with Γ having a subgraph as in Figure 8 (b). Since $a(w) = 4$, we must have $\mathcal{L}(w_5) \cap \{s, t\} \neq \emptyset$ by the assumption of $\mathcal{L}(w_4) \subseteq \{s, y_1\}$ and that on the type of W, hence it reduces to the case (ii) or (iii).

Case (i) implies that $w = u \cdot tst \cdot rvuyts \cdot w_5$, hence uw can be obtained from w by a right $\{t, u\}$-star operation. Case (ii) implies that $w = y_1sur \cdot tv \cdot uyy_1ty \cdot w_5$ and hence $\mathcal{L}(w) \supseteq \{s, u, r, y_1\}$, contradicting the assumption of $\mathcal{L}(w) = \{s, u, r\}$. In the case (iii), we have $W = \widetilde{E}_7$. It can be shown that w, uw form a primitive pair and hence $uw \underset{L}{\sim} w$ by Lemma 1.8. Finally, in the case (iv), it can be shown that $w = sur \cdot tv \cdot uyt \cdot sy_1ytuvr \cdot w_6$, where $w_6 \in \{r_1, r_1r_2\}$ with $r_1, r_2 \in S \setminus \{v\}$ satisfying $rr_1 \neq r_1r$ and $r_1r_2 \neq r_2r_1$ if $W = \widetilde{E}_8$, and $w_6 = 1$ if otherwise, hence $a(w) = 3$, contradicting the assumption of $a(w) = 4$.

Hence the result is proved in the case of Figure 7 (d).

Next assume that we are in the case of Figure 7 (e) for some $y \in \mathcal{L}(w_2)$ but not in the case of Figure 7 (b)–(d) for any $y \in \mathcal{L}(w_2)$. Then $w = srutvuy \cdot w_3$ for some $1 \neq w_3 \in W$ with any $z \in \mathcal{L}(w_3)$ satisfying $zy \neq yz$. We claim $u \leqslant w_3$. For otherwise, we would have $w_3 = y'$ with $W = \widetilde{E}_6$ as in Figure 7 (f). But this would imply $a(w) = 3$, a contradiction. Hence we must have $u \leqslant w_3$. Write $w = srutvuy \cdot z \cdot u \cdot z'$ for some $z, z' \in W$, where $u \not\leqslant z$, and $\ell(z)$ is smallest possible with this property. Then $\mathcal{R}(z) \subseteq \{t, v, y\}$, and $\mathcal{L}(yz) = \{y\}$, forcing $z = 1$. We have $w = srutvuyu \cdot z' = uy \cdot srutvuy \cdot z'$. Hence uw can be obtained from w by a left $\{u, y\}$-star operation. So $uw \underset{L}{\sim} w$. □

7.8. *Proof of Lemma 7.3.* Write $w = sts \cdot w_1$ for some $1 \neq w_1 \in W$. Then any $r \in I := \mathcal{L}(w_1)$ satisfies that either $tr \neq rt$, $sr = rs$, or $tr = rt$, $sr \neq rs$ and that $ru = ur$ for any $r, u \in I$ by the assumption on the type of W. If $\ell(w_I) \geqslant 3$ then one can check easily that either tw, w, or sw, w form a primitive pair. Now assume $\ell(w_I) \leqslant 2$. Then by re-labelling s, t if necessary, Γ has a subgraph as in Figure 9 (a) if $I = \{r\}$, and as in Figure 9 (b) or (c) if $I = \{r, v\}$ with $\ell(w_I) = 2$.

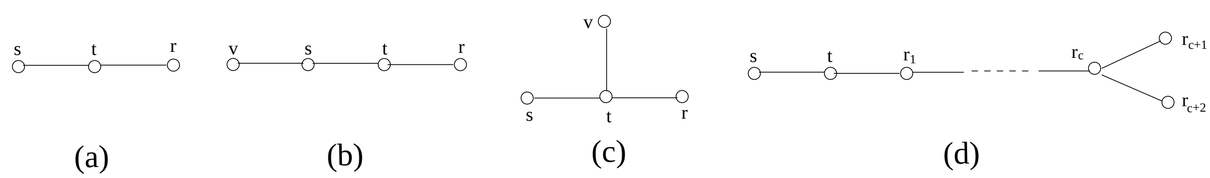


Figure 9.

First assume $s, t \not\leqslant w_1$. By the assumption of $a(w) = 4$ and Lemma 7.5, we see (by re-labelling s, t if necessary) that in the case of either (a) or (b) in Figure 9, w_1 has an expression $w_1 = r_1 r_2 \cdots r_c r_{c+1} r_{c+2} \cdot w_2$ for some $w_2 \in W$ and $r_i \in S$, $1 \leqslant i \leqslant c+2$, with Γ having a subgraph as in Figure 9 (d) (hence $r_1 = r$ in the case of Figure 9 (a), and $r_1 \in \{r, v\}$ in the case of Figure 9 (b)). Then either sw, w, or tw, w form a primitive pair. We have an expression $w_1 = sts \cdot rv \cdot w_2$ with some $w_2 \in W$ in the case of Figure 9 (c). Again, sw, w form a primitive pair.

Next assume either $s \leqslant w_1$ or $t \leqslant w_1$. By re-labelling s, t if necessary, we may write $w = sts \cdot x \cdot t \cdot y$ with some $x, y \in W$, where $s, t \not\leqslant x$, and $\ell(x)$ is smallest possible with this property. Then $x \neq 1$. Any $z \in \mathcal{R}(x)$ satisfies $tz \neq zt$. We have $\mathcal{L}(x) = \{r\}$ if t is not a branching node of

Γ (hence Γ has a subgraph as in Figure 9 (a) or (b)), and $\mathcal{L}(x) \subseteq \{r, v\}$ if t is a branching node of Γ as in Figure 9 (c). When $|\mathcal{L}(x)| = 1$, we have either $x = r$, or $x = r_1 r_2 \cdots r_c r_{c+1} r_{c+2} r_c \cdots r_2 r_1$ with Γ having a subgraph as in Figure 9 (d), where $r_1 = r$. When $\mathcal{L}(x) = \{r, v\}$, we have $w = sts \cdot rv \cdot w_2$ with some $w_2 \in W$. If $x = r$ then tw can be obtained from w by a right $\{t, r\}$-star operation, In any of the other cases, the elements sw, w form a primitive pair. $\square$

Remark 7.9. The arguments given in this section actually show that the conclusion of Proposition 6.2 is valid for W being any Coxeter group of type E_i or $\widetilde{E}_i$, $i = 6, 7, 8$. The conclusion of Proposition 6.2 can be further extended to the case where the group W is any Weyl or affine Weyl group of simply-laced type (i.e., types A, D, E, $\widetilde{A}$, $\widetilde{D}$ and $\widetilde{E}$), and the element $w \in W$ satisfies $a(w) \leqslant 6$ (see [31]). Of course, the proof in this more general case need be refined.

Appendix.

We display certain graphs for the group $\widetilde{E}_6$ in Figures A–D. These graphs are mainly for the description of all the distinguished involutions in the left cells of $\widetilde{E}_6$ in $W_{(4)}$ (see 6.10 for the explanation of the graphs).

By forgetting the labels of all the edges, these graphs become the left cell graphs of $\widetilde{E}_6$ in the set $W_{(4)}$ (i.e., the ones of the form $\mathcal{M}_L(x)$ for some $x \in W_{(4)}$). In this case, each vertex (or rather, each box) in the graphs represents a left cell, say L, inside the box, we record the set $\mathcal{R}(L)$. For example, we have $\mathcal{R}(L) = \{\mathbf{1{,}3}\}$ if L is the left-most vertex $\boxed{\mathbf{13}}$ in Figure A. Two vertices, say L, L', are joined by an edge, if the left cell L' can be obtained from L (and vice versa) by a right star operation. For example, in Figure A, the left-most vertex $\boxed{\mathbf{13}}$ is joined with a vertex $\boxed{\mathbf{14}}$ by an edge since the corresponding left cells can be obtained from each other by a right $\{\mathbf{3{,}4}\}$-star operation. The sets M_k, $k = 1, 2$, in (5.2.1)–(5.2.2) can be obtained from those left cell graphs easily. For example, we have the element **1312** in the set M_1 which corresponds to the vertex $\boxed{\mathbf{123}}$ in Figure A. Now the element in M_1 corresponding to the right-most vertex $\boxed{\mathbf{02}}$ in Figure A should be **131243542042**, which is obtained from **1312** by using the path:

$$\boxed{\mathbf{123}} - \boxed{\mathbf{14}} - \boxed{\mathbf{34}} - \boxed{\mathbf{35}} - \boxed{\mathbf{45}} - \boxed{\mathbf{25}} - \boxed{\mathbf{05}} - \boxed{\mathbf{04}} - \boxed{\mathbf{02}}$$

and hence by applying the following right star operations in turn: $\{\mathbf{3{,}4}\}$ (or $\{\mathbf{2{,}4}\}$), $\{\mathbf{1{,}3}\}$, $\{\mathbf{4{,}5}\}$, $\{\mathbf{3{,}4}\}$, $\{\mathbf{2{,}4}\}$, $\{\mathbf{0{,}2}\}$, $\{\mathbf{4{,}5}\}$, $\{\mathbf{2{,}4}\}$.

The corresponding graphs for the groups $\widetilde{E}_7$, $\widetilde{E}_8$ can be found at website:

http://www.math.ecnu.edu.cn/~jyshi

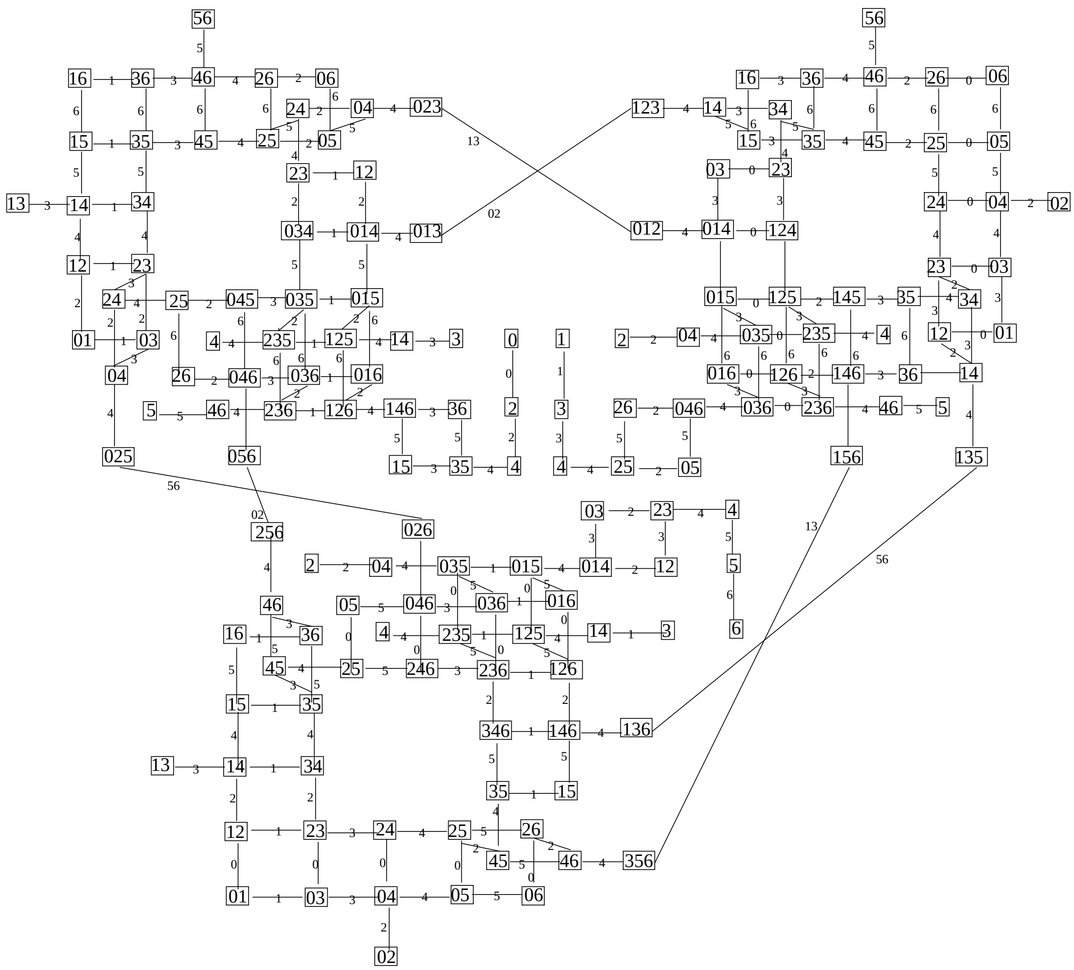


Figure A

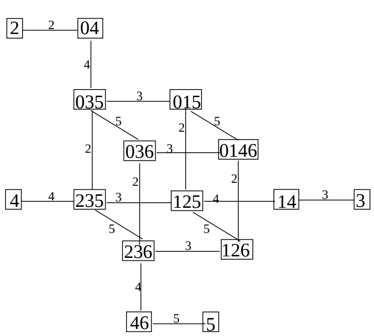


Figure B

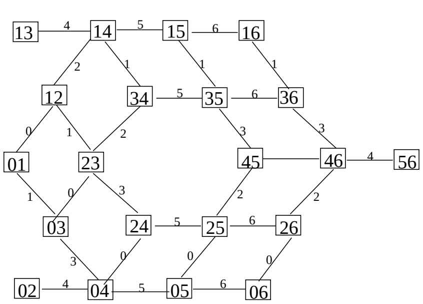


Figure C

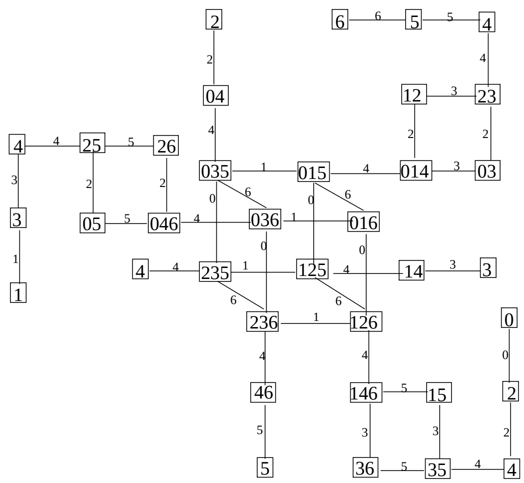


Figure D

The following is the list of certain choosing elements in $E(4)$ $(= E_{\min}(4))$ for $W = \widetilde{E}_6$, the elements are grouped according to the left cell L they lie in, while the left cells L are represented by $\mathcal{R}(L)$ and are arranged in the row order from top to bottom and also from left to right in each row according to their positions in the corresponding left cell graph $\mathcal{M}_L$. Any element of $E(4)$ not occurring in the list can be obtained from some element in the list by some graph automorphism of $\widetilde{E}_6$.

$\mathcal{M}_L$ in Figure A (The choosing elements belonging to certain left cells at northwest):

56: **2023456245, 4032452645, 4540324565, 5654032456**;

16: **20234562431, 40324526431, 45403245631, 02504234561**;

36: **2023456243, 4032452643, 4540324563, 0250423456**;

46: **202345624, 403245264, 454032456**; **26**: **20234562, 40324526**;

06: **2023456**; **24**: **202342, 043424**; **04**: **20234**; **023**: **2023**;

35: **202345243, 043424543, 045432453, 020542345**; **45**: **20234524, 04342454, 04543245**;

25: **2023452, 0434245**; **05**: **202345**; **23**: **34302**; **12**: **131042, 343102**;

034: **3430**; **014**: **13104, 34310**; **013**: **1310**; **035**: **34305, 45430**;

015: **131045, 343105, 454310**; **4**: **3430524, 4543024**; **235**: **343052, 454302**;

125: **3430521, 4543021, 1310452**; **14**: **34305214, 45430214, 13104524**;

3: **343052143, 454302143, 131045243**; **016**: **1310456, 3431056, 4543106, 5654310**;

126: **13104562, 34310562, 45431062, 56543102**;

$\boxed{146}$: **131045624, 343105624, 454310624, 565431024;**

$\boxed{36}$: **1310456243, 3431056243, 4543106243, 5654310243;**

$\boxed{35}$: **13104562435, 34310562435, 45431062435, 56543102435;**

$\boxed{4}$: **131045624354, 243105624354, 454310624354, 565431024354;**

$\boxed{2}$: **1310456243542, 2431056243542, 4543106243542, 5654310243542;**

$\boxed{0}$: **13104562435420, 24310562435420, 45431062435420, 56543102435420.**

From the above list, we see that, for example, the left cell labelled by $\boxed{56}$ contains four elements of $E(4)$ (listed immediately after the notation $\boxed{56}$:); there are two left cells in the list, both labelled by $\boxed{4}$, the first (resp., second) one lies in the same column as $\boxed{56}$ (resp., $\boxed{0}$) in Figure A, which contains two (resp., four) elements of $E(4)$. For the elements of $E(4)$ belonging to the left cells in Figure A but not in the above list, we can describe them by applying certain graph automorphisms of $\widetilde{E}_6$. For example, the left cell labelled by $\boxed{56}$ at northeast of Figure A contains four elements of $E(4)$, which can be obtained from those in the line of $\boxed{56}$ in the list by applying the graph automorphism ψ_{01} of $\widetilde{E}_6$, where ψ_{01} interchanges **0** and **1**.

$\mathcal{M}_L$ in Figure B (The choosing elements belonging to certain left cells at north and northwest):

$\boxed{2}$: **01463542;** $\boxed{04}$: **0146354;** $\boxed{035}$: **014635;** $\boxed{036}$: **01463;**

$\boxed{0146}$: **0146;** $\boxed{4}$: **01463524;** $\boxed{235}$: **0146352.**

$\mathcal{M}_L$ in Figure C (The choosing elements belonging to certain left cells at north):

$\boxed{13}$: **014625431;** $\boxed{14}$: **0146254314;** $\boxed{15}$: **01462543145;** $\boxed{16}$: **014625431456;**

$\boxed{34}$: **014625434;** $\boxed{35}$: **0146254345;** $\boxed{36}$: **01462543456.**

$\mathcal{M}_L$ in Figure D (The choosing elements belonging to certain left cells at southeast):

$\boxed{016}$: **01462543456210;** $\boxed{4}$: **014625434524;** $\boxed{235}$: **01462543452;**

$\boxed{236}$: **014625434562;** $\boxed{126}$: **0146254345621;** $\boxed{0}$: **0146254345621435420;**

$\boxed{46}$: **0146254345624;** $\boxed{146}$: **01462543456214;** $\boxed{15}$: **014625434562145;**

$\boxed{2}$: **014625434562143542;** $\boxed{5}$: **01462543456245;** $\boxed{35}$: **0146254345621435;**

$\boxed{4}$: **01462543456214354.**

We see that any left cell occurring as a vertex of the left cell graph $\mathcal{M}_L$ in Figure B, C or D contains a unique element of $E(4)$.

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