THE CELLS OF THE AFFINE WEYL GROUP $\widetilde{C}_n$ IN A CERTAIN QUASI-SPLIT CASE, II

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Abstract. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_{2n-1}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$. Let $\widetilde{\ell}$ be the length function of $\widetilde{A}_{2n-1}$. The main results of the paper are to prove the left-connectedness of any left cell of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$ in the set E_λ for any nice $\lambda \in \Lambda_{2n}$, to prove all the partitions $(2n-k, k)$ with $1 \leqslant k \leqslant n$ being nice and to describe all the cells of $(\widetilde{C}_n, \widetilde{\ell})$ in the set $E_{(2n-k,k)}$.

§0. Introduction.

0.1. This is a continuation for the study of Kazhdan-Lusztig cells in the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$ in my previous paper [10].

Let $\mathbb{Z}$ (respectively, $\mathbb{N}$, $\mathbb{P}$) be the set of all integers (respectively, non-negative integers, positive integers). For any $i \leqslant j$ in $\mathbb{Z}$, denote by $[i, j]$ the set $\{i, i+1, ..., j\}$. Denote $[1, j]$, $[0, j]$ simply by $[j]$, $(j]$ respectively. Let W be a Coxeter group with S the Coxeter generator set. Lusztig defined a *weighted function* L on W, called (W, L) a *weighted Coxeter group* (see 1.1) and extended the concepts of left, right and two-sided cells from an ordinary Coxeter group to a weighted Coxeter group (see [3], [7]). Each cell of (W, L) provides a representation of (W, L) and the associated Hecke algebra. It is a big project for the explicit description of cells in any weighted Coxeter group.

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0.2. For any $n > 1$, consider the affine Weyl group $\widetilde{A}_{2n-1}$ with the Coxeter generator set $\widetilde{S} = \{s_i \mid i \in (2n-1]\}$, where $s_i^2 = 1$, $s_i s_j = s_j s_i$ if $j \not\equiv i \pm 1 \,(\mathrm{mod}\ 2n)$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for any $i, j \in (2n-1]$ (we stipulate $s_{2n} = s_0$). Let $\widetilde{\ell}_{2n-1}$ be the length function of $(\widetilde{A}_{2n-1}, \widetilde{S})$.

Let α be the group automorphism of $\widetilde{A}_{2n-1}$ determined by setting $\alpha(s_i) = s_{2n-i}$ for $i \in (2n-1]$. Then the affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n-1}$ under α with the Coxeter generator set $S = \{t_i \mid i \in (n]\}$, where $t_i = s_i s_{2n-i}$ for any $i \in [n-1]$, $t_0 = s_0$ and $t_n = s_n$. The restriction to $\widetilde{C}_n$ of $\widetilde{\ell}_{2n-1}$ is a weighted function on $(\widetilde{C}_n, S)$. Hence $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ forms a weighted Coxeter group.

It is known that there is a surjective map ψ from $\widetilde{A}_{2n-1}$ to the set Λ_{2n} of partitions of $2n$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_{2n-1}$ to Λ_{2n} (see [6, Theorem 6] and [8, Theorem 17.4 and Proposition 5.15]). Let $E_\lambda := \psi^{-1}(\lambda) \cap \widetilde{C}_n$ for $\lambda \in \Lambda_{2n}$.

0.3. In our previous paper [10], we described all the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ in the sets $E_{\mathbf{k1^{2n-k}}}$ and $E_{\mathbf{h21^{2n-h-2}}}$ for all $k \in [2n]$ and $h \in [2, 2n-2]$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$. In the present paper, we define two kinds of partitions in Λ_{2n}, called a dual-symmetrizable partition and a nice partition respectively (see 3.13). We prove that any nice partition must be dual-symmetrizable (see Lemma 3.5) and conjecture that the converse should also be true (see Conjecture 3.14). We prove that any left cell of $\widetilde{C}_n$ in E_λ is left-connected if $\lambda \in \Lambda_{2n}$ is nice (Theorem 3.15). We give some detailed investigation on the set $E_{(2n-k,k)}$ for any $k \in [n]$. We prove that all the partitions $(2n-k, k)$, $k \in [n]$, are nice (see Theorem 4.12), that the set $E_{(2n-k,k)}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$ (see Theorem 4.13), and that the number of left cells contained in $E_{(2n-k,k)}$ is $2^{n-m}n!$ if $k = 2m$ is even and $2^{n-m-1}n!$ if $k = 2m+1$ is odd (see Theorem 4.12).

0.4. The most difficulty part in proving our results is to show the left-connectedness of a left cell in E_λ for our considered partition λ. The set Ω (see 3.2) plays a crucial role in our proof. Each $w \in \Omega$ determines a tabloid $T(w)$. Any $w \in \widetilde{C}_n \cap \Omega$ determines a $2n$-self-dual tabloid (see Lemma 3.5). Fix a left cell Γ of $\widetilde{C}_n$. We first prove that the set $\Gamma \cap \Omega$ is contained in some left-connected component of $E_{\psi(\Gamma)}$ (see Theorem 3.12). This implies that any left cell of E_λ is left-connected for any nice $\lambda \in \Lambda_{2n}$

(see Theorem 3.15). We prove by a step-by-step reduction in Section 4 that any partition of the form $(2n-k,k)$, $k\in[n]$, is nice (see Lemma 4.11). This proves the left-connectedness for any left cell of $\widetilde{C}_n$ in $E_{(2n-k,k)}$. Then the number of left cells of $\widetilde{C}_n$ contained in $E_{(2n-k,k)}$ can be obtained simply by counting the number of all the $2n$-self-dual tabloids corresponding to a fixed symmetric composition $\mathbf{a}$ with $\zeta(\mathbf{a})^\vee=(2n-k,k)$ (see Theorem 4.12). We conclude that $E_{(2n-k,k)}$ forms a single two-sided cell of $\widetilde{C}_n$ by showing the two-sided-connectedness of $E_{(2n-k,k)}$ (see Theorem 4.13).

It is shown that the composition $\mathbf{a}:=\xi(T(w))$ is symmetric for any $w\in\Omega\cap\widetilde{C}_n$. Conjecture 3.14 states that if $\lambda\in\Lambda_{2n}$ is such that there is some symmetric composition $\mathbf{a}$ of $2n$ satisfying $\lambda=\zeta(\mathbf{a})^\vee$, then λ is nice. We expect that our arguments in Section 4 could be extended to verify this conjecture.

We would like to mention that the successive star-operations applied in Sections 3-4 (e.g., the elements w', y so obtained from w in (3.11.4) and 4.4 respectively) are essentially the iterated star operations defined in [8, Chapter 8]. This is a generalization of Robinson-Schensted inserting algorithm on the symmetric group (see [8, Section 21.2]) and is one of powerful tools in getting our results.

0.5. The contents of the paper are organized as follows. In Section 1, we collect some concepts and known results concerning cells of a weighted Coxeter group. Then we concentrate ourselves to the weighted Coxeter group $(\widetilde{C}_n,\widetilde{\ell}_{2n-1})$ in Section 2, many useful results and technical tools are provided there. In Section 3, we study the properties for the set $\Omega\cap\widetilde{C}_n$ and prove that any left cell of $\widetilde{C}_n$ in E_λ is left-connected if $\lambda\in\Lambda_{2n}$ is nice. Finally, we study all the cells of $\widetilde{C}_n$ in $E_{(2n-k,k)}$ for any $k\in[n]$ in Section 4.

§1. Cells in Coxeter groups.

In this section, we collect some concepts and results concerning cells of a weighted Coxeter group, all but Lemma 1.5 follow Lusztig in [7], while Lemma 1.5 is a result in [10].

1.1. Let (W,S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w=s_1s_2\cdots s_r\in W$ with $s_i\in S$ is called *reduced* if $r=\ell(w)$. By a *weight function* on W, we mean a map $L:W\longrightarrow\mathbb{Z}$ satisfying that $L(s)=L(t)$ for any $s,t\in S$ conjugate in W and that $L(w)=L(s_1)+$

$L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ in W. Call (W, L) is a *weighted Coxeter group*.

We say that a weighted Coxeter group (W, L) is in the *split* case if $L = \ell$.

Suppose that there exists a group automorphism α of W with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J on S, let w_J be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function ℓ is a weight function on W^α. We say that the weighted Coxeter group (W^α, ℓ) is in the *quasi-split* case.

1.2. Let $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on a weighted Coxeter group (W, L) defined in [7]. The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (respectively, $\underset{R}{\sim}$, $\underset{LR}{\sim}$). The corresponding equivalence classes in W are called *left cells* (respectively, *right cells, two-sided cells*) of W.

1.3. For $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) = \mathcal{L}(w)$) (see [7, Lemma 8.6]).

1.4. In [7, Chapter 13], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ in terms of structural coefficients of the Hecke algebra associated to (W, L).

In [7, Chapters 14-16], Lusztig proved the following results when W is either a finite or an affine Coxeter group and when (W, L) is either in the split case or in the quasi-split case.

(1) $y \underset{LR}{\leqslant} w$ in W implies $a(w) \leqslant a(y)$. Hence $y \underset{LR}{\sim} w$ in W implies $a(w) = a(y)$.

(2) If $w, y \in W$ satisfy $a(w) = a(y)$ and $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$, $y \underset{LR}{\leqslant} w$) then $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$, $y \underset{LR}{\sim} w$).

For any $X \subset W$, denote $X^{-1} := \{x^{-1} \mid x \in X\}$.

Lemma 1.5. (see [10, Lemma 1.7]) *Suppose that W is either a finite or an affine Coxeter group and that (W, L) is either in the split case or in the quasi-split case.*

Let E be a non-empty subset of W satisfying the following conditions:

(a) There exists some $k \in \mathbb{N}$ with $a(x) = k$ for any $x \in E$;

(b) E is a union of some left cells of W;

(c) $E^{-1} = E$.

Then E is a union of some two-sided cells of W.

§2. The affine Weyl groups $\widetilde{A}_{2n-1}$ and $\widetilde{C}_n$.

From now on, we restrict our attention to the weighted Coxeter groups $(\widetilde{A}_{2n-1}, \widetilde{\ell})$ and $(\widetilde{C}_n, \widetilde{\ell})$, where $\widetilde{\ell} = \widetilde{\ell}_{2n-1}$ is the length function of the affine Weyl group $\widetilde{A}_{2n-1}$.

2.1. The affine Weyl group $\widetilde{A}_{2n-1}$ can be realized as the following permutation group on the set $\mathbb{Z}$ (see [5, Subsection 3.6] and [8, Subsection 4.1)]:

$$\widetilde{A}_{2n-1} = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \;\middle|\; (i+2n)w = (i)w + 2n, \sum_{i=1}^{2n} (i)w = \sum_{i=1}^{2n} i \right\}.$$

The Coxeter generator set $\widetilde{S} = \{s_i \mid i \in (2n-1]\}$ of $\widetilde{A}_{2n-1}$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{2n}, \\ t+1, & \text{if } t \equiv i \pmod{2n}, \\ t-1, & \text{if } t \equiv i+1 \pmod{2n}, \end{cases}$$

for any $t \in \mathbb{Z}$ and $i \in (2n-1]$. Any $w \in \widetilde{A}_{2n-1}$ can be realized as a $\mathbb{Z}$-indexed monomial matrix $A_w = (a_{ij})_{i,j \in \mathbb{Z}}$, where a_{ij} is 1 if $j = (i)w$ and 0 if otherwise. The row (respectively, column) indices of A_w increase from top to bottom (respectively, from left to right). We can conveniently use some familiar operations in linear algebra on the matrix A_w. For example, the matrix $A_{w^{-1}}$ is just the transposition of A_w; while $A_{s_i w}$ (respectively, A_{ws_i}) can be obtained from A_w by transposing the $(2nq+i)$th and the $(2nq+i+1)$th rows (respectively, columns) for all $q \in \mathbb{Z}$.

Let α be the group automorphism of $\widetilde{A}_{2n-1}$ determined by $\alpha(s_i) = s_{2n-i}$ for $i \in (2n-1]$. In terms of matrix form, for any $w \in \widetilde{A}_{2n-1}$, the matrix $A_{\alpha(w)}$ can be obtained from the matrix A_w by rotating with the angle π around the point $(qn + \frac{1}{2}, qn + \frac{1}{2})$ for any $q \in \mathbb{Z}$, where we identify A_w with a plane and the positions (i, j), $i, j \in \mathbb{Z}$, of A_w are identified with the corresponding integer lattice points.

The automorphism α gives rise to a permutation on the set Π^l (respectively, Π^r, Π^t) of left cells (respectively, right cells, two-sided cells) of $\widetilde{A}_{2n-1}$.

The affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n-1}$ under α, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$\widetilde{C}_n = \{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+2n)w = (i)w + 2n, (i)w + (1-i)w = 1, \ \forall\, i \in \mathbb{Z} \}$$

with the Coxeter generator set $S=\{t_i \mid i\in(n]\}$, where $t_i=s_is_{2n-i}$ for $i\in[n-1]$, $t_0=s_0$ and $t_n=s_n$. For the sake of convenience, we define s_i and t_j for any $i,j\in\mathbb{Z}$ by setting s_{2qn+b} to be s_b and $t_{2pn\pm a}$ to be t_a for any $p,q\in\mathbb{Z}$ and $b\in(2n-1]$ and $a\in(n]$. In terms of matrix, an element $w\in\widetilde{A}_{2n-1}$ is in $\widetilde{C}_n$ if and only if the matrix form A_w of w is centrally symmetric at the point $(qn+\frac{1}{2},qn+\frac{1}{2})$ for any $q\in\mathbb{Z}$.

Let $\widetilde{\ell}$, ℓ be the length functions on the Coxeter systems $(\widetilde{A}_{2n-1},\widetilde{S})$, $(\widetilde{C}_n,S)$, respectively. For any $x\in\widetilde{A}_{2n-1}$ and $k\in\mathbb{Z}$, let $m_k(x)={}^{\#}\{i\in\mathbb{Z}\mid i<k \text{ and } (i)x>(k)x\}$. Then the formulae for the functions $\widetilde{\ell}$ and ℓ are as follows.

Lemma 2.2. (see [10, Proposition 2.4]) *For any $w\in\widetilde{A}_{2n-1}$ and $x\in\widetilde{C}_n$, we have*

(1) $\widetilde{\ell}(w)=\sum_{1\leqslant i<j\leqslant 2n}\left|\left\lfloor\frac{(j)w-(i)w}{2n}\right\rfloor\right|=\sum_{k=1}^{2n}m_k(w)$;

(2) $\ell(x)=\frac{1}{2}(\widetilde{\ell}(x)+m_1(x)+m_{n+1}(x))$,

where $\lfloor a\rfloor$ is the largest integer not larger than a, and $|a|$ is the absolute value of a for any $a\in\mathbb{Q}$.

2.3. Fix $m\in\mathbb{P}$. By a partition of m, we mean an r-tuple $\lambda:=(\lambda_1,\lambda_2,...,\lambda_r)$ of weakly decreasing positive integers $\lambda_1\geqslant\cdots\geqslant\lambda_r$ with $\sum_{k=1}^r\lambda_k=m$ for some $r\in\mathbb{P}$. λ_i is called a *part* of λ. Let Λ_m be the set of all partitions of m.

For any $\lambda=(\lambda_1,...,\lambda_r)\in\Lambda_m$, define $\lambda^\vee=(\mu_1,...,\mu_{\lambda_1})$ by setting $\mu_j={}^{\#}\{k\in[r]\mid\lambda_k\geqslant j\}$ for any $j\in[\lambda_1]$, call $\lambda^\vee$ the *dual partition* of λ.

For any $\lambda=(\lambda_1,\lambda_2,...,\lambda_r)$ and $\mu=(\mu_1,\mu_2,...,\mu_t)$ in Λ_m, we write $\lambda\leqslant\mu$ if $\lambda_1+\cdots+\lambda_k\leqslant\mu_1+\cdots+\mu_k$ for any $1\leqslant k\leqslant\min\{r,t\}$. This defines a partial order on Λ_m. In particular, $\lambda\leqslant\mu$ if and only if $\mu^\vee\leqslant\lambda^\vee$.

2.4. Let $P=(E,\preceq)$ be a partial ordered set (or a poset in short) with the cardinal $|E|$ of the set E being $m\in\mathbb{P}$. By a chain (respectively, antichain) in P, we mean a sequence $a_1,a_2,...,a_r$ in E satisfying $a_1\prec a_2\prec\cdots\prec a_r$ (respectively, neither $a_i\prec a_j$ nor $a_j\prec a_i$ holds for any $i\neq j$ in $[r]$). We usually identify a chain (respectively, antichain) $a_1,a_2,...,a_r$ in E with the corresponding subset $\{a_1,a_2,...,a_r\}$. Fix $k\in[m]$. By a k-chain-family of P, we mean a subset $X=\bigcup_{i=1}^k X_i$ of E with X_i a chain for any $i\in[k]$. Let $d_k(P)$ be the maximally possible cardinal of a k-chain-family in P. Then there is some $t\in[m]$ with $d_1(P)<d_2(P)<\cdots<d_t(P)=m$. Let $\lambda_1(P)=d_1(P)$ and $\lambda_k(P)=d_k(P)-d_{k-1}(P)$ for any $k\in[2,t]$. Then $\psi(P):=(\lambda_1(P),\lambda_2(P),...,\lambda_t(P))\in\Lambda_m$ by a result of C. Greene in [2].

Fix $w \in \widetilde{A}_{2n-1}$. For any $i \neq j$ in $[2n]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both inequalities $2pn + i > 2qn + j$ and $(2pn + i)w < (2qn + j)w$ hold. In the matrix form of w, this means that the position $(2qn+j, (2qn+j)w)$ is located at the northeastern of the position $(2pn+i, (2pn+i)w)$. This determines a poset $P_w := ([2n], \preceq_w)$. A chain (respectively, an antichain) in P_w is called a *w-chain* (respectively, a *w-antichain*). We have that $\psi(w) := (\lambda_1(P_w), \lambda_2(P_w), ..., \lambda_r(P_w)) \in \Lambda_{2n}$ and that $w \mapsto \psi(w)$ is a surjective map from the set $\widetilde{A}_{2n-1}$ to Λ_{2n} by [8, Corollary 5.13]. $i \neq j$ in $[2n]$ are called *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* if otherwise. It is easily seen that $i < j$ in $[2n]$ are w-uncomparable if and only if $(i)w < (j)w < (i)w + 2n$.

For any $a \in \mathbb{Z}$, denote by $\langle a \rangle$ the unique integer in $[2n]$ satisfying $a \equiv \langle a \rangle \pmod{2n}$. In the subsequent discussion, we sometimes use the notation $i \prec_w j$, the phrase of i, j being " w-comparable " or " w-uncomparable " for some $i, j \in \mathbb{Z}$, which just mean that $\langle i \rangle$ and $\langle j \rangle$ satisfy the corresponding relation.

2.5. Let $\widetilde{\ell}$, ℓ be the length functions on the Coxeter systems $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_{2n-1}, \widetilde{\ell})$ is in the split case, while $(\widetilde{C}_n, \widetilde{\ell})$ is in the quasi-split case (see [7, Lemma 16.2]).

Let $\leqslant$, $\leqslant_C$ be the Bruhat-Chevalley orders on the Coxeter systems $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. Since the condition $x \leqslant_C y$ is equivalent to $x \leqslant y$ for any $x, y \in \widetilde{C}_n$, it will cause no confusion if we use the notation $\leqslant$ in the place of $\leqslant_C$. Hence from now on we shall use $\leqslant$ for both $\leqslant$ and $\leqslant_C$.

Let $\widetilde{\mathcal{L}}(x) = \{s \in \widetilde{S} \mid sx < x\}$ and $\widetilde{\mathcal{R}}(x) = \{s \in \widetilde{S} \mid xs < x\}$ for $x \in \widetilde{A}_{2n-1}$ and let $\mathcal{L}(y) = \{t \in S \mid ty < y\}$ and $\mathcal{R}(y) = \{t \in S \mid yt < y\}$ for $y \in \widetilde{C}_n$.

Lemma 2.6. (see [10, Corollary 2.6]) *For any $x \in \widetilde{C}_n$ and $i \in (n]$,*

$$\begin{aligned} s_i \in \widetilde{\mathcal{L}}(x) \iff & \quad s_{2n-i} \in \widetilde{\mathcal{L}}(x) && \iff t_i \in \mathcal{L}(x) \\ \iff & \quad (i)x > (i+1)x && \iff (2n+1-i)x < (2n-i)x, \\ s_i \in \widetilde{\mathcal{R}}(x) \iff & \quad s_{2n-i} \in \widetilde{\mathcal{R}}(x) && \iff t_i \in \mathcal{R}(x) \\ \iff & \quad (i)x^{-1} > (i+1)x^{-1} && \iff (2n+1-i)x^{-1} < (2n-i)x^{-1} \end{aligned}$$

If $x \in \widetilde{A}_{2n-1}$ and $s \in \widetilde{\mathcal{L}}(x)$ and $t \in \widetilde{\mathcal{R}}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$ by [8, Lemma 5.5 and Corollary 5.6]. This implies by Lemma 2.6 that if $x \in \widetilde{C}_n$ and $s \in \mathcal{L}(x)$ and $t \in \mathcal{R}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$.

Lemma 2.7. *Let $x, y \in \widetilde{C}_n$ and $x', y' \in \widetilde{A}_{2n-1}$.*

(1) $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{A}_{2n-1}$ (see [7, Lemma 16.14]).

(2) $x' \underset{LR}{\leqslant} y'$ if and only if $\psi(y') \leqslant \psi(x')$. The set $\psi^{-1}(\lambda)$ forms a two-sided cell of $\widetilde{A}_{2n-1}$ for any $\lambda \in \Lambda_{2n}$ (see [6, Theorem 6] and [8, Theorem 17.4] and [9, Theorem B]).

By Lemma 2.7 (1), we can just use the notation $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to $\widetilde{A}_{2n-1}$ or $\widetilde{C}_n$.

2.8. A non-empty subset E of an affine Weyl group $W = (W, S)$ is called *left-connected*, (respectively, *right-connected*) if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that $x_{i-1}x_i^{-1} \in S$ (respectively, $x_i^{-1}x_{i-1} \in S$) for every $i \in [r]$. E is called *two-sided-connected* if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that either $x_{i-1}x_i^{-1}$ or $x_i^{-1}x_{i-1}$ is in S for every $i \in [r]$.

Geometrically, the elements of an affine Weyl group W can be identified with the alcoves of a certain euclidean space V (see [4]). Thus a left-connected set of W is just such an alcove set E in V that for any $A, A' \in E$, there is a sequence $A_0 = A, A_1, ..., A_r = A'$ in E, where A_{i-1} and A_i share a common facet of codimension 1 in V for any $i \in [r]$.

Let $F \subseteq E$ in W. Call F a *left-connected component* of E, if F is a maximal left-connected subset of E. One can define a right-connected component and a two-sided-connected component of E similarly.

For any $\lambda \in \Lambda_{2n}$, denote $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

Lemma 2.9. (see [10, Lemma 2.18]) *Let $\lambda \in \Lambda_{2n}$.*

(1) Any left- (respectively, right-, two-sided-) connected component of $\psi^{-1}(\lambda)$ is contained in some left (respectively, right, two-sided) cell of $\widetilde{A}_{2n-1}$.

(2) Any left- (respectively, right-, two-sided-) connected component of E_λ is contained in some left (respectively, right, two-sided) cell of $\widetilde{C}_n$.

(3) The set E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$.

Corollary 2.10. (see [10, Corollary 2.19]) *Let $x, y, x', y' \in \widetilde{A}_{2n-1}$ satisfy $x, y \in E_\lambda$ and $x', y' \in \psi^{-1}(\lambda)$ for some $\lambda \in \Lambda_{2n}$.*

(1) If $\ell(y) = \ell(x) + \ell(yx^{-1})$ *then* x, y *are in the same left-connected component of* E_λ *and hence* $x \underset{L}{\sim} y$.

(2) If $\ell(y) = \ell(x) + \ell(x^{-1}y)$ *then* x, y *are in the same right-connected component of* E_λ *and hence* $x \underset{R}{\sim} y$.

(3) If $\widetilde{\ell}(y') = \widetilde{\ell}(x') + \widetilde{\ell}(y'x'^{-1})$ *then* x', y' *are in the same left-connected component of* $\psi^{-1}(\lambda)$ *and hence* $x' \underset{L}{\sim} y'$.

(4) If $\widetilde{\ell}(y') = \widetilde{\ell}(x') + \widetilde{\ell}(x'^{-1}y')$ *then* x', y' *are in the same right-connected component of* $\psi^{-1}(\lambda)$ *and hence* $x' \underset{R}{\sim} y'$.

2.11 $i, j \in [2n]$ are called $2n$*-dual*, if $i + j = 2n + 1$; in this case, we denote $j = \bar{i}$ (hence $i = \bar{j}$ also). Fix $w \in \widetilde{C}_n$. $i \in [2n]$ is called w*-wild* if i and $\bar{i}$ are w-comparable and w*-tame* if otherwise. $i \in [2n]$ is called *a* w*-wild head* (respectively, *a* w*-tame head*), if i is w-wild (respectively, w-tame) with $(\bar{i})w < (i)w$. In this case, call $\bar{i}$ *a* w*-wild tail* (respectively, *a* w*-tame tail*). In the subsequent discussion, we sometimes say that some $i \in \mathbb{Z}$ is w-wild or w-tame, which just means that the integer $\langle i \rangle$ is such.

The results in Lemmas 2.12-2.13 below can be checked easily:

Lemma 2.12. (see [10, Lemma 3.2]) *Fix* $w \in \widetilde{C}_n$. *Let* $i, j, k \in [2n]$.

(i) $j \prec_w k$ *if and only if* $\bar{k} \prec_w \bar{j}$;

Now suppose that $j \neq k$ *are* w*-wild heads and* i *is* w*-tame.*

(ii) $\bar{j} \prec_w k$ *if and only if* $\bar{j}, k$ *are* w*-comparable.*

(iii) If $\bar{j}, k$ *are* w*-uncomparable then so are* j, k *(respectively,* $\bar{j}, \bar{k}$*);*

(iv) i *and* k *are* w*-comparable if and only if* $i \prec_w k$.

(v) $\{j, i, \bar{j}\}$ *is a* w*-chain if and only if* j *is* w*-comparable with both* i *and* $\bar{i}$*;*

(vi) $\{j, k, \bar{j}, \bar{k}\}$ *is a* w*-chain if and only if* j, k *are* w*-comparable.*

Lemma 2.13. *Let* $w \in \widetilde{C}_n$ *and* $t > 1$.

(1) Let $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_t$ *be a* w*-chain and let* $h \leqslant l$ *in* $[t]$.

(1a) If j_h *is a* w*-wild head, then* j_l *is a* w*-wild head.*

(1b) If j_l *is a* w*-wild tail, then* j_h *is a* w*-wild tail.*

(1c) If both j_h *and* j_l *are* w*-tame, then* j_c *with* $c \in [h, l]$ *either all are* w*-tame heads, or all are* w*-tame tails.*

(2) Let $i_1, i_2, ..., i_{2m} \in [a+1, a+2m]$ *satisfy* $(i_1)w < (i_2)w < \cdots < (i_{2m})w$ *for some* $m \in [n]$ *with* $a \in \{-m, n-m\}$.

(2a) i_h *is either a w-wild tail or a w-tame integer for any* $h \in [m]$.

(2b) i_l *is either a w-wild head or a w-tame integer for any* $l \in [m+1, 2m]$.

(2c) $\langle i_{2m+1-h}\rangle = \overline{\langle i_h\rangle}$ *for any* $h \in [2m]$.

§3. The set Ω.

In the present section, we define a set Ω of certain finite posets, which includes a subset of $\widetilde{A}_{2n-1}$ under a certain identification. We characterize the elements of Ω in $\widetilde{C}_n$ (see Lemma 3.5). The main result of the section is to show that for any $\lambda \in \Lambda_{2n}$ and any left cell Γ of $\widetilde{C}_n$ in the set E_λ, the set $\Gamma \cap \Omega$ is either empty or contained in some left-connected component of E_λ (see Theorem 3.12). This further implies that for any nice partition $\lambda \in \Lambda_{2n}$, any left cell of $\widetilde{C}_n$ in E_λ is left-connected (see 3.13 and Theorem 3.15).

3.1. Fix $m \in \mathbb{P}$. A *generalized tabloid* (or a *tabloid* in short) of rank m is, by definition, an r-tuple $\mathbf{T} = (T_1, T_2, ..., T_r)$ with some $r \in \mathbb{P}$ such that T_j, $j \in [r]$, are pairwise disjoint subsets of $\mathbb{P}$ with $\sum_{i=1}^r |T_i| = m$. By a composition of m, we mean an r-tuple $(a_1, a_2, ..., a_r)$ with some $a_1, ..., a_r, r \in \mathbb{P}$ such that $\sum_{i=1}^r a_i = m$. Let $\widetilde{\Lambda}_m$ be the set of all compositions of m. We have $\xi(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|) \in \widetilde{\Lambda}_m$. Let $\mathcal{C}_m$ be the set of all tabloids of rank m.

For any $\mathbf{a} = (a_1, ..., a_r) \in \widetilde{\Lambda}_m$, let $i_1, i_2, ..., i_r$ be a permutation of $1, 2, ..., r$ such that $a_{i_1} \geqslant a_{i_2} \geqslant \cdots \geqslant a_{i_r}$. Then $\zeta(\mathbf{a}) := (a_{i_1}, a_{i_2}, ..., a_{i_r}) \in \Lambda_m$. Clearly, both $\xi : \mathcal{C}_m \longrightarrow \widetilde{\Lambda}_m$ and $\zeta : \widetilde{\Lambda}_m \longrightarrow \Lambda_m$ are surjective maps.

3.2. Let Ω_m be the set of all posets $P = (E, \preceq)$ with $E \subset \mathbb{P}$ and $|E| = m$ such that there is a set partition $E = E_1 \dot\cup E_2 \dot\cup \cdots \dot\cup E_r$ satisfying:

(i) $a \prec b$ for any $a \in E_i$ and $b \in E_j$ with $i < j$ in $[r]$;

(ii) E_i is a maximal antichain in E for any $i \in [r]$.

Define $T(P) := (E_1, E_2, ..., E_r)$. Then $T(P) \in \mathcal{C}_m$.

Denote $\Omega = \bigcup_{m \in \mathbb{P}} \Omega_m$. By a result of C. Greene in [2], we see that the partition $\zeta\xi(T(P))$ is the dual of $\psi(P)$ for any $P \in \Omega$.

By identifying any $w \in \widetilde{A}_{2n-1}$ with the poset $P_{w^{-1}} := ([2n], \prec_{w^{-1}})$, we can regard w as an element of Ω_{2n} and further of Ω if $P_{w^{-1}} \in \Omega_{2n}$.

In the most cases of the subsequent discussion, when we mention an element w of

Ω, we mean that w is an element in $\widetilde{A}_{2n-1}$, or even in $\widetilde{C}_n$, with $P_{w^{-1}} \in \Omega_{2n}$.

The following known result will be crucial in subsequent discussion.

Lemma 3.3. (see [8, Lemma 19.4.6 and Propositions 19.4.7-19.4.8])

(1) Suppose that $y, w \in \widetilde{A}_{2n-1} \cap \Omega$ *satisfy* $\xi(T(y)) = \xi(T(w))$*. Then* $y \underset{L}{\sim} w$ *if and only if* $T(y) = T(w)$*.*

(2) For any $\mathbf{a} \in \widetilde{\Lambda}_{2n}$*, let* $\lambda = \zeta(\mathbf{a})^\vee$*. Then there exists a bijective map from the set* Π^l_λ *of all left cells of* $\widetilde{A}_{2n-1}$ *in* $\psi^{-1}(\lambda)$ *to the set* $\xi^{-1}(\mathbf{a})$*.*

3.4. Fix $m \in [2n]$. Denote $\mathbf{a}^{\rm op} = (a_r, ..., a_2, a_1)$ for $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_m$. Call $\mathbf{a}$ *symmetric*, if $\mathbf{a}^{\rm op} = \mathbf{a}$.

Denote $\overline{E} = \{\bar{i} \mid i \in E\}$ for any $E \subseteq [2n]$ (see 2.11). Denote $\overline{\mathbf{T}} = (\overline{T_1}, \overline{T_2}, ..., \overline{T_r})$ and $\mathbf{T}^{\rm op} = (T_r, ..., T_2, T_1)$ for any $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_m$. Then $\overline{\mathbf{T}}, \mathbf{T}^{\rm op} \in \mathcal{C}_m$. We say that $\mathbf{T} \in \mathcal{C}_m$ is $2n$*-self-dual*, if $\overline{\mathbf{T}}^{\rm op} = \mathbf{T}$.

If $\mathbf{T} \in \mathcal{C}_m$ is $2n$-self-dual then the composition $\xi(\mathbf{T})$ is symmetric.

Lemma 3.5. *(1) The tabloid* $T(w)$ *is* $2n$*-self-dual for any* $w \in \Omega \cap \widetilde{C}_n$*.*

(2) For any $2n$*-self-dual* $\mathbf{T} \in \mathcal{C}_{2n}$*, there exists some* $w \in \Omega \cap \widetilde{C}_n$ *satisfying* $T(w) = \mathbf{T}$*.*

(3) If $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_{2n}$ *is* $2n$*-self-dual with* $r = 2m+1$ *odd, then* $|T_{m+1}|$ *is even.*

Proof. Let $w \in \widetilde{C}_n$ and $E, E' \subset [2n]$. Denote $E \prec_w E'$, if $a \prec_w a'$ for any $a \in E$ and any $a' \in E'$. We see by Lemma 2.12 that E is a w-antichain if and only if such is $\overline{E}$ and that $E \prec_w E'$ if and only if $\overline{E'} \prec_w \overline{E}$. Fix $w \in \Omega \cap \widetilde{C}_n$. We see that all the w^{-1}-tame integers in $[2n]$ are pairwise w^{-1}-uncomparable and hence form a single w^{-1}-antichain whenever they exist. We also see that the elements in any maximal w^{-1}-antichain of $[2n]$ are either all w^{-1}-wild heads, or all w^{-1}-wild tails, or all w^{-1}-tame integers. This implies by Lemma 2.12 that $T(w)$ is $2n$-self-dual, (1) is proved. Given any $2n$-self-dual $\mathbf{T} \in \mathcal{C}_{2n}$. We want to find some $w \in \Omega \cap \widetilde{C}_n$ with $T(w) = \mathbf{T}$. Write $\mathbf{T} = (T_1, T_2, ..., T_r)$ with $T_i = \{a_{i1}, a_{i2}, ..., a_{in_i}\}$ for any $i \in [r]$, where $n_i = |T_i|$ and $a_{i1} < a_{i2} < \cdots < a_{in_i}$. Then by the assumption of $\mathbf{T}$ being $2n$-self-dual, we have $n_{r+1-i} = n_i$ and $a_{r+1-i,n_i+1-j} = \overline{a_{ij}}$ for any $i \in [r]$ and any $j \in [n_i]$. Assume $r \in \{2m, 2m+1\}$ with some $m \in \mathbb{N}$. Denote $\delta_h := \sum_{i=h}^r n_i$ for $h \in [r+1]$ with the convention that $\delta_{r+1} = 0$. Define $w \in \widetilde{A}_{2n-1}$ by setting, for any $l \in [2n]$,

$$(l)w=\begin{cases} a_{hj}-2n(m+1-h), & \text{if } l=\delta_{h+1}+j \text{ with } h\in[m] \text{ and } j\in[n_h],\\ \overline{a_{h,n_h+1-j}}+2n(m+1-h), & \text{if } l=\delta_{r+2-h}+j \text{ with } h\in[m] \text{ and } j\in[n_h],\\ a_{m+1,j}, & \text{if } r=2m+1 \text{ and } l=\delta_{m+2}+j \text{ with } j\in[n_{m+1}]. \end{cases}$$

It is easy to check that the element w is in the set $\Omega\cap\widetilde{C}_n$ and satisfies $T(w)=\mathbf{T}$. This proves (2). Finally, (3) follows by the fact that $\overline{T_{m+1}}=T_{m+1}$ and $\bar{i}\neq i$ for any $i\in[2n]$. $\square$

By the results in [8, Subsection 19.4], we see that for any $w\in\widetilde{A}_{2n-1}$, there always exists some $y\in\Omega\cap\widetilde{A}_{2n-1}$ satisfying $y\underset{L}{\sim}w$. Comparing with this, for any $w'\in\widetilde{C}_n$, there does not always exist any $y'\in\Omega\cap\widetilde{C}_n$ satisfying $y'\underset{L}{\sim}w'$. For, there might exist no any symmetric $\mathbf{a}\in\widetilde{\Lambda}_{2n}$ satisfying $\zeta(\mathbf{a})^\vee=\psi(w')$.

3.6. For any $w\in\widetilde{C}_n$ and $t_i\in\mathcal{L}(w)$, the relation $\psi(t_iw)\leqslant\psi(w)$ holds in general by Lemmas 2.6 and 2.7. By [8, Lemma 5.8], we have $\psi(t_iw)=\psi(w)$ if one of the following cases occurs:

(a) $i\in[2,n-1]$ and, either $(i)w<(i-1)w<(i+1)w$ or $(i+1)w<(i-1)w<(i)w$;

(b) $i\in[n-2]$ and, either $(i)w<(i+2)w<(i+1)w$ or $(i+1)w<(i+2)w<(i)w$;

(c) $i=0$ and, either $(1)w<(2)w<(0)w$ or $(0)w<(2)w<(1)w$;

(d) $i=n$ and, either $(n)w<(n-1)w<(n+1)w$ or $(n+1)w<(n-1)w<(n)w$;

(e) $|(i)w-(i+1)w|>2n$.

The transformation $w\mapsto t_iw$ is called a *left star operation* on w in any of the cases (a)-(d). $j\mapsto(j)t_i$ is a poset isomorphism from $([2n],\prec_{t_iw})$ to $([2n],\prec_w)$ in the case (e). Hence t_iw and w are in the same left-connected component of $E_{\psi(w)}$ in any of the cases (a)-(e).

Let X and Y be two subsets of $\mathbb{Z}$. We write $X<Y$ (respectively, $X<_w Y$, $X\prec_w Y$) if $i<j$ (respectively, $(i)w<(j)w$, $i\prec_w j$) for any $i\in X$ and any $j\in Y$. In the case of $X<_w Y$, denote $d_w(X,Y):=\min\{(j)w-(i)w\mid i\in X, j\in Y\}$. Now assume $X,Y\subset[a+1,a+2n]$ for some $a\in\mathbb{Z}$. The relation $X\prec_w Y$ implies $X<_w Y$, but the converse is not true in general. However, in either of the cases (i)-(ii) below:

(i) $d_w(X,Y)>2n$;

(ii) $X>Y$.

the relation $X<_w Y$ in $[a+1,a+2n]$ does imply $X\prec_w Y$.

For any $w \in \widetilde{C}_n$, denote by K_w the left-connected component of $E_{\psi(w)}$ containing w.

Lemma 3.7. *Let $w \in E_\lambda$ for some $\lambda \in \Lambda_{2n}$. Assume that $E_1, E_2 \subset [2n]$ satisfy the conditions (1)-(2) below.*

(1) $m := |E_2| > 0$ and $E_1 \dot\cup E_2 = [2n]$ such that all elements of E_2 are w-wild heads.

(2) $E_1 \prec_w E_2$.

If $m < n$, then for any $p \in \mathbb{N}$, there exists some $w_p \in K_w$ satisfying the condition (a) below.

(a) $\langle([m])w_p\rangle = \langle(E_2)w\rangle$ and $[m+1, 2n] \prec_{w_p} [m]$ with $2np < d_{w_p}([m+1, 2n], [m]) < 2n(p+1)$.

If $m = n$, then for any $p \in \mathbb{N}$, there exists some $w_p \in K_w$ satisfying one of the conditions (a′)-(b′) below.

(a′) $\langle([n])w_p\rangle = \langle(E_2)w\rangle$ and $[n+1, 2n] \prec_{w_p} [n]$ with $4np < d_{w_p}([n+1, 2n], [n]) < (4p+2)n$.

(b′) $\langle([n+1, 2n])w_p\rangle = \langle(E_2)w\rangle$ and $[n] \prec_{w_p} [n+1, 2n]$ with $4pn+2n < d_{w_p}([n], [n+1, 2n]) < 4n(p+1)$.

Proof. We have $m \leqslant n$ and $\overline{E_2} \subseteq E_1$ by the assumption (1) on E_2. To show our result, we need only to deal with the case of $p = 0$.

First assume $m < n$. Let $E_1' = E_1 - \overline{E_2}$. Then $|E_1'| = 2(n-m) > 0$ is even and $\overline{E_1'} = E_1'$ and $d_w(E_1, E_2) = d_w(E_1', E_2)$. Write $d_w(E_1', E_2) = 2nq + r$ with some $q \in \mathbb{N}$ and $r \in [2n-1]$ (note that $2n \nmid d_w(E_1', E_2)$). There are uniquely determined order-preserving bijections $\tau : E_2 \longrightarrow [m]$ and $\tau' : E_1' \longrightarrow [m+1, 2n-m]$. Let $w_0 \in \widetilde{C}_n$ be given by the requirements that $(\tau(j))w_0 = (j)w - 2nq$ for any $j \in E_2$ and $(\tau'(h))w_0 = (h)w$ for any $h \in E_1'$, where we do not display the values $(l)w_0$ for $l \in [2n-m+1, 2n]$ since they are determined by the equations $(\overline{\tau(l)})w_0 = \overline{(\tau(l))w_0}$ for any $l \in E_2$ (similar treatment for those in the remaining part of the section). Then it is easily seen that w_0 can be obtained from w by successively left-multiplying some t_i's in the case of 3.6 (e) (meaning that $w_0 = t_{j_a} t_{j_{a-1}} \cdots t_{j_1} w$ for some $a \in \mathbb{N}$ and some $j_h \in (n]$ such that $|(j_h)x_{h-1} - (j_h+1)x_{h-1}| > 2n$ for every $h \in [a]$, where $x_h := t_{j_h} t_{j_{h-1}} \cdots t_{j_1} w$ for any $h \in (a]$) and hence $w_0 \in K_w$. Clearly, w_0 satisfies the condition (a) in the case of $p = 0$.

Next assume $m = n$. Then $E_1 = \overline{E_2}$. Write $d_w(\overline{E_2}, E_2) = 4nq + r$ with some $q \in \mathbb{N}$ and $r \in [4n-1]$. When $r \in [2n-1]$, let $\tau : E_2 \longrightarrow [n]$ be the uniquely determined order-preserving bijection and let $w_0 \in \widetilde{C}_n$ be given by the requirements that $(\tau(j))w_0 = (j)w - 2nq$ for any $j \in E_2$. When $r \in [2n+1, 4n-1]$, let $\tau' : E_2 \longrightarrow [n+1, 2n]$ be the uniquely determined order-preserving bijection and let $w_0 \in \widetilde{C}_n$ be given by the requirements that $(\tau'(j))w_0 = (j)w - 2nq$ for any $j \in E_2$. In either case, w_0 can be obtained from w by successively left-multiplying some t_i's in the case of 3.6 (e), hence $w_0 \in K_w$. Clearly, w_0 satisfies the condition (a′) or (b′) in the case of $p = 0$. □

3.8. Let $w \in E_\lambda$ for some $\lambda \in \Lambda_{2n}$. Assume that $E_1, E_2, E_3 \subset [2n]$ satisfy the conditions (i)-(iii) below.

(i) $E_1 \dot\cup E_2 \dot\cup E_3 \dot\cup \overline{E_2} \dot\cup \overline{E_3} = [2n]$ such that $E_2 \dot\cup E_3$ consists of some w-wild heads in $[2n]$, where the case $E_1 = \emptyset$ and/or $E_3 = \emptyset$ is allowed;

(ii) If $E_1 \neq \emptyset$ then $E_1 \prec_w E_2 \prec_w E_3$; if $E_1 = \emptyset$ then $\overline{E_2} \prec_w E_2 \prec_w E_3$;

(iii) $E_2 = \{a_1, a_2, ..., a_r\}$ is a w-antichain with $a_1 < a_2 < \cdots < a_r$ for some $r > 1$.

Let $|E_i| = m_i$ for $i \in [3]$. Denote $E'_3 = [m_3]$, $E'_2 = [m_3+1, m_3+m_2]$ and $E'_1 = [m_3+m_2+1, 2n-m_3-m_2]$. There are uniquely determined order-preserving bijections $\tau_i : E_i \longrightarrow E'_i$, $i \in [3]$. Let $w_1 \in \widetilde{C}_n$ be given by the requirements that $(\tau_i(j))w_1 = (j)w$ for any $i \in [3]$ and $j \in E_i$. Then w_1 can be obtained from w by successively left-multiplying some t_i's in the case of 3.6 (e). We see that w_1 is in K_w and satisfies the conditions (i′)-(iii′) below.

(i′) $E'_1 \dot\cup E'_2 \dot\cup E'_3 \dot\cup \overline{E'_2} \dot\cup \overline{E'_3} = [2n]$ such that $E'_2 \dot\cup E'_3$ consists of some w_1-wild heads in $[2n]$, where the case $E'_1 = \emptyset$ and/or $E'_3 = \emptyset$ is allowed;

(ii′) If $E'_1 \neq \emptyset$ then $E'_1 > E'_2 > E'_3$ and $E'_1 <_{w_1} E'_2 <_{w_1} E'_3$; if $E'_1 = \emptyset$ then $\overline{E'_2} > E'_2 > E'_3$ and $\overline{E'_2} <_{w_1} E'_2 <_{w_1} E'_3$;

(iii′) $E'_2 = \{a+1, a+2, ..., a+r\}$ is a w_1-antichain with $a = |E'_3|$.

Let $w_2 = t_{a+2n} \cdots t_{a+r+1} t_{a+r} w_1$ (see 2.1). Then for any $l \in [2n]$, we have

$$(3.8.1) \qquad (l)w_2 = \begin{cases} (l)w_1, & \text{if } l \notin E'_2 \cup \overline{E'_2}, \\ (l-1)w_1, & \text{if } l \in [a+2, a+r], \\ (a+r)w_1 - 2n, & \text{if } l = a+1. \end{cases}$$

From (3.8.1), we have $((a+1)w_2, (a+2)w_2, ..., (a+r)w_2) = ((a+r)w_1 - 2n, (a+1)w_1, (a+2)w_1, ..., (a+r-1)w_1)$. Comparing with the sequences of the column indexes modulo $2n$ for the entries 1 in the $(a+1)$th, $(a+2)$th,..., $(a+r)$th rows of the matrix forms of w_1, w_2, it looks likely that the sequence $\langle(a+1)w_1\rangle, \langle(a+2)w_1\rangle, ..., \langle(a+r)w_1\rangle$ is cyclicly permuted to $\langle(a+r)w_1\rangle, \langle(a+1)w_1\rangle, \langle(a+2)w_1\rangle, ..., \langle(a+r-1)w_1\rangle$ as w_1 is transformed to w_2. We have $E'_1 <_{w_2} E'_2 <_{w_2} E'_3$ if

$$E'_1 \neq \emptyset \ \text{ and } \ d_{w_1}(E'_1, E'_2) > 2n - (a+r)w_1 + (a+1)w_1. \tag{3.8.2}$$

and $\overline{E'_2} <_{w_2} E'_2 <_{w_2} E'_3$ if

$$E'_1 = \emptyset \ \text{ and } \ (a+r)w_1 > 3n. \tag{3.8.3}$$

When (3.8.2) (respectively, (3.8.3)) holds, we have $d_{w_2}(E'_1, E'_2) < d_{w_1}(E'_1, E'_2)$ (respectively, $d_{w_2}(\overline{E'_2}, E'_2) < d_{w_1}(\overline{E'_2}, E'_2)$) and $d_{w_2}(E'_2, E'_3) > d_{w_1}(E'_2, E'_3)$ and that $w_2 \in K_{w_1}$ since w_2 can be obtained from w_1 by successively left-multiplying some t_i's in the case of 3.6 (e).

Call each of the transformations $w_1 \mapsto w_2$ and $w_2 \mapsto w_1$ *an admissible E'_2-move* if one of the conditions (3.8.2) and (3.8.3) holds. More precisely, call the transformation $w_1 \mapsto w_2$ a *back admissible E'_2-move*, and $w_2 \mapsto w_1$ a *forward admissible E'_2-move.* By successively applying back (respectively, forward) admissible E'_2-moves on w_1 whenever they are applicable, we can " move " the entries 1 in the ith rows for all $i \in E'_2 \cup \overline{E'_2}$ close to (respectively, away from) the point $(n+\frac{1}{2}, n+\frac{1}{2})$, but with the entries 1 in the jth rows for all $j \in [2n] - E'_2 \cup \overline{E'_2}$ fixed, such that the resulting element w_3 is in K_{w_1} and satisfies the conditions (i′)-(iii′) above with w_3 in the place of w_1.

3.9. Let $w \in \Omega \cap E_\lambda$ for some $\lambda \in \Lambda_{2n}$. Then $T(w) = (T_1, T_2, ..., T_r) \in \mathcal{C}_{2n}$ and $\xi(T(w)) = \mathbf{a} := (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{2n}$ with $\mathbf{a}$ symmetric and with $\overline{T_i} = T_{r+1-i}$ for any $i \in [r]$ by Lemma 3.5. Let $E_i = [a_r + a_{r-1} + \cdots + a_{i+1} + 1, a_r + a_{r-1} + \cdots + a_{i+1} + a_i]$ for $i \in [r]$ with the convention that $E_r = [a_r]$. Assume

$$E_1 <_w E_2 <_w \cdots <_w E_r \ \text{ and } \ \langle(E_i)w\rangle = T_i \ \text{ for any } \ i \in [r]. \tag{3.9.1}$$

If $r = 2m$ is even, then E_i is a maximal w-antichain consisting of some w-wild heads for any $i \in [m+1, 2m]$. If $r = 2m+1$ is odd, then E_{m+1} is a maximal w-antichain consisting of all w-tame integers in $[2n]$ and satisfies $\overline{E_{m+1}} = E_{m+1}$, and E_i is a maximal w-antichain consisting of some w-wild heads for any $i \in [m+2, 2m+1]$.

An element $w \in \Omega \cap \widetilde{C}_n$ is called *standard* if w with $T(w) = (T_1, ..., T_r)$ and $\xi(T(w)) = (a_1, ..., a_r)$ satisfies the condition (3.9.1). A standard element w of $\Omega \cap \widetilde{C}_n$ is called *minimal* if there is no any back admissible E_i-move, $i \in \left[\left\lceil \frac{r+2}{2} \right\rceil, r\right]$, applicable to w, where the notation $\lceil x \rceil$ stands for the smallest integer not smaller that x for any rational number x. It is easily seen that for any $2n$-self-dual $\mathbf{T} \in \mathcal{C}_{2n}$, there exists a **unique minimal standard element** in $T^{-1}(\mathbf{T}) \cap \widetilde{C}_n$.

Lemma 3.10. *Let $w, y \in \widetilde{C}_n$.*

(1) If y is obtained from w by some admissible E-moves with E ranging over some maximal w-antichains of $[2n]$ each of those w-antichains consists of some w-wild heads, then w and y are in the same left-connected component of $E_{\psi(w)}$.

(2) If $w, y \in \Omega \cap E_\lambda$ with $T(w) = T(y)$ for some $\lambda \in \Lambda_{2n}$, then w, y are in the same left-connected component of E_λ.

Proof. (1) follows by the definition of an admissible E-move. Now consider (2). By successively left-multiplying some t_i's in the case of 3.6 (e) and some back admissible moves on w, y, we can transform w, y to some standard minimal elements w', y' in $\Omega \cap E_\lambda$, respectively. Hence $w' \in K_w$ and $y' \in K_y$ by 3.6 and 3.8. Since $T(w') = T(w) = T(y) = T(y')$, we have $w' = y'$ by 3.9. This proves (2). □

Let $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{2n}$ be symmetric and let $\mathbf{a}' = (a'_1, a'_2, ..., a'_r)$ be defined by setting

$$a'_l = \begin{cases} a_l, & \text{if } l \notin [r] - \{j, j+1, r-j, r+1-j\}, \\ a_{l+1}, & \text{if } l \in \{j, r-j\}, \\ a_{l-1}, & \text{if } l \in \{j+1, r+1-j\}. \end{cases}$$

for some $j \in \left[\lceil \frac{r}{2} \rceil + 1, r-1\right]$ with $a_j \neq a_{j+1}$. Then $\mathbf{a}'$ is also symmetric.

We say that $\mathbf{a}$ and $\mathbf{a}'$ can be obtained from each other by *a simple neighboring-terms-transposition.* Let $\lambda = \zeta(\mathbf{a})^\vee$.

Clearly, any two symmetric $\mathbf{b}, \mathbf{b}' \in \widetilde{\Lambda}_{2n}$ with $\zeta(\mathbf{b}) = \zeta(\mathbf{b}')$ can be obtained from one to the other by a sequence of simple neighboring-terms-transpositions.

Lemma 3.11. *In the above setup, there are some $w, w'' \in \Omega \cap \widetilde{C}_n$ such that $\xi(T(w)) = \mathbf{a}$, that $\xi(T(w'')) = \mathbf{a}'$ and that $w'' \in K_w$.*

Proof. We may assume $a_j < a_{j+1}$ without loss of generality. By Lemma 3.5, we may take some $w \in E_\lambda \cap \Omega$ with $T(w) = (T_1, ..., T_r)$ such that there are some $E_1, E_2, E_3, E_4 \subset [2n]$ satisfying the conditions (i)-(iii) below.

(i) $E_1 \dot\cup E_2 \dot\cup E_3 \dot\cup E_4 \dot\cup \overline{E_2} \dot\cup \overline{E_3} \dot\cup \overline{E_4} = [2n]$ with $E_2 \dot\cup E_3 \dot\cup E_4$ consisting of some w-wild heads in $[2n]$, where $E_1 = \emptyset$ if and only if $j = r/2$ and, $E_4 = \emptyset$ if and only if $j = r-1$;

(ii) If $E_1 \neq \emptyset$ then $E_1 > E_2 > E_3 > E_4$ and $E_1 <_w E_2 <_w E_3 <_w E_4$ and $d_w(E_i, E_{i+1}) > 2n$ for any $i \in [3]$, where we regard $d_w(E_3, E_4) > 2n$ as an empty condition if $E_4 = \emptyset$. If $E_1 = \emptyset$ then $\overline{E_2} > E_2 > E_3 > E_4$ and $\overline{E_2} <_w E_2 <_w E_3 <_w E_4$ and $d_w(\overline{E_2}, E_2), d_w(E_2, E_3), d_w(E_3, E_4) > 2n$;

(iii) $E_2 = \{a+u+1, a+u+2, ..., a+u+v\}$ and $E_3 = \{a+1, a+2, ..., a+u\}$ with $v = a_j$, $u = a_{j+1}$ and $a = \sum_{k=j+2}^{r} a_k = |E_4|$ and

$$T_j = \{\langle (a+u+i)w\rangle \mid i \in [v]\} \quad \text{and} \quad T_{j+1} = \{\langle (a+j)w\rangle \mid j \in [u]\}. \tag{3.11.1}$$

Since both E_2 and E_3 are w-antichains and satisfy the conditions (i)-(iii), we have that

(iv) $(a+u+1)w < (a+1)w < (a+2)w < \cdots < (a+u)w < (a+1)w + 2n$ and $(a+u+1)w < (a+u+2)w < \cdots < (a+u+v)w < (a+u+1)w + 2n$.

Let $Q = (E_1 \cup E_4 \cup \overline{E_4}, \prec_w)$. By 3.2, we have
(v) $Q \in \Omega$ and

$$T(Q) = (T_1, ..., \widehat{T_{r-j}}, \widehat{T_{r-j+1}}, ..., \widehat{T_j}, \widehat{T_{j+1}}, ..., T_r), \tag{3.11.2}$$

where the notation $\widehat{T_i}$ stands for the deletion of the component T_i.

Define the set $\Delta(\mathbf{a}; j)$ of all the elements $w \in \widetilde{C}_n$ with $E_1, E_2, E_3, E_4 \subset [2n]$ satisfying the above conditions (i),(iii)-(v) together with the condition (ii′) below.

(ii′) If $E_1 \neq \emptyset$ then $E_1 > E_2 > E_3 > E_4$ and $E_1 <_w E_2$ and $E_3 <_w E_4$ and $d_w(E_i, E_{i+1}) > 2n$ for any $i \in \{1, 3\}$, where we regard $d_w(E_3, E_4) > 2n$ as an empty condition if $E_4 = \emptyset$. If $E_1 = \emptyset$ then $\overline{E_2} > E_2 > E_3 > E_4$ and $\overline{E_2} <_w E_2$ and $E_3 <_w E_4$ and $d_w(\overline{E_2}, E_2), d_w(E_3, E_4) > 2n$.

Since the condition (ii) implies (ii′), any $w \in E_\lambda \cap \Omega$ satisfying the conditions (i)-(v) belongs to the set $\Delta(\mathbf{a}; j)$.

For any $w \in \Delta(\mathbf{a}; j)$, consider the poset $P'_w := (E_2 \cup E_3, \prec_w)$. We have that $\psi(w) \leqslant \lambda$, that $\psi(P'_w) \leqslant \mathbf{2^v 1^{u-v}}$ (see 3.2), and that

(3.11.2) $\psi(w)=\lambda$ if and only if $\psi(P'_w)=\mathbf{2^v1^{u-v}}$.

Define a sequence $\xi_w: i_{a+u+1}, i_{a+u+2}, ..., i_{a+u+v}$ in the set $[a+1,a+u]$ recurrently as follows. Let $i_{a+u+1}=a+1$. Now take $p\in[2,v]$ and assume that all the i_{a+u+l}'s with $l\in[p-1]$ have been defined. Define i_{a+u+p} to be the smallest $k\in[a+1,a+u]-\{i_{a+u+l}\mid l\in[p-1]\}$ with $(a+u+p)w<(k)w$. The sequence ξ_w does not exist in general.

(3.11.3) The sequence ξ_w exists if and only if $\psi(P'_w)=\mathbf{2^v1^{u-v}}$.

For $w\in\Delta(\mathbf{a};j)$ with $\psi(P'_w)=\mathbf{2^v1^{u-v}}$ (hence $\psi(w)=\lambda$ by (3.11.2)), define $w'\in\widetilde{C}_n$ by the requirements that for any $l\in[2n]$,

$$(3.11.4)\qquad (l)w'=\begin{cases}(l)w, & \text{if } l\notin E_2\cup E_3\cup\overline{E_2}\cup\overline{E_3},\\ (i_{a+u+p})w, & \text{if } l=a+p \text{ for some } p\in[v],\\ (l-v)w, & \text{if } l-v\in[a+1,a+u]-\{i_{a+u+p}\mid p\in[v]\},\\ (a+u+p)w, & \text{if } l-v=i_{a+u+p} \text{ for some } p\in[v].\end{cases}$$

Then w' can be obtained from w by successively applying some left star operations. More precisely, denote $t_{b,c,j}:=t_{b+1}t_{b+2}\cdots\widehat{t_j}\cdots t_{b+c}$ for any $j\in[b+1,b+c]$, $b\in\mathbb{Z}$ and $c\in\mathbb{P}$, where the notation $\widehat{t_j}$ means the omission of the factor t_j. Then

$$w'=t_{a+v-1,u,i_{a+u+v}+v-1}\cdots t_{a+1,u,i_{a+u+2}+1}t_{a,u,i_{a+u+1}}w.$$

So $w'\in K_w$ by 3.6 and hence $w'\underset{L}{\sim}w$ by Lemma 2.10.

Let $E'_2=\{a+v+1,a+v+2,...,a+v+u\}$ and $E'_3=\{a+1,a+2,...,a+v\}$. Then $(a+v+1)w'<(a+v+2)w'<\cdots<(a+v+u)w'$ and $(a+1)w'<(a+2)w'<\cdots<(a+v)w'<(a+1)w'+2n$. Hence E'_3 is a w'-antichain. If E'_2 is also a w'-antichain (i.e., $(a+v+u)w'<(a+v+1)w'+2n$), then define $w''\in\widetilde{C}_n$ by the requirements that for any $l\in[2n]$,

$$(l)w''=\begin{cases}(l)w', & \text{if } l\in E_1\cup E'_2\cup\overline{E'_2},\\ (l)w'+2n, & \text{if } l\in E'_3\cup E_4.\end{cases}$$

Then $\psi(w'')=\psi(w')$ and $\ell(w'')=\ell(w')+\ell(w''{w'}^{-1})$ by Lemma 2.2. We have $w''\in\Omega$ with $\xi(T(w''))=\mathbf{a}'$. Now assume that E'_2 is not a w'-antichain. Since $v<u$, we have $a+p\notin\{i_{a+u+l}\mid l\in[v]\}$ for some $p\in[u]$. Take such an integer

$a+p$ with p largest possible. By the construction of the sequence ξ_w, we have $((a+v+1)w', (a+v+u)w') = ((a+u+1)w, (a+p)w)$ and $2n < (a+v+u)w' - (a+v+1)w' = (a+p)w - (a+u+1)w$ by our assumption on E'_2. We claim that

$$p = u. \tag{3.11.5}$$

For otherwise, we would have $p < u$ and $i_{a+u+v} = a+u$. By the construction of the sequence ξ_w, this would imply $(a+u+v)w > (a+p)w$ and further $(a+u+v)w-(a+u+1)w > 2n$, contradicting the assumption that E_2 is a w-antichain. The claim is proved. Let $w_1 = t_{a+2n}t_{a+2n-1}\cdots t_{a+u}w$ and $y = t_{2n}t_{2n-1}\cdots t_{a+u}w$. Then we have that for any $l \in [2n]$,

$$(l)w_1 = \begin{cases} (l)w, & \text{if } l \notin E_3 \cup \overline{E_3}, \\ (l-1)w, & \text{if } l \in [a+2, a+u], \\ (a+u)w - 2n, & \text{if } l = a+1. \end{cases} \tag{3.11.6}$$

and that $\ell(y) = \ell(w) - \ell(wy^{-1}) = \ell(w_1) - \ell(w_1y^{-1})$ by Lemma 2.6. Then w_1 is in $\Delta(\mathbf{a}; j)$, which can be obtained from y by successively left-multiplying some t_i's in the case of 3.6 (e) and so $w_1 \in K_y$. We can define a sequence $\xi_{w_1} : i'_{a+u+1}, i'_{a+u+2}, ..., i'_{a+u+v}$ from the poset $P'_{w_1} = (E_2 \cup E_3, \preceq_{w_1})$ in the same way as ξ_w from $P'_w = (E_2 \cup E_3, \preceq_w)$. We claim that ξ_{w_1} does exist. For, we have $i'_{a+u+1} = a+1$ since $(a+1)w_1 = (a+u)w - 2n > (a+u+1)w = (a+u+1)w_1$. This implies that $i'_{a+u+q} \leqslant i_{a+u+q}+1$ for any $q \in [2, v]$ by the construction of the sequence ξ_{w_1} and by the facts that $(a+l)w_1 = (a+l-1)w$ and $(a+u+m)w_1 = (a+u+m)w$ for any $l \in [2, u]$ and $m \in [v]$. So the sequence ξ_{w_1} does exist by (3.11.5). The claim is proved.

By (3.11.3), the above claim implies $\psi(P'_{w_1}) = \mathbf{2^v 1^{u-v}}$ and further $\psi(w_1) = \lambda$ by (3.11.2). Since $y \in K_w$ by Corollary 2.10, this implies $w_1 \in K_w$.

We define w'_1, w''_1 from w_1 in the same way as w', w'' from w. If E'_2 is a w'_1-antichain, then w''_1 is in $\Omega \cap K_w$ and satisfies $\xi(T(w''_1)) = \mathbf{a}'$. If E'_2 is not a w'_1-antichain, then we can find some $w_2 \in \Delta(\mathbf{a}; j) \cap K_w$ from w_1 in the same way as w_1 from w. By applying induction on $(a+u)w - (a+u+1)w > 0$ and by noting that $(a+u)w-(a+u+1)w > (a+u)w_1-(a+u+1)w_1 > (a+u)w_2-(a+u+1)w_2 > \cdots > 0$, we can eventually find some w_q, $q \geqslant 1$, in $\Delta(\mathbf{a}; j) \cap K_w$ and define w'_q, w''_q from w_q in the same way as w', w'' from w such that the set E'_2 is w'_q-antichain and that w''_q is in $\Omega \cap K_w$ and satisfies $\xi(T(w''_q)) = \mathbf{a}'$. So our proof is complete. $\square$

Theorem 3.12. *Let* Γ *be a left cell of* $\widetilde{C}_n$.

(1) The set $\Gamma\cap\Omega$ *is non- empty if and only if there is some symmetric* $\mathbf{a}\in\widetilde{\Lambda}_{2n}$ *such that* $\zeta(\mathbf{a})^\vee=\psi(\Gamma)$.

(2) The set $\Gamma\cap\Omega$ *is contained in some left-connected component of* $E_{\psi(\Gamma)}$ *if it is non-empty.*

Proof. The assertion (1) follows by Lemmas 2.7, 3.3 and 3.5. For (2), take any $w,y\in\Gamma\cap\Omega$. Then the compositions $\mathbf{a}:=\xi(T(w))$ and $\mathbf{a}':=\xi(T(y))$ are both symmetric and satisfy $\zeta(\mathbf{a})=\zeta(\mathbf{a}')$ by Lemmas 2.7 and 3.5. Hence $\mathbf{a}'$ can be obtained from $\mathbf{a}$ by successively applying some simple neighboring-terms-transpositions (see the definition preceding Lemma 3.11). So $y\in K_w$ by Lemmas 2.7, 3.3, 3.10 (2) and 3.11. The assertion (2) is proved. □

3.13. A partition $\lambda\in\Lambda_{2n}$ is called *dual-symmetrizable*, if there exists some symmetric $\mathbf{a}\in\widetilde{\Lambda}_{2n}$ satisfying $\zeta(\mathbf{a})^\vee=\lambda$, and called *nice* if $\Omega\cap K_w\neq\emptyset$ for any $w\in E_\lambda$. By Lemma 3.5, we see that any nice $\lambda\in\Lambda_{2n}$ is dual-symmetrizable.

We conjecture that the converse also holds.

Conjecture 3.14. *Any dual-symmetrizable* $\lambda\in\Lambda_{2n}$ *is nice.*

Theorem 3.15. *Let* $\lambda\in\Lambda_{2n}$ *be nice. Then any left cell of* $\widetilde{C}_n$ *in* E_λ *is left-connected.*

Proof. Let Γ be a left cell of $\widetilde{C}_n$ in E_λ. We must show that any $w,w'\in\Gamma$ are in the same left-connected component of Γ. Since λ is nice, we can take some $y\in\Omega\cap K_w$ and $y'\in\Omega\cap K_{w'}$. Then $y,y'\in\Omega\cap\Gamma$ by Lemma 2.9. This implies that y,y' are in the same left-connected component of Γ by Theorem 3.12 and Lemma 2.9. Our result follows. □

§4. The cells of $\widetilde{C}_n$ in the set $E_{(2n-k,k)}$.

In the present section, we shall study the cells of $\widetilde{C}_n$ in the set $E_{(2n-k,k)}$ for any $k\in[n]$. The main results are Theorems 4.12 and 4.13. The crucial step in the section is to prove that Conjecture 3.14 holds in the case of $\lambda=(2n-k,k)$.

First we give a brief description for the elements in $E_{(2n-k,k)}$.

Lemma 4.1. *Let* $k\in[n]$ *and* $w\in\widetilde{C}_n$.

(1) w *is in* $E_{(2n-k,k)}$ *if and only if the following two conditions hold:*

(1a) The maximal length of a w*-chain in* $[2n]$ *is* $2n-k$;

(1b) Any maximal w-antichain in $[2n]$ has the cardinal $\leqslant 2$.

(2) Let $w \in E_{(2n-k,k)}$ be such that the set E of all the w-tame heads in $[2n]$ is non-empty. Then both E and $\overline{E}$ are w-chains. The set F of all the w-wild heads in $[2n]$ can be partitioned into at most two parts, say F_1 and F_2, such that the elements of $F_1 \cup E \cup \overline{F_2}$ are pairwise not $2n$-dual and comprise a w-chain of length n.

Proof. The implication " $\Longrightarrow$ " in (1) is obvious. For the implication " $\Longleftarrow$ " in (1), let $\psi(w) = \lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{2n}$. Then the condition (1a) implies $\lambda_1 = 2n - k$. We have $r \geqslant 2$ by the assumption $k \in [n]$ and $r \leqslant 2$ by the condition (1b). Hence $r = 2$ and $\lambda_2 = 2n - (2n - k) = k$. So (1) is proved. Then (2) follows by Lemma 2.12 and by the facts that any $i \in E$ is w-uncomparable with any $j \in \overline{E}$ and that $[2n]$ can be partitioned into exactly two w-chains. □

Denote by $\text{wh}_w(\gamma)$ (respectively, $\text{tm}_w(\gamma)$, $\text{wt}_w(\gamma)$) the number of w-wild heads (respectively, w-tame integers, w-wild tails) in a w-chain γ for any $w \in \widetilde{C}_n$.

Given $w \in E_\lambda$ with $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t) \in \Lambda_{2n}$ and $\lambda_1 < 2n$. In 4.2-4.6, we shall transform the element w in several steps, each step proceeds by successively left-multiplying some t_i's, most of them being in the cases of 3.6 (a)-(e), such that all the intermediate elements, including the resulting element w', are in the set K_w and that w' has some special form. For the sake of simplifying the notation, we denote some intermediate elements again by w from time to time.

4.2. Since $w \in E_\lambda$, we can choose some w-chain $\gamma : j_1, j_2, ..., j_{\lambda_1}$ in $\mathbb{Z}$ with $j_1 < j_2 < \cdots < j_{\lambda_1}$. Let $r = \text{wh}_w(\gamma) + \text{tm}_w(\gamma)$. We see by Lemma 2.13 that for any $a \in [\lambda_1]$, j_a is a w-wild tail if and only if $a \in [r+1, \lambda_1]$. We may assume the following conditions (4.2a)-(4.2b) on the w-chain γ at the beginning.

(4.2a) $\text{wh}_w(\gamma) \geqslant \text{wt}_w(\gamma)$.

For otherwise, $\text{wh}_w(\gamma) < \text{wt}_w(\gamma)$. Then we replace γ by $\overline{\gamma}$, the latter is obtained from γ by replacing each term d of γ by its $2n$-dual $\overline{d} := 2n + 1 - d$ and then by reversing the order of the resulting terms. Since $\overline{d}$ is a w-wild head (respectively, a w-wild tail) if and only if d is a w-wild tail (respectively, a w-wild head), we see that $\overline{\gamma}$ is a w-chain and satisfies $\text{wh}_w(\overline{\gamma}) = \text{wt}_w(\gamma) > \text{wh}_w(\gamma) = \text{wt}_w(\overline{\gamma})$.

By (4.2a), we have

$$r \in \left[\left\lceil \frac{\lambda_1}{2} \right\rceil, n\right] \tag{4.2.1}$$

since $r = \mathrm{wh}_w(\gamma) + \mathrm{tm}_w(\gamma)$.

(4.2b) $0 < j_{a+1} - j_a < 2n$ for any $a \in [\lambda_1 - 1]$.

For otherwise, say $j_{a+1} - j_a > 2n$ for some $a \in [\lambda_1 - 1]$. Let $\gamma' : j'_1, j'_2, ..., j'_{\lambda_1}$ be the sequence $j_1 + 2n, j_2 + 2n, ..., j_a + 2n, j_{a+1}, ..., j_{\lambda_1}$. Then γ' is also a w-chain with an additional property that $j'_{\lambda_1} - j'_1 < j_{\lambda_1} - j_1$. By applying induction on $j_{\lambda_1} - j_1 \geqslant \lambda_1 - 1$, we can eventually get a w-chain $\gamma'' : j''_1, j''_2, ..., j''_{\lambda_1}$ satisfying $0 < j''_{a+1} - j''_a < 2n$ for any $a \in [\lambda_1 - 1]$. By our construction, we have $\mathrm{wh}_w(\gamma'') = \mathrm{wh}_w(\gamma)$ and $\mathrm{wt}_w(\gamma'') = \mathrm{wt}_w(\gamma)$, so the validity of (4.2a) on γ implies that on γ''.

4.3. Now we want to transform w to some $w' \in K_w$ such that there exist some w'-chain $\gamma' : i_1, i_2, ..., i_{\lambda_1}$ in $\mathbb{Z}$ and some $r' \in [\lambda_1]$ satisfying the conditions (4.3a)-(4.3b) below.

(4.3a) $i_{a+1} - i_a = 1$ for any $a \in [r' - 1]$;

(4.3b) $r' := \mathrm{wh}_{w'}(\gamma') + \mathrm{tm}_{w'}(\gamma') \geqslant r \geqslant \lceil \frac{\lambda_1}{2} \rceil$.

If $r = 1$ then we take w' to be w and hence there is nothing to do. Now assume that $r > 1$ and that there is some $a \in [r-1]$ with $j_{a+1} - j_a > 1$. We may take such a number a smallest possible. Consider the number $j_a + 1$. We have either $(j_a + 1)w > (j_a)w$ or $(j_a + 1)w < (j_{a+1})w$ (that is, we never have $(j_{a+1})w < (j_a + 1)w < (j_a)w$) by the assumption of $w \in E_\lambda$. When $(j_a + 1)w > (j_a)w$, let b be the smallest integer in $[a]$ with $(j_b)w < (j_a + 1)w$. Let $y = t_{j_1} t_{j_2} \cdots \widehat{t_{j_b}} \cdots t_{j_a} w$, where the notation $\widehat{t}$ means the omission of the factor t. Then y is obtained from w by successively applying certain left star operations and hence $y \in K_w$, where there is a y-chain of length λ_1 with $j_1 + 1, j_2 + 1, ..., j_a + 1, j_{a+1}$ as its first $a + 1$ terms. When $(j_a + 1)w < (j_{a+1})w$, there are two possibilities:

(1) $\langle j_a + 1 \rangle = \overline{\langle j_a \rangle}$;

(2) $\langle j_a + 1 \rangle \neq \overline{\langle j_a \rangle}$.

In the case (1), we see that $j_a + i$, $i \in [a]$, are all w-wild tails with $\langle j_a + i \rangle = \overline{\langle j_{a+1-i} \rangle}$, hence $j_{a+1} - j_a > a$. Let $J = \{t_{j_1}, t_{j_2}, ..., t_{j_a}\}$ and $I = J - \{t_{j_a}\}$ and $y = w_J w_I w$. If j_{a+1} is a w-wild head, then we have $(j_a)w - (j_a + 1)w > 2n$ by the facts

$(j_{a+1})w-\frac{1}{2}((j_a)w+(j_a+1)w) > n$ and $(j_a)w > (j_{a+1})w$. So, if $(j_a)w-(j_a+1)w < 2n$ then j_{a+1} must be w-tame, in this case, let H_1 be the set of all the w-tame integers in $[j_a+1, j_a+n]$ and let H_2 be the set of all the w-tame integers in $[j_a-n+1, j_a]$. Then each of H_1 and H_2 forms a w-chain and the equality $\langle H_2\rangle = \overline{\langle H_1\rangle}$ holds by the assumption $w \in E_{(2n-k,k)}$ and by Lemma 4.1(2), where we define $\langle H\rangle := \{\langle h\rangle \mid h \in H\}$ and $\overline{H'} = \{\overline{h'} \mid h' \in H'\}$ for any $H \subset \mathbb{Z}$ and $H' \subseteq [2n]$. By Lemmas 4.1 (2) and 2.12, we see that there is some w-chain $j'_1, j'_2, ..., j'_{a'-1}, j_a, j_{a+1}, j'_{a'+2}, ..., j'_n$ of length n whose terms are pairwise not $2n$-dual modulo $2n$ and $j'_1, j'_2, ..., j'_{a'-1}, j_a$ are all w-wild heads and $H_1 = \{j_{a+1}, j'_{a'+2}, ..., j'_{a'+c}\}$ and $j'_{a'+c+1}, ..., j'_n$ are all w-wild tails, where $c = |H_1|$. So in either case, we have $\ell(y) = \ell(w) - \ell(w_J w_I)$ and $y \in E_{(2n-k,k)}$ by Lemmas 2.6 and 4.1, hence $y \in K_w$ by Corollary 2.10, such that there exists a y-chain γ' of length $2n-k$ with the first $a+1$ terms being $j_1+a, j_2+a, ..., j_a+a, j_{a+1}$ and with $\mathrm{wh}_y(\gamma') + \mathrm{tm}_y(\gamma') \geqslant r$. Note that in the case of $(j_a)w - (j_a+1)w < 2n$, we can obtain $y = w_J w_I w$ from w by successively left-multiplying some t_i's, though not all in the case of 3.6 (a)-(e), but still having $y \in K_w$, as shown above.

In the case (2), there exists some $i \in [j_a+2, j_{a+1}]$ such that $(j_a+1)w > (j_a+2)w > \cdots > (i-1)w < (i)w$ (since $(j_{a+1})w > (j_a+1)w$ by the assumption). Let h be the smallest integer in $[j_1, i-1]$ with $(h)w < (i)w$ and let $y = t_{j_1}t_{j_1+1}\cdots\widehat{t_h}\cdots t_{i-1}w$. In this case, y is obtained from w by successively applying certain left star operations, hence $y \in K_w$ and there is a y-chain γ' of length $2n-k$ with $\mathrm{wh}_y(\gamma') + \mathrm{tm}_y(\gamma') \geqslant r$ and with the first $a+1$ terms being $j_1+1, j_2+1, ..., j_a+1, j_{a+1}$ if $i < j_{a+1}$ and $j_1+1, j_2+1, ..., j_a+1, h+1$ if $i = j_{a+1}$. By applying induction first on $a \geqslant 1$ and then on $j_{a+1} - j_a \geqslant 1$, we can eventually get a required element w' in K_w.

4.4. By the result in 4.3, we may assume that $w \in E_{(2n-k,k)}$ has a w-chain γ : $j_1, j_2, ..., j_{2n-k}$ satisfying (4.2a) and $j_{a+1} - j_a = 1$ for any $a \in [r-1]$, where $r = \mathrm{wh}_w(\gamma)+\mathrm{tm}_w(\gamma)$. Let $c \in \mathbb{P}$ be the smallest number satisfying $\langle j_r+c+1\rangle = \overline{\langle j_r+c\rangle}$. We want to transform w to some $w' \in K_w$ such that there exists some w'-chain γ' of length $2n-k$ with the first r terms being $j_1+c, j_2+c, ..., j_r+c$ and with $r = \mathrm{wh}_{w'}(\gamma') + \mathrm{tm}_{w'}(\gamma')$. If $c = 0$ then we can take w' to be w. Now assume $c > 0$. If $(j_r+1)w > (j_r)w$ then j_r+1 is either a w-wild head or a w-tame integer. Let i be the smallest integer in $[r]$ with $(j_i)w < (j_r+1)w$. Let $y = t_{j_1}t_{j_2}\cdots\widehat{t_{j_i}}\cdots t_{j_r}w$. Then y is obtained from w by successively applying certain left star operations (hence

$y \in K_w$) and there exists some y-chain β of length $2n-k$ with the first r terms being $j_1+1, j_2+1, ..., j_r+1$ and with $r = \text{wh}_y(\beta) + \text{tm}_y(\beta)$. If $(j_r+1)w < (j_r)w$, then there are two possibilities:

(1) There exists some $a \in [c]$ such that $(j_r)w > (j_r+1)w > \cdots > (j_r+a-1)w < (j_r+a)w$;

(2) $(j_r)w > (j_r+1)w > \cdots > (j_r+c)w$.

In the case (1), let j be the smallest number in $[j_1, j_r+a-1]$ such that $(j)w < (j_r+a)w$. Let $y = t_{j_1}t_{j_2}\cdots\widehat{t_j}\cdots t_{j_r+a-1}w$.

In the case (2), we claim that $(j_r+c+1)w > (j_r+c-1)w$. To show this, we need only to prove that $(j_r+c+1)w > (j_r)w$ under the assumption in (2). For otherwise, $(j_r+c+1)w < (j_r)w$. Then $j_1, j_2, ..., j_r, j_r+c+1, j_r+c+2, ..., j_r+2c+r$ is a w-chain of length $2r+c$. Since $w \in E_{(2n-k,k)}$, we have $2r+c \leqslant 2n-k$. But $r \geqslant \frac{1}{2}(2n-k)$ by (4.2.1), a contradiction. This proves that $(j_r+c+1)w > (j_r+c-1)w$. Let j be the smallest number in $[j_1, j_r+c-1]$ such that $(j)w < (j_r+c+1)w$. Let $y = t_{j_1}t_{j_2}\cdots\widehat{t_j}\cdots t_{j_r+c-1}t_{j_r+c}w$. Then y is obtained from w by successively applying some left star operations (hence $y \in K_w$) and there exists some y-chain whose first r terms are $j_1+1, j_2+1, ..., j_r+1$ none of them is a y-wild tail. By applying induction on $c \geqslant 0$, we can eventually get a required element w'.

4.5. By the result in 4.4, we may assume that $w \in E_{(2n-k,k)}$ has a w-chain γ of length $2n-k$ satisfying (4.2a), together with the following conditions:

(i) the first r terms of γ are $a+1, a+2, ..., a+r$ for some $a \in \mathbb{Z}$, where $r = \text{wh}_w(\gamma) + \text{tm}_w(\gamma) \in \left[\lceil\frac{2n-k}{2}\rceil, n\right]$;

(ii) $\overline{a+r+1} = a+r$;

Clearly, the w-chain $a+1, a+2, ..., a+r$ is the longest one among all w-chains with $a+r$ the last term. Moreover, $(a+1, a+2, ..., a+r)$ could be either $(n+1-r, n+2-r, ..., n)$ or $(2n+1-r, 2n+2-r, ..., 2n)$. By the symmetry, we may assume without loss of generality that

(iii) $(a+1, a+2, ..., a+r) = (n+1-r, n+2-r, ..., n)$.

So we have that

(4.5.1) $n+1-r, n+2-r, ..., n$ forms a w-chain, the longest one among all w-chains with n the last term.

Let us describe the w-chain γ. We have $2n-k \leqslant 2r$ by (4.2.1). If $n+1-$

$r, n+2-r, ..., n$ are all w-wild heads then $2n-k = 2r$ and the w-chain γ could be $n+1-r, n+2-r, ..., n, n+1, ..., n+r$. Now assume $2n-k < 2r$. Hence n is a w-tame tail. We have $r \leqslant 2n-k$ in general since $r, k \in [n]$ by (4.2.1). The equality $r = 2n-k$ holds if and only if $r = k = n$ and $w = w_J$ with $J = \{t_1, t_2, ..., t_{n-1}\}$. Now assume $r < 2n-k$.

We claim that $(2r+1+k-n)w < (n)w < (2r+k-n)w$. We have $(n)w < (2r+k-n)w$ by the assumption that the length of the longest w-chain is $2n-k$. If $(2r+1+k-n)w > (n)w$ then let j be the largest number in $[2r+1+k-n, n+r]$ with $(j)w > (n)w$. Let $j'_i = 2n+1-j_i$ for any $i \in [r+1, 2n-k]$. Then $(j'_{r+1})w > (n+1)w > (2n+1-j)w$ by the fact $(j)w > (n)w > (j_{r+1})w$. So $j'_{2n-k}, j'_{2n-k-1}, ..., j'_{r+1}, 2n+1-j, 2n+2-j, ..., n$ forms a w-chain of length $n+j-k-r$ which is greater than r, contradicting (4.5.1). The claim is proved.

So far we have proved that any $x \in E_{(2n-k,k)}$ can be transformed into $X_{(2n-k,k)} \cap K_x$, where $X_{(2n-k,k)}$ is the set of all $w \in E_{(2n-k,k)}$ with the w-chain γ in (4.5.2) or (4.5.3) for some $r \in [\lceil \frac{2n-k}{2} \rceil, n]$.

(4.5.2) $\quad n+1-r,\ n+2-r,\ ...,\ n,\ 2r+1+k-n,\ \ 2r+2+k-n,\ \ ...,\ n+r.$

(4.5.3) $\quad 1-r,\quad 2-r,\quad ...,\ 0,\ 2r+1+k-2n,\ 2r+2+k-2n,\ ...,\ r.$

4.6. Fix $w \in X_{(2n-k,k)}$ with a w-chain γ as in (4.5.2). Then $(3n-2r-k+1)w < (n+1)w < (3n-2r-k)w$ by the fact $(2r+1+k-n)w < (n)w < (2r+k-n)w$. Denote $q(w) := (n+1)-(3n-2r-k+1) = 2r+k-2n$ which is in $\mathbb{N}$. Suppose $q(w) > 1$. Then n is a w-tame tail and $3n-2r-k+1 < n$. Let $w_1 = t_{n+1-r}t_{n+2-r} \cdots \widehat{t_{3n-2r-k+1}} \cdots t_n w$. Then w_1 is obtained from w by successively applying some left star operations, hence $w_1 \in K_w$. There is a w_1-chain in (4.6.1) below.

(4.6.1) $\qquad n+2-r, n+3-r, ..., n, 2r-1+k-n, 2r+k-n, ..., n+r-1.$

Clearly, (4.6.1) can be obtained from (4.5.2) by replacing r by $r-1$. Hence $w_1 \in X_{(2n-k,k)}$ with $q(w_1) = 2(r-1)+k-2n = q(w)-2 < q(w)$. If $q(w_1) > 1$, then we can find some $w_2 \in X_{(2n-k,k)} \cap K_{w_1}$ from w_1 by the same way as w_1 from w such that $q(w_2) < q(w_1)$. Recurrently, we can eventually find $w_a \in X_{(2n-k,k)} \cap K_w$ with some $a \in \mathbb{P}$ such that $q(w_a) \in \{0, 1\}$.

For $w \in X_{(2n-k,k)}$ with a w-chain γ as in (4.5.2), if $q(w) = 0$, then $k = 2m$ is even and $r = n-m$; if $q(w) = 1$, then $k = 2m+1$ is odd and $r = n-m$. Hence, by symmetry between (4.5.2) and (4.5.3), we have proved that any $x \in E_{(2n-k,k)}$ can be transformed into $Y_{(2n-k,k)} \cap K_x$, where $Y_{(2n-k,k)}$ is the set of all $w \in E_{(2n-k,k)}$ with the w-chain γ in one of (4.6.2)-(4.6.5) below.

(4.6.2) $m+1, \quad m+2, \quad ..., n, n+1, n+2, ..., 2n-m.$

(4.6.3) $m+1-n, m+2-n, ..., 0, 1, \quad 2, \quad ..., n-m.$

(4.6.4) $m+1, \quad m+2, \quad ..., n, n+2, n+3, ..., 2n-m.$

(4.6.5) $m+1-n, m+2-n, ..., 0, 2, \quad 3, \quad ..., n-m.$

4.7. By the processes in 4.2-4.6, we transform any $x \in E_{(2n-k,k)}$ to some $w \in K_x$ such that there is a w-chain $\gamma : j_1, j_2, ..., j_{2n-k}$ which is either in one of (4.6.2) and (4.6.4) and satisfies $(2r+k+1-n)w < (n)w < (2r+k-n)w$, or in one of (4.6.3) and (4.6.5) and satisfies $(2r+k+1)w < (2n)w < (2r+k)w$, where $k \in \{2m, 2m+1\}$ and $r = n-m \in \left[\lceil \frac{2n-k}{2} \rceil, n\right]$. Let $i_1, i_2, ..., i_{2n-2r}$ be in $[r+1-n, n-r]$ (respectively, in $[r+1, 2n-r]$) satisfy the relation

$$(4.7.1) \qquad (i_1)w > (i_2)w > \cdots > (i_{2n-2r})w.$$

Now we define a sequence $l_1, l_2, ..., l_{2n-2r}$ in $[2n-k]$ as follows. Let l_1 be the smallest integer a in $[2n-k]$ such that $0 < (j_a)w - (i_1)w < 2n$. Recurrently, suppose that we have defined all the integers $l_1, l_2, .., l_h$ for some $h \in [2n-2r]$. If $h < 2n-2r$ then we define l_{h+1} to be the smallest integer b in $[2n-k] - \{l_c \mid c \in [h]\}$ such that $0 < (j_b)w - (i_{h+1})w < 2n$.

Lemma 4.8. *Let $w \in E_{(2n-k,k)}$ be with a w-chain γ in one of (4.6.2)-(4.6.5). Then in the setup of 4.7, the integers $l_1, l_2, ..., l_{2n-2r}$ are well defined and satisfy the relation $l_1 < l_2 < \cdots < l_{2n-2r}$.*

Proof. By the assumption of $w \in E_{(2n-k,k)}$, we see that

(4.8.1) there exists some $a_h \in [2n-k]$ satisfying $0 < (j_{a_h})w - (i_h)w < 2n$ for any $h \in [2n-2r]$.

The existence of the integer l_1 follows by (4.8.1). Now assume that $h \in [2, 2n-2r]$ and that we have found all the integers $l_1, l_2, ..., l_{h-1}$ and have proved the relation $l_1 < l_2 < \cdots < l_{h-1}$. By the definition of the l_a's, we see that

(4.8.2) for any $a \in [h-1]$ with $l_a > 1$, we have either that $(j_{l_a-1})w - (i_a)w > 2n$, or that $0 < (j_{l_a-1})w - (i_a)w < 2n$ and $l_{a-1} = l_a - 1$.

By repeatedly applying (4.8.2), we get that

(4.8.3) for any $a \in [h-1]$ with $l_a > 1$, there exists some $b \in [a]$ such that $l_c - 1 = l_{c-1}$ and $0 < (j_{l_c-1})w - (i_c)w < 2n$ for any $c \in [b+1, a]$ and $(j_{l_b-1})w - (i_b)w > 2n$ whenever $l_b > 1$.

We claim that

(4.8.4) there must exist some $a \in [l_{h-1}+1, 2n-k]$ such that $0 < (j_a)w - (i_h)w < 2n$.

For otherwise, there would exist some $c \in [l_{h-1}, 2n-k]$ such that $(j_b)w - (i_h)w > 2n$ for any $b \in [l_{h-1}+1, c]$ and $(j_d)w - (i_h)w < 0$ for any $d \in [c+1, 2n-k]$. By (4.8.1), we must have $c = l_{h-1}$ and $0 < (j_{l_{h-1}})w - (i_h)w < 2n$. So by (4.8.3), there exists some $e \in [l_{h-1}]$ such that $l_d - 1 = l_{d-1}$ and $0 < (j_{l_d-1})w - (i_d)w < 2n$ for any $d \in [e+1, l_{h-1}]$ and $(j_{l_e-1})w - (i_e)w > 2n$ whenever $l_e > 1$. In this case, we claim that

(4.8.5) $i_h > i_{h-1} > \cdots > i_e$.

For otherwise, there would exist some $f \in [e, h-1]$ with $i_f > i_{f+1}$. Then $\{i_f, i_{f+1}, j_{l_f}\}$ would form a w-antichain, contradicting the assumption of $w \in E_{(2n-k,k)}$ by Lemma 4.1. The claim (4.8.5) is proved. By (4.8.5) together with the validity for one of (4.6.2)-(4.6.5), we see that

$$j_1, j_2, ..., j_{l_e-1}, i_e, i_{e+1}, ..., i_{h-1}, i_h, j_{l_{h-1}+1}, j_{l_{h-1}+2}, ..., j_{2n-k}$$

forms a w-chain of length $2n+1-k$, contradicting the assumption of $w \in E_{(2n-k,k)}$. This proves the claim (4.8.4). Hence the existence of the integer l_h follows by (4.8.4) immediately. Clearly, $l_h > l_{h-1}$. So our result follows by induction. □

4.9. Let $w \in E_{(2n-k,k)}$ be provided with the w-chain γ of the form in one of (4.6.2)-(4.6.5). By symmetry, we need only to consider the case where γ is in (4.6.2) or (4.6.4). In the setup of 4.7, let $[r+1-n, n-r] = E_1 \cup E_0 \cup E_{-1}$, where $E_1 = \{j \in [r+1-n, n-r] \mid j \text{ is a } w\text{-wild head}\}$, $E_{-1} = \{j \in [r+1-n, n-r] \mid j \text{ is a } w\text{-wild tail}\}$ and $E_0 = \{j \in [r+1-n, n-r] \mid j \text{ is } w\text{-tame}\}$. We have $\langle E_{-1}\rangle = \overline{\langle E_1\rangle}$ and $\langle E_0\rangle = \overline{\langle E_0\rangle}$.

Lemma 4.10. *Let $w \in E_{(2n-k,k)}$ be with the w-chain γ in (4.6.2) or (4.6.4). Then in the setup of 4.7 and 4.9, j_{l_a} is a w-wild head for any $i_a \in E_1$.*

Proof. The following two facts about the element w can be checked easily.

(i) $(j_{l_a})w > (i_a)w > 0$ for any $i_a \in E_1$.

(ii) For any $b \in [2n-k]$, the integer j_b is a w-wild head if and only if either that $j_b < n$, or that $j_b = n$ and γ is in (4.6.2).

By the fact (i), to show our result, we need only to consider the case where $(i_a)w \in [n]$ for some $i_a \in E_1$. If $(n)w \in [n]$ (i.e., the w-chain γ is in (4.6.4)), then any $i_a \in E_1$ with $(i_a)w \in [n]$ is w-uncomparable with $n+1$, hence $n \prec_w i_a$ by Lemmas 4.1 (2) and 2.12 (iv). This implies $j_{l_a} < n$ and so j_{l_a} is a w-wild head by the fact (ii).

Now assume $(n)w \notin [n]$ (i.e., the w-chain γ is in (4.6.2)). Hence $(n)w > n$. If $(n)w > 2n$, then $(n+1)w \leqslant 0$, hence $n+1 \prec_w i_a$ for any $i_a \in E_1$ by the fact (i). This implies that $j_{l_a} \leqslant n$ and hence j_{l_a} is a w-wild head for any $i_a \in E_1$ by the fact (ii).

Now assume $(n)w \in [n+1, 2n]$. Hence n is a w-wild head.

If $E_0 \neq \emptyset$, then we claim that $n+1 \prec_w i_a$ for any $i_a \in E_1$. For, any element of $E_0 \cap [n-r]$ is w-uncomparable with n. This implies by Lemma 4.1 (2) that n is w-comparable with any element of $\langle E_0 \cap [r+1-n, 0]\rangle$, hence $i_b \prec_w n$ for any $i_b \in \langle E_0 \cap [r+1-n, 0]\rangle$ by Lemma 2.12 (iv). But this is equivalent to that $n+1 \prec_w i_b$ for any $i_b \in E_0 \cap [n-r]$ by Lemma 2.12 (i). For any $i_a \in E_1$, if $(i_a)w < (n)w$, then i_a is w-uncomparable with n by the fact (i) and the assumption of $(n)w \in [n+1, 2n]$, so i_a must be w-comparable with any element of $E_0 \cap [n-r]$ by Lemma 4.1 (2), hence $i_b \prec_w i_a$ for any $i_b \in E_0 \cap [n-r]$ by Lemma 2.12 (iv), and further $n+1 \prec_w i_a$. The claim is proved. We see from this claim that $j_{l_a} \leqslant n$ for any $i_a \in E_1$ and hence j_{l_a} is a w-wild head by the fact (ii).

Now assume $E_0 = \emptyset$. Suppose that there exists some $i_a \in E_1$ such that j_{l_a} is not a w-wild head. Then $j_{l_a} > n$ by the assumption of $(n)w \in [n+1, 2n]$. We have $(i_a)w > 0$ by the fact (i). So i_a is w-uncomparable with n again by the assumption of $(n)w \in [n+1, 2n]$ and the fact $n - i_a \in [2n-1]$. By the definition of the l_b's in 4.7 and by the fact (4.8.3), we see that $n = j_{l_c}$ for some $c < a$ and that there exists some $d \in [c]$ such that $l_e - 1 = l_{e-1}$ and $0 < (j_{l_e-1})w - (i_e)w < 2n$ for any $e \in [d+1, a]$ and $(j_{l_d-1})w - (i_d)w > 2n$ whenever $l_d > 1$. By the same argument as that for the claim (4.8.5) with a, d in the place of h, e respectively, we can show that $i_a > i_{a-1} > \cdots > i_d$, hence the sequence

$$j_1, j_2, ..., j_{l_d-1}, i_d, i_{d+1}, ..., i_c, i_{c+1}, ..., i_a, \overline{i_a}, ..., \overline{i_{c+1}}, \overline{i_c}, ..., \overline{i_{d+1}}, \overline{i_d}, \overline{j_{l_d-1}}, ..., \overline{j_2}, \overline{j_1}$$

forms a w-chain of length $2r+2(a-c)=2n-k+2(a-c)>2n-k$, contradicting the assumption of $w\in E_{(2n-k,k)}$.

This proves our result. □

Lemma 4.11. *Let $w\in E_{(2n-k,k)}$ be with the w-chain γ in one of (4.6.2)- (4.6.5). Then $\Omega\cap K_w\neq\emptyset$.*

Proof. By symmetry, we need only to consider the case where the w-chain γ is in (4.6.2) or (4.6.4). Keep the setup of 4.7 and 4.9 for w. Define $y\in\widetilde{C}_n$ by the requirements that $(j_a)y=(j_a)w+2nq_a$ for any $a\in[r]$ and $(i_b)y=(i_b)w+2nq_{l_b}$ for any $b\in[2n-2r]$ with i_b a w-wild head and $(i_c)y=(i_c)w$ for any $c\in[2n-2r]$ with i_c w-tame, where $q_1,q_2,...,q_r$ is a strictly decreasing sequence of integers with $q_r>0$ if k is even and $q_r=0$ if k is odd. By Lemmas 2.2 and 4.10, we have $\ell(y)=\ell(w)+\ell(yw^{-1})$ and $y\in E_{(2n-k,k)}$. Hence $y\in K_w$ by Corollary 2.10 (1). If there is no w-tame integer in $i_1,i_2,...,i_{2n-2r}$, then $y\in\Omega$ by our construction, the result is proved in this case.

Now assume that there are some w-tame integers in $i_1,i_2,...,i_{2n-2r}$. In this case, we see from the proof of Lemma 4.10 that there exists some $c\in[n-r-1]$ such that i_a is a w-wild head for any $a\in[c]$ and that i_b is a w-tame tail for any $b\in[c+1,n-r]$ and that i_e is either a w-tame head or a w-wild tail for any $e\in[n-r+1,2n-2r]$. Let τ be the bijective map from the set $E:=\{i_a,j_b\mid a\in[n-r],b\in[r]\}$ to the set $[n]$ such that if k is even then

$$\begin{aligned}&(\tau(j_1),\tau(j_2),...,\tau(j_{l_1-1}),\tau(i_1),\tau(j_{l_1}),\tau(j_{l_1+1}),\tau(j_{l_1+2}),...,\tau(j_{l_2-1}),\tau(i_2),\tau(j_{l_2}),\\&\quad\tau(j_{l_2+1}),...,\tau(j_{l_c-1}),\tau(i_c),\tau(j_{l_c}),\tau(j_{l_c+1}),...,\tau(j_r),\tau(i_c+1),\tau(i_c+2),...,\tau(i_{n-r}))\\=&(1,2,...,n).\end{aligned}$$

and that if k is odd then

$$\begin{aligned}&(\tau(j_1),\tau(j_2),...,\tau(j_{l_1-1}),\tau(i_1),\tau(j_{l_1}),\tau(j_{l_1+1}),\tau(j_{l_1+2}),...,\tau(j_{l_2-1}),\tau(i_2),\tau(j_{l_2}),\tau(j_{l_2+1}),\\&\quad...,\tau(j_{l_c-1}),\tau(i_c),\tau(j_{l_c}),\tau(j_{l_c+1}),...,\tau(j_{r-1}),\tau(i_c+1),\tau(i_c+2),...,\tau(i_{n-r}),\tau(j_r))\\=&(1,2,...,n).\end{aligned}$$

Define $z\in\widetilde{C}_n$ by the requirement that $(\tau(a))z=(a)y$ for any $a\in E$. Then z can be obtained from y by successively left-multiplying some t_i's in the case of 3.6 (e).

Hence $z \in K_y$. There is some $h \in [n]$ such that $h-1$ is a z-wild head whenever $h > 1$ and that $h, h+1, ..., n$ are all z-tame tails and form a z-chain. When $n-h = 2p$ is even, define $x \in \widetilde{C}_n$ by the requirement that $((1)x, (2)x, ..., (n)x)$ is equal to

$$\begin{aligned}&((1)z+2np, (2)z+2np, ..., (h-1)z+2np, (h)z+2np, (n+1)z+2np,\\&(h+1)z+2n(p-1), (n+2)z+2n(p-1), (h+2)z+2n(p-2),\\&(n+3)z+2n(p-2), ..., (h+p-1)z+2n, (n+p)z+2n, (h+p)z).\end{aligned}$$

When $n-h = 2p-1$ is odd, define $x \in \widetilde{C}_n$ by the requirement that $((1)x, (2)x, ..., (n)x)$ is equal to

$$\begin{aligned}&((1)z+2np, (2)z+2np, ..., (h-1)z+2np, (h)z+2np, (n+1)z+2np,\\&(h+1)z+2n(p-1), (n+2)z+2n(p-1), (h+2)z+2n(p-2),\\&(n+3)z+2n(p-2), ..., (h+p-1)z+2n, (n+p)z+2n).\end{aligned}$$

We see by Lemma 2.2 that $\ell(x) = \ell(z) + \ell(xz^{-1})$ and that $x \in E_{(2n-k,k)}$. Hence $x \in K_z$ by Corollary 2.10 (1).

In either case, we have $x \in \Omega \cap K_w$, hence $\Omega \cap K_w \neq \emptyset$. □

Theorem 4.12. *(1) Any left cell of $\widetilde{C}_n$ in $E_{(2n-k,k)}$ is left-connected.*

(2) The number of left cells of $\widetilde{C}_n$ in $E_{(2n-k,k)}$ is $2^{n-m}n!$ if $k = 2m$ is even and $2^{n-m-1}n!$ if $k = 2m+1$ is odd.

Proof. (1) Let Γ be a left cell of $\widetilde{C}_n$ in $E_{(2n-k,k)}$. Take any $w, w' \in \Gamma$. By 1.4 (2), we see that the left-connected component of Γ containing w is just the set K_w. Hence we need only to show that $w' \in K_w$. By the processes (4.2)-(4.6) and Lemma 4.11, we can find some $y \in \Omega \cap K_w$ and $y' \in \Omega \cap K_{w'}$. Since $y \underset{L}{\sim} w \underset{L}{\sim} w' \underset{L}{\sim} y'$ by Lemma 2.9, we have $y' \in K_y$ by Theorem 3.12. This implies $w' \in K_w$, as required.

(2) Fix a symmetric $\mathbf{a} = (a_1, a_2, ..., a_{2n-k}) \in \widetilde{\Lambda}_{2n}$ with $\zeta(\mathbf{a})^\vee = (2n-k, k)$. By Lemmas 3.5, 3.3 and 2.7, we see that the number of left cells of $\widetilde{C}_n$ in $E_{(2n-k,k)}$ is equal to the number of $2n$-self-dual $\mathbf{T} \in \mathcal{C}_{2n}$ with $\xi(\mathbf{T}) = \mathbf{a}$.

Denote $q := \lfloor \frac{2n-k}{2} \rfloor$. Any $2n$-self-dual tabloid $\mathbf{T} = (T_1, T_2, ..., T_{2n-k}) \in \xi^{-1}(\mathbf{a})$ is determined entirely by its first q components by the facts that $T_i = \overline{T_{2n-k+1-i}}$ for any $i \in [q]$ and that $T_{q+1} = [2n] - \bigcup_{i=1}^{q}(T_i \cup \overline{T_i})$ is a union of some $2n$-dual pairs if $k = 2m+1$ is odd. Since the elements of $\bigcup_{i=1}^{q} T_i$ are pairwise not $2n$-dual, the number of the choices for T_1 is $2^{a_1}\binom{n}{a_1}$. Recurrently, when $T_1, T_2, ..., T_{h-1}$ have been chosen for $h \in [q]$, the number of the choices for T_h is $2^{a_h}\binom{n-a_1-\cdots-a_{h-1}}{a_h}$. So our result follows by the following two facts: (i) Among $a_1, a_2, ..., a_q$, the number 2 occurs m times, while 1 occurs $q-m$ times; (ii) The sum $a_1+\cdots+a_q$ is equal to n if $k = 2m$ and $n-1$ if $k = 2m+1$. □

Theorem 4.13. *The set $E_{(2n-k,k)}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$ for any $k \in [n]$.*

Proof. Let

$$w_0 = \begin{cases} w_{S-\{t_m\}}, & \text{if } k = 2m \text{ is even,} \\ w_J w_I w_K, & \text{if } k = 2m+1 \text{ is odd.} \end{cases}$$

where $K = S - \{t_0\}$, $I = K - \{t_{n-1}\}$ and $J = I - \{t_m, t_n\}$. Then $w_0 \in E_{(2n-k,k)}$. Let $Z_{(2n-k,k)} = \{w_0 \cdot x \in E_{(2n-k,k)} \mid x \in \widetilde{C}_n\}$. Clearly, $Z_{(2n-k,k)}$ is a right-connected subset of $E_{(2n-k,k)}$. Let $\mathbf{a} = (a_1, a_2, ..., a_{2n-k}) \in \widetilde{\Lambda}_{2n}$ be such that $a_i = a_{2n+1-k-i} = 1$ and $a_j = 2$ for $i \in [n-2m]$ and $j \in [n-2m+1, n]$ if $k = 2m$ is even and that $a_i = a_{2n+1-k-i} = 1$ and $a_j = 2$ for $i \in [n-2m-1]$ and $j \in [n-2m, n]$ if $k = 2m+1$ is odd. Clearly, $\mathbf{a}$ is symmetric with $\zeta(\mathbf{a})^\vee = (2n-k, k)$.

By Theorem 4.12 and Lemmas 2.7, 3.3, 3.5, to show our result, we need only to find some $w \in Z_{(2n-k,k)} \cap \Omega$ with $T(w) = \mathbf{T}$ for any $2n$-self-dual $\mathbf{T} = (T_1, T_2, ..., T_{2n-k}) \in \mathcal{C}_{2n}$ with $\xi(\mathbf{T}) = \mathbf{a}$.

A $2n$-self-dual $\mathbf{T} = (T_1, ..., T_{2n-k}) \in \xi^{-1}(\mathbf{a})$ is determined uniquely by the part $(T_{n-m+1}, T_{n-m+2}, ..., T_{2n-2m})$ if $k = 2m$ and by $(T_{n-m}, T_{n-m+1}, ..., T_{2n-2m-1})$ if $k = 2m+1$. We define an element w of $\widetilde{C}_n$ for a given $2n$-self-dual $\mathbf{T} = (T_1, ..., T_{2n-k}) \in \xi^{-1}(\mathbf{a})$ as follows.

First assume that $k = 2m$ and that $(T_{n-m+1}, T_{n-m+2}, ..., T_{2n-2m})$ is equal to

$$(\{c_{n-m+1}, d_{n-m+1}\}, \{c_{n-m+2}, d_{n-m+2}\}, ..., \{c_n, d_n\}, \{d_{n+1}\}, \{d_{n+2}\}, ..., \{d_{2n-2m}\}),$$

where $c_i < d_i$ in $[2n]$ for any $i \in [n-m+1, n]$. Then we define $w \in \widetilde{C}_n$ by the requirement that

$$\begin{aligned}&((n)w,(n-1)w,(n-2)w,(n-3)w,(n-4)w,(n-5)w,...,\\&\qquad(n-2m+2)w,(n-2m+1)w,(n-2m)w,(n-2m-1)w,...,(1)w)\\=&(d_{n-m+1}+2n,d_{n-m+2}+2n\cdot 2,d_{n-m+3}+2n\cdot 3,...,d_n+2nm,\\&\quad d_{n+1}+2n(m+1),d_{n+2}+2n(m+2),...,d_{2n-2m}+2n(n-m),\\&\quad c_{n-m+1}+2n,c_{n-m+2}+2n\cdot 2,c_{n-m+3}+2n\cdot 3,...,c_n+2nm)\end{aligned}$$

Next assume that $k=2m+1$ and that $(T_{n-m},T_{n-m+1},...,T_{2n-2m-1})$ is equal to

$$(\{c_{n-m},d_{n-m}\},\{c_{n-m+1},d_{n-m+1}\},...,\{c_n,d_n\},\{d_{n+1}\},\{d_{n+2}\},...,\{d_{2n-2m-1}\}),$$

where $c_i<d_i$ for any $i\in[n-m,n]$; in particular, $\overline{d_{n-m}}=c_{n-m}\in[n]$ by Lemma 3.5. Then we define $w\in\widetilde{C}_n$ by the requirement that

$$\begin{aligned}&((n)w,(n-1)w,(n-2)w,(n-3)w,(n-4)w,(n-5)w,...,\\&\qquad(n-2m+2)w,(n-2m+1)w,(n-2m)w,(n-2m-1)w,...,(1)w)\\=&(c_{n-m},d_{n-m+1}+2n,d_{n-m+2}+2n\cdot 2,d_{n-m+3}+2n\cdot 3,...,d_n+2nm,\\&\quad d_{n+1}+2n(m+1),d_{n+2}+2n(m+2),...,d_{2n-2m-1}+2n(n-m-1),\\&\quad c_{n-m+1}+2n,c_{n-m+2}+2n\cdot 2,c_{n-m+3}+2n\cdot 3,...,c_n+2nm)\end{aligned}$$

In either case, we have $w\in Z_{(2n-k,k)}\cap\Omega$ with $T(w)=\mathbf{T}$. Hence our result follows. □

Remark 4.14. In dealing with the case (1) of 4.3, we have to apply Lemma 4.1 (2), hence all the results in the present section are only valid for the set $E_{(2n-k,k)}$. If one can deduce some results for all the nice partitions of $2n$ which could replace Lemma 4.1 (2) in our proof, then this would be a good progress in approaching Conjecture 3.14.

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