SOME LEFT CELLS IN THE AFFINE WEYL GROUPS $\widetilde{E}_6$

Jian-yi Shi *, Xi-gou Zhang †

* Department of Mathematics, East China Normal University, Shanghai, 200062, China
† College of Mathematics and Information Science, JXNU, Nanchang, 330022, China

Abstract. The aim of the present paper is to describe all the left cells L of the affine Weyl groups $\widetilde{E}_6$ with $\mathbf{a}(L) \leqslant 11$. We find a representative set of those left cells which occur as the vertices of the corresponding left cell graphs. The main technical tools are the algorithm designed by the first-named author and the various right primitive pairs. Some interesting empirical phenomenon is observed concerning isomorphic left cell graphs in different two-sided cells of $\widetilde{E}_6$.

Let W be a Coxeter group with S its Coxeter generator set. In [10], Kazhdan and Lusztig introduced the concept of left, right and two-sided cells in W in order to construct representations of W and the associated Hecke algebra $\mathcal{H}$. Later Lusztig raised a theme in [14] for the description of all the cells in an affine Weyl group W_a. Lusztig defined a function $\mathbf{a} : W \to \mathbb{N} \cup \{\infty\}$ in [16], which is upper-bounded and is constant on any two-sided cell of W_a. Lusztig also introduced distinguished involutions of W in [17] which play an important role in the representations of W and $\mathcal{H}$. A remarkable fact is that any left cell of W_a contains a unique distinguished involution.

The cells (in particular, the left cells) L of W_a have been studied extensively by many people. They were described explicitly in the following cases:

(i) $W_a \in \{\widetilde{A}_n, \widetilde{B}_m, \widetilde{C}_l, \widetilde{D}_k, \widetilde{F}_4, \widetilde{G}_2 \mid n \geqslant 1, m = 3, 4, l = 2, 3, 4, k = 4, 5\}$ (see [21], [15], [16], [1], [8], [3], [26], [27], [28], [36], [7]);

(ii) $\mathbf{a}(L)$ is either $\frac{1}{2}|\Phi|$ or $\leqslant 4$, where Φ is the root system of the Weyl group associated to W_a (see [23], [14], [11], [20], [4], [5], [6], [31]);

(iii) $\mathbf{a}(L) = 5, 6, 7, 8$ in $\widetilde{E}_7$ (see [12], [34], [35], [37]) and $\mathbf{a}(L) = 5, 6$ in $\widetilde{E}_8$ (see [9])

(iv) L containing a fully-commutative element of W_a (see [29]).

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When either rank$(W_a) \leqslant 4$ or in any of the cases (ii)-(iv), all the distinguished involutions contained in those left cells L were described.

For any $k \in \mathbb{N}$, let $W_{(k)} = \{w \in W_a \mid \mathbf{a}(w) = k\}$. Then $W_{(k)}$ is a union of some two-sided cells of W_a. In the present paper, we shall describe all the left cells in the set $W_{(i)}$ with $i \leqslant 11$ for the affine Weyl group $\widetilde{E}_6$.

The main tool in describing the left cells is Algorithm 3.6, which was designed in [25] and improved in [28] by the first-named author. We apply it to find a representative set $E(\Omega)$ for all the left cells (or an *l.c.r. set* for brevity) of W_a in a two-sided cell Ω. $E(\Omega)$ is given in terms of left cell graphs $\mathcal{M}_L(x)$, $x \in P(\Omega)$, for a certain subset $P(\Omega)$ of Ω as follows.

(i) $P(\Omega) \subseteq E(\Omega)$;

(ii) There exists a bijective map $\psi : E(\Omega) \to \cup_{x \in P(\Omega)} M_L(x)$ ($M_L(x)$ is the vertex set of $\mathcal{M}_L(x)$) such that for any $y \in E(\Omega)$, $\psi(y)$ is the left cell of W_a containing y, and that there exist a unique $x \in P(\Omega)$ and a path $L_0 = \psi(x), L_1, ..., L_r = \psi(y)$ in $\mathcal{M}_L(x)$, where $\{L_{i-1}, L_i\}$ is a string for any $1 \leqslant i \leqslant r$ (see 2.1).

The main technical difficulty for doing this is in applying Processes **B** and **C** since the jointed relation x—y and the value $\mathbf{a}(z)$ for $x, y, z \in W_a$ are hard to be determined in general. To avoid this obstruction, we manage to find a number of (right) primitive pairs.

By expressing the elements of the group $\widetilde{E}_6$ in their alcove forms, we use the computer programme GAP to execute Algorithm 3.6. However, finding various primitive pairs is a flexible and technical task, which has to be done by hands. It becomes extremely difficult to draw a left cell graph when the number of its vertices is beyond one thousand. Hence we can work out all the left cells L of $\widetilde{E}_6$ only with $\mathbf{a}(L) \leqslant 11$ by the techniques presented here. All the remaining left cells L of $\widetilde{E}_6$ satisfy $\mathbf{a}(L) \in \{12, 13, 15, 16, 20, 25, 36\}$ by Theorem 1.6. The left cells of $\widetilde{E}_6$ with $\mathbf{a}(L) = 36$ (resp., $\mathbf{a}(L) = 25$) have been completely (resp., partially) described by the first-named author in [23] (resp., [30]) in an entirely different way. We are seeking some more techniques for describing the remaining left cells of $\widetilde{E}_6$.

Unlike what we did in the rank $\leqslant 5$ cases, $\widetilde{E}_6$ is the lowest rank group we have dealt with so far for which we have to apply process **C** to enlarge the set P in order to get an l.c.r. set of $\widetilde{E}_6$ in some two-sided cell Ω (e.g., when $\Omega = W_{(9)}$, see 4.11). In other words, in the rank $\leqslant 5$ cases, an l.c.r. set in any two-sided cell of the concerned groups can be obtained by only applying processes **A** and **B**, while in some two-sided cell (say $W_{(9)}$ for example) of $\widetilde{E}_6$, an l.c.r. set can't be obtained without applying process **C**.

An empirical phenomenon is interesting: if $\Omega, \Omega', \Omega''$ are three two-sided cells of $\widetilde{E}_6$ with $\Omega \underset{LR}{\leqslant} \Omega' \underset{LR}{\leqslant} \Omega''$ and $\mathbf{a}(\Omega) \leqslant 11$, and if Ω, Ω'' have isomorphic left cell graphs $\mathcal{M}_L, \mathcal{M}''_L$, respectively, then Ω' has a left cell graph $\mathcal{M}'_L$ satisfying $\mathcal{M}'_L \cong \mathcal{M}_L$. We wonder if such a phenomenon occurs in general. A direct check shows that this is the case when $W_a \in \{\widetilde{C}_4, \widetilde{F}_4\}$ (see [27], [28]).

The contents of the paper are organized as follows. Sections 1–3 are served as preliminaries, we collect some concept, terms and known results there. We introduce Kazhdan–Lusztig cells and affine Weyl groups in Section 1, star operations, primitive pairs and generalized τ-invariants in Section 2, and an algorithm for finding an l.c.r. set in a two-sided cell in Section 3. Then in Sections 4-5, we concentrate our attention on the affine Weyl groups $\widetilde{E}_6$, where we find out an l.c.r. set for any two-sided cell Ω of $\widetilde{E}_6$ with $\mathbf{a}(\Omega) \leqslant 11$ and deduce some related results. All the left cell graphs are displayed in Appendix.

§1. Kazhdan-Lusztig cells.

1.1. Let $\mathbb{N}$ (resp., $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$) be the set of all the non-negative integers (resp., integers, real numbers, complex numbers). Let W be a Coxeter group with S its distinguished generator set. Denote by $\leqslant$ (resp., l) the Bruhat-Chevalley order (resp., the length function) on W. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of all the Laurent polynomials in an indeterminate u with integer coefficients. The Hecke algebra $\mathcal{H}$ of W over $\mathcal{A}$ has two $\mathcal{A}$-bases $\{T_x\}_{x \in W}$ and $\{C_w\}_{w \in W}$ satisfying the following relations

$$\begin{aligned} &T_w T_{w'} = T_{ww'} && \text{if } l(ww') = l(w) + l(w'), \\ &T_s^2 = (u^{-1} - u)T_s + T_1 && \text{for } s \in S, \end{aligned}$$

and

$$C_w = \sum_{y \leqslant w} u^{l(w)-l(y)} P_{y,w}(u^{-2}) T_y,$$

where $P_{y,w} \in \mathbb{Z}[u]$ satisfies that $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \not\leqslant w$ and $\deg P_{y,w} \leqslant (1/2)(l(w) - l(y) - 1)$ if $y < w$. The $P_{y,w}$'s are known as *Kazhdan-Lusztig polynomials* (see [10]).

1.2. For $y, w \in W$ with $y < w$, denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(l(w)-l(y)-1)}$ in $P_{y,w}$. The elements y and w are called *jointed*, written y—w, if $\mu(y, w) \neq 0$. To any $x \in W$, we associate two subsets of S:

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

1.3. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on W defined as in [10], and let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation on W determined by $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$). The corresponding equivalence classes of W are called *left* (resp., *right*, *two-sided*) *cells* of W. $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) induces a partial order on the set of left (resp., right, two-sided) cells of W.

The following results come from [10, Proposition 2.4]:

(1) If $x \underset{L}{\leqslant} y$ in W then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$. In particular, if $x \underset{L}{\sim} y$ in W then $\mathcal{R}(x) = \mathcal{R}(y)$.

(2) If $x \underset{R}{\leqslant} y$ in W then $\mathcal{L}(x) \supseteq \mathcal{L}(y)$. In particular, if $x \underset{R}{\sim} y$ in W then $\mathcal{L}(x) = \mathcal{L}(y)$.

1.4. Define $h_{x,y,z} \in \mathcal{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z$$

for any $x, y, z \in W$. In [16], Lusztig defined a function $a : W \to \mathbb{N} \cup \{\infty\}$ by setting

$$\mathbf{a}(z) = \min\{k \in \mathbb{N} \mid u^k h_{x,y,z} \in \mathbb{Z}[u],\ \forall x, y \in W\} \quad \text{for any } z \in W$$

with the convention that $\mathbf{a}(z) = \infty$ if the minimum on the RHS of the above equation does not exist.

1.5. An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, reductive algebraic group over $\mathbb{C}$. Fix a maximal torus T of G, let X be the character group of T and let $\Phi \subset X$ be the root system of G with $\Delta = \{\alpha_1, ..., \alpha_l\}$ a choice of simple root system. Then $E = X \otimes_Z \mathbb{R}$ is a euclidean space with an inner product$\langle\ ,\ \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i, $1 \leqslant i \leqslant l$. Denote by N the group of all the translations $T_\lambda : x \mapsto x + \lambda$ on E with λ ranging over X. Then the semidirect product $W_a = N \rtimes W_0$ of W_0 with N is an *affine Weyl group.* Let K be the type dual to the type of G. Then the type of W_a is $\widetilde{K}$. In the case where no danger of confusion causes, W_a is denoted simply by its type $\widetilde{K}$. Let $w \mapsto \bar{w}$ be the canonical homomorphism from W_a to $W_0 \cong W_a/N$.

The following properties of the function a on (W_a, S) were proved by Lusztig:

(1) $x \underset{LR}{\leqslant} y \Longrightarrow \mathbf{a}(x) \geqslant \mathbf{a}(y)$. In particular, $x \underset{LR}{\sim} y \Longrightarrow \mathbf{a}(x) = \mathbf{a}(y)$. So we may define the value $\mathbf{a}(\Gamma)$ for a left (resp., right, two-sided) cell Γ of W_a to be the common value $\mathbf{a}(x)$ of all the $x \in \Gamma$ (see [16]).

(2) $\mathbf{a}(x) = \mathbf{a}(y)$ and $x \underset{L}{\leqslant} y$ (resp., $x \underset{R}{\leqslant} y$) $\Longrightarrow x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) (see [17]).

(3) Let $\delta(z) = \deg P_{e,z}$ for $z \in W_a$, where e is the identity of the group W_a. Define

$$\mathcal{D} = \{w \in W_a \mid l(w) = 2\delta(w) + \mathbf{a}(w)\}.$$

Then $\mathcal{D}$ is a finite set of involutions. Each left (resp., right) cell of W_a contains a unique element of $\mathcal{D}$ (called a *distinguished involution* of W_a by Lusztig, see [17]).

(4) For any $I \subsetneq S$, let w_I be the longest element in the subgroup W_I of W_a generated by I (note that W_I is always finite). Then $w_I \in \mathcal{D}$ and $\mathbf{a}(w_I) = l(w_I)$ (see [16]).

Let $W_{(i)} = \{w \in W_a \mid \mathbf{a}(w) = i\}$ for any $i \in \mathbb{N}$. Then the set $W_{(i)}$ is a union of some two-sided cells of W_a by (1).

(5) If $W_{(i)}$ contains an element of the form w_I for some $I \subset S$, then the set $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W_a (by (1)–(2)).

Call $s \in S$ *special* if the group $W_{S-\{s\}}$ has the maximum possible order among all the standard parabolic subgroups of W_a of the form W_I, $I \subsetneq S$. For $s \in S$, let

$$Y_s = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s\}\}.$$

Then Lusztig and Xi proved in [19] that

(6) Let $s \in S$ be special. Then $\Omega \cap Y_s$ is non-empty and forms a single left cell of W_a for any two-sided cell Ω of W_a.

Lusztig also proved that

Theorem 1.6. (see [18, Theorem 4.8]) *Let an algebraic group G and an affine Weyl group W_a be as in 1.5. Then there exists a bijective map $\mathbf{u} \mapsto c(\mathbf{u})$ from the set $\mathfrak{U}(G)$ of unipotent conjugacy classes in G to the set* Cell(W) *of two-sided cells in W which satisfies the equation $\mathbf{a}(c(\mathbf{u})) = \dim \mathcal{B}_u$, where u is any element in $\mathbf{u}$, and $\dim \mathcal{B}_u$ is the dimension of the variety of all the Borel subgroups of G containing u.*

1.7. Keep the notation in 1.5. Let $-\alpha_0$ be the highest short root in Φ. Denote $s_0 = s_{\alpha_0} T_{-\alpha_0}$, where s_{α_0} is the reflection in E with respect to α_0. Then $S = S_0 \cup \{s_0\}$ forms a Coxeter generator set of W_a.

The alcove form of an element $w \in W_a$ is, by definition, a Φ-tuple $(k(w,\alpha))_{\alpha\in\Phi}$ over $\mathbb{Z}$ determined by the following conditions.

(a) $k(w,-\alpha) = -k(w,\alpha)$ for any $\alpha \in \Phi$;

(b) $k(e,\alpha) = 0$ for any $\alpha \in \Phi$;

(c) If $w' = ws_i$ $(0 \leqslant i \leqslant l)$, then

$$k(w',\alpha) = k(w,(\alpha)\overline{s}_i) + \varepsilon(\alpha,i) \quad \text{with} \quad \varepsilon(\alpha,i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

where $\overline{s}_i = s_i$ if $1 \leqslant i \leqslant l$, and $\overline{s}_0 = s_{\alpha_0}$.

By condition (a), we can also denote the alcove form of $w \in W_a$ by a Φ^+-tuple $(k(w,\alpha))_{\alpha\in\Phi^+}$, where Φ^+ is the positive root system of Φ containing Δ.

Condition (c) defines a set of operators $\{s_i \mid 0 \leqslant i \leqslant l\}$ on the alcove form of $w \in W_a$:

$$s_i: \quad (k(w;\alpha))_{\alpha\in\Phi} \longmapsto (k(w;(\alpha)\overline{s}_i) + \varepsilon(\alpha,i))_{\alpha\in\Phi}.$$

These operators could be described graphically (see [31] for the type $\widetilde{E}_6$).

For $w, w' \in W_a$, w' is called a *left extension* of w if $l(w') = l(w) + l(w'w^{-1})$.

Then the following results were shown by the first-named author:

Proposition 1.8. (see [28]) *Let $w \in W_a$.*

(1) $l(w) = \Sigma_{\alpha\in\Phi^+}|k(w,\alpha)|$, where the notation $|x|$ stands for the absolute value of $x \in \mathbb{Z}$;

(2) $\mathcal{R}(w) = \{s_i \mid k(w,\alpha_i) < 0\}$;

(3) w' is a left extension of w if and only if the inequalities $k(w',\alpha)k(w,\alpha) \geqslant 0$ and $|k(w',\alpha)| \geqslant |k(w,\alpha)|$ hold for any $\alpha \in \Phi^+$.

§2. Graphs, strings and generalized τ-invariants.

In the present section, we assume that (W_a, S) is an irreducible affine Weyl group of simply-laced type, that is, the order $o(st)$ of the product st is not greater than 3 for any $s, t \in S$, or equivalently, W_a is of type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$.

2.1. Given $s \neq t$ in S with $o(st) = 3$, a set of the form $\{ys, yst\}$ is called a (right) $\{s,t\}$-*string* (or a *string* in short), if $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

We have the following result.

Proposition 2.2. (see [25]) *For* $s, t \in S$ *with* $o(st) = 3$*, let* $\{x_1, x_2\}$ *and* $\{y_1, y_2\}$ *be two* $\{s, t\}$*-strings. Then*

(1) $x_1 — y_1 \iff x_2 — y_2$*;*

(2) $x_1 \underset{L}{\sim} y_1 \iff x_2 \underset{L}{\sim} y_2$.

2.3. x is obtained from w by a (right) $\{s, t\}$*-star operation* (or a *star operation* in short), if $\{x, w\}$ is an $\{s, t\}$-string. Note that the resulting element x for an $\{s, t\}$-star operation on w is always unique whenever it exists.

Two elements $x, y \in W_a$ form a (right) *primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, \ldots, x_r$ and $y_0 = y, y_1, \ldots, y_r$ in W_a such that the following conditions are satisfied:

(a) For every $1 \leqslant i \leqslant r$, there exist some $s_i, t_i \in S$ with $o(s_i t_i) = 3$ such that both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are $\{s_i, t_i\}$-strings.

(b) $x_i — y_i$ for some (and then for all, under the condition (a)) $0 \leqslant i \leqslant r$ (see [10]).

(c) Either $\mathcal{R}(x_0) \nsubseteq \mathcal{R}(y_0)$ and $\mathcal{R}(y_r) \nsubseteq \mathcal{R}(x_r)$, or $\mathcal{R}(y_0) \nsubseteq \mathcal{R}(x_0)$ and $\mathcal{R}(x_r) \nsubseteq \mathcal{R}(y_r)$ hold.

Proposition 2.4. (see [25)] $x \underset{R}{\sim} y$ if $\{x, y\}$ is a primitive pair.

In order to describe the left cells of W_a, we need introduce the concept of a left cell graph.

2.5. By a graph $\mathcal{M}$, we mean a set M of vertices together with a set of edges, where each edge is a two-element subset of M, and each vertex is labeled by some subset of S. A graph is *finite* if it contains a finite number of vertices, and is *infinite* otherwise.

By a path $\mathcal{P}$ in a graph $\mathcal{M}$, we mean a sequence of vertices $z_0, z_1, \ldots, z_r$ in M with some $r > 0$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathcal{M}$ for any $1 \leqslant i \leqslant r$. In this case, we say that the length of $\mathcal{P}$ is r.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two graphs with the vertex sets M and M' respectively. They are called *isomorphic*, written $\mathcal{M} \cong \mathcal{M}'$, if there exists a bijection $\eta : M \longrightarrow M'$ satisfying that

(a) The labels of $\eta(x)$ and x are the same for any $x \in M$.

(b) For $x, y \in M$, $\{x, y\}$ is an edge of $\mathcal{M}$ if and only if $\{\eta(x), \eta(y)\}$ is an edge of $\mathcal{M}'$.

This is an equivalence relation on graphs.

2.6. For any $x \in W_a$, denote by $M(x)$ the set of all such elements $y \in W_a$ that there are $x = x_0, x_1, \ldots, x_r = y$ in W_a with some $r \geqslant 0$, where $\{x_{i-1}, x_i\}$ is a string for every $1 \leqslant i \leqslant r$.

Define a *graph* $\mathcal{M}(x)$ associated to an element $x \in W_a$ as follows. Its vertex set is $M(x)$;

its edge set consists of all the two-element subsets in $M(x)$ each of which forms a string; each $y \in M(x)$ is labeled by the set $\mathcal{R}(y)$.

A *left cell graph* $\mathcal{M}_L(x)$ associated to $x \in W_a$ is by definition a graph, whose vertex set $M_L(x)$ consists of all the left cells Γ of W_a with $\Gamma \cap M(x) \neq \emptyset$; $\Gamma \neq \Gamma'$ in $M_L(x)$ are jointed by an edge, if there is an edge $\{y, y'\}$ of $\mathcal{M}(x)$ with $y \in M(x) \cap \Gamma$ and $y' \in M(x) \cap \Gamma'$; each $\Gamma \in M_L(x)$ is labeled by the set $\mathcal{R}(\Gamma)$ (see 1.5 (1)).

Clearly, both $\mathcal{M}(x)$ and $\mathcal{M}_L(x)$ are connected graphs for any $x \in W$.

2.7. We say that $x, x' \in W_a$ have the same (right) *generalized τ-invariants*, if for any path $z_0 = x, z_1, \ldots, z_r$ in $\mathcal{M}(x)$, there is a path $z'_0 = x', z'_1, \ldots, z'_r$ in $\mathcal{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every $0 \leqslant i \leqslant r$, and if the same condition holds when the roles of x and x' are interchanged.

Then the following result is known.

Proposition 2.8. (see [25] [33]) *Suppose $x \underset{L}{\sim} y$ in W_a. Then*

(a) x, y have the same generalized τ-invariants.

(b) $\mathcal{M}_L(x) \cong \mathcal{M}_L(y)$.

§3. An algorithm for finding an l.c.r. set of W_a in a two-sided cell.

3.1. A subset $K \subset W_a$ is called a *representative set for the left cells* (or an *l.c.r set* for brevity) of W_a (resp., of W_a in a two-sided cell Ω), if $|K \cap \Gamma| = 1$ for any left cell Γ of W_a (resp., of W_a in Ω), where the notation $|X|$ stands for the cardinality of a set X.

Obviously, the set $\mathcal{D}$ (see. 1.5 (3)) is an l.c.r. set of W_a. But it is not easy to find the whole set $\mathcal{D}$ of W_a directly in general since it may involve the complicated computation of Kazhdan-Lusztig polynomials. The set $\mathcal{D} \cap \Omega$ could be found in a relative easier way once an l.c.r. set of W_a in a two-sided cell Ω has been given (see Remark 4.15 (2), [31, Section 6] and [13]).

We shall obtain an l.c.r. set $E(\Omega)$ of W_a in a two-sided cell Ω from a certain subset $P(\Omega)$ of Ω such that

(i) $P(\Omega) \subseteq E(\Omega)$;

(ii) There exists a bijective map $\psi : E(\Omega) \to \cup_{x \in P(\Omega)} M_L(x)$ ($M_L(x)$ is the vertex set of $\mathcal{M}_L(x)$) such that for any $y \in E(\Omega)$, $\psi(y)$ is the left cell of W_a containing y, and that there exist some element $x \in P(\Omega)$ and some path $L_0 = \psi(x), L_1, ..., L_r = \psi(y)$ in $\mathcal{M}_L(x)$, where $\{\psi^{-1}(L_{i-1}), \psi^{-1}(L_i)\}$ is a string for any $1 \leqslant i \leqslant r$.

Note that such an l.c.r. set $E(\Omega)$ of W_a in Ω can be easily obtained from the set $P(\Omega)$ and the corresponding left cell graphs $\mathcal{M}_L(x)$, $x \in P(\Omega)$. However, $E(\Omega)$ is not uniquely determined by these data in general. It is so if and only if the set $\cup_{x\in P(\Omega)} M(x)$ is distinguished (see 3.4).

The first-named author designed an algorithm for finding an l.c.r. set of W_a in a two-sided cell, which is based on the following

Theorem 3.2. (see [25, Theorem 3.1]) *Let Ω be a two-sided cell of W_a. Then a non-empty subset $E \subset \Omega$ is an l.c.r. set of W_a in Ω, if E satisfies the following conditions:*

(1) $x \underset{L}{\not\sim} y$ for any $x \neq y$ in E;

(2) For any $y \in W_a$, if there exists some $x \in E$ satisfying that y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $\mathbf{a}(y) = \mathbf{a}(x)$, then there exists some $z \in E$ with $y \underset{L}{\sim} z$.

3.3. We know that the relations y—x and $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ hold if and only if one of the following cases occurs:

(1) $\{x, y\}$ is a string;

(2) $y = x \cdot s$ for some $s \in S$ with $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$, where by the notation $a = b \cdot c$ $(a, b, c \in W_a)$, we mean $a = bc$ and $l(a) = l(b) + l(c)$;

(3) $y < x$ and y—x and $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$.

3.4. A subset $P \subset W_a$ is called *distinguished* if $P \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in P. For a two-sided cell Ω of W_a and $\emptyset \neq P \subseteq \Omega$, consider the following processes on P (see [25]).

(**A**) Find a distinguished subset Q of the largest possible cardinality from the set $\bigcup_{x\in P} M(x)$.

(**B**) Let $B_x = \{y \in W_a \mid y = x \cdot s \notin M(x)$ for some $s \in S$ with $\mathbf{a}(y) = \mathbf{a}(x)\}$ for any $x \in P$. Find a distinguished subset Q of the largest possible cardinality from the set $P \cup (\cup_{x\in P} B_x)$.

(**C**) Let $C_x = \{y \in W_a \mid y < x; y$—$x; \mathcal{R}(y) \supsetneqq \mathcal{R}(x); \mathbf{a}(y) = \mathbf{a}(x)\}$ for any $x \in P$. Find a distinguished subset Q of the largest possible cardinality from the set $P \cup (\cup_{x\in P} C_x)$.

3.5. A subset P of W_a is **A**-*saturated* (resp., **B**-*saturated*, **C**-*saturated*), if the Process **A** (resp., **B**, **C**) on P cannot produce any element z with $z \underset{L}{\not\sim} x$ for any $x \in P$.

Clearly, a set of the form $\bigcup_{x\in K} M(x)$ for any $K \subseteq W_a$ is always **A**-saturated.

It follows from Theorem 3.2 that an l.c.r. set of W_a in a two-sided cell Ω is exactly a distinguished subset of Ω which is **ABC**-saturated simultaneously. In order to get such a subset, we apply the following algorithm.

3.6. Algorithm (see [28, Algorithm 2.7]).

(1) Find a non-empty subset P of Ω (It is usual to take P distinguished and consisting of elements of the form w_I, $I \subset S$, whenever it is possible);

(2) Perform Processes **A**, **B** and **C** alternately on P until the resulting distinguished set cannot be further enlarged by any of these processes.

§4. The left cells of the affine Weyl group $\widetilde{E}_6$.

In this section, we shall explicitly describe all the left cells of the affine Weyl group $W_a = \widetilde{E}_6$ in all the two-sided cells Ω with $\mathbf{a}(\Omega) \leqslant 11$. We shall find an l.c.r. set in virtue of left cell graphs for each of such two-sided cells by applying Algorithm 3.6 (in the way explained in 3.1). This will be achieved by expressing the elements of W_a in their alcove forms and then in virtue of the computer programme GAP. The work is hard in applying Process **B** and is even harder in applying Process **C** since it is not easy to determine the joint relations and the a-values for the related elements in general. To avoid such difficult points, we try to find various primitive pairs.

4.1. The Coxeter graph of the group $\widetilde{E}_6$ is as follows.

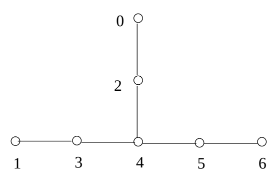


Fig. 1. The Coxeter graph of $\widetilde{E}_6$

Recall the notation $W_{(i)}$ for $i \geqslant 0$ in 1.5. By Theorem 1.6, $W_{(i)}$ is a single two-sided cell of $\widetilde{E}_6$ if $i \in \{0, 1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20, 25, 36\}$, and is a union of two two-sided cells of $\widetilde{E}_6$ if $i \in \{4, 6\}$ (see [2, Chapter 13]).

For $i \in \mathbb{N}$, denote by $P(i)$ the set of all the elements of the form w_I in $W_{(i)}$ for some $I \subset S$.

For the sake of simplifying the notation, denote by $\mathbf{i}$ (bold-faced) the reflection s_i corresponding to the vertex in Fig. 1.

4.2. For any $i \neq j$ in $\{1, 0, 6\}$, let ψ_{ij} be the unique automorphism of $\widetilde{E}_6$ which stabilizes the set S and transposes $\mathbf{i}$ and $\mathbf{j}$. For example, we have

$$(\psi_{10}(\mathbf{0}), \psi_{10}(\mathbf{1}), \psi_{10}(\mathbf{2}), \psi_{10}(\mathbf{3}), \psi_{10}(\mathbf{4}), \psi_{10}(\mathbf{5}), \psi_{10}(\mathbf{6})) = (\mathbf{1}, \mathbf{0}, \mathbf{3}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}).$$

Then ψ_{ij} preserves the value of the function a and the joint relation on elements, i.e., for $x, y, z \in \widetilde{E}_6$, we have $\mathbf{a}(\psi(z)) = \mathbf{a}(z)$, and x—y if and only if $\psi_{ij}(x)$—$\psi_{ij}(y)$. So ψ_{ij} stabilizes the set $W_{(k)}$ for any $k \geqslant 0$ and permutes the left (respectively, right, two-sided) cells of $\widetilde{E}_6$.

4.3. The two-sided cell $W_{(0)}$ consists of a single element: the identity element of the group $\widetilde{E}_6$. The two-sided cell $W_{(1)}$ consists of all the non-identity elements of $\widetilde{E}_6$ each of which has a unique reduced expression. The set $E(W_{(1)}) = S$ forms an l.c.r. set of $W_{(1)}$ (see [14]). The left cell graph of $W_{(1)}$ is isomorphic to Graph A (see Appendix).

4.4. Concerning the two-sided cell $W_{(2)}$, we take

$$P(2) = \{\mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{12}, \mathbf{10}, \mathbf{32}, \mathbf{30}, \mathbf{35}, \mathbf{36}, \mathbf{40}, \mathbf{46}, \mathbf{25}, \mathbf{26}, \mathbf{05}, \mathbf{06}\}.$$

The graph $\mathcal{M}(\mathbf{12})$ is infinite. Take a connected subgraph $\mathcal{M}'(\mathbf{12})$ from $\mathcal{M}(\mathbf{12})$ as in Graoh B with the vertex labeled by $\boxed{\mathbf{12}}$ being the element $\mathbf{12}$ (see Appendix). Then its vertex set $M'(\mathbf{12})$ is distinguished by Proposition 2.8, and is also **ABC**-saturated. So

$$E(W_{(2)}) = M'(\mathbf{12})$$

forms an l.c.r. set of $W_{(2)}$ by Theorem 3.2 (in the subsequent discussion, we shall frequently apply Proposition 2.8 and Theorem 3.2 but without mentioning them explicitly).

4.5. Concerning the two-sided cell $W_{(3)}$, we have

$$P(3) = \{\mathbf{131}, \mathbf{343}, \mathbf{424}, \mathbf{454}, \mathbf{202}, \mathbf{565}, \mathbf{146}, \mathbf{140}, \mathbf{150}, \mathbf{152}, \mathbf{162}, \mathbf{160}, \mathbf{352}, \mathbf{350}, \mathbf{362}, \mathbf{360}, \mathbf{460}\}.$$

Consider the graph $\mathcal{M}(\mathbf{131})$ (see Graph C in Appendix). Its vertex set $M(\mathbf{131})$ is distinguished, and is also **A**-saturated, but not **B**-saturated. Take $x = \mathbf{131420} \in M(\mathbf{131})$ and $x' = x \cdot \mathbf{5}$. We see from Fig. 2 (a) that $\{x, x'\}$ forms a primitive pair. Hence $x' \in W_{(3)}$ by Proposition 2.2 and 1.5 (1) (in the subsequent discussion, we shall frequently apply Proposition 2.2 and 1.5 (1) to primitive pairs but without mentioning them explicitly). The graph $\mathcal{M}(x')$ is isomorphic to the graph $\mathcal{M}(\mathbf{140})$ (see Graph D in Appendix). By 1.5 (5) and Proposition 2.3, the sets $M(x')$ and $M(\mathbf{140})$ represent the same set of left cells in $W_{(3)}$ since both contain a vertex labeled by $\boxed{\mathbf{140}}$. The set

$$E(W_{(3)}) = M(\mathbf{131}) \cup M(\mathbf{140})$$

is distinguished and also **ABC**-saturated. Thus it forms an l.c.r. set of $W_{(3)}$.

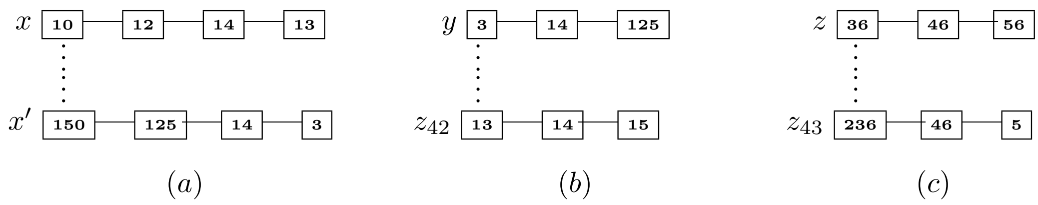


Fig. 2. The primitive pairs $\{x, x'\}$, $\{y, z_{42}\}$, $\{z, z_{43}\}$

4.6. There are two two-sided cells in $W_{(4)}$ (see 4.1). Take

$$P(4) = \{\,\mathbf{1312}, \mathbf{1310}, \mathbf{1315}, \mathbf{1316}, \mathbf{3430}, \mathbf{3436}, \mathbf{4540}, \mathbf{4541}, \mathbf{2421}, \mathbf{2426}, \mathbf{0201}, \mathbf{0203}, \mathbf{0205}, \mathbf{0206}, \mathbf{5652}, \mathbf{5650}, \mathbf{5651}, \mathbf{5653}, \mathbf{1046}\,\}.$$

The graph $\mathcal{M}(\mathbf{1312})$ is infinite. Take a subgraph $\mathcal{M}'(\mathbf{1312})$ in $\mathcal{M}(\mathbf{1312})$ with the vertex labeled by $\boxed{\mathbf{1312}}$ being the element $\mathbf{1312}$ (see Graph E in Appendix). Then its vertex set $\mathcal{M}'(\mathbf{1312})$ is distinguished, and also **ABC**-saturated. Let $W^1_{(4)}$ be the two-sided cell of $\widetilde{E}_6$ containing the element $\mathbf{1312}$. Then the set

$$E(W^1_{(4)}) = M'(\mathbf{1312})$$

forms an l.c.r. set of $W^1_{(4)}$.

Since there is no vertex in $\mathcal{M}'(\mathbf{1312})$ with the label $\boxed{\mathbf{1046}}$, we have $z_{41} = \mathbf{1046} \notin W^1_{(4)}$. Let $W^2_{(4)}$ be the two-sided cell of $\widetilde{E}_6$ containing the element z_{41}. The graph $\mathcal{M}(z_{41})$ is displayed as Graph F (see Appendix), whose vertex set $M(z_{41})$ is distinguished and also **A**-saturated, but not **B**-saturated. Let $y = z_{41} \cdot \mathbf{5243} \in M(z_{41})$ and $z_{42} = y \cdot \mathbf{1}$. Thus we see from Fig. 2 (b) that $\{y, z_{42}\}$ forms a primitive pair. Hence $z_{42} \in W^2_{(4)}$. The graph $\mathcal{M}(z_{42})$ is displayed as Graph C (see Appendix). But the set $M(z_{41}) \cup M(z_{42})$ is still not **B**-saturated. Let $z = z_{42} \cdot \mathbf{4156} \in M(z_{42})$ and $z_{43} = z \cdot \mathbf{2}$. Then we see from Fig. 2 (c) that $\{z, z_{43}\}$ forms a primitive pair. So $z_{43} \in W^2_{(4)}$. The graph $\mathcal{M}(z_{43})$ is displayed as Graph D (see Appendix). The set

$$E(W^2_{(4)}) = \cup_{i=1}^{3} M(z_{4i})$$

is distinguished and also **ABC**-saturated. Thus it forms an l.c.r. set of $W^2_{(4)}$.

The results in 4.6 were obtained in our previous paper [31].

4.7. The set $W_{(5)}$ forms a single two-sided cell of W by 4.1. Take

$$P(5) = \{\,\mathbf{13125}, \mathbf{13126}, \mathbf{13105}, \mathbf{13106}, \mathbf{56512}, \mathbf{56510}, \mathbf{56523}, \mathbf{56530}, \mathbf{02016}, \mathbf{02015}, \mathbf{02036}, \mathbf{02035}, \mathbf{34306}, \mathbf{24216}, \mathbf{45410}\,\}.$$

The graph $\mathcal{M}(\mathbf{13125})$ is infinite. Take a subgraph $\mathcal{M}'(\mathbf{13125})$ of $\mathcal{M}(\mathbf{13125})$ (see Graph G in Appendix) whose vertex set

$$E(W_{(5)}) = M'(\mathbf{13125})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(5)}$.

4.8. There are two two-sided cells in $W_{(6)}$ by 4.1. Take

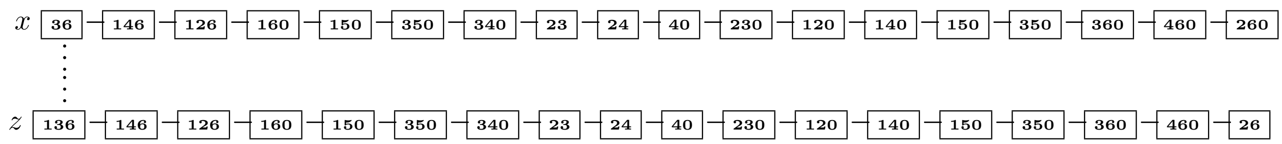
$$P(6) = \{\, w_{134}, w_{456}, w_{024}, w_{234}, w_{245}, w_{345}, w_{1356}, w_{1302}, w_{0256} \,\},$$

where the notation $w_{ij\cdots k}$ stands for the longest element in the subgroup of $\widetilde{E}_6$ generated by $\mathbf{i}, \mathbf{j}, ..., \mathbf{k}$. The graph $\mathcal{M}(z_{61})$ with $z_{61} = w_{134}$ is infinite, Take a subgraph $\mathcal{M}'(z_{61})$ of $\mathcal{M}(z_{61})$ (see Graph H in Appendix) whose vertex set

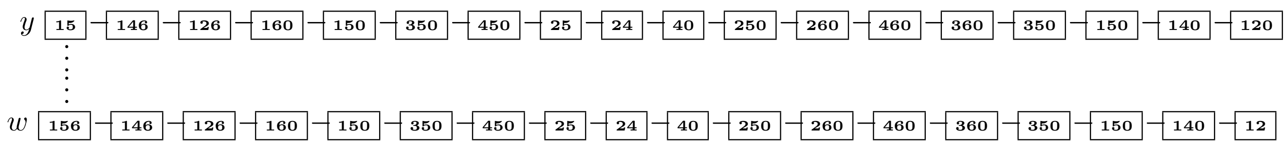
$$E(W^1_{(6)}) = M'(z_{61})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of the two-sided cell $W^1_{(6)}$ containing z_{61}.

Since no vertex in $\mathcal{M}'(z_{61})$ is labeled by $\boxed{\mathbf{1356}}$, $\boxed{\mathbf{1320}}$ or $\boxed{\mathbf{0256}}$, the elements $z_{62} = w_{1356}$, $z_{63} = w_{1302}$, $z_{64} = w_{0256}$ in $P(6)$ are in the two-sided cell $W^2_{(6)} = W_{(6)} - W^1_{(6)}$. The graphs $\mathcal{M}(z_{62})$, $\mathcal{M}(z_{63})$, $\mathcal{M}(z_{64})$ are displayed as Graphs I, $\psi_{06}(I)$, $\psi_{01}(I)$, respectively (see Appendix), where the graph $\psi_{ij}(I)$ is obtained from I by applying the automorphism ψ_{ij} of $\widetilde{E}_6$ (see 4.2). Take $x = z_{62} \cdot \mathbf{425434120456243}$ and $y = z_{62} \cdot \mathbf{425434120456245}$ in $M(z_{62})$ and let $z = x \cdot \mathbf{1}$, $w = y \cdot \mathbf{6}$.



(a)



(b)

Fig. 3. The primitive pairs $\{x, z\}$ and $\{y, w\}$

We see from Fig. 3 (a)–(b) that both $\{x,z\}$ and $\{y,w\}$ are primitive pairs. We have $\mathcal{M}(z)\cong\mathcal{M}(z_{63})$ (resp., $\mathcal{M}(w)\cong\mathcal{M}(z_{64})$) since $\mathbf{a}(z)=\mathbf{a}(z_{63})=6$ (resp., $\mathbf{a}(w)=\mathbf{a}(z_{64})=6$) and both graphs contain a vertex of the label $\boxed{\mathbf{0256}}$ (resp., $\boxed{\mathbf{1302}}$). The set

$$E(W^2_{(6)})=\cup_{i=2}^4 M(z_{6i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W^2_{(6)}$.

By 4.1, we see that $W_{(i)}$ is a single two-sided cell of $\widetilde{E}_6$ for any $i>6$ with $W_{(i)}\neq\emptyset$.

4.9. Consider the two-sided cell $W_{(7)}$ of W. Take

$$P(7)=\{\, w_{1346}, w_{1340}, w_{0246}, w_{0241}, w_{4561}, w_{4560}, w_{2346}, w_{2451}, w_{3450}, w_{13562}, w_{13560}, w_{13025}, w_{13026}, w_{02561}, w_{02563}\,\}.$$

The graph $\mathcal{M}(z_{71})$ with $z_{71}=w_{1346}$ is infinite. Take a subgraph $\mathcal{M}'(z_{71})$ of $\mathcal{M}(z_{71})$ (see Graph J in Appendix) whose vertex set $M'(z_{71})$ is distinguished. Let $x=z_{71}\cdot\mathbf{5464}\in M'(z_{71})$ and $x'=x\cdot\mathbf{2}$. We see from Fig. 4 (a) that $\{x,x'\}$ forms a primitive pair. Take $y=w_{1340}\mathbf{20}$ in $M(w_{1340})$ and $y'=y\cdot\mathbf{3}$. Also, take $z=w_{4560}\mathbf{20}$ in $M(w_{4560})$ and

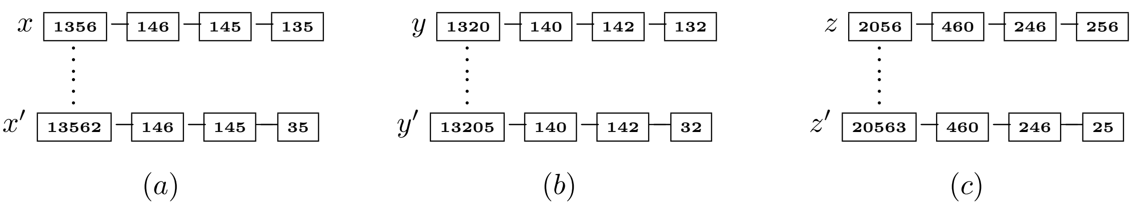


Fig. 4. The primitive pairs $\{x,x'\}$, $\{y,y'\}$, $\{z,z'\}$

$z'=z\cdot\mathbf{3}$. Then by Fig. 4 (b)–(c), both $\{y,y'\}$ and $\{z,z'\}$ are primitive pairs. Since they all contain an element with the label $\boxed{\mathbf{1340}}$, the sets $M(z_{71})$, $M(w_{1340})$, $M(w_{4560})$ represent the same set of left cells of $\widetilde{E}_6$ by 1.5 (5), i.e., for any left cell L of $\widetilde{E}_6$, the intersections $M(z_{71})\cap L$, $M(w_{1340})\cap L$, $M(w_{4560})\cap L$ are either all empty or all non-empty. Let $z_{72}=w_{02563}$, $z_{73}=w_{13025}$ and $z_{74}=w_{13562}$. The graph $\mathcal{M}(z_{74})$ is displayed as Graph K (see Appendix), while $\mathcal{M}(z_{72})$, $\mathcal{M}(z_{73})$ can be obtained from $\mathcal{M}(z_{74})$ by applying ψ_{01}, ψ_{06}, respectively (We denote them by $\psi_{01}(K)$, $\psi_{06}(K)$, respectively). Then the graph $\mathcal{M}(z')$ (resp., $\mathcal{M}(y')$, $\mathcal{M}(x')$) is isomorphic to $\mathcal{M}(z_{72})$ (resp., $\mathcal{M}(z_{73})$, $\mathcal{M}(z_{74})$) since both contain a vertex with the label $\boxed{\mathbf{20563}}$ (resp., $\boxed{\mathbf{13205}}$, $\boxed{\mathbf{13562}}$). The set

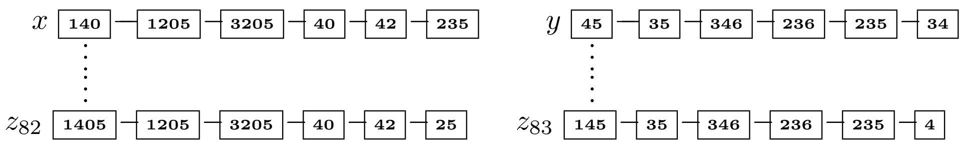
$$E(W_{(7)}) = M'(z_{71}) \cup \left(\cup_{i=2}^{4} M(z_{7i})\right)$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(7)}$.

4.10. Next consider the two-sided cell $W_{(8)}$. Take

$$P(8) = \{w_{13406}, w_{45601}, w_{02416}\}.$$

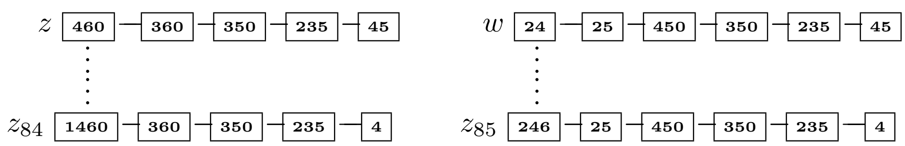
The graph $\mathcal{M}(z_{81})$ with $z_{81} = w_{13406}$ is infinite. Take a subgraph $\mathcal{M}'(z_{81})$ of $\mathcal{M}(z_{81})$ as Graph $L3$ with the vertex labeled by $\boxed{\mathbf{13460}}$ being the element z_{81} (see Appendix). Let $x = z_{81} \cdot \mathbf{52420} \in M'(z_{81})$ and $z_{82} = x \cdot \mathbf{5}$. We see from Fig. 5 (a) that $\{x, z_{82}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{82})$ is displayed as Graph J (see Appendix). Take $y = z_{82} \cdot \mathbf{065345} \in M(z_{82})$ and $z_{83} = y \cdot \mathbf{1}$. We see from Fig. 5 (b) that $\{y, z_{83}\}$ forms a primitive pair. Take $z = z_{82} \cdot \mathbf{23456}$



(a) (b)

Fig. 5. The primitive pairs $\{x, z_{82}\}$, $\{y, z_{83}\}$

and $w = z_{82} \cdot \mathbf{065234614520425142}$ in $M(z_{82})$. Let $z_{84} = z \cdot \mathbf{1}$ and $z_{85} = w \cdot \mathbf{6}$. We see from Fig. 6 (a)–(b) that both $\{z, z_{84}\}$ and $\{w, z_{85}\}$ are primitive pairs. The graphs $\mathcal{M}(z_{83})$, $\mathcal{M}(z_{84})$, $\mathcal{M}(z_{85})$ are displayed as Graphs K, $\psi_{06}(K)$, $\psi_{01}(K)$, respectively (see Appendix).



(a) (b)

Fig. 6. The primitive pairs $\{z, z_{84}\}$, $\{w, z_{85}\}$

Then the set

$$E(W_{(8)}) = M'(z_{81}) \cup \left(\cup_{i=2}^{5} M(z_{8i})\right)$$

is distinguished and also **ABC**-saturated, thus it forms an l.c.r. set of $W_{(8)}$.

4.11. Consider the two-sided cell $W_{(9)}$. Take

$$P(9) = \{w_{130256}\}.$$

The graph $\mathcal{M}(z_{91})$ with $z_{91} = w_{130256}$ is displayed as Graph $L1$ (see Appendix). Take $x = z_{91} \cdot \mathbf{42534} \in M(z_{91})$ and $z_{92} = x \cdot \mathbf{3}$. Then we see from Fig. 7 (a) that $\{x, z_{92}\}$ forms a primitive

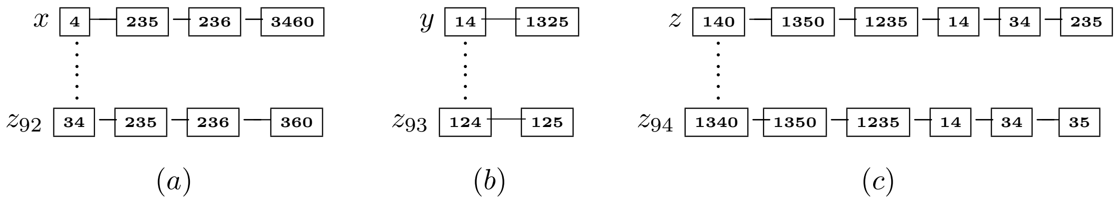


Fig. 7. The primitive pairs $\{x, z_{92}\}$, $\{y, z_{93}\}$, $\{z, z_{94}\}$

pair. The graph $\mathcal{M}(z_{92})$ is displayed as Graph $L3$ (see Appendix). Let $y = z_{92} \cdot \mathbf{1} \in M(z_{92})$ and $z_{93} = y \cdot \mathbf{2}$. We see from Fig, 7 (b) that $\{y, z_{93}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{93})$ is displayed as Graph $L2$ (see Appendix). Take $z = z_{92} \cdot \mathbf{1404} \in M(z_{92})$ and $z_{94} = z \cdot \mathbf{3}$. We see from Fig. 7 (c) that $\{z, z_{94}\}$ forms a primitive pair. The graph $\mathcal{M}(z_{94})$ is displayed as Graph J (see Appendix). The set $\cup_{i=1}^{4} M(z_{9i})$ is distinguished and **AB**-saturated, but not **C**-saturated. Take

$$w = z_{91} \cdot \mathbf{423542365413024} \cdot \mathbf{5} \cdot \mathbf{620}, \qquad u = \mathbf{561302454324130245632450 24} \cdot \mathbf{3} \cdot \mathbf{1},$$

$$z_{95} = z_{91} \cdot \mathbf{423542365413024} \cdot \mathbf{620}, \qquad z_{96} = \mathbf{56130245432413024563245024} \cdot \mathbf{1},$$

$$v = z_{91} \cdot \mathbf{423456453241} \cdot \mathbf{3} \cdot \mathbf{0245}, \qquad z_{97} = z_{91} \cdot \mathbf{423456453241} \cdot \mathbf{0245}.$$

in $M(z_{94})$. The graphs $\mathcal{M}(z_{95})$, $\mathcal{M}(z_{96})$, $\mathcal{M}(z_{97})$ are displayed as Graphs K, $\psi_{06}(K)$, $\psi_{01}(K)$, respectively (see Appendix). By Fig. 8 (a)–(c), we see that $\{w, z_{95}\}$, $\{u, z_{96}\}$ and $\{v, z_{97}\}$) are all primitive pairs. The set

$$E(W_{(9)}) = \cup_{i=1}^{7} M(z_{9i})$$

is distinguished and **ABC**-saturated, so it forms an l.c.r. set of $W_{(9)}$.

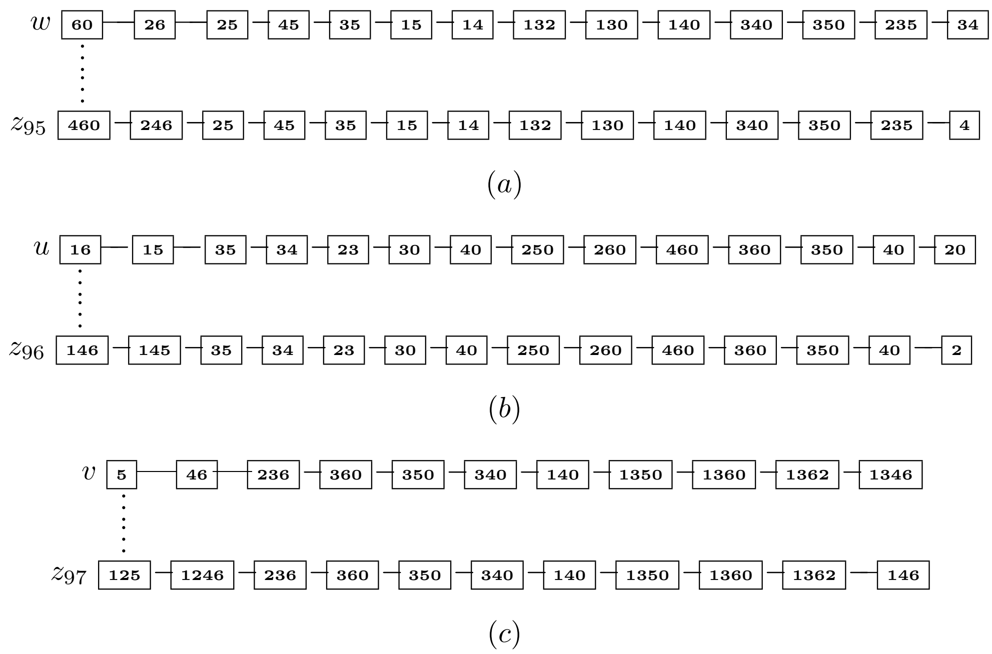


Fig. 8. The primitive pairs $\{w, z_{95}\}$, $\{u, z_{96}\}$, $\{v, z_{97}\}$

4.12. In the two-sided cell $W_{(10)}$, take

$$P(10) = \{w_{1345}, w_{1342}, w_{0243}, w_{0245}, w_{3456}, w_{2456}\}.$$

The graph $\mathcal{M}(z_{10,1})$ with $z_{10,1} = w_{1342}$ is displayed as Graph P (see [32 Fig. 19]). Its vertex set $M(z_{10,1})$ is not $\mathbf{B}$-saturated. Take

$x = z_{10,1} \cdot \mathbf{054652543}$, $\quad y = z_{10,1} \cdot \mathbf{05465342403541324534}$,

$z = z_{10,1} \cdot \mathbf{05465342405253413542}$, $\quad w = z_{10,1} \cdot \mathbf{543204146412454624234132463}$,

$u = z_{10,1} \cdot \mathbf{02456452434153520451424}$

in $M(z_{10,1})$. Let $z_{10,2} = x \cdot \mathbf{0}$, $z_{10,3} = y \cdot \mathbf{0}$, $z_{10,4} = z \cdot \mathbf{4}$, $z_{10,5} = w \cdot \mathbf{0}$, $z_{10,6} = u \cdot \mathbf{6}$.

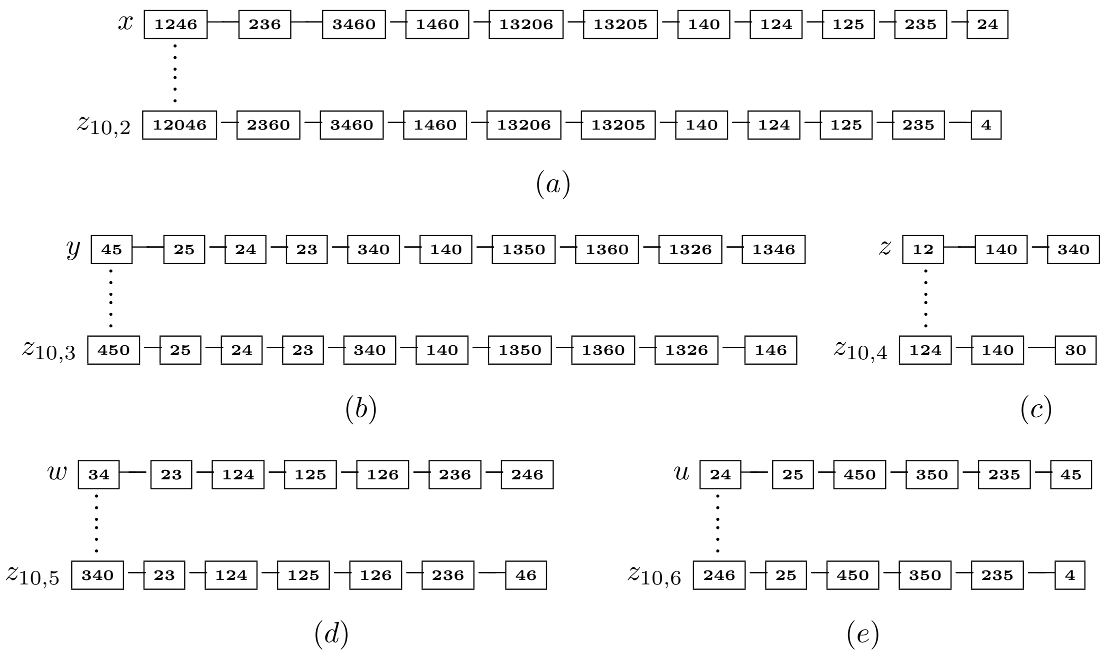


Fig. 9. The primitive pairs $\{x, z_{10,2}\}$, $\{y, z_{10,3}\}$, $\{z, z_{10,4}\}$, $\{w, z_{10,5}\}$, $\{u, z_{10,6}\}$

The graphs $\mathcal{M}(z_{10,2})$, $\mathcal{M}(z_{10,3})$, $\mathcal{M}(z_{10,4})$, $\mathcal{M}(z_{10,5})$ and $\mathcal{M}(z_{10,6})$ are displayed as Graphs $L3$, J, K, $\psi_{06}(K)$, $\psi_{01}(K)$, respectively (see Appendix). From the graphs in Fig. 9 (a)-(e), we see that $\{x, z_{10,2}\}$, $\{y, z_{10,3}\}$, $\{z, z_{10,4}\}$, $\{w, z_{10,5}\}$ and $\{u, z_{10,6}\}$ are all primitive pairs. The set

$$E(W_{(10)}) = \cup_{i=1}^{6} M(z_{10,i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(10)}$.

4.13. In the two-sided cell $W_{(11)}$, take

$$P(11) = \{w_{13450}, w_{13426}, w_{02436}, w_{02451}, w_{34560}, w_{24561}\}.$$

The graph $\mathcal{M}(z_{11,1})$ with $z_{11,1} = w_{13426}$ is displayed as Graph Q (see [32, Fig. 20]). Its vertex set $M(z_{11,1})$ is not **B**-saturated. Take

$$x = z_{11,1} \cdot \mathbf{02543}, \qquad w = z_{11,1} \cdot \mathbf{024534234046354345134240323546 53450},$$

$$y = z_{11,1} \cdot \mathbf{5432414640214542642341624 3453}, \quad z = z_{11,1} \cdot \mathbf{54324146254204254342132546 52145}$$

in $M(z_{11,1})$. Let $z_{11,2} = x \cdot \mathbf{6}$, $z_{11,3} = y \cdot \mathbf{0}$, $z_{11,4} = z \cdot \mathbf{0}$, $z_{11,5} = w \cdot \mathbf{1}$. From Fig. 10 (a)–(d),

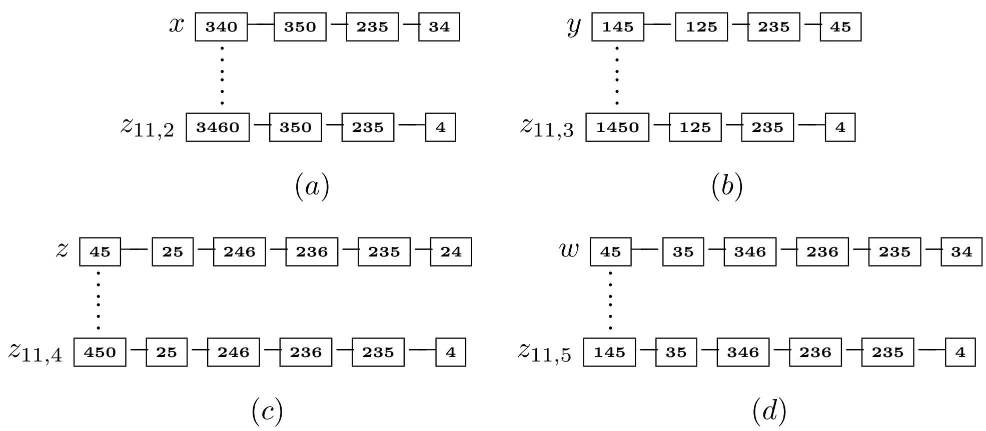


Fig. 10. The primitive pairs $\{x, z_{11,2}\}$, $\{y, z_{11,3}\}$, $\{z, z_{11,4}\}$, $\{w, z_{11,5}\}$

we see that $\{x, z_{11,2}\}$, $\{y, z_{11,3}\}$, $\{z, z_{11,4}\}$ and $\{w, z_{11,5}\}$ are all primitive pairs. The graphs $\mathcal{M}(z_{11,2})$, $\mathcal{M}(z_{11,5})$, $\mathcal{M}(z_{11,3})$, $\mathcal{M}(z_{11,4})$ are displayed as Graphs $L1$, K, $\psi_{06}(K)$, $\psi_{01}(K)$, respectively (see Appendix). We may check that the set

$$E(W_{(11)}) = \cup_{i=1}^{5} M(z_{11,i})$$

is distinguished and also **ABC**-saturated, hence it forms an l.c.r. set of $W_{(11)}$.

§5. Summary of our results.

5.1. Keep the notation in 1.5–1.6 with G the reductive algebraic group of type E_6. For a unipotent class $\mathbf{u}$ of G, let $c(\mathbf{u})$ be the corresponding two-sided cell of $\widetilde{E}_6$. Denote by $n(c(\mathbf{u}))$ the number of left cells of $\widetilde{E}_6$ contained in $c(\mathbf{u})$ and by $E(c(\mathbf{u}))$ an l.c.r. set of $\widetilde{E}_6$ in $c(\mathbf{u})$. Then the l.c.r. sets for the two-sided cells of $\widetilde{E}_6$ obtained in Section 4 can be displayed into Table 1 below, where the unipotent conjugacy classes of G are parameterized as in [2, Chapter 13].

5.2. Consider the graphs displayed in Table 2. We have $\mathcal{M}_L(x) \cong \mathcal{M}(x)$ for $x \notin \Xi := \{\mathbf{12}, \mathbf{1312}, \mathbf{13125}, z_{61}, z_{71}, z_{81}\}$; while for any $x \in \Xi$, the graph $\mathcal{M}(x)$ is infinite. Owing to its larger size, we divide the graph H into two parts $H1$ and $H2$. The graphs P and Q are too large to be displayed in the paper, which can be found in the website of the first-named author (see [32, Figs. 19-20]). We denote by $n(M)$ the cardinality of the vertex set M of a graph $\mathcal{M}$. By 4.1, 4.5 and 4.7, we see that the automorphism ψ_{ij} of $\widetilde{E}_6$ (see 4.2) stabilizes each two-sided cell Ω of $\widetilde{E}_6$. So ψ_{ij} gives rise to a permutation on the left cells of $\widetilde{E}_6$ in Ω and further to a permutation on the left-cell graphs in Ω. Indeed, ψ_{ij} stabilizes all the left-cell graphs of $\widetilde{E}_6$ except for those in Table 3, where ψ_{ij} transposes two members in each pair.

Unipotent class of G	$c(\mathbf{u})$	$n(c(\mathbf{u}))$	$E(c(\mathbf{u}))$
E_6	$W_{(0)}$	1	$\{e\}$
$E_6(a_1)$	$W_{(1)}$	7	S
D_5	$W_{(2)}$	27	$M'(\mathbf{12})$
$E_6(a_3)$	$W_{(3)}$	57	$M(\mathbf{131}) \cup M(\mathbf{140})$
A_5	$W^1_{(4)}$	162	$M'(\mathbf{1312})$
$D_5(a_1)$	$W^2_{(4)}$	72	$\cup_{i=1}^3 M(z_{4i})$
$A_4 + A_1$	$W_{(5)}$	216	$M'(\mathbf{13125})$
A_4	$W^1_{(6)}$	432	$M'(z_{61})$
D_4	$W^2_{(6)}$	270	$\cup_{i=2}^4 M(z_{6i})$
$D_4(a_1)$	$W_{(7)}$	540	$M'(z_{71}) \cup (\cup_{i=2}^4 M(z_{7i}))$
$A_3 + A_1$	$W_{(8)}$	675	$M'(z_{81}) \cup (\cup_{i=2}^5 M(z_{8i}))$
$2A_2 + A_1$	$W_{(9)}$	720	$\cup_{i=1}^7 M(z_{9i})$
A_3	$W_{(10)}$	1890	$\cup_{i=1}^6 M(z_{10,i})$
$A_2 + 2A_1$	$W_{(11)}$	2160	$\cup_{i=1}^5 M(z_{11,i})$

Table 1

Graphs $\mathcal{M}$	Left-cell graphs isomorphic to $\mathcal{M}$	$n(M)$
$\boxed{\emptyset}$	$\mathcal{M}_L(e)$	1
A	$\mathcal{M}_L(\mathbf{1})$	7
B	$\mathcal{M}_L(\mathbf{12})$	27
C	$\mathcal{M}_L(\mathbf{131})$, $\mathcal{M}_L(z_{42})$	21
D	$\mathcal{M}_L(\mathbf{140})$, $\mathcal{M}_L(z_{43})$	36
E	$\mathcal{M}_L(\mathbf{1312})$	162
F	$\mathcal{M}_L(z_{41})$	15
G	$\mathcal{M}_L(\mathbf{13125})$	216
H	$\mathcal{M}_L(z_{61})$	432
I	$\mathcal{M}_L(z_{62})$	90
$\psi_{06}(I)$	$\mathcal{M}_L(z_{63})$	90
$\psi_{01}(I)$	$\mathcal{M}_L(z_{64})$	90
J	$\mathcal{M}_L(z_{71})$, $\mathcal{M}_L(z_{82})$, $\mathcal{M}_L(z_{94})$, $\mathcal{M}_L(z_{10,3})$	300
K	$\mathcal{M}_L(z_{74})$, $\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{10,4})$, $\mathcal{M}_L(z_{11,5})$	80
$\psi_{06}(K)$	$\mathcal{M}_L(z_{73})$, $\mathcal{M}_L(z_{84})$, $\mathcal{M}_L(z_{96})$, $\mathcal{M}_L(z_{10,5})$, $\mathcal{M}_L(z_{11,3})$	80
$\psi_{01}(K)$	$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{85})$, $\mathcal{M}_L(z_{97})$, $\mathcal{M}_L(z_{10,6})$, $\mathcal{M}_L(z_{11,4})$	80
$L1$	$\mathcal{M}_L(z_{91})$, $\mathcal{M}_L(z_{11,2})$	10
$L2$	$\mathcal{M}_L(z_{93})$	35
$L3$	$\mathcal{M}_L(z_{81})$, $\mathcal{M}_L(z_{92})$, $\mathcal{M}_L(z_{10,2})$	135
P	$\mathcal{M}_L(z_{10,1})$	1215
Q	$\mathcal{M}_L(z_{11,1})$	1910

Table 2

5.3. In Table 4, we list the position of $L(z)$ as a vertex of the left-cell graph $\mathcal{M}_L(z)$, where $L(z)$ is the left cell of $\widetilde{E}_6$ containing the element z. In the most cases, such a position is determined uniquely by the label of the vertex $L(z)$; when there exist some other vertices of $\mathcal{M}_L(z)$ sharing the same label as $L(z)$, we need some additional data to distinguish $L(z)$ from the others: either the label of some adjacent vertex, or the label of some distance-2 vertex. Hence, by " **450**, distance 2 to **560** ", we mean that the vertex $L(z_{10,3})$ of $\mathcal{M}_L(z_{10,3})$ is labeled by **450** and that there is a path of length 2 connecting $L(z_{10,3})$ and a vertex labeled by **560** (see 2.5). Also, by " **1450**, adjacent to **1250** ", we mean that the vertex $L(z_{82})$ of $\mathcal{M}_L(z_{82})$ is labeled by **1450** and that there is a vertex of $\mathcal{M}_L(z_{82})$ labeled by **1250** adjacent to $L(z_{82})$.

Transposed by ψ_{10}	Transposed by ψ_{16}	Transposed by ψ_{06}
$\mathcal{M}_L(z_{62})$, $\mathcal{M}_L(z_{64})$	$\mathcal{M}_L(z_{63})$, $\mathcal{M}_L(z_{64})$	$\mathcal{M}_L(z_{62})$, $\mathcal{M}_L(z_{63})$
$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{74})$	$\mathcal{M}_L(z_{72})$, $\mathcal{M}_L(z_{73})$	$\mathcal{M}_L(z_{73})$, $\mathcal{M}_L(z_{74})$
$\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{85})$	$\mathcal{M}_L(z_{84})$, $\mathcal{M}_L(z_{85})$	$\mathcal{M}_L(z_{83})$, $\mathcal{M}_L(z_{84})$
$\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{97})$	$\mathcal{M}_L(z_{96})$, $\mathcal{M}_L(z_{97})$	$\mathcal{M}_L(z_{95})$, $\mathcal{M}_L(z_{96})$
$\mathcal{M}_L(z_{10,3})$, $\mathcal{M}_L(z_{10,4})$	$\mathcal{M}_L(z_{10,4})$, $\mathcal{M}_L(z_{10,5})$	$\mathcal{M}_L(z_{10,3})$, $\mathcal{M}_L(z_{10,5})$
$\mathcal{M}_L(z_{11,3})$, $\mathcal{M}_L(z_{11,4})$	$\mathcal{M}_L(z_{11,3})$, $\mathcal{M}_L(z_{11,5})$	$\mathcal{M}_L(z_{11,4})$, $\mathcal{M}_L(z_{11,5})$

Table 3

z	Position of $L(z)$ in $\mathcal{M}_L(z)$	z	Position of $L(z)$ in $\mathcal{M}_L(z)$
z_{61}	**134**	z_{62}	**1356**
z_{63}	**1230**	z_{64}	**2560**
z_{71}	**1346**	z_{72}	**23560**
z_{73}	**12350**	z_{74}	**12356**
z_{81}	**13460**	z_{82}	**1450**, adjacent to **1250**
z_{83}	**145**	z_{84}	**1460**, adjacent to **360**
z_{85}	**246**	z_{91}	**123560**
z_{92}	**34**, adjacent to **235**	z_{93}	**124**
z_{94}	**1340**	z_{95}	**460**, adjacent to **246**
z_{96}	**146**, adjacent to **145**	z_{97}	**125**, adjacent to **1246**
$z_{10,1}$	**1234**	$z_{10,2}$	**12460**
$z_{10.3}$	**450**, distance 2 to **560**	$z_{10,4}$	**124**
$z_{10,5}$	**340**	$z_{10,6}$	**246**
$z_{11,1}$	**12346**	$z_{11,2}$	**3460**
$z_{11,3}$	**1450**	$z_{11,4}$	**450**
$z_{11,5}$	**145**		

Table 4

5.4. As above, let $n(\Omega)$ denote the number of left cells of $\widetilde{E}_6$ in any two-sided cell Ω. According to the result of the first-named author in [23], we have $n(W_{(36)}) = |W_0| = 2^7 3^4 5$ each left cell in $W_{(36)}$ forms a single sign type in the sense of [22]. In [30], the first-named author proved the inequality $n(W_{(25)}) \leqslant |W_0|/2$ and conjectured that the equality should hold. The numbers

$n(W_{(k)})$ with $k \in \{12, 13, 15, 16, 20\}$ haven't yet been calculated so far. We conjecture that $n(W_{(20)}) = |W_0|/4$.

5.5. In [31, Section 6], we described all the distinguished involutions d of $\widetilde{E}_6$ with $\mathbf{a}(d) \leqslant 4$. Based on our results on the left cells Γ of $\widetilde{E}_6$ with $5 \leqslant \mathbf{a}(\Gamma) \leqslant 11$ and the result of the first-named author in [24, Proposition 5.12], Z. X. Liu found all the distinguished involutions d of $\widetilde{E}_6$ with $5 \leqslant \mathbf{a}(d) \leqslant 11$ in her Master thesis [13] by applying the same techniques as that in [31].

5.6. From Table 2, we see that in $\widetilde{E}_6$, if $\Omega, \Omega', \Omega''$ are three two-sided cells with $\Omega \underset{LR}{\leqslant} \Omega' \underset{LR}{\leqslant} \Omega''$ and $\mathbf{a}(\Omega) \leqslant 11$, and if Ω, Ω'' have left cell graphs $\mathcal{M}_L, \mathcal{M}''_L$, respectively with $\mathcal{M}_L \cong \mathcal{M}''_L$, then Ω' has a left cell graph $\mathcal{M}'_L$ satisfying $\mathcal{M}'_L \cong \mathcal{M}_L$. We wonder if this is still the case without the restrictive condition of $\mathbf{a}(\Omega) \leqslant 11$, or further, if it holds in general. A direct check shows that this is the case when $W_a = \widetilde{C}_4, \widetilde{F}_4$ (see [27], [28]).

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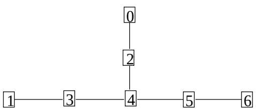
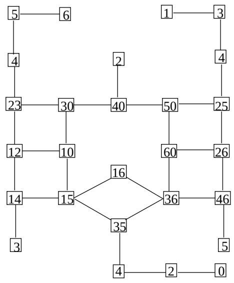
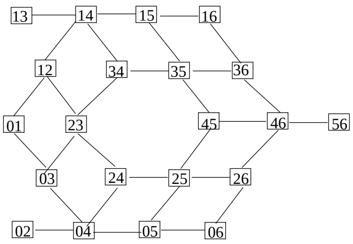
Appendix.



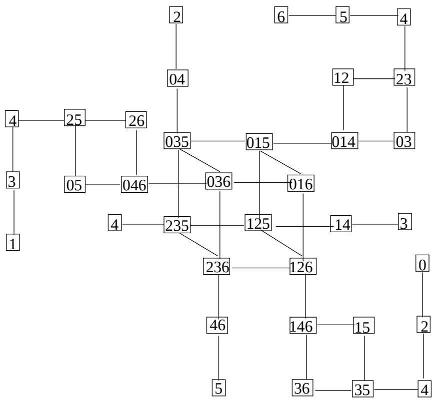
Graph A



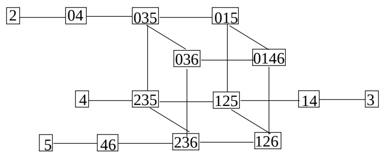
Graph B



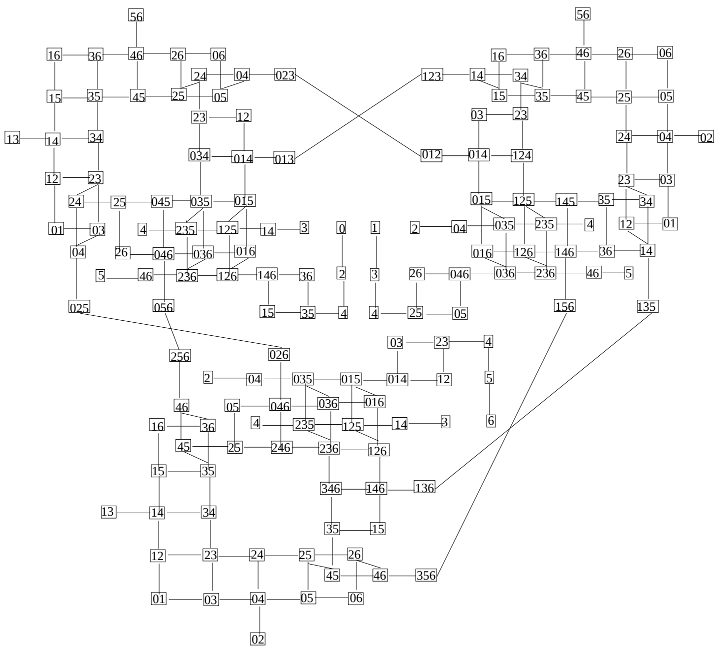
Graph C



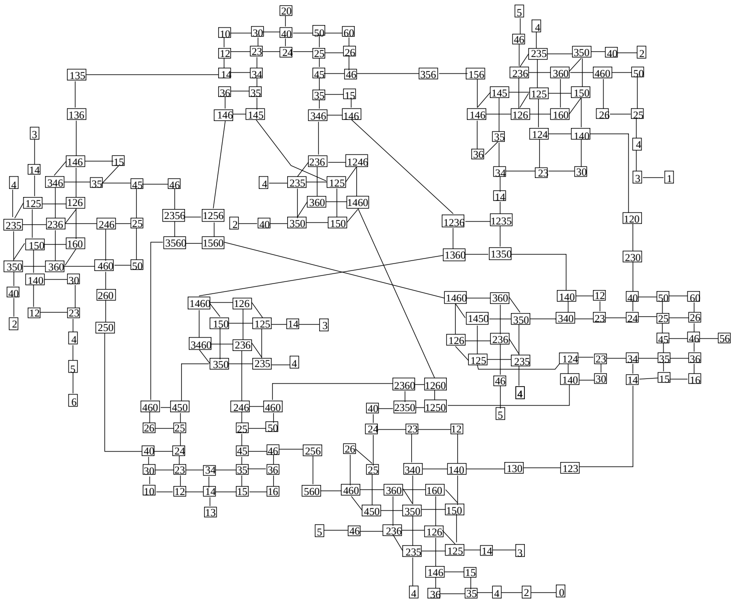
Graph D



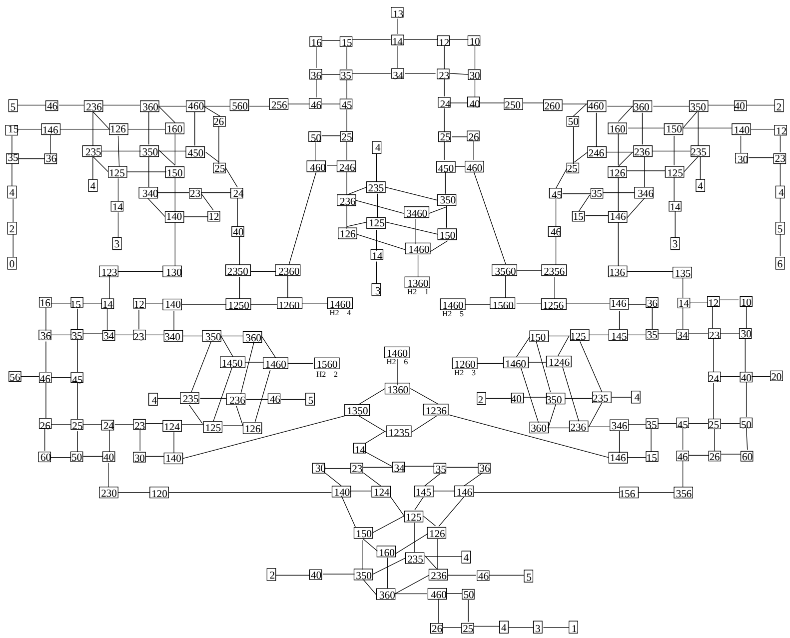
Graph F



Graph E

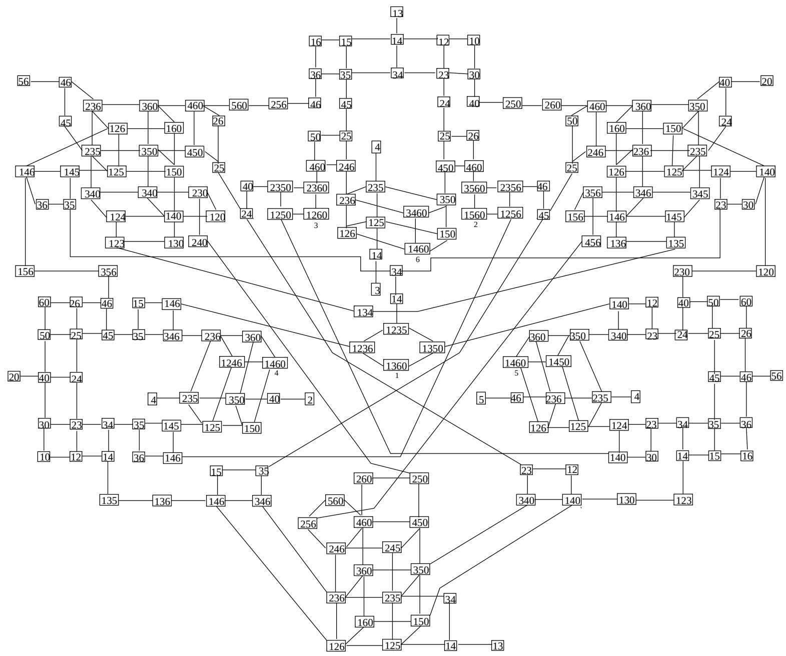


Graph G

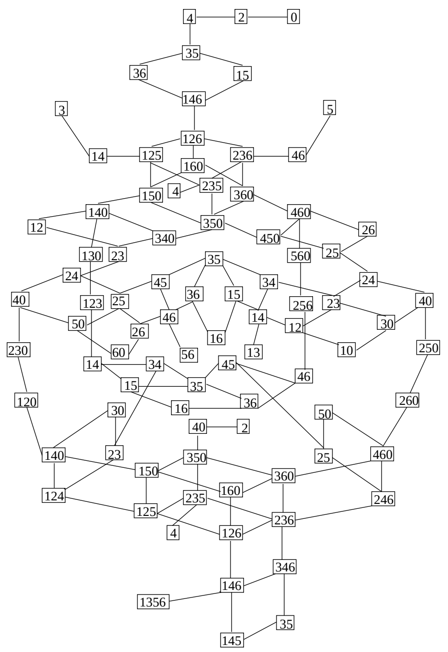


Graph H (part $H1$)

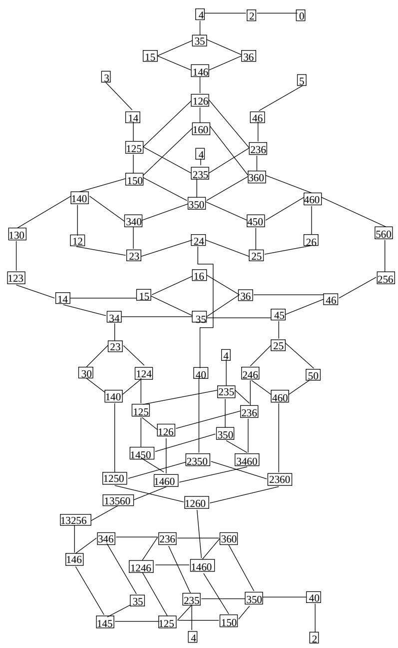
where by a node in part $H1$ with an underneath label " H2 i ", we mean that it is identified with that in part $H2$ with the underneath label i. For example, the node **1360** ($H2$ 1) in part $H1$ is identified with the node **1360** (1) in part $H2$.



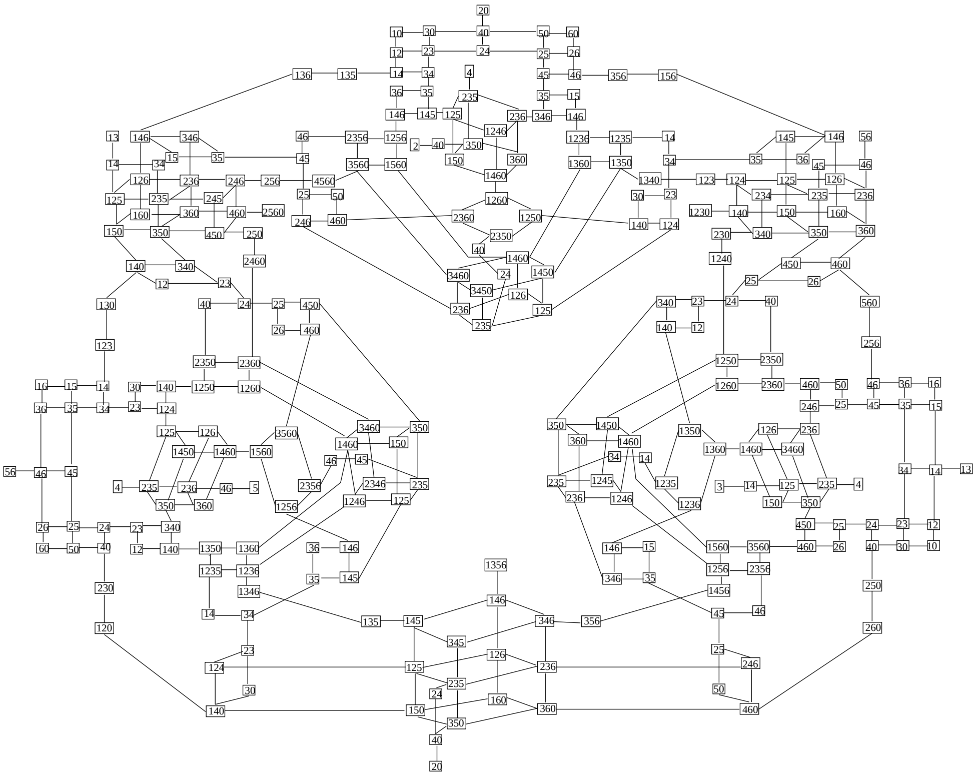
Graph H (part $H2$)



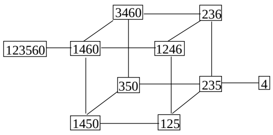
Graph I



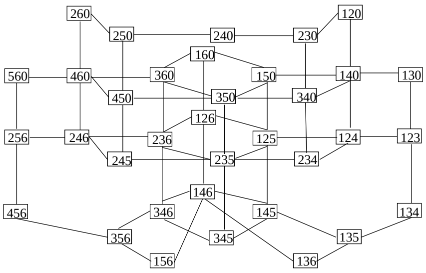
Graph K



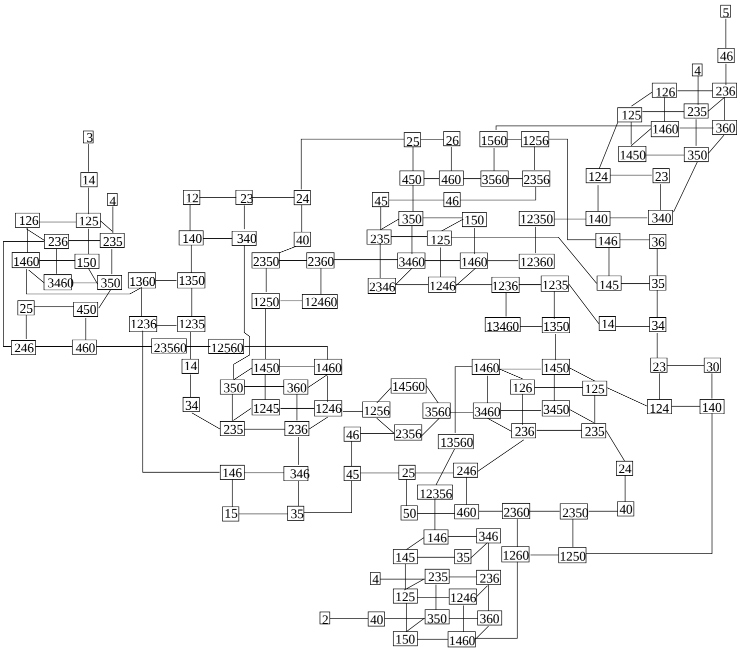
Graph J



Graph $L1$



Graph $L2$



Graph $L3$