AUTOMORPHISM GROUPS OF THE IMPRIMITIVE COMPLEX REFLECTION GROUPS

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Abstract. We describe the group of all reflection-preserving automorphisms of an imprimitive complex reflection group. We also study some properties of this automorphism group.

§0. Introduction.

Let $\mathbb{N}$ (respectively, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$) be the set of all positive integers (respectively, integers, real numbers, complex numbers). For any $k \leqslant n$ in $\mathbb{N}$, denote $[k, n] := \{k, k+1, ..., n\}$ and $[n] := [1, n]$. Shephard and Todd classified all finite complex reflection groups (see [5]). There are two families of such groups: primitive and imprimitive. For any $m, p, n \in \mathbb{N}$ with $p \mid m$ (reading " p divides m "), let $G(m, p, n)$ be the group consisting of all $n \times n$ monomial matrices whose non-zero entries $a_1, ..., a_n$ are the mth roots of unity with $\left(\prod_{i=1}^n a_i\right)^{m/p} = 1$. In [2], Cohen proved that any irreducible imprimitive reflection group is isomorphic to some $G(m, p, n)$ (see [2, 2.4]). We see that $G(m, p, n)$ is a Coxeter group if either $m \leqslant 2$ or $(p, n) = (m, 2)$.

By an automorphism ϕ of a reflection group G, we mean that ϕ is an automorphism of the group G as an abstract group which sends any reflection of G to a reflection. In the

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present paper, when we mention an automorphism of G, we always mean that G is regarded as a reflection group. Denote by $\mathrm{Aut}(G)$ the group consisting of all automorphisms of G.

The aim of the present paper is to describe the group $\mathrm{Aut}(m,p,n) := \mathrm{Aut}(G(m,p,n))$. Set

$$\mathrm{Int}(m,p,n) := \{\tau_g \mid g \in G(m,p,n)\},$$

where $\tau_g : x \mapsto gxg^{-1}$ is the inner automorphism of $G(m,p,n)$ determined by g. The structure of $\mathrm{Aut}(m,p,n)$ is well known in the case where $G(m,p,n)$ is a Coxeter group. More precisely, when $m \leqslant 2$, we have

$$G(m,p,n) \in \{A_h, B_k, D_l \mid h \geqslant 1, k \geqslant 2, l \geqslant 4\}$$

and

$$\mathrm{Aut}(m,p,n) \cong \mathrm{Int}(m,p,n) \cdot \Gamma$$

with Γ the graph automorphism group of $G(m,p,n)$. On the other hand, we have $G(m,m,2) = I_2(m)$, the dihedral group generated by two reflections s_α, s_β, where α, β are two unitary vectors in a plane with inner product $(\alpha,\beta) = -\cos(\pi/m)$. Then $\mathrm{Aut}(m,m,2)$ consists of all transformations which sends s_α to any reflection $s_{\alpha'}$ of $I_2(m)$ and s_β to another reflection $s_{\beta'}$ satisfying $(\alpha',\beta') = \cos(k\pi/m)$ for some $1 \leqslant k < m$ with $\gcd(k,m) = 1$ (see [4]).

So we need only consider the case of $m > 2$ and $n > 1$ and $(p,n) \neq (m,2)$ for $\mathrm{Aut}(m,p,n)$ in this paper. Our results can be stated briefly as follows. Set

$$\mathrm{Int}(m,1,n)_p = \{\tau_g^{(p)} \mid g \in G(m,1,n)\},$$

where $\tau_g^{(p)}$ denotes the restriction of τ_g to $G(m,p,n)$. For any $k \in [m]$ with $\gcd(k,m) = 1$, the transformation $\psi_k : (a_{ij}) \mapsto (a_{ij}^k)$ on $G(m,p,n)$ is in $\mathrm{Aut}(m,p,n)$ (see Lemma 2.5). Let

$$\Psi(m) := \{\psi_k \mid k \in [m], \gcd(k,m) = 1\}.$$

We have

$$\mathrm{Aut}(m,p,n) = \mathrm{Int}(m,1,n)_p \rtimes \Psi(m)$$

for $(m,p,n) \notin \{(3,3,3),(4,2,2)\}$. In particular, we have

$$\mathrm{Aut}(m,p,n) = \mathrm{Int}(m,p,n) \rtimes \Psi(m)$$

if $\gcd(p,n) = 1$. We also determine the structure of $\mathrm{Aut}(m,p,n)$ in the exceptional cases (see Theorem 6.1). As a consequence, we get the order of $\mathrm{Aut}(m,p,n)$ in all cases (see Proposition 6.2).

The above description for the group $\mathrm{Aut}(m,p,n)$ is also applicable to the most cases of $G(m,p,n)$ being a Coxeter group (see Remark 6.3).

We also study some properties of $\mathrm{Aut}(m,p,n)$. Among others, we give an explicit description of its centre (see 6.4-6.9).

The contents of the paper are organized as follows. In Section 1, we collect some concepts and results for later use. We study some general properties of $\mathrm{Aut}(m,p,n)$ in Section 2. In Sections 3–5, we describe $\mathrm{Aut}(m,p,n)$ explicitly in three cases: $p = 1$ and $p = m$ and $p \in [2, m-1]$ separately, one case in each section. In Section 6, we study some properties of $\mathrm{Aut}(m,p,n)$.

§1. Preliminaries.

1.1. Let V be a Hermitian space of dimension n. A *reflection* in V is a unitary transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* in V is a finite group generated by reflections in V. A reflection group G is called a *real group* or a *Coxeter group* if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C}\otimes_{\mathbb{R}} V_0 \to V$ is bijective. Call G a *complex group* otherwise (according to this definition, a real group is not complex).

1.2. A reflection group G in V is called *imprimitive* if G acts on V irreducibly and if V is a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ of nontrivial proper subspaces V_i $(i \in [t])$ of V such that G permutes the set $\{V_i \mid i \in [t]\}$. In this situation, the family $\{V_i \mid i \in [t]\}$ is called *a system of imprimitivity* for G. Cohen [2] showed that any imprimitive complex reflection group is isomorphic to $G(m,p,n)$ for some $m,p,n \in \mathbb{N}$ with $p \mid m$ and $m > 2$ and $n > 1$ and $(p,n) \neq (m,2)$; he also showed that $G(m,p,n)$ $(p \mid m$ and $n \geqslant 2)$ has a unique system of imprimitivity if it is irreducible under the natural action on $\mathbb{C}^n$ and

$$(m,p,n) \notin \{(2,1,2),(4,4,2),(3,3,3),(2,2,4)\} \quad \text{(see [2, Lemma 2.7])}.$$

The group $G(1,1,n)$ $(n \geqslant 2)$ is reducible and hence is not imprimitive.

In this paper, when the group $G(m,p,n)$ is mentioned, we always assume $p \mid m$ and $m > 2$ and $n > 1$ and $(p,n) \neq (m,2)$ unless otherwise specified.

1.3. Any $w \in G(m,p,n)$ can be expressed in the form $w = [a_1,...,a_n|\sigma]$ with some $\sigma \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the symmetric group on the set $[n]$ and $a_i \in \mathbb{Z}$ for $i \in [n]$, such that the entry of w in the $(k,(k)\sigma)$-position is $\exp\left((2\pi a_k\sqrt{-1})/m\right)$ for $k \in [n]$. We have $p \mid \Sigma_{k=1}^n a_k$.

An element $w = [a_1,...,a_n|\sigma]$ of $G(m,p,n)$ is a reflection if one of the following conditions holds:

(1) $\sigma = (i,j)$ is a transposition of i and j for some $i \neq j$ in $[n]$ and $a_i + a_j \equiv 0$ and $a_k \equiv 0 \pmod m$ for $k \neq i,j$. In this case, denote w by $s(i,j;a_i)$ and call it *a reflection of type I*. Clearly, any reflection of type I has order 2. We also have $s(i,j;a_i) = s(j,i;-a_i)$.

All reflections of type I are contained in the subgroup $G(m,m,n)$ of $G(m,p,n)$.

(2) $\sigma=1$, and there exists some $k\in[n]$ with $a_k\not\equiv 0$ and $a_i\equiv 0 \pmod m$ for all $i\in[n]\setminus\{k\}$. In this case, denote w by $s(k;a_k)$, and call it *a reflection of type II*. The reflection $s(k;a_k)$ is a diagonal matrix with order $m/\gcd(m,a_k)$. Such reflections exist only when $p<m$.

By [1], we know that $G(m,p,n)$ has a generating set S_0 consisting of

(i) $n+1$ reflections: s_0, s_1' and s_i for $i\in[n-1]$ if $p\in[2,m-1]$;

(ii) n reflections: s_0 and s_i for $i\in[n-1]$ if $p=1$;

(iii) n reflections: s_1' and s_i for $i\in[n-1]$ if $p=m$,

where $s_0=s(1;p)$ and $s_1'=s(1,2;-1)$ and $s_i=s(i,i+1;0)$.

1.4. Let G be a reflection group. Following Shi in [6, 1.9], a *presentation of G by generators and relations* (or just *a presentation of G* in short) is by definition a pair (S,P), where

(1) S is a finite generating set for G which consists of reflections, and S has minimally possible cardinality with this property.

(2) P is a finite set of relations on S, and any other relation on S is a consequence of the relations in P.

We say that S is a *generating reflection set* of G if S satisfies (1).

1.5. For $i\neq j$ and $i'\neq j'$ in $[n]$, and $k,k',l\in\mathbb{Z}$ with $m\nmid l$, denote $t=s(i,j;k)$, $t'=s(i',j';k')$ and $s=s(i';l)$. Then we have

$$\begin{cases} tt'=t't, & \text{if } \{i,j\}\cap\{i',j'\}=\emptyset,\\ tt't\cdots=t'tt'\cdots\ \ (m/\gcd(k-k',m)\text{ factors on each side}), & \text{if } (i,j)=(i',j'),\\ tt't\cdots=t'tt'\cdots\ \ (m/\gcd(k+k',m)\text{ factors on each side}), & \text{if } (i,j)=(j',i'),\\ tt't=t'tt', & \text{otherwise.}\end{cases}$$

$$\begin{cases} ts=st, & \text{if } i'\notin\{i,j\},\\ stst=tsts, & \text{if } i'\in\{i,j\}.\end{cases}$$

From the above relations, we see that two non-commuting reflections $r,r'\in G(m,p,n)$ satisfy the relation $rr'rr'=r'rr'r$ if and only if either exactly one of r,r' has type

II, or $r = s(i,j;k)$ and $r' = s(i,j;k')$ for some $i \neq j$ in $[n]$ and some $k, k' \in \mathbb{Z}$ with $m/\gcd(k-k',m) = 4$. This fact will be useful in the subsequent discussion.

1.6. Denote by $o(s)$ the order of $s \in G(m,p,n)$. We have a presentation (S_0, P_0) of the group $G(m,p,n)$, where S_0 is the generating reflection set as in 1.3 and P_0 is a relation set on S_0 given as follows (see [1]).

(1) When $p = 1$, the set P_0 consists of the relations: $o(s_0) = m$ and $o(s_i) = 2$ for $i \in [n-1]$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$; $s_i s_j = s_j s_i$ for $|i-j| > 1$; $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

(2) When $p = m$, the set P_0 consists of the relations: $o(s'_1) = o(s_i) = 2$ for $i \in [n-1]$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$; $s_i s_j = s_j s_i$ for $|i-j| > 1$; $s'_1 s_i = s_i s'_1$ for $i > 2$; $s'_1 s_2 s'_1 = s_2 s'_1 s_2$; $o(s'_1 s_1) = m$; $s'_1 s_1 s_2 s'_1 s_1 s_2 = s_2 s'_1 s_1 s_2 s'_1 s_1$.

(3) When $p \in [2, m-1]$, the set P_0 consists of the relations: $o(s'_1) = o(s_i) = 2$ for $i \in [n-1]$; $o(s_0) = m/p$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$; $s_i s_j = s_j s_i$ for $|i-j| > 1$; $s'_1 s_i = s_i s'_1$ for $i > 2$; $s'_1 s_2 s'_1 = s_2 s'_1 s_2$; $o(s'_1 s_1) = m$; $s'_1 s_1 s_2 s'_1 s_1 s_2 = s_2 s'_1 s_1 s_2 s'_1 s_1$; $s_0 s'_1 s_0 s'_1 = s'_1 s_0 s'_1 s_0$; $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$; $s_0 s'_1 s_1 = s'_1 s_1 s_0$; $(s'_1 s_1)^{p-1} = s_0^{-1} s_1 s_0 s'_1$.

§2. Automorphisms of a reflection group.

2.1. Denote by $\mathrm{Aut}(G)$ the automorphism group of G. The aim of this paper is to describe the automorphism group $\mathrm{Aut}(m,p,n) := \mathrm{Aut}(G(m,p,n))$ of the group $G(m,p,n)$.

Lemma 2.2. *(see* [7, 2.10 and Lemma 2.1], [8, Lemma 2.2]*) Let S be a generating reflection set of the group $G(m,p,n)$.*

(1) If $p = 1$, then S consists of $n-1$ reflections of type I and one reflection of type II and order m;

(2) If $p = m$, then S consists of n reflections of type I;

(3) If $p \in [2, m-1]$, then S consists of n reflections of type I and one reflection of type II and order m/p.

Denote by $|X|$ the cardinality of a set X.

Lemma 2.3. *Let S_0 be the generating reflection set of $G(m,p,n)$ as in 1.3. Then for any $\eta\in\mathrm{Aut}(m,p,n)$, the image of S_0 under η can be displayed as follows:*

(1) When $p=1$, the n-tuple $(\eta(s_0),\eta(s_1),...,\eta(s_{n-1}))$ is equal to

$$(s((1)\sigma;k),s((1)\sigma,(2)\sigma;k_1),...,s((n-1)\sigma,(n)\sigma;k_{n-1})) \tag{2.3.1}$$

for some $\sigma\in\mathcal{S}_n$ and $k,k_1,...,k_{n-1}\in\mathbb{Z}$ with k coprime to m.

(2) When $p=m$ and $(m,m,n)\neq(3,3,3)$, the n-tuple $(\eta(s_1'),\eta(s_1),...,\eta(s_{n-1}))$ is equal to

$$(s((1)\sigma,(2)\sigma;k_1'),s((1)\sigma,(2)\sigma;k_1),...,s((n-1)\sigma,(n)\sigma;k_{n-1})) \tag{2.3.2}$$

for some $\sigma\in\mathcal{S}_n$ and $k_1',k_1,...,k_{n-1}\in\mathbb{Z}$ with $\gcd(k_1-k_1',m)=1$.

(3) When $p\in[2,m-1]$ and $(m,p,n)\neq(4,2,2)$, the $(n+1)$-tuple $(\eta(s_0),\eta(s_1'),\eta(s_1),...,\eta(s_{n-1}))$ is equal to

$$(s((1)\sigma;pk),s((1)\sigma,(2)\sigma;k_1'),s((1)\sigma,(2)\sigma;k_1),...,s((n-1)\sigma,(n)\sigma;k_{n-1})) \tag{2.3.3}$$

for some $\sigma\in\mathcal{S}_n$ and $k,k_1',k_1,...,k_{n-1}\in\mathbb{Z}$ such that $\gcd(k_1-k_1',m)=1$ and $k_1-k_1'\equiv k$ ($\mathrm{mod}\ m/p$) (hence $\gcd(k,m/p)=1$).

Proof. Let (S_0,P_0) be the presentation of the group $G(m,p,n)$ as in 1.6. Then for any $\eta\in\mathrm{Aut}(m,p,n)$, the pair $(\eta(S_0),\eta(P_0))$ is again a presentation of $G(m,p,n)$, where $\eta(P_0)$ is the relation set on $\eta(S_0)$ which is obtained from P_0 by substituting the elements of S_0 by the corresponding elements of $\eta(S_0)$.

(1) By the relation $o(\eta(s_0))=o(s_0)=m>2$ in 1.6, we see that $\eta(s_0)$ is a reflection of type II. Now by Lemma 2.2 (1), the images of all reflections in $S_0\setminus\{s_0\}$ must be of type I. We claim that there exist some permutation $h_1,h_2,...,h_n$ of $1,2,...,n$ and some

$k, k_1, ..., k_{n-1} \in \mathbb{Z}$ with $\gcd(k, m) = 1$ such that $\eta(s_l) = s(h_l, h_{l+1}; k_l)$ for $l \in [n-1]$ and $\eta(s_0) = s(h_1; k)$. This can be seen by 1.5 and the relations

(i) $\eta(s_i)\eta(s_{i+1})\eta(s_i) = \eta(s_{i+1})\eta(s_i)\eta(s_{i+1})$ for $i \in [n-2]$;

(ii) $\eta(s_i)\eta(s_j) = \eta(s_j)\eta(s_i)$ for $i, j \in [0, n-1]$ with $|i-j| > 1$;

(iii) $\eta(s_0)\eta(s_1)\eta(s_0)\eta(s_1) = \eta(s_1)\eta(s_0)\eta(s_1)\eta(s_0)$.

So (1) is proved by taking $\sigma \in \mathcal{S}_n$ with $(j)\sigma = h_j$ for $j \in [n]$.

(2) Recall that all reflections in $G(m, m, n)$ are of type I. By 1.6, we have the relations

(i) $\eta(s_i)\eta(s_{i+1})\eta(s_i) = \eta(s_{i+1})\eta(s_i)\eta(s_{i+1})$ for $i \in [n-2]$;

(ii) $\eta(s_i)\eta(s_j) = \eta(s_j)\eta(s_i)$ for $i, j \in [n-1]$ with $|i-j| > 1$;

(iii) $o(\eta(s'_1)\eta(s_1)) = m \geqslant 3$;

(iv) $\eta(s'_1)\eta(s_2)\eta(s'_1) = \eta(s_2)\eta(s'_1)\eta(s_2)$;

(v) $\eta(s'_1)\eta(s_l) = \eta(s_l)\eta(s'_1)$ for $l \in [3, n-1]$.

Then by the assumption that $(m, m, n) \neq (3, 3, 3)$, there is a unique system of imprimitivity of $G(m, m, n)$ which is necessarily fixed by any automorphism (see 1.2). So we see by 1.5 that there exist some permutation $h_1, h_2, ..., h_n$ of $1, 2, ..., n$ and some $k'_1, k_1, ..., k_{n-1} \in \mathbb{Z}$ with $\gcd(k_1 - k'_1, m) = 1$ such that $\eta(s_i) = s(h_i, h_{i+1}; k_i)$ for $i \in [n-1]$ and $\eta(s'_1) = s(h_1, h_2; k'_1)$. So we get (2) by taking $\sigma \in \mathcal{S}_n$ with $(j)\sigma = h_i$ for $j \in [n]$.

(3) We claim that $\eta(s_0)$ is of type II. For otherwise, we would have $o(\eta(s_0)) = m/p = 2$ and exactly one reflection (say t) of type II in the set $\Delta = \{\eta(s'_1), \eta(s_i) \mid i \in [n-1]\}$ by Lemma 2.2 (3). If $n > 2$, then there is also some $t' \in \Delta \setminus \{t\}$ with $\{t, t'\} \neq \{\eta(s_1), \eta(s'_1)\}$ and $tt' \neq t't$ by the assumption of $n > 2$. By 1.5, we would have $tt'tt' = t'tt't$, which gives rise to a contradiction by 1.6 (3). If $n = 2$, then $S_0 = \{s_0, s_1, s'_1\}$. By Lemma 2.2 (3) and the symmetry of s_1, s'_1 in S_0, we may assume that $\eta(s_0)$ and $\eta(s'_1)$ have type I, and $\eta(s_1)$ has type II without loss of generality. So $\eta(s_1) = s(h_1; k)$ and $\eta(s_0) = s(h_1, h_2; k_1)$ and $\eta(s'_1) = s(h_1, h_2; k'_1)$ for some permutation h_1, h_2 of $1, 2$ and some $k_1, k'_1, k \in \mathbb{Z}$. Hence $m = o(\eta(s_1)\eta(s'_1)) = 4$ by 1.5–1.6, which would imply $(m, p, n) = (4, 2, 2)$, contradicting

our assumption. So the claim is proved.

Now that $\eta(s_0)$ is of type II. Then all reflections in Δ are of type I by Lemma 2.2 (3). By the same arguments as that in (1)-(2), we see by 1.5 and 1.6 (3) that there are some permutation $h_1,...,h_n$ of $1,...,n$ and some $k,k_1',k_1,...,k_{n-1}\in\mathbb{Z}$ with $\gcd(k,m/p)=1$ and $\gcd(k_1-k_1',m)=1$ such that $\eta(s_i)=s(h_i,h_{i+1};k_i)$ for $i\in[n-1]$, that $\eta(s_0)=s(h_1;pk)$ and that $\eta(s_1')=s(h_1,h_2;k_1')$. Furthermore, by the relation

$$(\eta(s_1')\eta(s_1))^{p-1}=\eta(s_0)^{-1}\eta(s_1)\eta(s_0)\eta(s_1'),$$

we have $k_1-k_1'\equiv k\ (\bmod\ m/p)$. Hence we get (3) by taking $\sigma\in\mathcal{S}_n$ with $(i)\sigma=h_i$ for $i\in[n]$. □

2.4. Set

$$\Phi(m):=\{i\in[m-1]\mid\gcd(i,m)=1\}.$$

Then $\Phi(m)$ is a multiplicative group of order $\phi(m)$, an Euler number. For any $k\in\Phi(m)$ and any $n\times n$ matrix $w=(a_{ij})$, define $\psi_k(w)=(a_{ij}^k)$. In particular, when $w=[a_1,...,a_n|\sigma]\in G(m,p,n)$, we have $\psi_k(w)=[ka_1,...,ka_n|\sigma]\in G(m,p,n)$. So ψ_k can be regarded as a transformation on $G(m,p,n)$ (we adopt such a viewpoint from now on).

Lemma 2.5. *(1) $\psi_k\in\mathrm{Aut}(m,p,n)$ for any $k\in\Phi(m)$.*

(2) $\Psi(m):=\{\psi_k\mid k\in\Phi(m)\}$ forms a subgroup of $\mathrm{Aut}(m,p,n)$ of order $\phi(m)$.

Proof. Since $G(m,p,n)$ consists of monomial matrices, we have $\psi_k(wy)=\psi_k(w)\psi_k(y)$ for any $w,y\in G(m,p,n)$. By the condition $\gcd(k,m)=1$, there exists some $j\in\Phi(m)$ with $kj\equiv1\ (\bmod\ m)$. So $\psi_k\psi_j=\psi_j\psi_k=\psi_1$ is the identity transformation on $G(m,p,n)$. By the description of reflections in 1.3, we see that ψ_k stabilizes the reflection set of $G(m,p,n)$. So $\psi_k\in\mathrm{Aut}(m,p,n)$. Hence (1) is proved and (2) follows by noting that $\psi:k\mapsto\psi_k$ is an injective group homomorphism from $\Phi(m)$ to $\mathrm{Aut}(m,p,n)$ with the image $\Psi(m)$. □

2.6. For any $g \in G(m,p,n)$, define

$$\tau_g : G(m,p,n) \to G(m,p,n)$$

by setting $\tau_g(x) = gxg^{-1}$ for any $x \in G(m,p,n)$. Then τ_g is an inner automorphism of $G(m,p,n)$ which stabilizes the reflection set of $G(m,p,n)$. Hence $\tau_g \in \mathrm{Aut}(m,p,n)$. Let

$$\mathrm{Int}(m,p,n) = \{\tau_g \mid g \in G(m,p,n)\}.$$

By a well-known result in group theory, we get

Lemma 2.7. $\mathrm{Int}(m,p,n)$ *is a normal subgroup of* $\mathrm{Aut}(m,p,n)$.

Lemma 2.8. $\mathrm{Int}(m,p,n) \cap \Psi(m) = 1$.

Proof. Assume $\tau \in \mathrm{Int}(m,p,n) \cap \Psi(m)$. Then there exist some $g = [a_1,...,a_n|\sigma] \in G(m,p,n)$ and $k \in \Phi(m)$ with $\tau = \tau_g = \psi_k$. For any $x = [b_1,...,b_n|\sigma'] \in G(m,p,n)$, we have $\tau_g(x) = [c_1,...,c_n|\sigma\sigma'\sigma^{-1}]$ for some $c_1,...,c_n \in \mathbb{Z}$ and $\psi_k(x) = [kb_1,...,kb_n|\sigma']$. The equation $\tau_g = \psi_k$ implies that $\sigma\sigma'\sigma^{-1} = \sigma'$ for any $\sigma' \in \mathcal{S}_n$, i.e., σ is in the centre of $\mathcal{S}_n$. If $n > 2$ then $\sigma = 1$, hence g is diagonal. Take any diagonal $x = [b_1,...,b_n|1] \in G(m,p,n)$ with $b_1,...,b_n$ not all zero. We have

$$[b_1,...,b_n|1] = \tau_g(x) = \psi_k(x) = [kb_1,...,kb_n|1].$$

This implies $k = 1$ and hence $\tau = 1$, as required.

It remains to consider the case where $n = 2$ and $\sigma = (12)$. Then $p < m$ by the assumption at the end of 1.2. The equation $\tau_g(x) = \psi_k(x)$ for any $x = [b_1,b_2|1] \in G(m,p,n)$ amounts to the equation system: $kb_1 \equiv b_2$ and $kb_2 \equiv b_1 (\bmod m)$ for any $b_1,b_2 \in \mathbb{Z}$ with $p \mid (b_1+b_2)$. But the latter does not always hold by observing the case of $b_1 = 0$ and $b_2 = p$. So $\tau_g \neq \psi_k$ in this case.

So our result is proved. $\square$

2.9. For any $g \in G(m,1,n)$, the inner automorphism τ_g of $G(m,1,n)$ stabilizes the normal subgroup $G(m,p,n)$ of $G(m,1,n)$, with the restriction $\tau_g^{(p)} := \tau_g|_{G(m,p,n)}$ being in $\mathrm{Aut}(m,p,n)$. Denote

$$\mathrm{Int}(m,1,n)_p = \{\tau_g^{(p)} \mid g \in G(m,1,n)\},$$

which forms a subgroup of $\mathrm{Aut}(m,p,n)$ normalized by $\Psi(m)$. We can show

$$\mathrm{Int}(m,1,n)_p \cap \Psi(m) = 1$$

by the argument similar to that for Lemma 2.8, hence

$$\mathrm{Int}(m,1,n)_p \Psi(m) = \mathrm{Int}(m,1,n)_p \rtimes \Psi(m).$$

Denote $\iota = \tau_s^{(p)}$ with $s = s(1;1) \in G(m,1,n)$.

Lemma 2.10. *For any $x, y \in G(m,1,n)$ and any divisor $p \in \mathbb{N}$ of m, we have $\tau_x^{(p)} = \tau_y^{(p)}$ if and only if $\tau_x = \tau_y$.*

Proof. We need only show that $\tau_x^{(p)} = \tau_y^{(p)}$ implies $\tau_x = \tau_y$. Now $\tau_x^{(p)} = \tau_y^{(p)}$ if and only if $\tau_x(g) = \tau_y(g)$ for any $g \in G(m,p,n)$. The latter holds if and only if $y^{-1}x$ lies in the centralizer $Z_{G(m,1,n)}(G(m,p,n))$ of $G(m,p,n)$ in $G(m,1,n)$. Thus to show the equality $\tau_x = \tau_y$, we need only show that $Z_{G(m,1,n)}(G(m,p,n))$ consists of scalar matrices.

Take any $z = [z_1, ..., z_n|\sigma] \in Z_{G(m,1,n)}(G(m,p,n))$ with some $z_1, ..., z_n \in \mathbb{Z}$ and $\sigma \in \mathcal{S}_n$. By the equations

$$\tau_z(s(i, i+1; 0)) = s(i, i+1; 0) \qquad \text{for all } \; i \in [n-1], \tag{2.10.1}$$

we see that σ lies in the centre of $\mathcal{S}_n$. We claim $\sigma = 1$ (i.e., z is diagonal). It is obvious in the case $n > 2$. If $n = 2$ and $\sigma = (1,2)$, that is, $z = [z_1, z_2|(1,2)]$. Then the equations $\tau_z(s(1,2;k)) = s(1,2;k)$ with $k = 0, 1$, imply that

$$z_1 \equiv z_2 \pmod{m} \quad \text{and} \quad z_1 \equiv z_2 + 2 \pmod{m}.$$

This is impossible by our assumption of $m > 2$. Hence the claim is proved and so z is diagonal. Then (2.10.1) further implies that

$$z_1 \equiv z_2 \equiv \cdots \equiv z_n \pmod{m}.$$

So z is a scalar matrix. Hence our conclusion follows. □

§3. The Group $\mathrm{Aut}(m, 1, n)$.

Theorem 3.1. *$\mathrm{Aut}(m, 1, n) = \mathrm{Int}(m, 1, n) \rtimes \Psi(m)$.*

Proof. The group $\mathrm{Aut}(m, 1, n)$ has a normal subgroup $\mathrm{Int}(m, 1, n)$ and a subgroup $\Psi(m)$ by Lemmas 2.5 and 2.7. So $\mathrm{Aut}(m, 1, n)$ has a subgroup

$$G := \mathrm{Int}(m, 1, n)\Psi(m) = \mathrm{Int}(m, 1, n) \rtimes \Psi(m)$$

by Lemma 2.8.

Take any $\eta \in \mathrm{Aut}(m, 1, n)$. Then by (2.3.1), the image of the generating set S_0 of $G(m, 1, n)$ under η is as follows:

$$(\eta(s_0), \eta(s_1), ..., \eta(s_{n-1})) = (s((1)\sigma; k), s((1)\sigma, (2)\sigma; k_1), ..., s((n-1)\sigma, (n)\sigma; k_{n-1}))$$

for some $\sigma \in \mathcal{S}_n$ and some integers $k, k_1, ..., k_{n-1}$ with k coprime to m. Identify σ with $[0, ..., 0|\sigma] \in G(m, 1, n)$. Then

$$((\tau_\sigma\eta)(s_0), (\tau_\sigma\eta)(s_1), ..., (\tau_\sigma\eta)(s_{n-1})) = (s(1; k), s(1, 2; k_1), ..., s(n-1, n; k_{n-1})).$$

There exists $w := [p_1, ..., p_n|1] \in G(m, 1, n)$ satisfying that $p_j \in [m]$ for $j \in [n]$, and $p_i - p_{i+1} \equiv -k_i \pmod{m}$ for $i \in [n-1]$. Then

$$((\tau_w\tau_\sigma\eta)(s_0), (\tau_w\tau_\sigma\eta)(s_1), ..., (\tau_w\tau_\sigma\eta)(s_{n-1})) = (s(1; k), s_1, ..., s_{n-1}).$$

Since $\gcd(k,m)=1$, there exists a unique $c\in\Phi(m)$ with $kc\equiv 1 \pmod{m}$. Then $\psi_c\tau_w\tau_\sigma\eta=1$. Hence $\eta=\tau_{\sigma^{-1}}\tau_{w^{-1}}\psi_c^{-1}\in G$. So our equation is proved. □

§**4. The Group** Aut(m,m,n)**.**

We shall describe Aut(m,m,n) in two cases: $(m,m,n)=(3,3,3)$ and $(m,m,n)\neq(3,3,3)$.

4.1. Let (S_0,P_0) be the presentation of $G(3,3,3)$ with $S_0=\{s_1',s_1,s_2\}$ and P_0 as in 1.3 and 1.6 (2). Define $\mu: S_0\to G(3,3,3)$ by setting $\mu(s_1')=s(2,3;-1)$, $\mu(s_1)=s_1$ and $\mu(s_2)=s_2$. We see that $\mu(S_0)$ is a generating reflection set of $G(3,3,3)$ and that all relations in P_0 remain valid when substituting s by $\mu(s)$ for all $s\in S_0$. So μ can be extended to an automorphism of $G(3,3,3)$ which is still denoted by μ.

Theorem 4.2. Aut$(3,3,3)=\langle\tau_{s_1},\mu,\psi_2\cdot\iota\rangle$.

Proof. This can be checked directly by GAP (see [3]). □

Theorem 4.3. *Let* $G_1:=\mathrm{Int}(m,m,n)\rtimes\Psi(m)$ *and* $G_2:=\mathrm{Int}(m,1,n)_m\rtimes\Psi(m)$*. Then* Aut$(m,m,n)=G_2$ *for any* $(m,m,n)\neq(3,3,3)$*. In particular, if* $\gcd(m,n)=1$ *then* Aut$(m,m,n)=G_1$*.*

Proof. By Lemmas 2.5, 2.7, 2.8 and 2.10, we have $G_1\subseteq G_2\subseteq \mathrm{Aut}(m,m,n)$.

Take $\eta\in\mathrm{Aut}(m,m,n)$. Then by (2.3.2), the image $\eta(S_0)$ of S_0 under η is as follows:

$$(\eta(s_1'),\eta(s_1),...,\eta(s_{n-1}))=(s((1)\sigma,(2)\sigma;k_1'),s((1)\sigma,(2)\sigma;k_1),...,s((n-1)\sigma,(n)\sigma;k_{n-1}))$$

for some $\sigma\in\mathcal{S}_n$ and $k_1',k_1,...,k_{n-1}\in\mathbb{Z}$ with $\gcd(k_1-k_1',m)=1$. As before, identifying σ with $[0,...,0|\sigma]\in G(m,m,n)$, we get

$$((\tau_\sigma\eta)(s_1'),(\tau_\sigma\eta)(s_1),...,(\tau_\sigma\eta)(s_{n-1}))=(s(1,2;k_1'),s(1,2;k_1),...,s(n-1,n;k_{n-1})).$$

If $\gcd(m,n)=1$, there exists a unique $w:=[p_1,...,p_n|1]\in G(m,m,n)$ satisfying that $p_j\in[m]$ for $j\in[n]$, and $p_i-p_{i+1}\equiv -k_i \pmod{m}$ for $i\in[n-1]$, and $m\mid\sum_{i=1}^n p_i$. Then we have

$$((\tau_w\tau_\sigma\eta)(s_1'), (\tau_w\tau_\sigma\eta)(s_1), ..., (\tau_w\tau_\sigma\eta)(s_{n-1})) = (s(1,2;k_1'-k_1), s_1, ..., s_{n-1}). \tag{4.3.1}$$

If $\gcd(m,n) > 1$, there exists $w' := [p_1', ..., p_n'|1] \in G(m,m,n)$ satisfying that $p_j' \in [m]$ for $j \in [n]$, and $p_i' - p_{i+1}' \equiv -k_i \pmod{m}$ for $i \in [2, n-1]$, and $m \mid \sum_{i=1}^n p_i'$. We have

$$\begin{aligned}&((\tau_{w'}\tau_\sigma\eta)(s_1'), (\tau_{w'}\tau_\sigma\eta)(s_1), ..., (\tau_{w'}\tau_\sigma\eta)(s_{n-1}))\\ =&(s(1,2;k_1'+(p_1'-p_2')), s(1,2;k_1+(p_1'-p_2')), s_2, ..., s_{n-1}).\end{aligned}$$

In this case,

$$\begin{aligned}&((\iota^{p_2'-p_1'-k_1}\tau_{w'}\tau_\sigma\eta)(s_1'), (\iota^{p_2'-p_1'-k_1}\tau_{w'}\tau_\sigma\eta)(s_1), ..., (\iota^{p_2'-p_1'-k_1}\tau_{w'}\tau_\sigma\eta)(s_{n-1}))\\ =&(s(1,2;k_1'-k_1), s_1, s_2, ..., s_{n-1}).\end{aligned} \tag{4.3.2}$$

In each of the cases (4.3.1) and (4.3.2), we have $\gcd(k_1'-k_1, m) = 1$, hence there exists a unique $c \in \Phi(m)$ such that $(k_1'-k_1)c \equiv -1 \pmod{m}$. Then

$$\eta = \begin{cases} \tau_{\sigma^{-1}}\tau_{w^{-1}}\psi_c^{-1} \in G_1 & \text{if } \gcd(m,n) = 1 \\ \tau_{\sigma^{-1}}\tau_{(w')^{-1}}\iota^{k_1+p_1'-p_2'}\psi_c^{-1} \in G_2 & \text{if } \gcd(m,n) > 1 \end{cases}$$

So our conclusion follows. □

§5. The group $\mathrm{Aut}(m,p,n)$ with $p \in [2, m-1]$.

In this section, we describe $\mathrm{Aut}(m,p,n)$ with $p \in [2, m-1]$. We shall deal with two cases: $(m,p,n) = (4,2,2)$ and $(m,p,n) \neq (4,2,2)$. Set $q = m/p$.

5.1. Let (S_0, P_0) be the presentation of $G(4,2,2)$ with $S_0 = \{s_0, s_1', s_1\}$ and P_0 as in 1.3 and 1.6 (3). Define $\nu : S_0 \to S_0$ by setting $\nu(s_0) = s_1$ and $\nu(s_1) = s_0$ and $\nu(s_1') = s_1'$. We see that all relations in P_0 remain valid when substituting s by $\nu(s)$ for all $s \in S_0$. So ν can be extended to an automorphism of $G(4,2,2)$ which is still denoted by ν.

Theorem 5.2. Aut$(4,2,2) = \langle \iota, \nu \rangle$.

Proof. This can be checked directly by GAP (see [3]). $\square$

Theorem 5.3. *Let* $G_1 := \mathrm{Int}(m,p,n) \rtimes \Psi(m)$ *and* $G_2 := \mathrm{Int}(m,1,n)_p \rtimes \Psi(m)$. *Then* $\mathrm{Aut}(m,p,n) = G_2$ *for any* $(m,p,n) \neq (4,2,2)$. *In particular, if* $\gcd(p,n) = 1$ *then* $\mathrm{Aut}(m,p,n) = G_1$.

Proof. By Lemmas 2.5, 2.7, 2.8 and 2.10, we have $G_1 \subseteq G_2 \subseteq \mathrm{Aut}(m,p,n)$.

Take $\eta \in \mathrm{Aut}(m,p,n)$. Then by (2.3.3), the image $\eta(S_0)$ of S_0 under η is as follows:

$$\begin{aligned}&(\eta(s_0), \eta(s_1'), \eta(s_1), ..., \eta(s_{n-1}))\\ =&(s((1)\sigma; pk), s((1)\sigma, (2)\sigma; k_1'), s((1)\sigma, (2)\sigma; k_1), ..., s((n-1)\sigma, (n)\sigma; k_{n-1}))\end{aligned}$$

for some $\sigma \in \mathcal{S}_n$ and some $k, k_1', k_1, ..., k_{n-1} \in \mathbb{Z}$ with $\gcd(k,q) = 1$ and $\gcd(k_1 - k_1', m) = 1$ and $k_1 - k_1' \equiv k \pmod q$. By identifying σ with $[0, ..., 0|\sigma] \in G(m,p,n)$, we get

$$\begin{aligned}&((\tau_\sigma\eta)(s_0), (\tau_\sigma\eta)(s_1'), (\tau_\sigma\eta)(s_1), ..., (\tau_\sigma\eta)(s_{n-1}))\\ =&(s(1; pk), s(1,2; k_1'), s(1,2; k_1), ..., s(n-1, n; k_{n-1})).\end{aligned}$$

When $\gcd(p,n) = 1$, there exists $w := [p_1, ..., p_n|1] \in G(m,p,n)$ satisfying that $p_j \in [m]$ for $j \in [n]$, and $p_i - p_{i+1} \equiv -k_i \pmod m$ for $i \in [n-1]$, and $p \mid \sum_{i=1}^n p_i$. Then

$$\begin{aligned}&((\tau_w\tau_\sigma\eta)(s_0), (\tau_w\tau_\sigma\eta)(s_1'), (\tau_w\tau_\sigma\eta)(s_1), ..., (\tau_w\tau_\sigma\eta)(s_{n-1}))\\ =&(s(1; pk), s(1,2; k_1' - k_1), s_1, ..., s_{n-1}).\end{aligned} \tag{5.3.1}$$

When $\gcd(p,n) > 1$, there exists $w' := [p_1', ..., p_n'|1] \in G(m,p,n)$ satisfying that $p_j' \in [m]$ for $j \in [n]$, and $p_i' - p_{i+1}' \equiv -k_i \pmod m$ for $i \in [2, n-1]$, and $p \mid \sum_{=1}^n p_i'$. Then

$$\begin{aligned}&((\tau_{w'}\tau_\sigma\eta)(s_0), (\tau_{w'}\tau_\sigma\eta)(s_1'), (\tau_{w'}\tau_\sigma\eta)(s_1), ..., (\tau_{w'}\tau_\sigma\eta)(s_{n-1}))\\ =&(s(1; pk), s(1,2; k_1' + (p_1' - p_2')), s(1,2; k_1 + (p_1' - p_2')), s_2, ..., s_{n-1}).\end{aligned}$$

In this case, let $\kappa=\iota^{p_2'-p_1'-k_1}\tau_{w'}\tau_\sigma\eta$. Then

$$(\kappa(s_0),\kappa(s_1'),\kappa(s_1),...,\kappa(s_{n-1}))=(s(1;pk),s(1,2;k_1'-k_1),s_1,s_2,...,s_{n-1}). \tag{5.3.2}$$

In each of the cases (5.3.1) and (5.3.2), we have $\gcd(k_1'-k_1,m)=1$, hence there exists a unique $c\in\Phi(m)$ with $(k_1-k_1')c\equiv 1\ (\bmod\ m)$. Hence $(k_1-k_1')c\equiv 1\ (\bmod\ q)$ as $q\mid m$. Since $k_1-k_1'\equiv k\ (\bmod\ q)$, we have $kc\equiv 1\ (\bmod\ q)$. Then

$$\eta=\begin{cases}\tau_{\sigma^{-1}}\tau_{w^{-1}}\psi_c^{-1}\in G_1 & \text{if } \gcd(p,n)=1\\ \tau_{\sigma^{-1}}\tau_{(w')^{-1}}\iota^{k_1+p_1'-p_2'}\psi_c^{-1}\in G_2 & \text{if } \gcd(p,n)>1\end{cases}$$

So our conclusion follows. □

Remark 5.4. Suppose that $(m,p,n)\neq(3,3,3),(4,2,2)$. We have

$$|\mathrm{Int}(m,p,n)|=|G(m,p,n)|/|Z(m,p,n)|=n!\,m^{n-1}/\gcd(n,p)$$

and

$$|\mathrm{Int}(m,1,n)_p|=|\mathrm{Int}(m,1,n)|=|G(m,1,n)|/|Z(m,1,n)|=n!\,m^{n-1};$$

the latter follows by Lemma 2.10, where $Z(m,p,n)$ is the centre of $G(m,p,n)$. Hence $\gcd(p,n)=1$ if and only if $\mathrm{Int}(m,p,n)=\mathrm{Int}(m,1,n)_p$. Let G_1, G_2 be given in Theorem 5.3 (respectively, Theorem 4.3). Then we see that $\gcd(p,n)=1$ if and only if $G_1=G_2$ if and only if $\iota\in G_1$.

§6. Some properties of $\mathrm{Aut}(m,p,n)$.

In this section, we shall study some properties of $\mathrm{Aut}(m,p,n)$. Theorem 6.1 summarizes the main results in Sections 3-5. Proposition 6.2 provides the order of $\mathrm{Aut}(m,p,n)$. In 6.4-6.6, we study the centre $Z(\mathrm{Aut}(m,p,n))$ of $\mathrm{Aut}(m,p,n)$ with $(m,p,n)\notin\{(3,3,3),(4,2,2)\}$. Then we study $\mathrm{Aut}(3,3,3)$ and $\mathrm{Aut}(4,2,2)$ in 6.7 and 6.8 respectively. Finally, the order of $Z(\mathrm{Aut}(m,p,n))$ is summarized in Corollary 6.9.

Theorem 6.1.

(1) $\mathrm{Aut}(m,p,n) = \mathrm{Int}(m,1,n)_p \rtimes \Psi(m)$ *if* $(m,p,n) \notin \{(3,3,3),(4,2,2)\}$. *In particular,* $\mathrm{Aut}(m,p,n) = \mathrm{Int}(m,p,n) \rtimes \Psi(m)$ *if* $\gcd(p,n) = 1$*;*

(2) $\mathrm{Aut}(3,3,3) = \langle \tau_{s_1}, \mu, \psi_2 \cdot \iota \rangle$*;*

(3) $\mathrm{Aut}(4,2,2) = \langle \iota, \nu \rangle$.

Proposition 6.2. $\mathrm{Aut}(m,p,n)$ *has order* $m^{n-1} \cdot n! \cdot \phi(m)$ *if* $(m,p,n) \notin \{(3,3,3),(4,2,2)\}$, 432 *if* $(m,p,n) = (3,3,3)$*, and* 48 *if* $(m,p,n) = (4,2,2)$.

Proof. The result in the case of $(m,p,n) \in \{(3,3,3),(4,2,2)\}$ can be checked by GAP (see [3]). Now assume $(m,p,n) \notin \{(3,3,3),(4,2,2)\}$. By Remark 5.4, we have $|\mathrm{Int}(m,1,n)_p| = n!\, m^{n-1}$. This implies the result by Theorem 6.1 (1). □

Remark 6.3. Recall the description of $\mathrm{Aut}(m,p,n)$ when $G(m,p,n)$ is a Coxeter group (see Introduction). We see that Theorem 6.1 and Proposition 6.2 also hold when $G(m,p,n)$ is a Coxeter group with $(m,p,n) \notin \{(2,2,4),(2,1,2),(1,1,2)\}$, where each of $G(2,2,4)$, $G(2,1,2)$ has more than one system of imprimitivity (see 1.2). We see that

$$\mathrm{Int}(2,1,4)_2 \rtimes \Psi(2) = \mathrm{Int}(2,1,4)_2$$

is a subgroup of $\mathrm{Aut}(2,2,4)$ of index 3 and hence $|\mathrm{Aut}(2,2,4)| = 576$. We also see that

$$\mathrm{Int}(2,1,2) \rtimes \Psi(2) = \mathrm{Int}(2,1,2)$$

is a subgroup of $\mathrm{Aut}(2,1,2)$ of index 2 and hence $|\mathrm{Aut}(2,1,2)| = 8$. Also, we have

$$\mathrm{Aut}(1,1,2) = \mathrm{Int}(1,1,2) = 1.$$

In 6.4-6.6, we assume $(m,p,n) \notin \{(3,3,3),(4,2,2)\}$.

Proposition 6.4. *$Z(\mathrm{Aut}(m,p,n))$ is trivial when $n>2$.*

Proof. Take $\eta\in Z(\mathrm{Aut}(m,p,n))$. Then $\eta\cdot\tau_g=\tau_g\cdot\eta$ for any $g\in S_0$ (see 1.3 for S_0). This implies that $\tau_{\eta(g)\cdot g^{-1}}$ is the identity automorphism of $G(m,p,n)$. Hence $\eta(g)\cdot g^{-1}\in Z(m,p,n)$, that is,

$$\eta(g)=[a_g,...,a_g|1]\cdot g \quad \text{for some } a_g\in[0,m-1] \text{ with } na_g\equiv 0 \pmod p. \tag{6.4.1}$$

By Lemma 2.3, we see that both g and $\eta(g)$ are reflections of $G(m,p,n)$ with the same type and order and hence the same eigenvalue multi-set. By the assumption $n>2$ and by comparing with the eigenvalue multi-sets on both sides of (6.4.1), we get $a_g\equiv 0 \pmod m$ for any $g\in S_0$. So $\eta=1$. The result follows. □

Next we consider $Z(\mathrm{Aut}(m,p,2))$. We deal with the cases of p being odd and even separately.

Proposition 6.5. *Assume p odd. Then we have*

$$Z(\mathrm{Aut}(m,p,2))=\begin{cases}\{1,\tau_{[0,m/2|1]},\tau_{s_1}\cdot\psi_{m-1},\tau_{[0,m/2|(12)]}\cdot\psi_{m-1}\}, & \textit{if } m \textit{ is even.}\\ \{1,\tau_{s_1}\cdot\psi_{m-1}\} & \textit{if } m \textit{ is odd.}\end{cases}$$

Proof. Since p is assumed odd, we have

$$\mathrm{Aut}(m,p,2)=\mathrm{Int}(m,p,2)\rtimes\Psi(m)$$

by Theorem 6.1 (1). Take $\eta\in Z(\mathrm{Aut}(m,p,2))$. Then η has a unique expression of the form $\tau_g\cdot\psi_c$ for some $g\in G(m,p,2)$ and some $c\in\Phi(m)$.

We have $\eta\cdot\tau_s=\tau_s\cdot\eta$ for any $s\in S_0$. This implies that $\tau_{g^{-1}s^{-1}g\psi_c(s)}=1$, that is, $g^{-1}s^{-1}g\psi_c(s)\in Z(G(m,p,2))$. Write $g=[a,b|\sigma]$ for some $\sigma\in\mathcal{S}_2$ and some $a,b\in[0,m-1]$ with $p\mid(a+b)$. Then by a direct computation with s ranging over S_0, we get that modulo m,

(i) $2a \equiv 2b$;

(ii) $2c \equiv 2$ and $pc \equiv p$ if $\sigma = 1$;

(iii) $2c \equiv -2$ and $pc \equiv -p$ if $\sigma = (12)$.

By (i), we have $a \equiv b \pmod{m}$ if m is odd, and $a \equiv b \pmod{m/2}$ if m is even. Let H be the set $\{[0,0|\sigma] \mid \sigma \in \mathcal{S}_2\}$ if m is odd, and $\{[0,0|\sigma],[0,m/2|\sigma] \mid \sigma \in \mathcal{S}_2\}$ if m is even. Then the condition (i) implies that $\tau_g \in \{\tau_s \mid s \in H\}$. By (ii)–(iii), we get $c \equiv 1 \pmod{m}$ if $\sigma = 1$, and $c \equiv -1 \pmod{m}$ if $\sigma = (12)$ by the assumption of p being odd.

So far we have proved that $Z(\mathrm{Aut}(m,p,2))$ is contained in

$$H_o := \{1, \tau_{s_1} \cdot \psi_{m-1}\}$$

if m is odd, and in

$$H_e := \{1, \tau_{s_1} \cdot \psi_{m-1}, \tau_{[0,m/2|1]}, \tau_{[0,m/2|(12)]} \cdot \psi_{m-1}\}$$

if m is even. Since H_o for m odd (or H_e for m even) is obviously in $Z(\mathrm{Aut}(m,p,2))$, our result follows. □

Proposition 6.6. *Assume p even. Then we have*

$$Z(\mathrm{Aut}(m,p,2)) = \left\{1,\ \tau^{(p)}_{[0,m/2|1]},\ \tau_{s_1}\cdot\psi_{m-1},\ \tau^{(p)}_{[0,m/2|(12)]}\cdot\psi_{m-1}\right\}.$$

Proof. Since p is assumed even, we have

$$\mathrm{Aut}(m,p,2) = \mathrm{Int}(m,1,2)_p \rtimes \Psi(m)$$

by Theorem 6.1 (2). Take $\eta \in Z(\mathrm{Aut}(m,p,2))$. Then η has a unique expression of the form $\tau^{(p)}_g \cdot \psi_c$ for some $g \in G(m,1,2)$ and some $c \in \Phi(m)$.

We have $\eta \cdot \tau^{(p)}_s = \tau^{(p)}_s \cdot \eta$ for any $s \in S_0$, where S_0 is the generating set of $G(m,1,n)$ as in 1.3. This implies that

$$\tau^{(p)}_{g^{-1}s^{-1}g\psi_c(s)} = 1, \quad \text{that is,} \quad g^{-1}s^{-1}g\psi_c(s) \in Z_{G(m,1,2)}(G(m,p,2)).$$

Write $g = [a, b|\sigma]$ for some $\sigma \in \mathcal{S}_2$ and some $a, b \in [0, m-1]$. Then by a direct computation with s ranging over S_0, we get that modulo m,

(i) $2a \equiv 2b$;

(ii) $c \equiv 1$ if $\sigma = 1$;

(iii) $c \equiv -1$ if $\sigma = (12)$.

By (i), we have $a \equiv b \pmod{m/2}$ since m is even. So $Z(\mathrm{Aut}(m, p, 2))$ is contained in

$$H := \{1, \tau_{s_1} \cdot \psi_{m-1}, \tau^{(p)}_{[0,m/2|1]}, \tau^{(p)}_{[0,m/2|(12)]} \cdot \psi_{m-1}\}.$$

Since H is obviously in $Z(\mathrm{Aut}(m, p, 2))$, our result follows. □

Finally we consider $\mathrm{Aut}(3, 3, 3)$ and $\mathrm{Aut}(4, 2, 2)$.

6.7. Recall the element $\mu \in \mathrm{Aut}(3, 3, 3)$ defined in 4.1. By Theorem 4.2, we have $\mathrm{Aut}(3, 3, 3) = \langle \tau_{s_1}, \psi_2 \cdot \iota, \mu \rangle$ with $o(\tau_{s_1}) = o(\psi_2 \cdot \iota) = 2$ and $o(\mu) = 3$. The group $\langle \tau_{s_1}, \psi_2 \cdot \iota \rangle$ is isomorphic to the dihedral group of order 12; the elements τ_{s_1} and μ commute; while $\langle \psi_2 \cdot \iota, \mu \rangle$ has order 48, which can be presented as

$$\langle \psi_2 \cdot \iota, \mu \mid (\psi_2 \cdot \iota)^2 = \mu^3 = ((\psi_2 \cdot \iota)\mu(\psi_2 \cdot \iota)\mu(\psi_2 \cdot \iota)\mu^{-1})^2 = 1 \rangle.$$

The centre of $\mathrm{Aut}(3, 3, 3)$ is trivial.

6.8. By Theorem 5.2, we have $\mathrm{Aut}(4, 2, 2) = \langle \iota, \nu \rangle$ with ν defined in 5.1, which can be presented as

$$\mathrm{Aut}(4, 2, 2) = \langle \iota, \nu \mid \iota^4 = \nu^2 = (\iota \cdot \nu)^6 = 1, (\iota \cdot \nu)^3 = (\nu \cdot \iota)^3 \rangle.$$

The centre of $\mathrm{Aut}(4, 2, 2)$ is a cycle group of order 2 and is generated by $(\iota \cdot \nu)^3$.

From 6.4-6.8, we get the following result immediately:

Corollary 6.9. *The cardinality of the group* $Z(\mathrm{Aut}(m,p,n))$ *is 1 if* $n > 2$ *and* $2 \cdot \gcd(m,2)/(1+\delta_{4,m}\delta_{2,p})$ *if* $n = 2$*, where* δ_{xy} *is 1 if* $x = y$ *and 0 otherwise.*

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