THE CELLS OF THE AFFINE WEYL GROUP $\widetilde{C}_n$ IN A CERTAIN QUASI-SPLIT CASE

Jian-yi Shi

Department of Mathematics
East China Normal University
Shanghai, 200241, P.R.China

Abstract. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_{2n-1}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$. Let $\widetilde{\ell}$ be the length function of $\widetilde{A}_{2n-1}$. We study the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$. The main results of the paper are to give an explicit description for all the cells of $(\widetilde{C}_n, \widetilde{\ell})$ corresponding to the partitions $\mathbf{k1^{2n-k}}$ and $\mathbf{h21^{2n-h-2}}$ for all $1 \leqslant k \leqslant 2n$ and $2 \leqslant h \leqslant 2n-2$, and also for all the cells of $(\widetilde{C}_3, \widetilde{\ell})$.

§0. Introduction.

0.1. In his book [8], Lusztig introduced a weighted Coxeter group (W, L), which is, by definition, a Coxeter system (W, S) together with a weight function $L : W \longrightarrow \mathbb{Z}$. He proposed a bundle of conjectures, intending to generalize many results on cells of W in the equal parameter case to the unequal parameters case. The most successful part for such a generalization is when (W, L) is in a certain quasi-split case, that is, W can be realized as the fixed point set of a finite or an affine Coxeter system $(\widetilde{W}, \widetilde{S})$ under a group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$, the weight function L is the restriction to W of the length function $\widetilde{\ell}$ of $\widetilde{W}$ (see [8, Chapter 16], [6], [2], [4]).

Key words and phrases. Affine Weyl group; weighted Coxeter group; quasi-split case; cells, partitions..

Supported by the NSF of China, the SFUDP of China, Shanghai Leading Academic Discipline Project (B407) and Program of Shanghai Subject Chief Scientist (11xd1402200)

Typeset by $\mathcal{AMS}$-TEX

0.2. For any $i \leqslant j$ in the integer set $\mathbb{Z}$, denote by $[i,j]$ the set $\{i, i+1, ..., j\}$. Denote $[1,j]$ simply by $[j]$. Fix an integer $n \geqslant 2$ and let $m \in \{2n-1, 2n, 2n+1\}$. Consider the quasi-split case where W is the affine Weyl group $\widetilde{C}_n$, realized as the fixed point set of the affine Weyl group $\widetilde{W} = \widetilde{A}_m$ under the group automorphism $\alpha = \alpha_{m,n}$ determined by $\alpha_{m,n}(s_i) = s_{2n-i}$ for $i \in [0,m]$ if $m \in \{2n-1, 2n\}$ and by $\alpha_{m,n}(s_i) = s_{2n+1-i}$ for $i \in [0,m]$ if $m = 2n+1$, where the Coxeter generator set $\widetilde{S} = \{s_i \mid i \in [0,m]\}$ of $\widetilde{A}_m$ satisfies $s_i^2 = 1$, $s_i s_j = s_j s_i$ if $j \not\equiv i \pm 1 \pmod{m+1}$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for any $i, j \in [0,m]$ (we stipulate $s_{m+1} = s_0$).

0.3. Let $\widetilde{\ell}_m$ be the length function of $(\widetilde{A}_m, \widetilde{S})$. All the cells of the weighted Coxeter group $(\widetilde{A}_m, \widetilde{\ell}_m)$ were described explicitly in [9], [7]. In the present paper, we only consider the case of $m = 2n-1$. Based on the results in [9], [7], we describe all the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ corresponding to the partitions $\mathbf{k1^{2n-k}}$ and $\mathbf{h21^{2n-h-2}}$ for all $k \in [2n]$ and $h \in [2, 2n-2]$ (see 2.2 and Theorems 4.9, 5.1). We also describe all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$ (see Theorem 6.1). In particular, we prove that all the left (respectively, two-sided) cells described in the paper are left- (respectively, two-sided-) connected (see 2.17). It was conjectured by Lusztig for the left-connectedness of left cells of affine Weyl groups in the split case (see [1]).

0.4. It is known that there is a surjective map ψ from $\widetilde{A}_{2n-1}$ to the set Λ_{2n} of partitions of $2n$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_{2n-1}$ to Λ_{2n} (see Lemma 2.16). Let $E_\lambda := \psi^{-1}(\lambda) \cap \widetilde{C}_n$ for $\lambda \in \Lambda_{2n}$. For all considered $\lambda \in \Lambda_{2n}$, we shall find such a finite subset F_λ of E_λ that the intersection of F_λ with any left-connected component of E_λ (see 2.17) is non-empty. Then we shall use the set F_λ to enumerate the left cells of $\widetilde{C}_n$ in E_λ in virtue of either generalized tabloids of rank $2n$ (see 3.5 and Lemma 3.6, $E_{\mathbf{k1^{2n-k}}}$ and $E_{\mathbf{h21^{2n-h-2}}}$ are treated in this way) or of generalized τ-invariants by regarding F_λ as a subset of $\widetilde{A}_{2n-1}$ (see 2.9 and Lemma 2.16, $E_{\mathbf{3^2}}$ and $E_{\mathbf{2^3}}$ are treated in this way). When E_λ is a single two-sided cell, we prove this fact by showing that E_λ is two-sided-connected; when E_λ is a union of two two-sided cells E'_λ, E''_λ, we prove this fact by Lemma 1.7 and by showing that each of E'_λ, E''_λ is two-sided-connected.

0.5. One may expect that the methods used in the present paper are applicable to describe the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ in E_λ for all the other $\lambda \in \Lambda_{2n}$, and further to describe all the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_m)$ with $m \in \{2n, 2n+1\}$. I conjecture that in any of the above quasi-split cases, each left (respectively, two-sided) cell of the weighted Coxeter group $\widetilde{C}_n$ is left- (respectively, two-sided-) connected.

0.6. From now on we denote $\widetilde{\ell}_{2n-1}$ simply by $\widetilde{\ell}$. The contents of the paper are organized as follows. In Section 1, we collect some concepts and known results concerning cells of a weighted Coxeter group. Then we concentrate ourselves to the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$ in Sections 2-3, many useful results and technical tools are provided there. We give an explicit description for all the cells of $(\widetilde{C}_n, \widetilde{\ell})$ corresponding to the partitions $\mathbf{k1^{2n-k}}$, $k \in [2n]$, and $\mathbf{h21^{2n-h-2}}$, $h \in [2, 2n-2]$, in Sections 4 and 5, respectively. Finally, we describe all the cells of $(\widetilde{C}_3, \widetilde{\ell})$ in Section 6.

§1. Cells in Coxeter groups.

In this section, we collect some concepts and results concerning cells of a weighted Coxeter group, all but Lemma 1.7 follow Lusztig in [8].

1.1. Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. By a *weight function* on W, we mean a map $L : W \longrightarrow \mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ in W. Call (W, L) is a *weighted Coxeter group.*

A weighted Coxeter group (W, L) is called in the *split* case if $L = \ell$.

Suppose that there exists a group automorphism $\alpha : W \longrightarrow W$ with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J on S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function $\ell : W \longrightarrow \mathbb{N} := \{0, 1, 2, ...\}$ is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the *quasi-split* case.

1.2. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. Denote $v_w = v^{L(w)}$ for any $w \in W$. Define a ring involution $a \mapsto \bar{a}$ of $\mathcal{A}$ by setting $\overline{\sum_i a_i v^i} = \sum_i a_i v^{-i}$ where $a_i \in \mathbb{Z}$ in the sum. Define $\mathcal{A}_{<m} = \{f \in \mathcal{A} \mid \deg f < m\}$ for any $m \in \mathbb{Z}$.

1.3. For any $w, x, y, z \in W$ and $s \in S$ with $sx < x < y < sy$, define $p_{z,w}, M^s_{x,y} \in \mathcal{A}$ recurrently by the following requirements:

$$p_{z,w} = 0 \text{ if } z \not\leqslant w,\ p_{w,w} = 1 \text{ and } p_{z,w} \in \mathcal{A}_{<0} \text{ if } z < w. \tag{1.3.1}$$

$$p_{z,w} = v_s^{\epsilon} p_{z,sw} + p_{sz,sw} - \sum_{\substack{z \leqslant z' < sw \\ sz' < z'}} M^s_{z',sw} p_{z,z'} \quad \text{for } z < w \text{ and } sw < w, \tag{1.3.2}$$

where $\epsilon = 1$ if $sz < z$, and -1 if $sz > z$ (see [8, The proof of Theorem 6.6]).

$$\sum_{\substack{x \leqslant z < y \\ sz < z}} M^s_{z,y} p_{x,z} \equiv v_s p_{x,y} \pmod{\mathcal{A}_{<0}}, \tag{1.3.3}$$

$$\overline{M^s_{x,y}} = M^s_{x,y}. \tag{1.3.4}$$

The condition (1.3.3) determines the coefficients of v^k in $M^s_{x,y}$ for all $k \geqslant 0$; then (1.3.4) determines all the other coefficients (see [8, Proposition 6.3]).

1.4. Define a preorder $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$) on W which is transitively generated by the relation $y \underset{L}{\leftarrow} w$ (respectively, $y \underset{R}{\leftarrow} w$), where $w < sw$, and either $y = sw$ or $M^s_{y,w} \neq 0$ (respectively, $w < ws$, and either $y = ws$ or $M^s_{y^{-1},w^{-1}} \neq 0$) holds for some $s \in S$. The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (respectively, $\underset{R}{\sim}$). The corresponding equivalence classes in W are called *left cells* (respectively, *right cells*) of W. Write $y \underset{LR}{\leqslant} w$ in W, if there exists a sequence $y_0 = y, y_1, \cdots, y_r = w$ in W with some $r \geqslant 0$ such that for every $i \in [r]$, either $y_{i-1} \underset{L}{\leqslant} y_i$ or $y_{i-1} \underset{R}{\leqslant} y_i$ holds.

The equivalence relation associated to the preorder $\underset{LR}{\leqslant}$ is denoted by $\underset{LR}{\sim}$ and the corresponding equivalence classes in W are called *two-sided cells* of W.

1.5. For $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) = \mathcal{L}(w)$) (see [8, Lemma 8.6]).

1.6. In [8, Chapter 13], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ in terms of structural coefficients of the Hecke algebra associated to (W, L).

In [8, Chapters 14-16], Lusztig proved the following results when W is either a finite or an affine Coxeter group and when (W, L) is either in the split case or in the quasi-split case.

(1) $y \underset{LR}{\leqslant} w$ in W implies $a(w) \leqslant a(y)$. Hence $y \underset{LR}{\sim} w$ in W implies $a(w) = a(y)$.

(2) If $w, y \in W$ satisfy $a(w) = a(y)$ and $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$, $y \underset{LR}{\leqslant} w$) then $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$, $y \underset{LR}{\sim} w$).

For any $X \subset W$, write $X^{-1} := \{x^{-1} \mid x \in X\}$.

Lemma 1.7. *Suppose that W is either a finite or an affine Coxeter group and that (W, L) is either in the split case or in the quasi-split case.*

Let E be a non-empty subset of W satisfying the following conditions:

(a) There exists some $k \in \mathbb{N}$ with $a(x) = k$ for any $x \in E$;

(b) E is a union of some left cells of W;

(c) $E^{-1} = E$.

Then E is a union of some two-sided cells of W.

Proof. By the conditions (b)-(c), we see that E is also a union of some right cells of W. Let $W_{(k)} = \{w \in W \mid a(w) = k\}$. Then $W_{(k)}$ is a union of some two-sided cells of W by 1.6 (1). If the result is false, then by the condition (c), there must exist some $x \in E$ and $y \in W_{(k)} \setminus E$ such that either $x \underset{L}{\leqslant} y$ or $y \underset{L}{\leqslant} x$. In either case, we have $x \underset{L}{\sim} y$ by 1.6 (2), contradicting the condition (b). This proves our result. $\square$

§2. The affine Weyl groups $\widetilde{A}_{2n-1}$ and $\widetilde{C}_n$.

From now on, we concentrate ourselves to the weighted Coxeter groups $(\widetilde{A}_{2n-1}, \widetilde{\ell})$ and $(\widetilde{C}_n, \widetilde{\ell})$, where $\widetilde{\ell}$ is the length function of the affine Weyl group $\widetilde{A}_{2n-1}$.

2.1. The affine Weyl group $\widetilde{A}_{2n-1}$ can be realized as the following permutation group on the set $\mathbb{Z}$ (see [5, Subsection 3.6] and [9, Subsection 4.1)]:

$$\widetilde{A}_{2n-1} = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \,\middle|\, (i+2n)w = (i)w + 2n, \sum_{i=1}^{2n} (i)w = \sum_{i=1}^{2n} i \right\}.$$

The Coxeter generator set $\widetilde{S} = \{s_i \mid i \in [0, 2n-1]\}$ of $\widetilde{A}_{2n-1}$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{2n}, \\ t+1, & \text{if } t \equiv i \pmod{2n}, \\ t-1, & \text{if } t \equiv i+1 \pmod{2n}, \end{cases}$$

for any $t \in \mathbb{Z}$ and $i \in [0, 2n-1]$. Any $w \in \widetilde{A}_{2n-1}$ can be realized as a $\mathbb{Z} \times \mathbb{Z}$ monomial matrix $A_w = (a_{ij})_{i,j \in \mathbb{Z}}$, where a_{ij} is 1 if $j = (i)w$ and 0 if otherwise. The row (respectively, column) indices of A_w are increasing from top to bottom (respectively, from left to right). We can conveniently use some familiar operations in linear algebra on the matrix A_w. For example, $A_{w^{-1}}$ is just the transposed matrix of A_w; $A_{s_i w}$ (respectively, A_{ws_i}) can be obtained from A_w by transposing the $(2nq+i)$th and the $(2nq+i+1)$th rows (respectively, columns) for all $q \in \mathbb{Z}$.

Let $\alpha : \widetilde{A}_{2n-1} \longrightarrow \widetilde{A}_{2n-1}$ be the group automorphism determined by $\alpha(s_i) = s_{2n-i}$ for $i \in [0, 2n-1]$. Then the affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n-1}$ under α, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$\widetilde{C}_n = \{w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+2n)w = (i)w + 2n, (i)w + (1-i)w = 1, \ \forall\, i \in \mathbb{Z}\}$$

with the Coxeter generator set $S = \{t_i \mid i \in [0, n]\}$, where $t_i = s_i s_{2n-i}$ for $i \in [n-1]$, $t_0 = s_0$ and $t_n = s_n$. For the sake of convenience, we define s_i

and t_j for any $i, j \in \mathbb{Z}$ by setting s_{2qn+b} to be s_b and $t_{2pn\pm a}$ to be t_a for any $p, q \in \mathbb{Z}$ and $b \in [0, 2n-1]$ and $a \in [0, n]$.

2.2. By a partition of a positive integer n, we mean an r-tuple $\lambda := (\lambda_1, \lambda_2, ..., \lambda_r)$ of weakly decreasing positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ with $\sum_{k=1}^{r} \lambda_k = n$ for some $r \geqslant 1$. λ_i is called a *part* of λ. We usually denote λ in the form $\mathbf{j_1^{k_1} j_2^{k_2} \cdots j_m^{k_m}}$ (boldfaced) with $j_1 > j_2 > \cdots > j_m \geqslant 1$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$ for $i \geqslant 1$. Let Λ_n be the set of all partitions of n. For example, $\mathbf{63^3 1^2}$ stands for the partition $(6, 3, 3, 3, 1, 1)$ of 17.

Fix $w \in \widetilde{A}_{2n-1}$. For any $i \neq j$ in $[2n]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both inequalities $2pn + i > 2qn + j$ and $(2pn + i)w < (2qn + j)w$ hold. In terms of matrix entries of w, this means that the entry 1 at the position $(2qn + j, (2qn + j)w)$ is located at the northeastern of the entry 1 at the position $(2pn + i, (2pn + i)w)$ (see Figure 1). This defines a partial order $\preceq_w$ on the set $[2n]$.

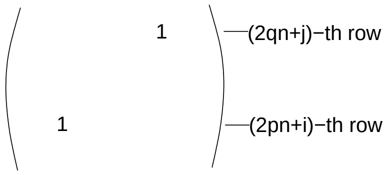


Figure 1

A sequence $a_1, a_2, ..., a_r$ in $[2n]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. Sometimes we identify a w-chain $a_1, a_2, ..., a_r$ with the corresponding set $\{a_1, a_2, ..., a_r\}$. For any $k \geqslant 1$, a k-w-chain-family is by definition a disjoint union $X = \cup_{i=1}^{k} X_i$ of k w-chains $X_1, ..., X_k$ in $[2n]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = 2n$. Let $\lambda_1 = d_1$ and $\lambda_{k+1} = d_{k+1} - d_k$ for $k \in [r-1]$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of Curtis Greene in [3]. Hence $w \mapsto \psi(w) := (\lambda_1, ..., \lambda_r)$ defines a map from the set $\widetilde{A}_{2n-1}$ to Λ_{2n}.

2.3. Let $\widetilde{\ell}$, ℓ be the length functions on the Coxeter systems $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_{2n-1}, \widetilde{\ell})$ is in the split case, while $(\widetilde{C}_n, \widetilde{\ell})$ is in the quasi-split case (see [8, Lemma 16.2]).

For any $x \in \widetilde{A}_{2n-1}$ and $k \in \mathbb{Z}$, let $m_k(x) = {}^{\#}\{i \in \mathbb{Z} \mid i < k \text{ and } (i)x \geqslant (k)x\}$. Then the formulae for the functions $\widetilde{\ell}$ and ℓ are as follows.

Proposition 2.4. *For any $w \in \widetilde{A}_{2n-1}$ and $x \in \widetilde{C}_n$, we have*

(1) $\widetilde{\ell}(w) = \sum_{1 \leqslant i < j \leqslant 2n} \left| \left\lfloor \frac{(j)w - (i)w}{2n} \right\rfloor \right| = \sum_{k=1}^{2n} m_k(w)$;

(2) $\ell(x) = \frac{1}{2}(\widetilde{\ell}(x) + m_1(x) + m_{n+1}(x))$,

where $\lfloor a \rfloor$ is the largest integer not larger than a, and $|a|$ is the absolute value of a for any $a \in \mathbb{Q}$.

Proof. The first equality of (1) is just the result in [9, Lemma 4.2.2], while the second equality of (1) follows by the facts that for any $1 \leqslant i < j \leqslant 2n$, at most one of $m_{ij}(w) := {}^{\#}\{k \in \mathbb{Z} \mid k \equiv i \pmod{2n} \text{ and } k < j \text{ and } (k)w > (j)w\}$ and $m_{ji}(w) := {}^{\#}\{k \in \mathbb{Z} \mid k \equiv j \pmod{2n} \text{ and } k < i \text{ and } (k)w > (i)w\}$ is strictly positive, that $\left| \left\lfloor \frac{(j)w - (i)w}{2n} \right\rfloor \right| = \max\{m_{ij}(w), m_{ji}(w)\}$ and that $m_k(w) = \sum_{i \in [2n] \setminus \{k\}} m_{ik}(w)$. The equality of (2) follows by the definition of t_i's in terms of s_j's. □

2.5. Let $\leqslant$, $\leqslant_C$ be the Bruhat-Chevalley orders on the Coxeter systems $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. Since the condition $x \leqslant_C y$ is equivalent to $x \leqslant y$ for any $x, y \in \widetilde{C}_n$, it will cause no confusion if we use the notation $\leqslant$ in the place of $\leqslant_C$. Hence from now on we shall use $\leqslant$ for both $\leqslant$ and $\leqslant_C$.

Let $\widetilde{\mathcal{L}}(x) = \{s \in \widetilde{S} \mid sx < x\}$ and $\widetilde{\mathcal{R}}(x) = \{s \in \widetilde{S} \mid xs < x\}$ for $x \in \widetilde{A}_{2n-1}$ and let $\mathcal{L}(y) = \{t \in S \mid ty < y\}$ and $\mathcal{R}(y) = \{t \in S \mid yt < y\}$ for $y \in \widetilde{C}_n$.

Corollary 2.6. *For any $x \in \widetilde{C}_n$ and $i \in [0, n]$,*

$$\begin{aligned}
s_i \in \widetilde{\mathcal{L}}(x) \iff{}& s_{2n-i} \in \widetilde{\mathcal{L}}(x) && \iff t_i \in \mathcal{L}(x) \\
\iff{}& (i)x > (i+1)x && \iff (2n+1-i)x < (2n-i)x, \\
s_i \in \widetilde{\mathcal{R}}(x) \iff{}& s_{2n-i} \in \widetilde{\mathcal{R}}(x) && \iff t_i \in \mathcal{R}(x) \\
\iff{}& (i)x^{-1} > (i+1)x^{-1} && \iff (2n+1-i)x^{-1} < (2n-i)x^{-1}
\end{aligned}$$

Proof. The equivalent conditions involving the s_i's follow by [9, Lemma 4.2.4], while those involving the t_j's follow by the expression of t_j in terms of s_i's and by Proposition 2.4. $\square$

2.7. Any $w \in \widetilde{C}_n$ is determined uniquely by the n-tuple $((1)w, (2)w, ..., (n)w)$. Hence we shall identify w with the n-tuple $((1)w, (2)w, ..., (n)w)$ and denote the latter by $[(1)w, (2)w, ..., (n)w]$ in such a sense.

Let $w = [a_1, a_2, ..., a_n]$ and $w' = t_i w = [a'_1, a'_2, ..., a'_n]$ and $w'' = wt_i = [a''_1, a''_2, ..., a''_n]$ be in $\widetilde{C}_n$. When $i \in [n-1]$, we have $a'_j = a_j$ for $j \in [n] \backslash \{i, i+1\}$, $a'_i = a_{i+1}$ and $a'_{i+1} = a_i$; when $i = 0$, we have $a'_j = a_j$ for $j \in [2, n]$ and $a'_1 = 1 - a_1$; when $i = n$, we have $a'_j = a_j$ for $j \in [n-1]$ and $a'_n = 2n+1-a_n$.

For any $a \in \mathbb{Z}$, denote by $\langle a \rangle$ the unique integer in $[2n]$ satisfying $a \equiv \langle a \rangle \pmod{2n}$. When $i \in [n-1]$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{i, i+1, 2n-i, 2n+1-i\}$, $a''_j = a_j + 1$ if $\langle a_j \rangle \in \{i, 2n-i\}$ and $a''_j = a_j - 1$ if $\langle a_j \rangle \in \{i+1, 2n+1-i\}$; when $i = 0$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{1, 2n\}$, $a''_j = a_j + 1$ if $\langle a_j \rangle = 2n$ and $a''_j = a_j - 1$ if $\langle a_j \rangle = 1$; when $i = n$, we have $a''_j = a_j$ if $\langle a_j \rangle \notin \{n, n+1\}$, $a''_j = a_j + 1$ if $\langle a_j \rangle = n$ and $a''_j = a_j - 1$ if $\langle a_j \rangle = n+1$.

Let η be the group automorphism of $\widetilde{C}_n$ determined by the condition $\eta(t_i) = t_{n-i}$ for any $i \in [0, n]$.

The following results provide some information from the expression $w = [a_1, a_2, ..., a_n] \in \widetilde{C}_n$.

Proposition 2.8. *Let* $w = [a_1, a_2, ..., a_n]$ *and* $w' = \eta(w) = [a'_1, a'_2, ..., a'_n]$ *be in* $\widetilde{C}_n$. *Let* $k \in [0, n]$. *Then*

(1) $t_k \in \mathcal{L}(w)$ *if and only if* $a_k > a_{k+1}$, *with the convention that* $a_0 = 1$ *and* $a_{n+1} = n$.

(2) Let $\langle a_i \rangle, \langle a_j \rangle \in \{k, k+1, 2n-k, 2n+1-k\}$ *for some* $i \neq j$ *in* $[n]$. *Then* $t_k \in \mathcal{R}(w)$ *if one of the following conditions holds:*

(i) either $a_j - a_i > 2n$, *or* $i > j$ *and* $a_j > a_i$ *if* $(\langle a_i \rangle, \langle a_j \rangle) \in \{(k, k+1), (2n-k, 2n+1-k)\}$;

(ii) $a_i + a_j < 1$ *if* $(\langle a_i \rangle, \langle a_j \rangle) = (k, 2n-k)$;

(iii) $a_i + a_j > 2n+1$ *if* $(\langle a_i\rangle, \langle a_j\rangle) = (2n+1-k, k+1)$.

(3) $a'_i = n+1-a_{n+1-i}$ *for any* $i \in [n]$.

Proof. (1) and (2) follow by 2.7 and Corollary 2.6. For (3), apply induction on $\ell(w) \geqslant 0$. It is trivial when $\ell(w) = 0$. If $\ell(w) > 0$, write $w = t_i y$ for some $t_i \in \mathcal{L}(w)$, then $\eta(w) = t_{n-i}\eta(y)$. We can describe the expression $y = [b_1, ..., b_n]$ from $w = [a_1, ..., a_n]$ by 2.7. Then $\eta(y) = [n+1-b_n, n+1-b_{n-1}, ..., n+1-b_1]$ by inductive hypothesis. Hence the expression $\eta(w) = [n+1-a_n, n+1-a_{n-1}, ..., n+1-a_1]$ can be checked from the equation $\eta(w) = t_{n-i}\eta(y)$ again by 2.7. The detailed proof is left to the readers. □

2.9. For any $i \in [0, 2n-1]$, let $\widetilde{D}_R(i)$ be the set of all $w \in \widetilde{A}_{2n-1}$ satisfying $|\{s_i, s_{i+1}\} \cap \widetilde{\mathcal{R}}(w)| = 1$. When $w \in \widetilde{D}_R(i)$, exactly one of ws_i and ws_{i+1} is in $\widetilde{D}_R(i)$, denote it by w^*, call the transformation from w to w^* a *right* $\{s_i, s_{i+1}\}$*-star operation* (or a *right star operation* in short) on w. For any $w \in \widetilde{A}_{2n-1}$, let $\widetilde{M}(w)$ be the set of all $y \in \widetilde{A}_{2n-1}$, where there exists a sequence $x_0 = w, x_1, ..., x_r = y$ in $\widetilde{A}_{2n-1}$ with some $r \geqslant 0$ such that for every $i \in [r]$, x_i is obtained from x_{i-1} by a right $\{s_{k_i}, s_{k_i+1}\}$-star operation with some $k_i \in \mathbb{Z}$. Define a graph $\widetilde{\mathcal{M}}(w)$ as follows. Its vertex set is $\widetilde{M}(w)$, each $x \in \widetilde{M}(w)$ is labeled by $\widetilde{\mathcal{R}}(x)$. Two vertices $x, y \in \widetilde{M}(w)$ are joined by a solid edge if y can be obtained from x by a right star operation. By a *path* in $\widetilde{\mathcal{M}}(w)$, we mean a sequence $x_0, x_1, ..., x_r$ in $\widetilde{M}(w)$ with some $r \geqslant 0$ such that x_{i-1} and x_i are joined by a solid edge for every $i \in [r]$. Two elements $w, y \in \widetilde{A}_{2n-1}$ are said to have *the same generalized τ-invariants*, if for any path $w_1 = w, w_2, ..., w_r$ in $\widetilde{\mathcal{M}}(w)$, there exists a path $y_1 = y, y_2, ..., y_r$ in $\widetilde{\mathcal{M}}(y)$ such that $\widetilde{\mathcal{R}}(w_i) = \widetilde{\mathcal{R}}(y_i)$ for every $i \in [r]$ and if the above condition still holds when the role of w and y are interchanged.

For any $i \in [0, n-1]$, let $D_R(i)$ be the set of all $w \in \widetilde{C}_n$ such that $|\{t_i, t_{i+1}\} \cap \mathcal{R}(w)| = 1$. Regarding $\widetilde{C}_n$ as a subset of $\widetilde{A}_{2n-1}$, $w \in \widetilde{C}_n$ is in $D_R(i)$ if and only if w is in $\widetilde{D}_R(i)$ if and only if w is in $\widetilde{D}_R(2n-i-1)$. When $w \in D_R(i)$, exactly one of wt_i and wt_{i+1} is in $D_R(i)$ unless that

$i \in \{0, n-1\}$ and $w = xy$ with $x, y \in \widetilde{C}_n$ satisfying $\mathcal{R}(x) \cap \{t_i, t_{i+1}\} = \emptyset$ and $y \in \{t_i t_{i+1}, t_{i+1} t_i\}$. In this excepted case, both wt_i and wt_{i+1} are in $D_R(i)$. When $|\{wt_i, wt_{i+1}\} \cap D_R(i)| = 1$, denote by w^* the unique element in $\{wt_i, wt_{i+1}\} \cap D_R(i)$, then w^* can be obtained from w by a pair of right star operations if $i \in [n-2]$ and, by a single right star operation if $w^* = wt_m$ with $i \in \{0, n-1\}$ and $m \in \{0, n\}$ and, by none of the above two ways if $i \in \{0, n-1\}$ and $w^* = wt_m$ with $m \in \{1, n-1\}$. When $\{wt_i, wt_{i+1}\} \subset D_R(i)$, define w_1^*, w_2^* by the conditions $\{w_1^*, w_2^*\} = \{wt_i, wt_{i+1}\}$ and $w_1^* < w_2^*$, then $x \in \{w_1^*, w_2^*\}$ can be obtained from w by one right star operation if $x = wt_m$ with $m \in \{0, n\}$ and, not by one or two right star operation if $x = wt_m$ with $m \in \{1, n-1\}$.

In the remaining part of the paper, when we mention a right star operation and the generalized τ-invariants on $w \in \widetilde{C}_n$, we always regard w as an element of $\widetilde{A}_{2n-1}$. We make such a convention once and forever.

Examples 2.10. (1) The elements $x = t_0$, $y = t_0 t_1$ and $z = t_0 t_1 t_0$ are in $D_R(0)$. We have $x^* = z^* = y$ and $y_1^* = x$ and $y_2^* = z$. The elements y, z can be obtained from one to another by one right $\{s_0, s_1\}$-star operation, and also by one right $\{s_0, s_{2n-1}\}$-star operation, but x, y can't be obtained from one to another by one or two right star operation.

(2) Assume $n > 2$. The elements $x = t_1$ and $y = t_1 t_2$ are in $D_R(1)$. We have $x^* = y$ and $y^* = x$. The elements x, y can be obtained from one to another by a right $\{s_1, s_2\}$-star operation followed by a right $\{s_{2n-2}, s_{2n-1}\}$-star operation.

2.11. For any $w \in \widetilde{C}_n$, define $M(w)$ to be the set of all $y \in \widetilde{C}_n$ such that there exists a sequence $x_0 = w, x_1, ..., x_r = y$ with some $r \geqslant 0$ such that for every $i \in [r]$, $x_i^{-1} x_{i-1} \in S$ and x_i can be obtained from x_{i-1} by one or two right star operation. Define a graph $\mathcal{M}(w)$ as follows. Its vertex set is $M(w)$. Each $x \in M(w)$ is labeled by $\mathcal{R}(x)$. Two elements $x, y \in M(w)$ are joined by a solid edge if $x^{-1} y \in S$ and x can be obtained from y by one or two right star operation.

It is easy to see that if $y, w \in \widetilde{C}_n$ have the same generalized τ-invariants, then for any path $w_1 = w, w_2, ..., w_r$ in $\mathcal{M}(w)$, there exists a path $y_1 = y, y_2, ..., y_r$ in $\mathcal{M}(y)$ such that $\mathcal{R}(w_i) = \mathcal{R}(y_i)$ for every $i \in [r]$ and the above condition still holds when the role of w and y are interchanged. In Section 6, the graphs $\mathcal{M}(w)$ with $w \in \widetilde{C}_3$ will be used to confirm that two elements of $\widetilde{C}_3$ have different generalized τ-invariants.

Example 2.12. In Figure 12 (see Section 6), the vertices $[3, 2, 0]$, $[4, 2, 0]$ and $[4, 1, -1]$ represent three elements $x, y, z \in \widetilde{C}_3$ with labels $\mathcal{R}(x) = \{t_0, t_2\}$ and $\mathcal{R}(y) = \{t_0, t_3\}$ and $\mathcal{R}(z) = \{t_1, t_3\}$, where we use a boldfaced letter $\mathbf{i}$ to denote the generator t_i, hence, for example, the notation $\boxed{\mathbf{13}}$ stands for the set $\{t_1, t_3\}$. x and y are joined by a solid edge since $x^{-1}y = t_3 \in S$ and y can be obtained from x by a right $\{s_2, s_3\}$-star operation.

$\mathcal{M}(y)$ and $\mathcal{M}(z)$ are two different graphs without any common vertex. We use a dashed edge to join the vertices y and z for indicating the fact that $y^{-1}z \in S$ but z can't be obtained from y by one or two right star operation.

We see that no two vertices in Figure 12 have the same generalized τ-invariants.

2.13. For any $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and $\mu = (\mu_1, \mu_2, ..., \mu_t)$ in Λ_{2n}, we write $\lambda \leqslant \mu$ if $\lambda_1 + \cdots + \lambda_k \leqslant \mu_1 + \cdots + \mu_k$ for any $1 \leqslant k \leqslant \min\{r, t\}$. This defines a partial order on Λ_{2n}. It is well known that if $x \in \widetilde{A}_{2n-1}$ and $s \in \widetilde{\mathcal{L}}(x)$ and $t \in \widetilde{\mathcal{R}}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$ (see [9, Lemma 5.5 and Corollary 5.6]). This implies by Corollary 2.6 that if $x \in \widetilde{C}_n$ and $s \in \mathcal{L}(x)$ and $t \in \mathcal{R}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$.

Let $\widetilde{a}$, a be the a-functions of the weighted Coxeter groups $(\widetilde{A}_{2n-1}, \widetilde{\ell})$,x $(\widetilde{C}_n, \widetilde{\ell})$, respectively (see 2.3 and 1.6).

Lemma 2.14. (see [8, Lemma 16.5]) *$a(z) = \widetilde{a}(z)$ for any $z \in \widetilde{C}_n$.*

Lemma 2.15. (see [8, Lemma 16.14]) *Let $x, y \in \widetilde{C}_n$. Then $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{A}_{2n-1}$.*

By Lemma 2.15, we can just use the notation $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$)

for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to the group $\widetilde{A}_{2n-1}$ or $\widetilde{C}_n$.

For any $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{2n}$, define $\mu = (\mu_1, ..., \mu_t) \in \Lambda_{2n}$ by setting $\mu_j = {}^{\#}\{k \in [r] \mid \lambda_k \geqslant j\}$ for any $j \geqslant 1$, call μ the *dual partition* of λ.

Lemma 2.16. *Let $x, y \in \widetilde{A}_{2n-1}$.*

(1) $x \underset{L}{\sim} y$ if and only if x, y have the same generalized τ-invariants (see [9, Theorem 16.1.2]).

(2) $x \underset{LR}{\leqslant} y$ if and only if $\psi(y) \leqslant \psi(x)$. The set $\psi^{-1}(\lambda)$ forms a two-sided cell of $\widetilde{A}_{2n-1}$ for any $\lambda \in \Lambda_{2n}$ (see [7, Theorem 6] and [9, Theorem 17.4] and [11, Theorem B]).

(3) $\widetilde{a}(x) = \sum_{i=1}^{t}(i-1)\mu_i$, where $(\mu_1, ..., \mu_t)$ is the dual partition of $\psi(x)$ (see [10, Subsection 6.27]).

2.17. A non-empty subset E of a Coxeter group $W = (W, S)$ is said *left-connected*, (respectively, *right-connected*) if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that $x_{i-1}x_i^{-1} \in S$ (respectively, $x_i^{-1}x_{i-1} \in S$) for every $i \in [r]$. E is said *two-sided-connected* if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that either $x_{i-1}x_i^{-1}$ or $x_i^{-1}x_{i-1}$ is in S for every $i \in [r]$.

Let $F \subseteq E$ in W. Call F a *left-connected component* of E, if F is a maximal left-connected subset of E. One can define a right-connected component and a two-sided-connected component of E similarly.

For any $\lambda \in \Lambda_{2n}$, denote $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

Lemma 2.18. *Let $\lambda \in \Lambda_{2n}$.*

(1) Any left- (respectively, right-, two-sided-) connected component of $\psi^{-1}(\lambda)$ is contained in some left (respectively, right, two-sided) cell of $\widetilde{A}_{2n-1}$.

(2) Any left- (respectively, right-, two-sided-) connected component of E_λ is contained in some left (respectively, right, two-sided) cell of $\widetilde{C}_n$.

(3) The set E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$.

Proof. The assertions (1)-(2) follow by 1.6 (1)-(2), Lemmas 2.14 and 2.16. Now we consider (3). By Lemmas 2.15-2.16, we see that E_λ is either empty or a union of some left cells of $\widetilde{C}_n$ with $E_\lambda^{-1} = E_\lambda$ for any $\lambda \in \Lambda_{2n}$. So the assertion (3) follows by Lemmas 2.16 (3) and 1.7. □

Corollary 2.19. *Let $x, y \in \widetilde{A}_{2n-1}$ satisfy $x, y \in \psi^{-1}(\lambda)$ for some $\lambda \in \Lambda_{2n}$.*

(1) If $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(yx^{-1})$ then x, y are in the same left-connected component of $\psi^{-1}(\lambda)$ and hence $x \underset{L}{\sim} y$.

(2) If $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(x^{-1}y)$ then x, y are in the same right-connected component of $\psi^{-1}(\lambda)$ and hence $x \underset{R}{\sim} y$.

Let $x, y \in \widetilde{C}_n$ be in E_λ for some $\lambda \in \Lambda_{2n}$.

(3) If $\ell(y) = \ell(x) + \ell(yx^{-1})$ then x, y are in the same left-connected component of E_λ and hence $x \underset{L}{\sim} y$.

(4) If $\ell(y) = \ell(x) + \ell(x^{-1}y)$ then x, y are in the same right-connected component of E_λ and hence $x \underset{R}{\sim} y$.

Proof. By symmetry, we need only to show (1) and (3).

(1) Let $yx^{-1} = s_{i_r} s_{i_{r-1}} \cdots s_{i_2} s_{i_1}$ be a reduced expression of yx^{-1} with $s_{i_j} \in \widetilde{S}$. Let $x_k = s_{i_k} s_{i_{k-1}} \cdots s_{i_2} s_{i_1} x$ for $k \in [0, r]$, where we stipulate $x_0 = x$. Then $\widetilde{\ell}(x_k) = \widetilde{\ell}(x_{k-1}) + 1$ for any $k \in [r]$. Hence $\psi(x) = \psi(x_0) \leqslant \psi(x_1) \leqslant \cdots \leqslant \psi(x_r) = \psi(y) = \psi(x)$ by 2.13. This implies that x, y are in the same left-connected component of $\psi^{-1}(\lambda)$. Hence $x \underset{L}{\sim} y$ by Lemma 2.18.

(3) Let $yx^{-1} = t_{i_r} t_{i_{r-1}} \cdots t_{i_1}$ be a reduced expression of yx^{-1} with $t_{i_j} \in S$. Let $x_k = t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x$ for $k \in [0, r]$, where we stipulate $x_0 = x$. Then $\ell(x_k) = \ell(x_{k-1}) + 1$ for any $k \in [r]$. Hence $\psi(x) = \psi(x_0) \leqslant \psi(x_1) \leqslant \cdots \leqslant \psi(x_r) = \psi(y) = \psi(x)$ by 2.13. This implies that x, y are in the same left-connected component of E_λ. Hence $x \underset{L}{\sim} y$ by Lemmas 2.15 and 2.18. □

§3. Partial order $\preceq_w$ on $[2n]$ determined by an element w.

In this section, we introduce two technical tools. One is a transformation on an element in 3.3, which is a crucial step in proving the left-connectedness of a left cell and in finding a representative set for the left cells of $\widetilde{C}_n$ in the set E_λ, $\lambda \in \Lambda_{2n}$. The other is the generalized tabloids in 3.5, by which we can check if two elements of $\widetilde{C}_n$ are in the same left cell.

3.1. $i, j \in [2n]$ are said *$2n$-dual*, if $i+j = 2n+1$; in this case, we denote $j = \bar{i}$ (hence $i = \bar{j}$ also). Recall the partial order $\preceq_w$ on $[2n]$ defined in 2.2 for any $w \in \widetilde{A}_{2n-1}$ and that $\widetilde{C}_n$ can be regarded as a subset of $\widetilde{A}_{2n-1}$ (see 2.1). Fix $w \in \widetilde{A}_{2n-1}$. $i \neq j$ in $[2n]$ are said *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* if otherwise. When $w \in \widetilde{C}_n$, $i \in [2n]$ is said *w-wild* if i and $\bar{i}$ are w-comparable and *w-tame* if otherwise. $i \in [2n]$ is said *a w-wild head* (respectively, *a w-tame head*), if i is w-wild (respectively, w-tame) with $(\bar{i})w < (i)w$.

It is easily seen that $i < j$ in $[2n]$ are w-uncomparable if and only if $(i)w < (j)w < (i)w + 2n$.

A subset $E \subseteq [2n]$ is a *w-chain*, if E is totally ordered with respect to $\preceq_w$, i.e., there is an expression $E = \{i_1, i_2, ..., i_r\}$ with $i_1 \prec_w i_2 \prec_w \cdots \prec_w i_r$.

Lemma 3.2. *Fix $w \in \widetilde{C}_n$. Let $i, j, k \in [2n]$.*

(i) $j \prec_w k$ if and only if $\bar{k} \prec_w \bar{j}$;

Now suppose that $j \neq k$ are w-wild heads and i is w-tame.

(ii) $\bar{j} \prec_w k$ if and only if $\bar{j}, k$ are w-comparable.

(iii) If $\bar{j}, k$ are w-uncomparable then so are j, k (respectively, $\bar{j}, \bar{k}$);

(iv) i and k are w-comparable if and only if $i \prec_w k$.

(v) $\{j, i, \bar{j}\}$ is a w-chain if and only if j is w-comparable with both i and $\bar{i}$;

(vi) $\{j, k, \bar{j}, \bar{k}\}$ is a w-chain if and only if j, k are w-comparable.

Proof. The assertions (i)-(iv) can be checked directly. Then (v) follows by (i) and (iv). Finally, (vi) is a simple consequence of (i)-(iii). □

3.3. Let

$$t_{i,j} = t_{i+j-1} t_{i+j-2} \cdots t_{i+1} t_i, \qquad \text{(3.3.1)}$$
$$d_{i,j} = t_{i-j+1} t_{i-j+2} \cdots t_{i-1} t_i.$$

for any $i, j \in \mathbb{Z}$ with $j > 0$. Suppose that $x \in \widetilde{C}_n$ and $i \in \mathbb{Z}$ satisfy $(i)x - 2n > (j)x$ for any $i < j \leqslant i + a$ with some $a \in [2n-1]$. Let $x' = t_{i,a} x$.

Then $\ell(x') = \ell(x) - a$ and $\psi(x) = \psi(x')$. Moreover, if $(i)x - 2n > (j)x$ for any $i < j < i + 2n$, let $x'' = t_{i,2n}x$, then

$$(k)x'' = \begin{cases} (k)x - 2n, & \text{if } k \equiv i \pmod{2n}, \\ (k)x + 2n, & \text{if } k \equiv 2n - i \pmod{2n}, \\ (k)x, & \text{if otherwise.} \end{cases}$$

for any $k \in \mathbb{Z}$, where x'' satisfies $\ell(x'') = \ell(x) - 2n$ and $\psi(x) = \psi(x'')$.

Fix $w \in \widetilde{C}_n$. Suppose that $E_1 = \{i_1, i_2, ..., i_a\}$ and $E_2 = \{j_1, j_2, ..., j_b\}$ are two subsets of $[2n]$ satisfying that

(i) $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$ with $a > 0$ and $b \geqslant 0$ and $a + b = n$;

(ii) the elements of $E_1 \cup E_2$ are pairwise not $2n$-dual;

(iii) $(\bar{k})w < (k)w$ for any $k \in E_1 \cup E_2$;

(iv) If $b > 0$ then $(i)w - (j)w > 2ln$ for any $i \in E_1$ and $j \in E_2$; if $b = 0$ then $(i)w > (2l + 1)n$ for any $i \in E_1$, where l is some positive integer.

By repeatedly left multiplying various elements of the form $t_{i,j}$ on w, we can obtain some $w' \in \widetilde{C}_n$ such that there are some $1 \leqslant k_1 < k_2 < \cdots < k_b \leqslant 2b$ (the latter is only an empty condition in the case of $b = 0$) satisfying that

(1) $\ell(w') = \ell(w) - \ell(ww'^{-1})$;

(2) If $b > 0$ then $[2b] = \{k_1, k_2, ..., k_b, 2b+1-k_1, 2b+1-k_2, ..., 2b+1-k_b\}$ and the map $\phi : \{j_1, j_2, ..., j_b, \bar{j}_1, \bar{j}_2, ..., \bar{j}_b\} \longrightarrow [2b]$ given by $\phi(j_m) = k_m$ and $\phi(\bar{j}_m) = 2b + 1 - k_m$ for $m \in [b]$ is an order-preserving bijection.

(3) $(p)w' = (i_p)w - 2l'n$ and $(a+k_q)w' = (j_q)w$ for any $p \in [a]$ and $q \in [b]$, where $l' \in \mathbb{Z}$ satisfy $l' \geqslant l$;

(4) $(\overline{\langle c\rangle})w' < (\langle c\rangle)w'$ for any $c \in [a] \cup \{a + k_m \mid m \in [b]\}$;

(5) If $b > 0$ then $0 < \min\{(c)w' - (a + k_m)w' \mid c \in [a], m \in [b]\} < 2n$; if $b = 0$ then $n < \min\{(c)w' \mid c \in [n]\} \leqslant 3n$.

We see by Lemma 3.2 that $\psi(w') = \psi(w)$ (denote the common partition by λ) and by Corollary 2.19 that w, w' are in the same left-connected component of E_λ.

Example 3.4. (a) Let

$$w=[8,30,4,-11,27,2]\in\widetilde{C}_6.$$

Then $E_1=\{2,5,9\}$ and $E_2=\{1,7,10\}$ satisfy the conditions (i)-(iv) in 3.3 with $n=6$ and $(a,b,l)=(3,3,1)$. Let $w'=t_{4,9}t_{5,8}t_{9,4}w$. Then

$$w'=[18,15,12,8,4,2]\in\widetilde{C}_6.$$

Hence w' satisfies the conditions (1)-(5) in 3.3 with $b>0$ and $\psi(w')=\psi(w)=\mathbf{93}$.

(b) Let

$$w=[20,30,-8,-11,27,-10]\in\widetilde{C}_6.$$

Then $E_1=\{1,2,5,7,9,10\}$ and $E_2=\emptyset$ satisfy the conditions (i)-(iv) in 3.3 with $n=6$ and $(a,b,l)=(6,0,1)$. Let $w'=t_{6,7}t_{6,7}t_{6,7}t_{7,6}t_{9,4}t_{10,3}w$. Then

$$w'=[8,18,15,11,12,9]\in\widetilde{C}_6.$$

Hence w' satisfies the conditions (1)-(5) in 3.3 with $b=0$ and $\psi(w')=\psi(w)=\mathbf{82^2}$.

3.5. By a *composition* of $2n$, we mean an r-tuple $(a_1,a_2,...,a_r)$ of positive integers $a_1,...,a_r$ with some $r\geqslant 1$ such that $\sum_{i=1}^r a_i=2n$. Let $\widetilde{\Lambda}_{2n}$ be the set of all compositions of $2n$. Clearly, $\Lambda_{2n}\subseteq\widetilde{\Lambda}_{2n}$.

A generalized tabloid of rank $2n$ is, by definition, an r-tuple $\mathbf{T}=(T_1,T_2,...,T_r)$ with some $r\in\mathbb{N}$ such that $[2n]$ is a disjoint union of its non-empty subsets T_j, $j\in[r]$. We have $\xi(\mathbf{T}):=(|T_1|,|T_2|,...,|T_r|)\in\widetilde{\Lambda}_{2n}$, where $|T_i|$ denotes the cardinal of the set T_i. Let $i_1,i_2,...,i_r$ be a permutation of $1,2,...,r$ such that $|T_{i_1}|\geqslant|T_{i_2}|\geqslant\cdots\geqslant|T_{i_r}|$. Then $\zeta(\mathbf{T}):=(|T_{i_1}|,|T_{i_2}|,...,|T_{i_r}|)\in\Lambda_{2n}$. Two generalized tabloids $\mathbf{T}=(T_1,...,T_r)$ and $\mathbf{T}'=(T'_1,...,T'_t)$ of $2n$ are said *equal* if and only if $r=t$ and $T_i=T'_i$ for any $i\in[r]$. Let $\mathcal{C}_{2n}$ be the set of all generalized tabloids of rank $2n$. Then both $\xi:\mathcal{C}_{2n}\longrightarrow\widetilde{\Lambda}_{2n}$ and $\zeta:\mathcal{C}_{2n}\longrightarrow\Lambda_{2n}$ are surjective maps.

Let Ω be the set of all $w\in\widetilde{A}_{2n-1}$ such that there is a generalized tabloid $\mathbf{T}=(T_1,T_2,...,T_r)\in\mathcal{C}_{2n}$ satisfying:

(i) For any $i < j$ in $[r]$, we have $\langle (a)w^{-1}\rangle \prec_w \langle (b)w^{-1}\rangle$ for any $a \in T_i$ and $b \in T_j$;

(ii) For any $i \in [r]$, $\langle (a)w^{-1}\rangle$ and $\langle (b)w^{-1}\rangle$ are w-uncomparable for any $a \neq b$ in T_i.

Clearly, $\mathbf{T}$ is determined entirely by $w \in \Omega$, denote $\mathbf{T}$ by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{2n}$ is surjective by [9, Proposition 19.1.2]. By a result of Curtis Greene in [3], we see that the partition $\zeta(T(w))$ is the dual of $\psi(w)$.

The following known result will be crucial in subsequent discussion.

Lemma 3.6. (see [9, Lemma 19.4.6]) *Suppose that $y, w \in \widetilde{A}_{2n-1}$ are two elements in Ω with $\xi(T(y)) = \xi(T(w))$. Then $y \underset{L}{\sim} w$ if and only if $T(y) = T(w)$.*

§4. The set $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$.

Recall that in 2.17 we defined the set E_λ for any $\lambda \in \Lambda_{2n}$. We have $E_\lambda^{-1} = E_\lambda$. The group automorphism η of $\widetilde{C}_n$ (see 2.7) stabilizes each E_λ for any $\lambda \in \Lambda_{2n}$.

In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$ for all $k \in [2n]$. $E_{\mathbf{1}^{\mathbf{2n}}}$ consists of the identity element of $\widetilde{C}_n$. In the subsequent discussion, we shall always assume $k > 1$ though the most results still hold without such a restriction.

4.1. First assume $k = 2m + 1$ odd with some $m \in \mathbb{N}$. Then $2n - k = 2l - 1$ is also odd with $m + l = n$. By Lemma 3.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$ if and only if w satisfies the condition (4.1.1) below.

(4.1.1) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m \in [2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$ and with either $\bar{i}_1, i_1 \prec_w j_1$ or $\bar{i}_l, i_l \prec_w j_1$.

Let F_1^o (respectively, F_2^o) be the set of all $w \in \widetilde{C}_n$ satisfying the condition (4.1.2) below.

(4.1.2) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m \in [2n]$

such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $0 < (j_{a+1})w - (j_a)w < 2n$ for any $a \in [m-1]$; (iii) $(i_1)w < (j_1)w < (\bar{i}_l)w + 2n$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$ (respectively, $(\bar{i}_l)w + 2n < (j_1)w < (i_1)w + 2n$ and $(i_1, i_2, ..., i_{l-1}, j_m, j_{m-1}, ..., j_1, i_l) = (n+1, n+2, ..., 2n)$).

4.2. Next assume $k = 2m$ even with some $m \geqslant 1$. Then $2n - k = 2l$ is also even with $m + l = n$. By Lemma 3.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{\mathbf{k1^{2n-k}}}$ if and only if w satisfies the condition (4.2.1) below.

(4.2.1) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m \in [2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$; (iii) j_1 is w-uncomparable with all of $i_a, \bar{i}_a$, $a \in [l]$.

If $m = n$ then (4.2.1) (iii) is an empty condition. Now assume $m < n$. Under the assumption of (4.2.1) (i)-(ii), the condition (4.2.1) (iii) is equivalent to that either $\bar{i}_1 < j_1 < i_1$ and $(\bar{i}_1)w < (j_1)w < (i_1)w$, or $i_l < j_1 < \bar{i}_l + 2n$ and $(i_l)w < (j_1)w < (\bar{i}_l)w + 2n$. Under the assumption of j_1 being a w-wild head, this is also equivalent to that either $\bar{i}_1 < j_1 \leqslant n$ and $n < (j_1)w < (i_1)w$, or $i_l < j_1 \leqslant 2n$ and $2n < (j_1)w < (\bar{i}_l)w + 2n$. Let $E'_{\mathbf{k1^{2n-k}}}$ (respectively, $E''_{\mathbf{k1^{2n-k}}}$) be the set of all $w \in E_{\mathbf{k1^{2n-k}}}$ such that $\bar{i}_1 < j_1 \leqslant n$ and $n < (j_1)w < (i_1)w$ (respectively, $i_l < j_1 \leqslant 2n$ and $2n < (j_1)w < (\bar{i}_l)w + 2n$). Then $E_{\mathbf{k1^{2n-k}}}$ is a disjoint union of the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$.

Let F_1^e (respectively, F_2^e) be the set of all $w \in \widetilde{C}_n$ satisfying the condition (4.2.2) below.

(4.2.2) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m \in [2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $0 < (j_{a+1})w - (j_a)w < 2n$ for any $a \in [m-1]$; (iii) $n < (j_1)w < (i_1)w$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ (respectively, $2n < (j_1)w < (\bar{i}_l)w + 2n$ and $(i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1) = (n+1, n+2, ..., 2n)$).

The sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ (respectively, F_1^e and F_2^e) can also be defined by the condition (4.2.1) (respectively, (4.2.2)) in the case of $m = n$ if we stipulate that $i_1 = (i_1)w = 2n+1$, $\bar{i}_1 = (\bar{i}_1)w = 0$, $i_l = (i_l)w = n$ and $\bar{i}_l = (\bar{i}_l)w = n+1$.

Clearly, $F_1^e \subset E'_{\mathbf{k1^{2n-k}}}$ and $F_2^e \subset E''_{\mathbf{k1^{2n-k}}}$. Also, $E'^{-1}_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}}$ and $E''^{-1}_{\mathbf{k1^{2n-k}}} = E''_{\mathbf{k1^{2n-k}}}$ if $k \in [2n-2]$ is even.

By 3.3 and 4.1-4.2, it is easily seen that

Lemma 4.3. *$F_1^\epsilon \cup F_2^\epsilon \subset E_{\mathbf{k1^{2n-k}}}$, where ϵ is o if k is odd and e if k is even. For any $w \in E_{\mathbf{k1^{2n-k}}}$, there exists some $w' \in F_1^\epsilon \cup F_2^\epsilon$ such that w', w are in the same left-connected component of $E_{\mathbf{k1^{2n-k}}}$.*

Recall the group automorphism η of $\widetilde{C}_n$ defined in 2.7.

Lemma 4.4. *Let ϵ be as in Lemma 4.3.*

(1) The map η interchanges the sets F_1^ϵ and F_2^ϵ.

(2) If $k \in [2n-2]$ is even, then $E'^{-1}_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}}$ and $E''^{-1}_{\mathbf{k1^{2n-k}}} = E''_{\mathbf{k1^{2n-k}}}$. The map η interchanges the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$.

(3) Each of F_1^ϵ and F_2^ϵ is contained in a right-connected component of $E_{\mathbf{k1^{2n-k}}}$.

Proof. The assertions (1)-(2) follow by Proposition 2.8 (3) and by the discussion in 4.1-4.2. Hence to show the assertion (3), we need only to show that F_1^ϵ is contained in a right-connected component of $E_{\mathbf{k1^{2n-k}}}$.

(I) First assume ϵ being o. Then $w^{(o)} := t_n w_J = [1, 2, ..., l-1, 2n+1-l, 2n-l, ..., n+2, n]$ is the unique shortest element in F_1^o, where $J = \{t_l, t_{l+1}, ..., t_n\}$. Take any $w \in F_1^o$. Keep the notation in (4.1.2).

(a) First assume $(j_a)w - (j_{a-1})w = 1$ for any $a \in [2, m]$. By the condition $(i_1)w < (j_1)w < (\bar{i}_l)w + 2n$, there exists a largest $b \in [l]$ satisfying $(i_b)w < (j_1)w$. If $b = l$ then w is one of two elements $w_1 = [n+1-l, n+2-l, ..., n-1, 2n, 2n-1, ..., n+l+1, n]$ and $w_2 = [n+1-l, n+2-l, ..., n-1, 3n-l, 3n-l-1, ..., 2n+1, n]$. Let $J = \{t_1, t_2, ..., t_{n-2}\}$, $J_1 = J \setminus \{t_{n-l}\}$, $I_1 = \{t_1, t_2, ..., t_{n-l-1}\}$ and $I = I_1 \cup \{t_0\}$. Then $w_2 = w_1 w_{I_1} w_I$ and $w_1 =$

$w^{(o)}w_Jw_{J_1}$ satisfy $\ell(w_2)=\ell(w_1)+\ell(w_{I_1}w_I)$ and $\ell(w_1)=\ell(w^{(o)})+\ell(w_Jw_{J_1})$ by Proposition 2.8 (2). Hence w_1, w_2 and $w^{(o)}$ are in the same right-connected component of $E_{\mathbf{k1^{2n-k}}}$ by Corollary 2.19. Now assume $b<l$. Then by the condition $(i_1)w<(j_1)w<(\bar{i}_l)w+2n$, we have $w=[1,2,...,l-b,n+1-b,n+2-b,...,n-1,2n+b-l,2n+b-l-1,...,n+b+1,n]$. Let $J=\{t_{l+1-b},t_{l+2-b},...,t_{n-2}\}$, $J_1=J\backslash\{t_{n-b}\}$. Then $w=w^{(o)}w_Jw_{J_1}$ satisfies $\ell(w)=\ell(w^{(o)})+\ell(w_Jw_{J_1})$ by Proposition 2.8 (2). Hence w and $w^{(o)}$ are in the same right-connected component of $E_{\mathbf{k1^{2n-k}}}$ by Corollary 2.19.

(b) Next assume $(j_a)w-(j_{a-1})w>1$ for some $a\in[2,m]$. Let a be the largest integer with such a property. Then $d:=(j_a)w-1$ should be congruent to one (say $(k)w$) of $(j_b)w$, $(\bar{j}_b)w$, $(i_c)w$, $(\bar{i}_c)w$, $(\bar{j}_a)w$ modulo $2n$ for some $b\in[a-1]$ and $c\in[l]$. When $d\not\equiv(\bar{j}_a)w$ modulo $2n$, let $y_1=wt_dt_{d+1}\cdots t_{m+d-a}$. Then for any $t\in\mathbb{Z}$, we have

$$(t)y_1=\begin{cases}(t)w-1, & \text{if } t\equiv(j_h)w\,(\text{mod } 2n)\ \text{ for some } h\in[a,m],\\ (t)w+1, & \text{if } t\equiv(\bar{j}_h)w\,(\text{mod } 2n)\ \text{ for some } h\in[a,m],\\ (t)w+(m+1-a), & \text{if } t\equiv(k)w\,(\text{mod } 2n),\\ (t)w-(m+1-a), & \text{if } t\equiv(\bar{k})w\,(\text{mod } 2n),\\ (t)w, & \text{if otherwise.}\end{cases}$$

We see that either $(j_h)w-(k)w>2n$ for all $h\in[a,m]$, or $k=i_f$ with $(j_a)w=(i_f)w+1$ for some $f\in[l]$ (hence $j_h<k$ for any $h\in[a,m]$ in the latter case) by the condition (4.1.2) on $w\in F_1^o$ and by the choice of a. We see by Corollary 2.6 and Proposition 2.8 (2) that $\ell(y_1)=\ell(w)-(m+1-a)$ and $y_1\in F_1^o$. When $d\equiv(\bar{j}_a)w$ (mod $2n$), we have $d\equiv n,0$ (mod $2n$) and $(j_a)w-(\bar{j}_a)w>2n$. In this case, let $y_1=ww_{J_1}w_J$ with $J=\{t_n,t_{n-1},...,t_{n+a-m}\}$ and $J_1=J\setminus\{t_n\}$ if $d\equiv n$ (mod $2n$) and $J=\{t_0,t_1,...,t_{m-a}\}$ and $J_1=J\setminus\{t_0\}$ if $d\equiv 0$ (mod $2n$). Again by Corollary 2.6 and Proposition 2.8 (2), we have $\ell(y_1)=\ell(w)-(w_{J_1}w_J)$ and $y_1\in F_1^o$. By applying induction on $p:=\ell(w)\geqslant\ell(w^{(o)})$, we see that there exists a sequence $y_0=w,y_1,...,y_r$ in F_1^o with some $r\geqslant 0$ such that $\ell(y_h)=\ell(y_{h-1})-\ell(y_{h-1}^{-1}y_h)$ for every $h\in[r]$

and that $(j_a)y_r - (j_{a-1})y_r = 1$ for any $a \in [2, m]$. This implies by Corollary 2.19 that $y_0, y_1, ..., y_r$ are in the same right-connected component of $E_{\mathbf{k1^{2n-k}}}$. Since y_r and $w^{(o)}$ are in the same right-connected component of $E_{\mathbf{k1^{2n-k}}}$ by (a), F_1^o is contained in a right-connected component of $E_{\mathbf{k1^{2n-k}}}$.

(II) Next assume ϵ being e. Then $w^{(e)} := w_J = [1, 2, ..., l, 2n - l, 2n - l - 1, ..., n + 1]$ is the unique shortest element in F_1^e with $J = \{t_{l+1}, t_{l+2}, ..., t_n\}$, which is also the unique element w in F_1^e satisfying the condition $(j_a)w - (j_{a-1})w = 1$ for any $a \in [2, m]$, where $l, m, j_1, j_2, ..., j_m$ are given as in (4.2.2). Now take any $w \in F_1^e \setminus \{w^{(e)}\}$. Then there exists some $a \in [2, m]$ with $(j_a)w - (j_{a-1})w > 1$. Let a be the largest integer with such a property. Hence by applying the same argument as that in (I) (b), we can find a sequence $y_0 = w, y_1, ..., y_r = w^{(e)}$ in F_1^e with some $r \geqslant 0$ such that $\ell(y_h) = \ell(y_{h-1}) - \ell(y_{h-1}^{-1}y_h)$ for every $h \in [r]$. This implies by Corollary 2.19 that F_1^e is contained in a right-connected component of $E_{\mathbf{k1^{2n-k}}}$. $\square$

Lemma 4.5. *For $k \in [2, 2n]$, $\epsilon \in \{o, e\}$ and $i = 1, 2$, let $F_i^\epsilon \subset E_{\mathbf{k1^{2n-k}}}$ be defined as in 4.1-4.2.*

(1) $|F_1^o| = |F_2^o| = 2^{m-1}n!/(n - m)!$ if $k = 2m + 1$.

(2) $|F_1^e| = |F_2^e| = 2^{m-1}n!/(n - m + 1)!$ if $k = 2m$.

Note: The above formulae can be combined into the following one:

$$|F_1^\epsilon| = |F_2^\epsilon| = 2^{\lfloor \frac{k}{2} \rfloor - 1}n!/(n - \lfloor \frac{k-1}{2} \rfloor)!$$

Proof. We have $|F_1^\epsilon| = |F_2^\epsilon|$ by Lemma 4.4.

First assume ϵ being o. Let us enumerate the set $F^o := F_1^o \cup F_2^o$.

Let G^o be the set of all $w \in \widetilde{C}_n$ satisfying the condition (4.1.2) but with the condition (iii) replaced by (iii)$'$ below:

(iii)$'$ $(i_1)w < (j_1)w < (i_1)w + 2n$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$.

We see that $F_1^o \subset G^o$ and that there exists a bijection $\lambda_x : G^o \setminus F_1^o \longrightarrow F_2^o$ defined by $\lambda_x(w) = xw$, where, if $l > 1$ then let $x = w_I w_{I_1} w_J w_{J_1} t_n$ with $J = \{t_l, t_{l+1}, ..., t_n\}$, $J_1 = J \setminus \{t_{n-1}, t_n\}$, $I = \{t_2, t_3, ..., t_{n-2}\}$ and $I_1 = I \setminus \{t_{l-1}\}$;

if $l = 1$ then $x = w_{J_1} w_J d_{n-1,n-1}$ (see (3.3.1)) with $J = \{t_2, t_3, ..., t_n\}$ and $J_1 = J \setminus \{t_n\}$.

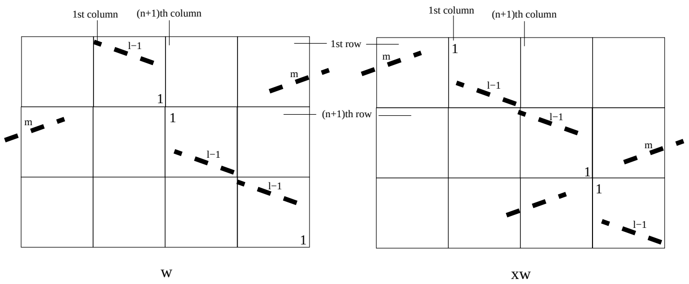
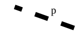
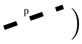


Figure 2

Figure 2 displays the corresponding parts for the matrix forms of w and xw in the case of $l > 1$, where the symbol  (respectively, ) stands for a rectangular submatrix with p rows each row has a unique non-zero entry, which is 1, the non-zero entries are going down to the right (respectively, to the left).

Note that in either case, we have $\ell(xw) = \ell(w) - \ell(x)$ and $\psi(w) = \psi(xw)$, hence $xw \underset{L}{\sim} w$ by Corollary 2.19 and Lemma 2.18.

Now we enumerate the set G^o. Any $w \in G^o$ is determined entirely by the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$, $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ under the conditions (4.1.2) (i)-(ii) and (iii)$'$. There are $\binom{n}{l} = \dfrac{n!}{l!(n-l)!}$ different choices for the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$ by the condition $n < (i_1)w < (i_2)w < \cdots < (i_l)w \leqslant 2n$. Once they are fixed, the numbers of different choices for $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ are $2m, 2(m-1), ..., 2$ in turn by the conditions (4.1.2) (i)-(ii), (iii)$'$ and the facts that $m+l = n$ and $b \not\equiv c, 2n-c \pmod{2n}$ for any $b \neq c$ in $\{(i_1)w, ..., (i_l)w, (j_1)w, ..., (j_m)w\}$. This implies that $|G^o| = \binom{n}{l} 2^m m!$ and hence (1) is proved by the facts $|F_1^o| = |F_2^o| = \frac{1}{2}|G^o|$ and $m + l = n$.

Next assume ϵ being e. We need only to enumerate the set F_1^e by Lemma 4.4. Then $w \in F_1^e$ is determined entirely by the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$, $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ under the condition (4.2.2). There are $\binom{n}{l+1}$ different choices for the integers $(j_1)w$, $(i_1)w$, $(i_2)w$, ..., $(i_l)w$ by the condition $n < (j_1)w < (i_1)w < (i_2)w < \cdots < (i_l)w \leqslant 2n$. Once they are fixed, the numbers of different choices for $(j_2)w$, $(j_3)w$, ..., $(j_m)w$ are $2(m-1), 2(m-2), ..., 2$ in turn by the condition (4.2.2) and the facts that $m+l=n$ and $b \not\equiv c, 2n-c \pmod{2n}$ for any $b \neq c$ in $\{(i_1)w, ..., (i_l)w, (j_1)w, ..., (j_m)w\}$. This implies that $|F_1^e| = \binom{n}{l+1}2^{m-1}(m-1)!$ and hence (2) is proved by the fact $m+l=n$. $\square$

Lemma 4.6. *No two elements of $F_1^\epsilon \cup F_2^\epsilon$ are in the same left cell of $\widetilde{C}_n$.*

Proof. First assume ϵ being o. Let $w \in F_1^o$ be as in (4.1.2). If $l=1$ then let $w'=w$; if $l>1$ then let $w' = w_{J_6}w_{J_7}t_n w_{J_1}w_{J_5}w_{J_3}w_{J_4}w_{J_1}w_{J_2}w$, where $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \setminus \{t_{l-1}\}$, $J_3 = \{t_0, t_1, ..., t_{m-1}\}$, $J_4 = J_3 \setminus \{t_0\}$, $J_5 = J_1 \setminus \{t_m\}$, $J_6 = \{t_l, t_{l+1}, ..., t_n\}$ and $J_7 = J_6 \setminus \{t_n, t_{n-1}\}$ (see Figure 3). Regarding w' as an element of $\widetilde{A}_{2n-1}$, we have $w' \in \Omega$ (see 3.5), which satisfies $\psi(w) = \psi(w')$ and $\ell(w') = \ell(w) + \ell(w'w^{-1})$, hence $w \underset{L}{\sim} w'$ by Corollary 2.19.

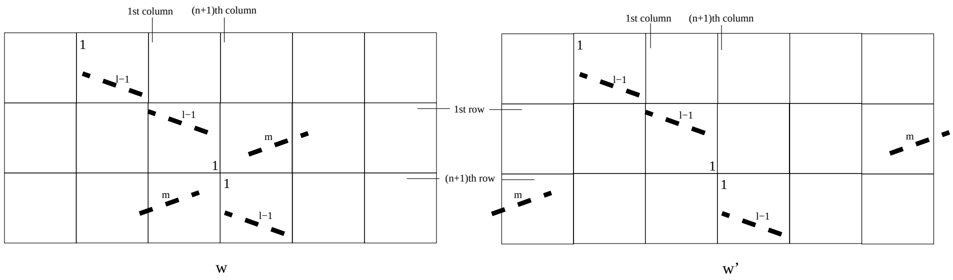


Figure 3

We see that $T(w') = (T_1, T_2, ..., T_{2m+1})$ with $T_c = (\langle(\bar{j}_{m+1-c})w\rangle)$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$ (see 2.7 for the notation $\langle q\rangle$ with $q \in \mathbb{Z}$).

Similarly, for $w \in F_2^o$ as in (4.1.2), we can find some element w' of $\widetilde{C}_n$ satisfying $w \underset{L}{\sim} w'$ and $w' \in \Omega$ as an element of $\widetilde{A}_{2n-1}$. We again get $T(w') = (T_1, T_2, ..., T_{2m+1})$ with $T_c = (\langle(\bar{j}_{m+1-c})w\rangle)$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$.

We see that the above $T(w')$ with $w' \underset{L}{\sim} w$ and $w' \in \Omega$ depends only on $w \in F_1^o \cup F_2^o$ but not on the choice of w' in Ω. So we can denote $T(w')$ by

$T(w)$. We claim that $T(w)$ should be pairwise different in $\mathcal{C}_{2n}$ as w ranges over $F_1^o \cup F_2^o$. For, recall that in the proof of Lemma 4.5, there is a bijective map τ from G^o to $F_1^o \cup F_2^o$ which satisfies $w \underset{L}{\sim} \tau(w)$ for any $w \in G^o$. We see that the $(2m+1)$-tuple $(T_1, T_2, ..., T_{2m+1})$, with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$, should be pairwise different in $\mathcal{C}_{2n}$ as w ranges over G^o. This proves our assertion in the case of ϵ being o by Lemma 3.6.

Next assume ϵ being e. If $m = n$, then $F_1^e \cup F_2^e \subseteq \Omega$. The set $\{T(w) \mid w \in F_1^e \cup F_2^e\}$ is in 1-1 correspondence with the set $\{(\{a_1\}, \{a_2\}, ..., \{a_{2n}\}) \mid \{a_1, a_2, ..., a_{2n}\} = [2n]; a_i + a_{2n+1-i} = 2n+1, \forall\ i \in [n]\}$. So our result in this case follows by Lemmas 3.6 and 2.15. Now assume $m < n$. Let $w \in F_1^e$ be as in (4.2.2). If $m = 1$ then let $w' = w_J w_{J_1} w$ with $J = \widetilde{S} \setminus \{s_n\}$ and $J_1 = J \setminus \{s_{n+1}\}$; if $m > 1$ then let $w' = w_{I_3} w_{I_4} w_{I_1} w_{I_2} w$ with $I_1 = \widetilde{S} \setminus \{s_{n-1}, s_n\}$, $I_2 = I_1 \setminus \{s_l\}$, $I_3 = \widetilde{S} \setminus \{s_{n+m-1}\}$ and $I_4 = I_3 \setminus \{s_{n+2m-1}\}$ (see Figure 4 in the case of $m > 1$). Then w' is in $\widetilde{A}_{2n-1}$, but not in $\widetilde{C}_n$. We have $w' \in \Omega$, which satisfies $\psi(w) = \psi(w')$ and $\widetilde{\ell}(w') = \widetilde{\ell}(w) + \widetilde{\ell}(w'w^{-1})$, hence $w \underset{L}{\sim} w'$ by Corollary 2.19.

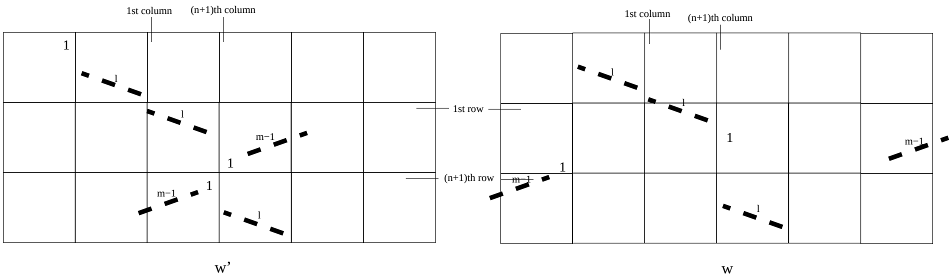


Figure 4

We have $T(w') = (T_1, T_2, ..., T_{2m}) \in \mathcal{C}_{2n}$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle(j_1)w\rangle, \langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m})w\rangle\}$ for $d \in [m+2, 2m]$.

Similarly, for $w \in F_2^e$ as in (4.2.2), we can find some $w' \in \widetilde{A}_{2n-1}$ satisfying $w \underset{L}{\sim} w'$ and $w' \in \Omega$. We again get $T(w') = (T_1, T_2, ..., T_{2m})$ with $T_c =$

$(\langle\bar{j}_{m+1-c})w\rangle$ for $c\in[m]$, $T_{m+1}=\{\langle(j_1)w\rangle,\langle(\bar{i}_a)w\rangle,\langle(i_a)w\rangle\mid a\in[l]\}$ and $T_d=\{\langle(j_{d-m})w\rangle\}$ for $d\in[m+2,2m]$.

Again, the above $T(w')$ with $w'\underset{L}{\sim}w$ and $w'\in\Omega$ depends only on $w\in F_1^o\cup F_2^o$ but not on the choice of w' in Ω. So we can denote $T(w')$ by $T(w)$. Then $T(w)$ should be pairwise different in $\mathcal{C}_{2n}$ as w ranges over F_1^e (respectively, F_2^e) by the proof of Lemma 4.5. We claim that $T(w)\in\mathcal{C}_{2n}$ for $w\in F_1^e$ should be different from that for $w\in F_2^e$. For, consider $T_m=\{\langle(\bar{j}_1)w\rangle\}$ in $T(w)=(T_1,...,T_{2m})$. If $w\in F_1^e$, then $\langle(\bar{j}_1)w\rangle\leqslant n$. If $w\in F_2^e$, then $n<\langle(i_l)w\rangle<\langle(\bar{j}_1)w\rangle$. Hence T_m for $w\in F_1^e$ should be different from that for $w\in F_2^e$. The claim is proved. So our assertion in the case of ϵ being e follows by Lemmas 3.6 and 2.15. □

Lemma 4.7. *The set $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$ forms a single two-sided cell of $\widetilde{C}_n$ if either $k\in[2n]$ is odd or $k=2n$. In particular, $E_{\mathbf{2n}}$ is the lowest two-sided cell of $\widetilde{C}_n$ under the partial order $\underset{LR}{\leqslant}$.*

Proof. First assume $k=2m+1\in[2n]$ odd. Let $w^{(o)}=t_nw_J$ and $y^{(o)}=t_0w_I$ with $J=\{t_l,t_{l+1},...,t_n\}$ and $I=\{t_0,t_1,...,t_{n-l}\}$, where $l=n-m$. Then $y^{(o)}=\eta(w^{(o)})$ by Proposition 2.8 (3). By (4.1.1)-(4.1.2) and the proof of Lemma 4.4, we see that any element of $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$ is in a two-sided-connected component of $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$ containing the element either $w^{(o)}$ or $y^{(o)}$. Thus by Lemma 2.18, in order to show our result, we need only to show that $w^{(o)}$ and $y^{(o)}$ are contained in the same two-sided-connected component of $E_{\mathbf{k}\mathbf{1}^{\mathbf{2n-k}}}$.

If $l=1$, then let $I_1=S\setminus\{t_{n-1},t_n\}$, $I_2=I_1\setminus\{t_0\}$, $I_3=S\setminus\{t_0,t_1\}$ and $I_4=I_3\setminus\{t_n\}$ and let $y_0=w^{(o)}$, $y_1=w_{I_2}w_{I_1}y_0$, $y_2=d_{n-1,n-1}y_1$, $y_3=y_2t_{1,n-1}$ and $y_4=y_3w_{I_4}w_{I_3}$; if $l>1$, then let $J_1=\{t_1,t_2,...,t_{n-2}\}$, $J_2=J_1\setminus\{t_{l-1}\}$, $J_3=\{t_0,t_1,...,t_m\}$, $J_4=J_3\setminus\{t_0\}$, $J_5=\{t_l,t_{l+1},...,t_n\}$, $J_6=J_5\setminus\{t_{n-1},t_n\}$, $J_7=\{t_2,t_3,...,t_{n-2}\}$ and $J_8=J_7\setminus\{t_{l-1}\}$. Let $y_0=w^{(o)}$, $y_1=w_{J_1}w_{J_2}y_0$, $y_2=t_0w_{J_3}w_{J_4}y_1$, $y_3=y_2t_nw_{J_6}w_{J_5}$ and $y_4=y_3w_{J_8}w_{J_7}$. In either case, we have $y_4=y^{(o)}$.

In Figure 5, we display the corresponding parts of the matrix forms for

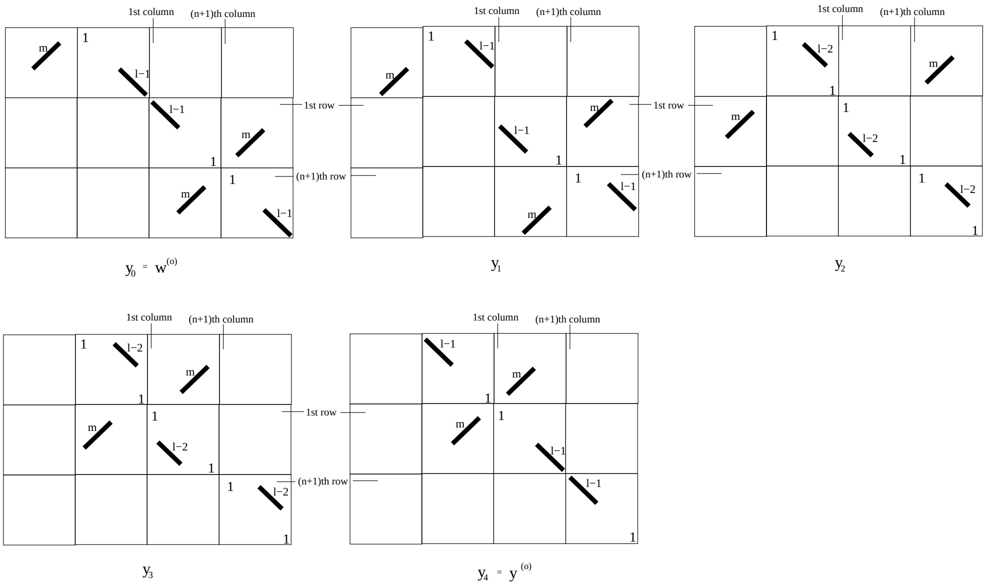


Figure 5

the elements $y_0, ..., y_4$ for $l > 1$, the notation $\overset{l}{\diagdown}$ stands for the $l \times l$ diagonal submatrix with all the diagonal entries being 1, while $\overset{m}{\diagup}$ stands for the $m \times m$ anti-diagonal submatrix with all the anti-diagonal entries being 1.

We have $y_i \in E_{\mathbf{k1^{2n-k}}}$ for any $i \in [0,4]$. Also, $\ell(y_1) = \ell(y_0) + \ell(y_1 y_0^{-1})$, $\ell(y_2) = \ell(y_1) + \ell(y_2 y_1^{-1})$, $\ell(y_3) = \ell(y_2) - \ell(y_2^{-1} y_3)$ and $\ell(y_4) = \ell(y_3) - \ell(y_3^{-1} y_4)$ (see Figure 5 for $l > 1$). This implies by Corollary 2.19 that $w^{(o)}$ and $y^{(o)}$ are contained in the same two-sided-connected component of $E_{\mathbf{k1^{2n-k}}}$.

Next assume $k = 2n$. By the part (II) in the proof of Lemma 4.4, we see that any element of $E_{\mathbf{2n}}$ is in a two-sided-connected component of $E_{\mathbf{2n}}$ containing the element either w_J or w_I with $J = \{t_1, t_2, ..., t_n\}$ and $I = \{t_0, t_1, ..., t_{n-1}\}$. Let $K = J \setminus \{t_n\}$ and let $y = w_K w_I w_J$. Then $y = w_I w_J w_K \in E_{\mathbf{2n}}$ satisfies $\ell(y) = \ell(w_J) + \ell(w_K w_I) = \ell(w_I) + \ell(w_J w_K)$. So w_J and w_I are contained in the same two-sided-connected component of $E_{\mathbf{2n}}$ by Corollary 2.19. Hence $E_{\mathbf{2n}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$ by Lemma 2.18, which is the lowest one under the partial order $\underset{LR}{\leqslant}$ by Lemmas 2.15-2.16. $\square$

In the proof of Lemma 4.7, we actually show that if $k \in [2n]$ is either odd or 2n then the set $E_{\mathbf{k1^{2n-k}}}$ is two-sided-connected. By 3.3, 4.2 and Lemmas 4.3-4.4, we see that if $k = 2m < 2n$ is even then each of the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is contained in some two-sided-connected component of $E_{\mathbf{k1^{2n-k}}}$. Now we have

Lemma 4.8. *If $k = 2m < 2n$ is even, then the set $E_{\mathbf{k1^{2n-k}}}$ has two two-sided-connected components $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$.*

Proof. Keep the notation in (4.2.1) for $w \in E_{\mathbf{k1^{2n-k}}}$. Denote the integer $j_a, \bar{j}_a, i_b, \bar{i}_b$ by $j'_a, \bar{j}'_a, i'_b, \bar{i}'_b$, resp,. $j''_a, \bar{j}''_a, i''_b, \bar{i}''_b$ for $a \in [m]$ and $b \in [l]$, according to w being $w' \in E'_{\mathbf{k1^{2n-k}}}$, resp., $w'' \in E''_{\mathbf{k1^{2n-k}}}$. Observe the following facts: If w'' is obtained from w' by left multiplying some $t \in S$, then $j''_1 = \langle (j'_1)t \rangle$ (see 2.7) and $(j''_1)w'' = (j'_1)w'$. If w'' is obtained from w' by right multiplying some $t \in S$, then $j''_1 = j'_1$ and $(j''_1)w'' = (j'_1)w't$.

We see that $w' \in E'_{\mathbf{k1^{2n-k}}}$ satisfies $\bar{i}'_1 < j'_1 \leqslant n$ and $n < (j'_1)w' < (i'_1)w'$, and that $w'' \in E''_{\mathbf{k1^{2n-k}}}$ satisfies $i''_l < j''_1 \leqslant 2n$ and $2n < (j''_1)w'' < (\bar{i}''_l)w''+2n$. Since $i''_l \geqslant n+1$ and $\bar{i}'_1 \geqslant 1$, we have $j'_1 \in [2, n]$ and $j''_1 \in [n+2, 2n]$, hence $j''_1 \neq \langle (j'_1)t \rangle$ for any $t \in S$. So no element of $E''_{\mathbf{k1^{2n-k}}}$ could be obtained from an element of $E'_{\mathbf{k1^{2n-k}}}$ by left multiplying some $t \in S$. On the other hand, since $(i'_1)w' \leqslant 2n$, we have $(j'_1)w' \leqslant 2n-1$ and $2n+1 \leqslant (j''_1)w''$, hence $(j''_1)w'' \neq (j'_1)w't$ for any $t \in S$. So no element of $E''_{\mathbf{k1^{2n-k}}}$ could be obtained from an element of $E'_{\mathbf{k1^{2n-k}}}$ by right multiplying some $t \in S$. This implies that $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ form two different two-sided-connected components of $E_{\mathbf{k1^{2n-k}}}$ by the fact that $E_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}} \cup E''_{\mathbf{k1^{2n-k}}}$. □

Theorem 4.9. *(1) If $k = 2m+1 \in [2n]$ is odd, then $E_{\mathbf{k1^{2n-k}}}$ is a two-sided cell of $\widetilde{C}_n$ containing $2^m n!/(n-m)!$ left cells.*

(2) $E_{\mathbf{2n}}$ is the lowest two-sided cell of $\widetilde{C}_n$ consists of $2^n n!$ left cells.

(3) If $k = 2m \in [2n-2]$ is even, then $E_{\mathbf{k1^{2n-k}}}$ is a union of two two-sided cells $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$ of $\widetilde{C}_n$, each of $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$ contains $2^{m-1} n!/(n-m+1)!$ left cells. The group automorphism η interchanges

$E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$.

(4) Each left (respectively, two-sided) cell of $\widetilde{C}_n$ *in* $E_{\mathbf{k1^{2n-k}}}$ *is left- (respectively, two-sided-) connected.*

(5) The set $E_{\mathbf{k1^{2n-k}}}$ *is infinite unless* $k = 1, 2$.

Proof. By Lemma 2.18, we see that E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$ for any $\lambda \in \Lambda_{2n}$. Hence the assertions (1)-(2) follow by Lemmas 4.3 and 4.5-4.7. For (3), we see by Lemmas 4.3-4.4 and 4.6 that each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ contains $2^{m-1}n!/(n-m+1)!$ left cells. By Lemmas 1.7, 4.4, 2.16 (3) and 2.14, we see that each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is a union of some two-sided cells of $\widetilde{C}_n$. On the other hand, each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is a two-sided-connected component of $E_{\mathbf{k1^{2n-k}}}$ by Lemma 4.8, which should be contained in some two-sided cell of $\widetilde{C}_n$ by Lemma 2.18. So each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ forms a single two-sided cell of $\widetilde{C}_n$. The last assertion of (3) follows by Lemma 4.4. This proves (3). The assertion (4) follows by Lemmas 4.3, 4.6 and the assertions (1)-(3). Finally, consider (5). If either $m > 1$ in $k = 2m$ or $m > 0$ in $k = 2m + 1$, then the number of the choices for the integer $(j_m)w$ in the condition (4.1.1) or (4.2.1) is infinite. On the other hand, $E_{\mathbf{1^{2n}}} = \{1\}$ and $E_{\mathbf{21^{2n-2}}} = \{t_0, t_n\}$. This proves (5). □

§5. The set $E_{(k,2,1,...,1)}$ with $(k, 2, 1, ..., 1) \in \Lambda_{2n}$.

In this section, we describe cells of $\widetilde{C}_n$ in the set $E_{(k,2,1,...,1)}$ with $(k, 2, 1, ..., 1) \in \Lambda_{2n}$. The main result is as follows.

Theorem 5.1. *Let* $\lambda = (2m, 2, 1, ...,)$ *and* $\mu = (2m+1, 2, 1, ..., 1)$ *be in* Λ_{2n}.

(1) The set E_λ *forms a single two-sided cell of* $\widetilde{C}_n$ *if* $m = n-1$ *and is a union of two two-sided cells (say* E'_λ *and* E''_λ*) of* $\widetilde{C}_n$ *if* $m < n-1$. *The set* E_μ *is a union of two two-sided cells (say* E'_μ *and* E''_μ*) of* $\widetilde{C}_n$.

(2) Let $n(\nu)$ *be the number of left cells of* $\widetilde{C}_n$ *in* E_ν *for* $\nu = \lambda, \mu$. *Then* $n(\lambda) = \frac{2^{m-1}n!(n+3-m)}{(n+1-m)!}$ *and* $n(\mu) = \frac{2^m \cdot n!}{(n-m)!}$.

Let $n'(\nu)$ *and* $n''(\nu)$ *be the numbers of left cells in* E'_ν *and* E''_ν *respectively for* $\nu = \lambda, \mu$. *Then* $\{n'(\lambda), n''(\lambda)\} = \left\{\frac{2^{m-1}n!}{(n+1-m)!}, \frac{2^{m-1}n!(n+2-m)}{(n+1-m)!}\right\}$ *and*

$n'(\mu) = n''(\mu) = \frac{2^{m-1}\cdot n!}{(n-m)!}$.

(3) Any left (respectively, two-sided) cell Γ *of* $\widetilde{C}_n$ *in* $E_\lambda \cup E_\mu$ *is left- (respectively, two-sided-) connected.*

(4) $E_{\mathbf{k21^{2n-k-2}}}$ *is infinite unless* $k = 2, 3$.

We shall prove Theorem 5.1 in the remaining part of the section.

5.2. We denote by λ the partition $(2m, 2, 1, ..., 1) \in \Lambda_{2n}$ with $m \geqslant 1$ until 5.10. Let $l = n - m - 1$. Then $w \in \widetilde{C}_n$ is in E_λ if and only if one of the conditions (a)-(c) below holds:

(a) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_m$, k, $i_1, i_2, ..., i_l$ in $[2n]$ with $j_1, j_2, ..., j_m$, k w-wild heads and $i_1, i_2, ..., i_l$ w-tame heads such that

(a1) $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$;

(a2) $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$;

(a3) j_1 (respectively, k) is w-comparable with none of $i_h, \bar{i}_h$, $h \in [l]$;

(a4) k is w-uncomparable with j_p for some $p \in [m]$.

Note that both (a2) and (a3) become empty condition if $m = n - 1$.

(b) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_m$, $i_1, i_2, ..., i_l, i_{l+1}$ in $[2n]$ with $j_1, j_2, ..., j_m$ w-wild heads and $i_1, i_2, ..., i_l, i_{l+1}$ w-tame heads such that

(b1) $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$;

(b2) $i_1 < i_2 < \cdots < i_l < i_{l+1}$ and $(i_1)w < (i_2)w < \cdots < (i_l)w < (i_{l+1})w$;

(b3) j_1 is w-comparable with at least one of $i_1, \bar{i}_1, i_{l+1}, \bar{i}_{l+1}$, but not with $i_h, \bar{i}_h$ simultaneously for any $h \in \{1, l+1\}$.

(c) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_{m-1}$, $i_1, i_2, ..., i_{l+2}$ in $[2n]$ with $j_1, j_2, ..., j_{m-1}$ w-wild heads and $i_1, i_2, ..., i_{l+2}$ w-tame heads such that

(c1) $\bar{j}_1 \prec_w i_q \prec_w i_p \prec_w j_1 \prec_w j_2 \prec_w \cdots \prec_w j_{m-1}$ for some $p, q \in [l+2]$ with $l + 2 \in \{p, q\}$;

(c2) $i_1 < i_2 < \cdots < i_l < i_{l+1}$ and $(i_1)w < (i_2)w < \cdots < (i_l)w < (i_{l+1})w$.

It is easily seen that for any $w_1 \in E_\lambda$ satisfying the condition (c), there exists some $w_2 \in E_\lambda$ satisfying the condition (b) such that w_1 and w_2 are in the same left-connected component of E_λ.

5.3. Let E'_λ be the set of all $w \in E_\lambda$ satisfying the condition (a) in 5.2 with one additional requirement that $\bar{k}$ and j_1 are w-uncomparable, that is, at least one of the following two cases occurs:

(a5) $\bar{i}_1 < j_1 < \bar{j}_1 < i_1$ and $(\bar{i}_1)w < (\bar{j}_1)w < (j_1)w < (i_1)w$ and $i_l - 2n < k - 2n < \bar{k} < \bar{i}_l$ and $(i_l)w - 2n < (\bar{k})w < (k)w - 2n < (\bar{i}_l)w$;

(a6) $\bar{i}_1 < k < \bar{k} < i_1$ and $(\bar{i}_1)w < (\bar{k})w < (k)w < (i_1)w$ and $i_l - 2n < j_1 - 2n < \bar{j}_1 < \bar{i}_l$ and $(i_l)w - 2n < (\bar{j}_1)w < (j_1)w - 2n < (\bar{i}_l)w$.

Let $E''_\lambda = E_\lambda \setminus E'_\lambda$.

Lemma 5.4. *$E'^{-1}_\lambda = E'_\lambda$ and $E''^{-1}_\lambda = E''_\lambda$ for $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$ with $m < n - 1$.*

Proof. By closely observing the matrix forms of elements, we see that if w is in E_λ and satisfies the conditions (a1)-(a5) (respectively, (a1)-(a4) and (a6)), then so does w^{-1}. Hence $E'^{-1}_\lambda = E'_\lambda$. Then we also have $E''^{-1}_\lambda = E''_\lambda$ by the fact $E^{-1}_\lambda = E_\lambda$. □

5.5. Let F'_λ be the set of all $w' \in \widetilde{C}_n$ satisfying the condition (a$'$) below.

(a$'$) Let $j_1, j_2, ..., j_m$, k be w'-wild heads and $i_1, i_2, ..., i_l$ w'-tame heads satisfying that

(a$'$1) either

(a$'$11) $(\bar{k}, \bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ with $0 < (k)w' - 2n < (\bar{i}_l)w' < (\bar{i}_{l-1})w' < \cdots < (\bar{i}_1)w' < (j_1)w' < (i_1)w'$ if $m < n - 1$ and with $0 < (k)w' - 2n \leqslant n < (j_1)w' \leqslant 2n$ if $m = n - 1$,

or

(a$'$12) $(\bar{k}, i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1) = (n + 1, n + 2, ..., 2n)$ with $n < (k)w' < (i_1)w' < (i_2)w' < \cdots < (i_l)w' < (j_1)w' < (\bar{i}_l)w' + 2n$ if $m < n - 1$ and with $0 < (j_1)w' - 2n \leqslant n < (k)w' \leqslant 2n$ if $m = n - 1$;

(a$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m - 1]$.

Let F''_λ be the set of all $w' \in \widetilde{C}_n$ satisfying one of the conditions (b$'$), (c$'$) below.

(b$'$) Let $j_1, j_2, ..., j_m$, k be w'-wild heads and $i_1, i_2, ..., i_l$ w'-tame heads satisfying that

(b$'$1) either

(b$'$11) $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1, k) = (1, 2, ..., n)$ with $n < (j_1)w' < (k)w' < (i_1)w'$ if $m < n-1$ and with $n < (j_1)w' < (k)w' \leqslant 2n$ if $m = n-1$,

or

(b$'$12) $(i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1, k) = (n+1, n+2, ..., 2n)$ with $2n < (j_1)w' < (k)w' < (\bar{i}_l)w' + 2n$ if $m < n-1$ and $2n < (j_1)w' < (k)w' \leqslant 3n$ if $m = n-1$;

(b$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(b$'$3) $(i_1)w' < (i_2)w' < \cdots < (i_l)w'$.

(c$'$) Let $j_1, j_2, ..., j_m$ be w'-wild heads and $i_1, i_2, ..., i_{l+1}$ w'-tame heads with $m < n-1$ such that

(c$'$1) either

(c$'$11) $(\bar{i}_{l+1}, \bar{i}_l, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ and $(i_1)w' < (j_1)w' \leqslant 2n$,

or

(c$'$12) $(i_1, i_2, ..., i_{l+1}, j_m, j_{m-1}, ..., j_1) = (n+1, n+2, ..., 2n)$ and $(\bar{i}_{l+1})w' + 2n < (j_1)w' \leqslant 3n$;

(c$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(c$'$3) $(i_1)w' < (i_2)w' < \cdots < (i_l)w' < (i_{l+1})w'$.

5.6. We have $F'_\lambda \subseteq E'_\lambda$ and $F''_\lambda \subseteq E''_\lambda$ by 5.2-5.3 and 5.5. Also, we see by 3.3 and Lemma 3.2 that any left-connected component of E'_λ (respectively, E''_λ) contains some element of F'_λ (respectively, F''_λ).

Recall the notation in 3.5. Let $\alpha = (1, ..., 1, 2, 2(n-m), 1, ..., 1) \in \widetilde{\Lambda}_{2n}$ with 2 its m-th component. Let $\mathcal{C}_\alpha$ be the set of all $\mathbf{T} = (T_1, T_2, ..., T_{2m}) \in \mathcal{C}_{2n}$ with $\xi(\mathbf{T}) = \alpha$.

Let $F_\lambda := F'_\lambda \cup F''_\lambda$. By the technique similar to that used in the proof of Lemma 4.6, we can find, for any $w' \in F_\lambda$, some $y \in \Omega$ satisfying $y \underset{L}{\sim} w'$ and $T(y) \in \mathcal{C}_\alpha$. Now we describe the generalized tabloid $T(y)$ as follows.

(1) If w' satisfies the condition (a$'$) or (b$'$) in 5.5, then

$$\begin{aligned}T(y)=\big(&\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(\bar{k})w'\rangle\},\\&\{\langle(j_1)w'\rangle,\langle(k)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l]\},\\&\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}\big)\,.\end{aligned}$$

where (i) $0<(k)w'-2n<(\bar{i}_l)w'$ and $n<(j_1)w'<(i_1)w'$
if $(\bar{k},\bar{i}_l,\bar{i}_{l-1},...,\bar{i}_1,j_m,j_{m-1},...,j_1)=(1,2,...,n)$;
(ii) $n<(k)w'<(i_1)w'$ and $2n<(j_1)w'<(\bar{i}_l)w'+2n$
if $(\bar{k},i_1,i_2,...,i_l,j_m,j_{m-1},...,j_1)=(n+1,n+2,...,2n)$;
(iii) $n<(j_1)w'<(k)w'<(i_1)w'$
if $(\bar{i}_l,\bar{i}_{l-1},...,\bar{i}_1,j_m,j_{m-1},...,j_1,k)=(1,2,...,n)$;
(iv) $2n<(j_1)w'<(k)w'<(\bar{i}_l)w'+2n$
if $(i_1,i_2,...,i_l,j_m,j_{m-1},...,j_1,k)=(n+1,n+2,...,2n)$.
Here we stipulate $(\bar{i}_l)w'=n+1$ and $(i_1)w'=2n+1$ in the case of $l=0$.

(2) If w' satisfies the condition (c′) in 5.5 with $(\bar{i}_{l+1},...,\bar{i}_1,j_m,...,j_1)=(1,2,...,n)$ and $(i_p)w'<(j_1)w'<(i_{p+1})w'$ for some $p\in[l+1]$ with the convention that $(i_{l+2})w'=2n+1$, then

$$\begin{aligned}T(y)=(&\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(i_p)w'\rangle\},\\&\{\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l+1]\}\setminus\{\langle(i_p)w'\rangle\},\\&\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\})\,.\end{aligned}$$

(3) If w' satisfies the condition (c′) in 5.5 with $(i_1,...,i_{l+1},j_m,...,j_1)=(n+1,n+2,...,2n)$ and $(\bar{i}_p)w'+2n<(j_1)w'<(\bar{i}_{p-1})w'+2n$ for some $p\in[l+1]$ with the convention that $(\bar{i}_0)w'=n+1$, then

$$\begin{aligned}T(y)=(&\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(\bar{i}_p)w'\rangle\},\\&\{\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l+1]\}\setminus\{\langle(\bar{i}_p)w'\rangle\},\\&\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\})\,.\end{aligned}$$

5.7. By Lemma 3.6, we see that the generalized tabloid $T(y)\in\mathcal{C}_\alpha$ given in 5.6 only depends on $w'\in F_\lambda$ but not on the choice of $y\in\Omega$. Hence we could denote $T(y)$ by $T(w')$. This defines a map $T : F_\lambda \longrightarrow \mathcal{C}_\alpha$. By 5.5-5.6, we see that $\mathbf{T} = (T_1, T_2, ..., T_{2m}) \in \mathcal{C}_\alpha$ is in the image of the map T if and only if $\mathbf{T}$ satisfies the following conditions:

(1) Let $T_i = \{p_i\}$ for $i \in [2m] \setminus \{m, m+1\}$. Then $\bar{p}_i = p_{2m+1-i}$ for any $i \in [m-1]$;

(2) By (1), we have $T_m \cup T_{m+1} = \{\bar{q}_{n+1-m}, \bar{q}_{n-m}, ..., \bar{q}_1, q_1, ..., q_{n-m}, q_{n+1-m}\}$ for some $\bar{q}_1 < q_1 < q_2 < \cdots < q_{n+1-m}$ in $[2n]$, and

$$(5.7.1) \quad T_m \in \{\{\bar{q}_1, q_{n+1-m}\}, \{\bar{q}_1, \bar{q}_2\}, \{q_{n-m}, q_{n+1-m}\}, \{\bar{q}_{i+1}, q_i\} \mid i \in [n-m]\}.$$

When the equivalent conditions hold, $T^{-1}(\mathbf{T})$ consists of a single element of F_λ if $T_m \in \{\{\bar{q}_1, \bar{q}_2\}, \{q_{n-m}, q_{n+1-m}\}\}$ (i.e., in the case (b$'$) of 5.5) and consists of two elements of F_λ if T_m is either equal to $\{\bar{q}_1, q_{n+1-m}\}$ (i.e., in the case (a$'$) of 5.5) or in the set $\{\{\bar{q}_{i+1}, q_i\} \mid i \in [n-m]\}$ (i.e., in the case (c$'$) of 5.5).

Suppose $m < n-1$. By 5.6, we see that $T(w') \neq T(w'')$ for any $w' \in F'_\lambda$ and any $w'' \in F''_\lambda$. This implies by 5.6 and Lemmas 3.6, 2.15-2.16 that each of the sets E'_λ and E''_λ is a union of some left cells of $\widetilde{C}_n$.

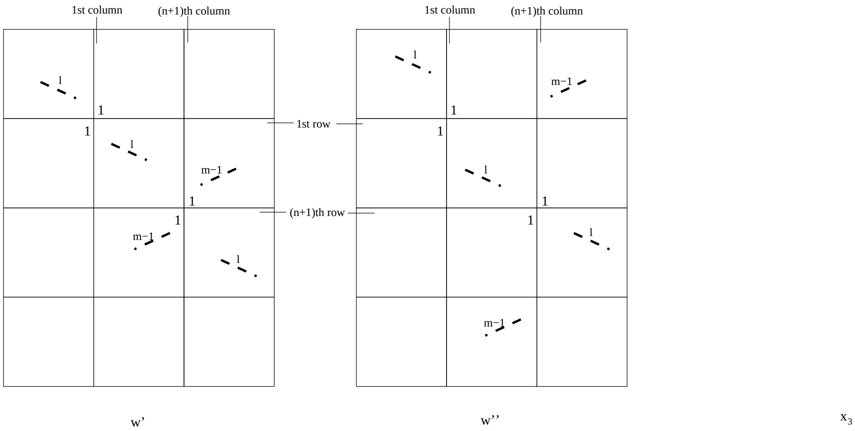
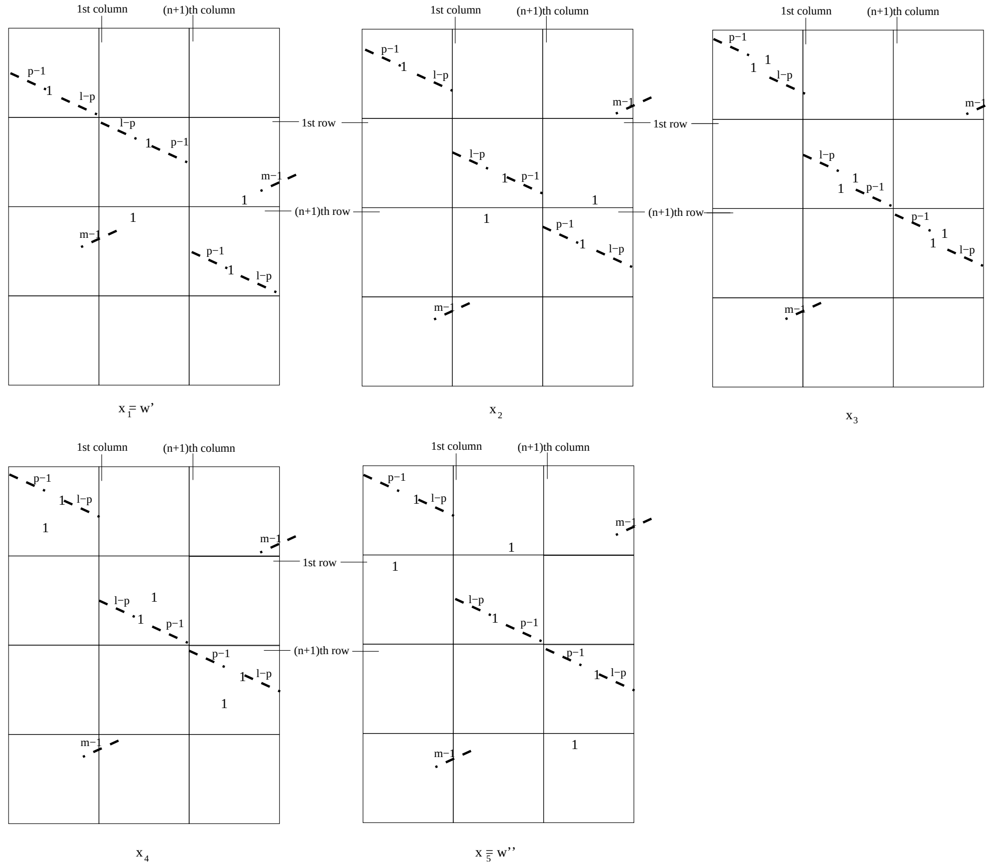


Figure 6

Let us first consider the case (a$'$) of 5.5. Let F'_i be the set of all $w' \in F'_\lambda$ satisfying the conditions (a$'$1i) and (a$'$2) for $i = 1, 2$. Then F'_λ is a disjoint union of F_1' and F_2'. Use the notation in 5.6 and in (2) above, $w' \in F_1'$ means that $\langle(\bar{k})w'\rangle$, $\langle(\bar{j}_1)w'\rangle$ in 5.6 are q_{n+1-m}, $\bar{q}_1$ in (2), respectively, while $w' \in F_2'$ means that $\langle(\bar{j}_1)w'\rangle$, $\langle(\bar{k})w'\rangle$ in 5.6 are q_{n+1-m}, $\bar{q}_1$ in (2), respectively. Take any $w' \in F_1'$ with the notation as in (a$'$) of 5.5. Let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \backslash \{t_{l+1}\}$, $J_3 = \{t_0, t_1, ..., t_{m-2}\}$, $J_4 = \{t_1, t_2, ..., t_{m-1}\}$. Let $J_j' = \eta(J_j)$ for $j \in [4]$. If $(j_2)w' < (k)w'$, let $w'' = w_{J_4}w_{J_3}w_{J_1}w_{J_2}w'$, then $w'' \in F_2'$ with $\ell(w'') = \ell(w') + \ell(w_{J_4}w_{J_3}w_{J_1}w_{J_2})$ (see Figure 6). If $(j_2)w' > (k)w'$, let $w'' = w_{J_2'}w_{J_1'}w_{J_3'}w_{J_4'}w'$, then $w'' \in F_2'$ with $\ell(w'') = \ell(w') - \ell(w_{J_2'}w_{J_1'}w_{J_3'}w_{J_4'})$. In either case, the elements w'' and w' are in the same left-connected component of E_λ' by Corollary 2.19 and hence $w'' \underset{L}{\sim} w'$. The above correspondence $w' \mapsto w''$ defines a bijective map from the set F_1' to F_2' with $T(w') = T(w'')$.



$x_1 = w'$ x_2 x_3

x_4 $x_5 = w''$

Figure 7

Next we consider the case (c$'$) of 5.5. Let F_i'' be the set of all $w' \in F_\lambda''$

satisfying the conditions (c$'1i$) and (c$'2$)-(c$'3$) for $i=1,2$. Then $F_1''\cap F_2''=\emptyset$. Use the notation in 5.6 and in (2) above, $w'\in F_1''$ means that $\langle(\bar{j}_1)w'\rangle$, $\langle(i_p)w'\rangle$ in 5.6 are $\bar{q}_{p+1}$, q_p in (2), respectively, while $w'\in F_2''$ means that $\langle(\bar{j}_1)w'\rangle$, $\langle(\bar{i}_p)w'\rangle$ in 5.6 are q_p, $\bar{q}_{p+1}$ in (2), respectively. Take any $w'\in F_1''$ with the notation as in (c$'$) of 5.5. Let $J_1=\{t_1,t_2,...,t_{n-2}\}$, $J_2=J_1\setminus\{t_l\}$, $J_3=\{t_0,t_1,...,t_{m-2}\}$, $J_4=J_3\setminus\{t_0\}$, $J_5=\{t_{n-1},t_{n-2},...,t_{n+1-p}\}$, $J_6=J_5\setminus\{t_{n-1}\}$, $J_7=\{t_m,t_{m+1},...,t_{n-2-p}\}$, $J_8=J_7\setminus\{t_{n-2-p}\}$ and $J_9=\{t_1,t_2,...,t_{m-1}\}$. Let $x_1=w'$, $x_2=w_{J_3}w_{J_4}w_{J_1}w_{J_2}x_1$, $x_3=w_{J_5}w_{J_6}t_nx_2$, $x_4=t_0w_{J_9}w_{J_4}w_{J_7}w_{J_8}x_3$. Let $w''=x_4$. Then $x_i\in E_\lambda''$ for $i\in[4]$ and $w''\in F_2''$ with $\ell(x_2)=\ell(x_1)+\ell(w_{J_3}w_{J_4}w_{J_1}w_{J_2})$, $\ell(x_3)=\ell(x_2)-\ell(w_{J_5}w_{J_6}t_n)$ and $\ell(x_4)=\ell(x_3)+\ell(t_0w_{J_9}w_{J_4}w_{J_7}w_{J_8})$ (see Figure 7). This implies by Corollary 2.19 that w', w'' are in the same left-connected component of E_λ'' and hence $w'\underset{L}{\sim}w''$. The correspondence $w'\mapsto w''$ defines a bijective map from the set F_1'' to F_2'' with $T(w')=T(w'')$.

From 5.6 and the above discussion, we conclude that

Lemma 5.8. *Let* $\lambda=(2m,2,1,...,1)\in\Lambda_{2n}$.

(1) Each of E_λ' *and* E_λ'' *is a union of some left cells of* $\widetilde{C}_n$ *if* $m<n-1$.

(2) Any left cell of $\widetilde{C}_n$ *in* E_λ *is left-connected.*

Now we consider the two-sided cells of $\widetilde{C}_n$ in E_λ.

Lemma 5.9. *Let* $\lambda=(2m,2,1,...,1)\in\Lambda_{2n}$.

(1) If $m<n-1$, *then each of* E_λ' *and* E_λ'' *is two-sided-connected and forms a two-sided cell of* $\widetilde{C}_n$.

(2) The set $E_{(2n-2,2)}$ *is two-sided-connected and forms a single two-sided cell of* $\widetilde{C}_n$.

Proof. By 1.6 (1)-(2), Lemmas 1.7, 5.4 and 5.8, to show our result, we need only to prove that each of E_λ' and E_λ'' is two-sided-connected if $m<n-1$ and that the set $E_{(2n-2,2)}$ is two-sided-connected.

(I) First assume $m<n-1$.

(Ia) E_λ' **is two-sided-connected.**

Let $w_1 = [0, 2, 3, ..., n-m, n+m, n+m-1, ..., n+2, n+1]$ and $w_2 = [0, -1, -2, ..., -m+1, m+1, m+2, ..., n-1, n+1]$ be in $\widetilde{C}_n$ (see Figure 8). Then $w_1, w_2 \in F'_\lambda$. Let F'_1, F'_2 be defined as in 5.7. Then

$$(5.9.1)\quad \eta(E'_\lambda) = E'_\lambda; \quad w_i \in F'_i,\ \eta(w_i) = w_{3-i},\ \eta(F'_i) = F'_{3-i} \text{ for } i = 1, 2.$$

by Proposition 2.8 (3). By 5.6, to show (Ia), we need only to prove that

(a) Any $x \in F'_i$ is in a right-connected component of E'_λ containing w_i for $i = 1, 2$;

(b) w_1 and w_2 are in the same two-sided-connected component of E'_λ.

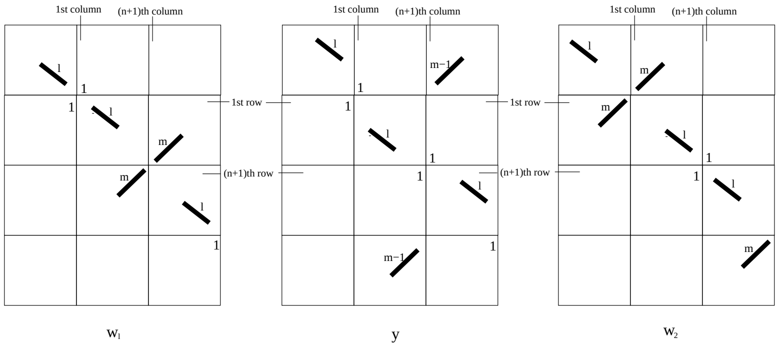


Figure 8

For (a), we need only to deal with the case of $i = 1$ by the fact (5.9.1), while the argument for this part is similar to that for Lemma 4.4 (3) (hence leaving it to the readers). Now let us prove (b). Let $J_1 = \{t_2, t_3, ..., t_{n-2}\}$, $J_2 = J_1 \backslash \{t_{n-m}\}$, $J_3 = \{t_0, t_1, ..., t_{m-1}\}$, $J_4 = J_3 \backslash \{t_1\}$, $J_5 = \{t_n, t_{n-1}, ..., t_{n+1-m}\}$, $J_6 = J_5 \setminus \{t_{n-1}\}$ and $y = w_{J_3} w_{J_4} w_{J_1} w_{J_2} w_1$. Then $y = w_2 w_{J_1} w_{J_2} w_{J_6} w_{J_5}$, which is in E'_λ and satisfies $\ell(y) = \ell(w_1) + \ell(w_{J_3} w_{J_4} w_{J_1} w_{J_2}) = \ell(w_2) + \ell(w_{J_1} w_{J_2} w_{J_6} w_{J_5})$ (see Figure 8). This proves (b) by Corollary 2.19.

(Ib) E''_λ **is two-sided-connected.**

Let $w_1 = [1, 2, ..., n-m-1, n+m, n+m-1, ..., n+1, n+m+1]$ and $w_2 = [-m, 0, -1, -2, ..., -m+1, m+2, m+3, ..., n]$ be in $\widetilde{C}_n$ (see Figure 9). Let F''_1, F''_2 be defined as in 5.7. Then

$$w_i \in F_i'', \quad \eta(w_i) = w_{3-i}, \quad \eta(F_i'') = F_{3-i}'' \ \text{ for } i = 1, 2. \tag{5.9.2}$$

by Proposition 2.8 (3). By 5.6, to show (Ib), we need only to prove that

(a) Any $x \in F_i''$ is in the right-connected component of E_λ'' containing w_i for $i = 1, 2$;

(b) w_1 and w_2 are in the same two-sided-connected component of E_λ''.

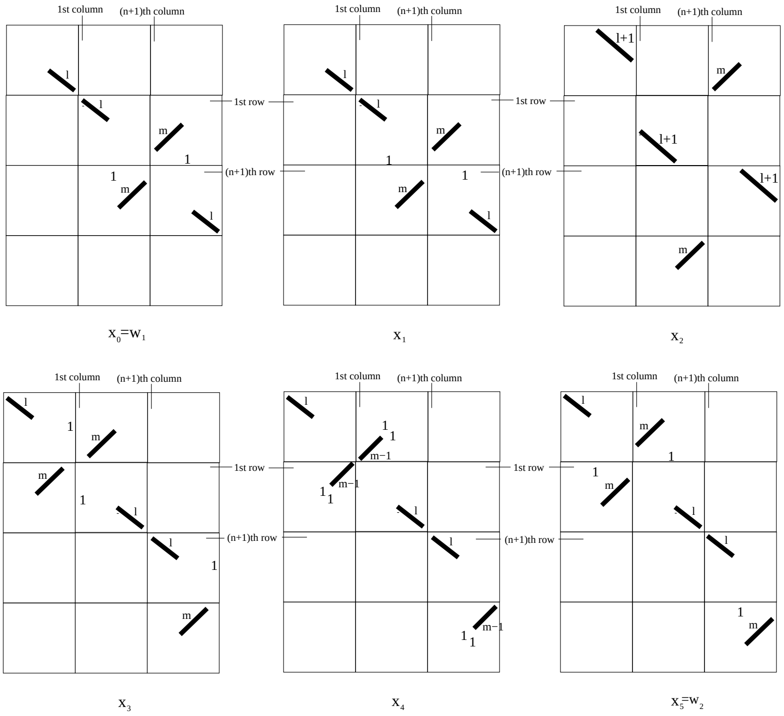


Figure 9

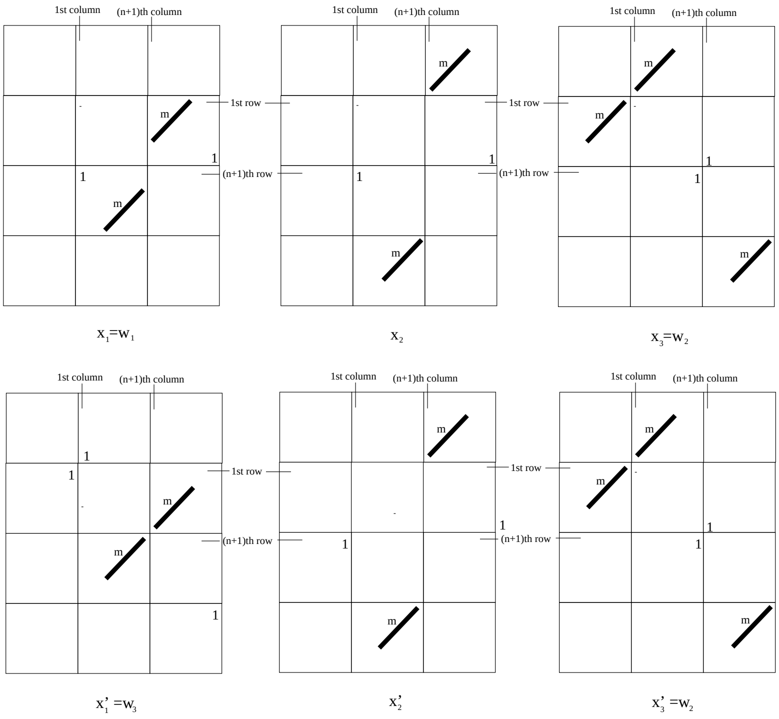
By (5.9.2), to prove (a), we need only to deal with the case of $i = 1$, the latter can be proved by the argument similar to that for Lemma 4.4 (3) (hence leaving it to the readers). Next let us prove (b).

Let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \setminus \{t_{n-m-1}\}$, $J_3 = \{t_0, t_1, ..., t_{m-1}\}$, $J_4 = J_3 \setminus \{t_0\}$, $J_5 = \{t_n, t_{n-1}, ..., t_{n-m+1}\}$, $J_6 = J_5 \setminus \{t_n\}$, $J_7 = \{t_2, t_3, ..., t_{n-1}\}$, $J_8 = J_7 \setminus \{t_{n-m}\}$ and $J_9 = J_4 \setminus \{t_1\}$. Let $x_0 = w_1$, $x_1 = t_n x_0$, $x_2 = w_{J_3} w_{J_4} w_{J_1} w_{J_2} x_1$, $x_3 = x_2 w_{J_6} w_{J_5} w_{J_8} w_{J_7}$, $x_4 = x_3 t_0 w_{J_9} w_{J_4}$ and $x_5 = w_{J_9} w_{J_4} x_4$. Then $x_5 = w_2$. We have $x_i \in E_\lambda''$ for any $i \in [0, 5]$ and $\ell(x_1) = \ell(x_0) - 1$,

$\ell(x_2)=\ell(x_1)+\ell(w_{J_3}w_{J_4}w_{J_1}w_{J_2})$, $\ell(x_3)=\ell(x_2)-\ell(w_{J_6}w_{J_5}w_{J_8}w_{J_7})$, $\ell(x_4)=\ell(x_3)+\ell(t_0w_{J_9}w_{J_4})$, $\ell(x_5)=\ell(x_4)-\ell(w_{J_9}w_{J_4})$ (see Figure 9).

By Corollary 2.19, we see that the elements x_{i-1},x_i are contained in the same two-sided-connected component of E''_λ for any $i\in[5]$. This proves (b) and hence E''_λ is two-sided connected.

(II) $E_{(2n-2,2)}$ **is two-sided-connected**.



$x_1=w_1$ x_2 $x_3=w_2$

$x'_1=w_3$ x'_2 $x'_3=w_2$

Figure 10

Denote $\lambda=(2n-2,2)$. Let $w_1=[2n-1,2n-2,...,n+1,2n]$, $w_2=[0,-1,...,-n+2,n+1]$, $w_3=[0,2n-1,2n-2,...,n+1]$ and $w_4=[-n+1,0,-1,...,-n+2]$ be in $\widetilde{C}_n$ (see Figure 10). Then $w_i\in F_\lambda:=F'_\lambda\cup F''_\lambda$ for $i\in[4]$. By the argument similar to that for Lemma 4.4 (3), we can show that any element of F_λ is in a right-connected component of E_λ containing w_i for some $i\in[4]$ (hence leaving it to the readers). Note that $\eta(w_i)=w_{5-i}$ for $i\in[4]$ by Proposition 2.8 (3). Thus to show that E_λ is two-sided-connected, we need only to prove that w_1,w_2 (respectively, w_2,w_3) are in the same two-sided-connected component of E_λ.

Let $x_1 = w_1$, $x_2 = w_{J_4}w_{J_2}x_1$, $x_3 = x_2w_{J_5}w_{J_3}$, $x'_1 = w_3$, $x'_2 = t_nw_{J_4}w_{J_2}w_{J_3}w_{J_1}x'_1$, $x'_3 = x'_2w_{J_1}w_{J_5}t_0w_{J_1}w_{J_3}$, where $J_1 = \{t_2, t_3, ..., t_{n-1}\}$, $J_2 = \{t_1, t_2, ..., t_{n-2}\}$, $J_3 = J_2 \cup \{t_{n-1}\}$, $J_4 = J_2 \cup \{t_0\}$ and $J_5 = J_1 \cup \{t_n\}$. Then $x_3 = x'_3 = w_2$ and $x_i, x'_i \in E_\lambda$ for $i \in [3]$ and $\ell(x_2) = \ell(x_1) + \ell(w_{J_4}w_{J_2}) = \ell(x_3) + \ell(w_{J_3}w_{J_5})$ and $\ell(x'_2) = \ell(x'_1) + \ell(t_nw_{J_4}w_{J_2}w_{J_3}w_{J_1}) = \ell(x'_3) + \ell(w_{J_3}w_{J_1}t_0w_{J_5}w_{J_1})$ (see Figure 10). This implies by Corollary 2.19 that w_1, w_2, respectively, w_2, w_3, are in the same two-sided-connected component of E_λ. So E_λ is two-sided-connected. □

Let $n(\lambda)$, $n'(\lambda)$, $n''(\lambda)$ be the numbers of left cells of $\widetilde{C}_n$ in the sets E_λ, E'_λ, E''_λ, respectively for $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$.

Lemma 5.10. $n(\lambda) = \frac{2^{m-1}n!(n+3-m)}{(n+1-m)!}$ *for* $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$. *In this case, if* $m < n-1$, *then* $n'(\lambda) = \frac{2^{m-1}n!}{(n+1-m)!}$ *and* $n''(\lambda) = \frac{2^{m-1}n!(n+2-m)}{(n+1-m)!}$.

Proof. We must enumerate the generalized tabloids $\mathbf{T} = (T_1, T_2, ..., T_{2m})$ in $\mathcal{C}_\alpha$ satisfying the conditions (1)-(2) of 5.7. The number of the choices for the set $E := T_m \cup T_{m+1} = \{\bar{q}_{n+1-m}, \bar{q}_{n-m}, ..., \bar{q}_1, q_1, ..., q_{n-m}, q_{n+1-m}\}$ with $\bar{q}_1 < q_1 < q_2 < \cdots < q_{n+1-m}$ in $[2n]$ is $\binom{n}{n+1-m}$. Once E is fixed, the number of the choices for the set T_m to satisfy (5.7.1) is $n+3-m$, while that for the $(m-1)$-tuple $(T_1, ..., T_{m-1})$ is $2^{m-1}(m-1)!$. We know that w' is in E'_λ (respectively, E''_λ) if and only if the m-th component $T_m(w')$ of $T(w')$ is $\{\bar{q}_1, q_{n+1-m}\}$ (respectively, satisfies (5.7.1) but is not $\{\bar{q}_1, q_{n+1-m}\}$). This proves our result. □

If $m > 1$, then the number of the choices for the integer $(j_m)w$ in the case (a) of 5.2 is infinite, hence $|E_{(2m,2,1,...,1)}| = \infty$. On the other hand, the set $E_{\mathbf{2^21^{2n-4}}} = \{t_0t_n, t_it_{i+1}\cdots t_j, t_jt_{j-1}\cdots t_i \mid 0 \leqslant i \leqslant j \leqslant n\} \setminus \{t_0, t_n\}$ is finite.

So far we have proved all the assertions of Theorem 5.1 involving the partition $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$.

5.11. Next let $\mu = (2m+1, 2, 1, ..., 1) \in \Lambda_{2n}$ with some $m \geqslant 1$. Let $l = n-m-1$. Then $w \in \widetilde{C}_n$ is in E_μ if and only if w satisfies the condition

(5.11.1) below.

(5.11.1) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_m$, k, $i_1, i_2, ..., i_l$ in $[2n]$ with $j_1, j_2, ..., j_m$, k w-wild heads and $i_1, i_2, ..., i_l$ w-tame heads such that (i) $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$; (ii) either $\bar{i}_1, i_1 \prec_w j_1$, or $\bar{i}_l, i_l \prec_w j_1$; (iii) $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (iv) k is w-comparable with none of $\bar{i}_h, i_h, j_q$ for any $h \in [l]$ and some $q \in [m]$.

According to the conditions (i)-(ii) and (iv), if k is w-comparable with j_p for some $p \in [m]$, then $k \prec_w j_p$ and $p > 1$ by Lemma 3.2. Thus under the assumption of the conditions (i)-(ii), the condition (iv) is equivalent to that k is w-comparable with none of $\bar{i}_h, i_h, j_1$ for any $h \in [l]$.

5.12. Under the condition (5.11.1) on $w \in E_\mu$, the following facts concerning w can be checked easily.

(a) If $\bar{i}_1, i_1 \prec_w j_1$, then $j_1 < \bar{i}_1 < i_1 < \bar{j}_1$ and $(\bar{j}_1)w < (\bar{i}_1)w < (i_1)w < (j_1)w$ and $i_l - 2n < k - 2n < \bar{k} < \bar{i}_l$ and $(i_l)w - 2n < (\bar{k})w < (k)w - 2n < (\bar{i}_l)w$;

(b) If $\bar{i}_l, i_l \prec_w j_1$, then $j_1 - 2n < i_l - 2n < \bar{i}_l < \bar{j}_1$ and $(\bar{j}_1)w < (i_l)w - 2n < (\bar{i}_l)w < (j_1)w - 2n$ and $\bar{i}_1 < k < \bar{k} < i_1$ and $(\bar{i}_1)w < (\bar{k})w < (k)w < (i_1)w$.

Let E'_μ (respectively, E''_μ) be the set of all $w \in E_\mu$ satisfying the condition (a) (respectively, (b)).

From the matrix forms of elements, we see that w is in E_μ and satisfies the condition (a) (respectively, (b)) if and only if so does w^{-1}. So by 5.11, we get

Lemma 5.13. *Let* $\mu = (2m+1, 2, 1, ..., 1) \in \Lambda_{2n}$ *with some* $m \geqslant 1$. *Then*

(1) ${E'_\mu}^{-1} = E'_\mu$ *and* ${E''_\mu}^{-1} = E''_\mu$.

(2) The group automorphism η *of* $\widetilde{C}_n$ *interchanges the sets* E'_μ *and* E''_μ.

(3) E_μ *is a disjoint union of subsets* E'_μ *and* E''_μ.

5.14. Let F'_μ (respectively, F''_μ) be the set of all $w' \in \widetilde{C}_n$ satisfying the condition (a$'$) (respectively, (b$'$)) below.

(a$'$) There exist w'-wild heads $j_1, j_2, ..., j_m$, k and w'-tame heads $i_1, i_2, ..., i_l$ satisfying that

(i) $(\bar{k}, \bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$;
(ii) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;
(iii) $(i_1)w' < (i_2)w' < \cdots < (i_l)w' < (\bar{k})w' + 2n \leqslant 2n$;
(iv) $(i_p)w' < (j_1)w' < (i_{p+1})w'$ for some $p \in [l]$ with the convention that $(i_{l+1})w' = (k)w'$.

(b′) There exist w'-wild heads $j_1, j_2, ..., j_m$, k and w'-tame heads $i_1, i_2, ..., i_l$ satisfying that
(i) $(\bar{k}, i_1, i_2, ..., i_{l-1}, j_m, j_{m-1}, ..., j_1, i_l) = (n+1, n+2, ..., 2n)$;
(ii) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;
(iii) $n < (k)w' < (i_1)w' < (i_2)w' < \cdots < (i_l)w'$;
(iv) $(\bar{i}_p)w' + 2n < (j_1)w' < (\bar{i}_{p-1})w' + 2n$ for some $p \in [l]$ with the convention that $(\bar{i}_0)w' = (k)w'$.

5.15. By 5.11-5.12 and 5.14, we have $F'_\mu \subseteq E'_\mu$ and $F''_\mu \subseteq E''_\mu$. Also, we see by 3.3 and Lemma 3.2 that any left-connected component of E'_μ (respectively, E''_μ) contains some element of F'_μ (respectively,F''_μ).

Let $\beta = (1, ..., 1, 2, 2l+1, 1, ..., 1) \in \widetilde{\Lambda}_{2n}$, where 2 is the $(m+1)$-th component of β. Let $\mathcal{C}_\beta$ be the set of all $\mathbf{T} = (T_1, T_2, ..., T_{2m+1}) \in \mathcal{C}_{2n}$ with $\xi(\mathbf{T}) = \beta$ (see 3.5).

Let $F_\mu := F'_\mu \cup F''_\mu$. By the technique similar to that used in the proof of Lemma 4.6, we can find, for any $w' \in F_\mu$, some $z \in \Omega$ satisfying $z \underset{L}{\sim} w'$ and $T(z) \in \mathcal{C}_\beta$. Now we describe, for any $w' \in F_\mu$, the generalized tabloid $T(z) \in \mathcal{C}_\beta$ as follows. If $w' \in F'_\mu$ is as in 5.14 (a′), then

$$\begin{aligned}(5.15.1)\quad T(z) = \big(&\{\langle(\bar{j}_m)w'\rangle\}, ..., \{\langle(\bar{j}_1)w'\rangle\}, \{\langle(\bar{k})w'\rangle, \langle(i_p)w'\rangle\},\\ &\{\langle(k)w'\rangle, \langle(j_1)w'\rangle, \langle(\bar{i}_h)w'\rangle, \langle(i_h)w'\rangle \mid h \in [l]\} \setminus \{\langle(i_p)w'\rangle\},\\ &\{\langle(j_2)w'\rangle\}, ..., \{\langle(j_m)w'\rangle\}\big)\,.\end{aligned}$$

where (i) $\langle(\bar{k})w'\rangle \in [(i_l)w' + 1, 2n]$; (ii) $p \in [l]$. If $p < l$, then $(\bar{i}_{p+1})w' < (\bar{j}_1)w' < (\bar{i}_p)w'$; if $p = l$ then $(\bar{j}_1)w'$ is one of the three cases: $(k)w' - 2n < (\bar{j}_1)w' < (\bar{i}_l)w'$, $1 \leqslant (\bar{j}_1)w' < (k)w' - 2n$, $(\bar{k})w' < (\bar{j}_1)w' \leqslant 0$.

If $w'\in F''_\mu$ is as in 5.14 (b′), then

$$\begin{aligned}(5.15.2)\quad T(z)=\big(&\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_1)w'\rangle\},\{\langle(\bar{k})w'\rangle,\langle(\bar{i}_p)w'\rangle\},\\&\{\langle(k)w'\rangle,\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l]\}\setminus\{\langle(\bar{i}_p)w'\rangle\},\\&\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}\big)\,.\end{aligned}$$

where (i) $\langle(\bar{k})w'\rangle\in[(\bar{i}_1)w'+1,n]$; (ii) $p\in[l]$. If $p>1$ then $(i_{p-1})w'<(\bar{j}_1)w'+2n<(i_p)w'$; if $p=1$ then $(\bar{j}_1)w'$ is one of the three cases: $(k)w'<(\bar{j}_1)w'+2n<(i_1)w'$, $n<(\bar{j}_1)w'+2n<(k)w'$, $(\bar{k})w'<(\bar{j}_1)w'+2n\leqslant n$.

5.16. We see that $T(z)$ only depends on $w'\in F_\mu$, but not on the choice of $z\in\Omega$, $z\underset{L}{\sim}w'$. So it makes sense to denote $T(z)$ by $T(w')$. This defines a map $T:F_\mu\longrightarrow\mathcal{C}_\beta$. By 5.14-5.15, we see that $\mathbf{T}=(T_1,T_2,...,T_{2m+1})\in\mathcal{C}_\beta$ is in the image of T if and only if $\mathbf{T}$ satisfies the following conditions:

(1) Let $T_i=\{p_i\}$ for $i\in[2m+1]\setminus\{m,m+1,m+2\}$. Then $\bar{p}_i=p_{2m+2-i}$ for $i\in[m-1]$;

(2) By (1), we have $\bigcup_{i=m}^{m+2}T_i=\{\bar{q}_{n-m+1},\bar{q}_{n-m},...,\bar{q}_1,q_1,q_2,...,q_{n-m+1}\}$ for some $\bar{q}_1<q_1<q_2<\cdots<q_{n-m+1}$ in $[2n]$. Then the ordered pair (T_m,T_{m+1}) is in the set $E_1\cup E_2\cup E_3\cup E_4$, where

$$\begin{aligned}E_1=&\{(\{\bar{q}_{i+1}\},\{q_{n-m+1},q_i\})\mid i\in[n-m-1]\}\\E_2=&\{(\{q_j\},\{\bar{q}_1,\bar{q}_{j+1}\})\mid j\in[2,n-m]\}\\E_3=&\{(\{\bar{q}_{n-m+1}\},\{q_{n-m},q_{n-m-1}\}),(\{q_{n-m+1}\},\{q_{n-m},q_{n-m-1}\})\},\\E_4=&\{(\{q_1\},\{\bar{q}_2,\bar{q}_3\}),(\{\bar{q}_1\},\{\bar{q}_2,\bar{q}_3\})\}\end{aligned}$$

5.17. Keep the notation in 5.14-5.15 and take $w'\in F_\mu$. Let $T_i(w')$ be the i-th component of $T(w')$ for any $i\in[2m+1]$. We see that w' is in F'_μ if and only if $(T_m(w'),T_{m+1}(w'))\in E_1\cup E_3$ and that w' is in F''_μ if and only if $(T_m(w'),T_{m+1}(w'))\in E_2\cup E_4$. This implies that the ordered pair $(T_m(w'),T_{m+1}(w'))$ (and hence the generalized tabloid $T(w')$) associated to $w'\in F'_\mu$ is always different from that associated to $w'\in F''_\mu$. So each of E'_μ and E''_μ is a union of some left cells of $\widetilde{C}_n$ by Lemmas 3.6 and 2.15. This further implies by Lemmas 2.14, 2.16, 5.13 and 1.7 that each of E'_μ and E''_μ is a union of some two-sided cells of $\widetilde{C}_n$. Let $w_1 = [0, 2, 3, ..., n-m-1, n+m+1, n+m, ..., n+2, n]$ and $w_2 = [1, -1, -2, ..., -m, m+2, m+3, ..., n-1, n+1]$ be in $\widetilde{C}_n$ (see Figure 11). Then $w_1 \in F'_\mu$ and $w_2 \in F''_\mu$. By the argument similar to that for Lemma 4.4 (3), we can prove that any $w \in F'_\mu$ (respectively, $w \in F''_\mu$) is in the right-connected component of E'_μ (respectively, E''_μ) containing w_1 (respectively, w_2) (the proof is left to the readers). So by 5.15 and Lemma 2.18, we conclude that

(5.17.1) Each of E'_μ and E''_μ is two-sided-connected and hence forms a single two-sided cell of $\widetilde{C}_n$.

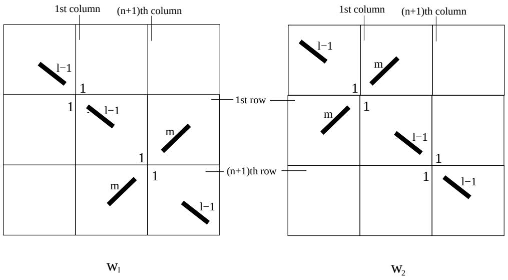


Figure 11

Assume that $\mathbf{T} = (T_1, T_2, ..., T_{2m+1}) \in \mathcal{C}_\beta$ satisfy the conditions (1)-(2) of 5.16. If $(T_m, T_{m+1}) \in E_1 \cup E_3$, then the element $w' \in F'_\mu$ with $T(w') = \mathbf{T}$ is uniquely determined by the conditions 5.14 (a$'$) and (5.15.1). This implies by 5.15 that any left cell Γ of $\widetilde{C}_n$ in E'_μ is left-connected. Since $E''_\mu = \eta(E'_\mu)$ by Lemma 5.13, any left cell Γ of $\widetilde{C}_n$ in E''_μ is also left-connected. So we conclude that

(5.17.2) All left cells of $\widetilde{C}_n$ in E_μ are left-connected.

5.18. Now we want to enumerate the left cells in E'_μ and E''_μ. Since $\eta(E'_\mu) = E''_\mu$, we need only deal with the set E'_μ. By 5.16 and Lemma 3.6, we need only to enumerate the generalized tabloids $\mathbf{T} = (T_1, T_2, ..., T_{2m+1})$ in $\mathcal{C}_\beta$ satisfying the conditions 5.16 (1)-(2) but with the condition $(T_m, T_{m+1}) \in E_1 \cup E_2 \cup E_3 \cup E_4$ replaced by $(T_m, T_{m+1}) \in E_1 \cup E_3$.

The number of the choices for the set

$$E := \bigcup_{i=m}^{m+2} T_i = \{\bar{q}_{n-m+1}, \bar{q}_{n-m}, ..., \bar{q}_1, q_1, q_2, ..., q_{n-m+1}\}$$

is $\binom{n}{n+1-m}$. Once E is fixed, the number of the choices for the triple (T_m, T_{m+1}, T_{m+2}) is $|E_1 \cup E_3| = n+1-m$, while the number of the choices for the $(m-1)$-tuple $(T_1, T_2, ..., T_{m-1})$ is $2^{m-1}(m-1)!$. Let $n(\mu)$, $n'(\mu)$, $n''(\mu)$ be defined as in Theorem 5.1 (3). Then we have

(5.18.1) $n'(\mu) = n''(\mu) = \frac{1}{2}n(\mu) = \binom{n}{n-m+1}(n+1-m)\cdot 2^{m-1}(m-1)! = \frac{2^{m-1}\cdot n!}{(n-m)!}$ for $\mu = (2m+1, 2, 1, ..., 1) \in \Lambda_{2n}$.

If $m > 1$, then the number of the choices for the integer $(j_m)w$ in the condition (5.11.1) is infinite, hence

(5.18.2) $|E_{(2m+1,2,1,...,1)}| = \infty$.

On the other hand, denote $p_{i,j} := t_i t_{i-1} \cdots t_1 t_0 t_n t_1 \cdots t_{j-1} t_j$ for $i, j \in [n]$ and $q_{i,j} = \eta(p_{i,j})$. Then $E'_{\mathbf{321^{2n-5}}} = \{q_{i,j} \mid i, j \in [n]\}$ and $E''_{\mathbf{321^{2n-5}}} = \{p_{i,j} \mid i, j \in [n]\}$. So we conclude that

(5.18.3) The set $E_{\mathbf{321^{2n-5}}} = E'_{\mathbf{321^{2n-5}}} \cup E''_{\mathbf{321^{2n-5}}}$ is finite.

So by (5.17.1)-(5.17.2) and (5.18.1)-(5.18.3), it is proved for all the assertions of Theorem 5.1 involving the partition $\mu = (2m+1, 2, 1, ..., m1) \in \Lambda_{2n}$.

§6. The cells in the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell})$.

As an application of Theorems 4.9 and 5.1, we shall describe all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell})$ in this section.

Recall the notation E_λ for $\lambda \in \Lambda_{2n}$ and $\eta : \widetilde{C}_n \longrightarrow \widetilde{C}_n$ defined before (see 2.17 and 2.7). Let $n(\lambda)$ be the number of left cells of $\widetilde{C}_n$ in E_λ. When E_λ is a union of two two-sided cells (say E'_λ, E''_λ) of $\widetilde{C}_n$, denote by $n'(\lambda)$, $n''(\lambda)$ the numbers of left cells of $\widetilde{C}_n$ in E'_λ, E''_λ, respectively.

The main result of the section is as follows.

Theorem 6.1. *In the weighted Coxeter group* $(\widetilde{C}_3, \widetilde{\ell})$, *we have*

(1) E_λ *is a single two-sided cell of* $\widetilde{C}_3$ *if* $\lambda \in \{\mathbf{6}, \mathbf{51}, \mathbf{42}, \mathbf{3^2}, \mathbf{31^3}, \mathbf{2^3}, \mathbf{1^6}\}$ *and is a union of two two-sided cells of* $\widetilde{C}_3$ *if* $\lambda \in \{\mathbf{41^2}, \mathbf{321}, \mathbf{2^21^2}, \mathbf{21^4}\}$*.* E_λ *is finite if* $\lambda \in \{\mathbf{1^6}, \mathbf{21^4}, \mathbf{2^21^2}, \mathbf{321}\}$*, and infinite if otherwise.*

(2) η *stabilizes the two-sided cells* $E'_{\mathbf{2^21^2}}$ *and* $E''_{\mathbf{2^21^2}}$*, and interchanges the following pairs of two-sided cells:* $E'_{\mathbf{41^2}}$*,* $E''_{\mathbf{41^2}}$*;* $E'_{\mathbf{321}}$*,* $E''_{\mathbf{321}}$*;* $E'_{\mathbf{21^4}}$*,* $E''_{\mathbf{21^4}}$*.*

(3) The numbers $n(\lambda)$ *for any* $\lambda \in \Lambda_6$ *are listed as follows.*

λ	$\mathbf{6}$	$\mathbf{51}$	$\mathbf{42}$	$\mathbf{41^2}$	$\mathbf{3^2}$	$\mathbf{321}$	$\mathbf{31^3}$	$\mathbf{2^3}$	$\mathbf{2^21^2}$	$\mathbf{21^4}$	$\mathbf{1^6}$
$n(\lambda)$	*48*	*24*	*24*	*12*	*12*	*6*	*6*	*8*	*5*	*2*	*1*

where $n'(\mathbf{41^2}) = n''(\mathbf{41^2}) = 6$*,* $n'(\mathbf{321}) = n''(\mathbf{321}) = 3$*,* $n'(\mathbf{2^21^2}) = 4$*,* $n''(\mathbf{2^21^2}) = n'(\mathbf{21^4}) = n''(\mathbf{21^4}) = 1$*.*

(4) Each left (respectively, two-sided) cell of $\widetilde{C}_3$ *is left- (respectively, two-sided-) connected.*

6.2. All the results in Theorem 6.1 follow by Theorems 4.9 and 5.1 except for those involving the partitions $\mathbf{3^2}$ and $\mathbf{2^3}$.

Consider the partial order $\preceq_w$ on $[6]$ with respect to a fixed $w \in \widetilde{C}_3$. The following equivalent conditions hold by Lemma 3.2:

(1) $\psi(w) = \mathbf{3^2}$ if and only if one of the conditions (1a)-(1c) holds for some pairwise not 6-dual $i, j, k \in [6]$:

(1a) i is w-tame and j, k are both w-wild heads such that $i \prec_w k$, that $\bar{i} \prec_w j$ and that j, k are w-uncomparable;

(1b) k is a w-wild head and i, j are both w-tame such that $j \prec_w i \prec_w k$ and that $\bar{k}$ is w-uncomparable with j;

(1c) i, j, k are all w-tame with $i \prec_w j \prec_w k$.

(2) $\psi(w) = \mathbf{2^3}$ if and only if one of the conditions (2a)-(2c) holds for some pairwise not 6-dual $i, j, k \in [6]$:

(2a) i, j, k are all w-wild heads which are pairwise w-uncomparable;

(2b) i is w-tame; j, k are both w-wild heads which are w-uncomparable; i is w-comparable with some element either in $\{j, k\}$, or in $\{\bar{j}, \bar{k}\}$ but not both;

(2c) k is a w-wild head and i,j are both w-tame heads such that $j \prec_w i$ and that k is w-uncomparable with $i,\bar{j}$.

Since $\{[6i-1,6i,3] \mid i \in \mathbb{Z}\setminus\{0\}\} \subset E_{\mathbf{3^2}}$ and $\{[3i+1,3i+2,3i+3] \mid i \in \mathbb{Z}\setminus\{0\}\} \subset E_{\mathbf{2^3}}$, we have

(6.2.1) $|E_{\mathbf{3^2}}| = |E_{\mathbf{2^3}}| = \infty$.

6.3. Let $F'_{\mathbf{3^2}}$ be the set of all $w' \in \widetilde{C}_3$ satisfying the condition (6.3.1) below.

(6.3.1) There exists some pairwise not 6-dual $i,j,k \in [6]$ such that one of the following conditions holds:

(a) i is a w'-tame head and j,k are both w'-wild heads such that (i) $j<k$ and $(j)w' < (k)w' < (j)w'+6$; (ii) $j<\bar{i}$ and $k<i$; (iii) $(\bar{i})w' < (j)w'$ and $(i)w' < (k)w'$; (iv) either $(k)w' < (\bar{i})w'+6$ or $(j)w' < (i)w'$;

(b) k is a w-wild head and i,j are both w-tame heads such that (i) $i<j$ and $(i)w' > (j)w'$; (ii) either $(k,i,j) = (4,5,6)$ and $6 < (4)w' < (1)w'+6$, or $(k,i,j) = (1,4,5)$ and $3 < (1)w' < (4)w'$;

(c) $w' = [3,2,1]$.

Then we see by 3.3 and 6.2 (1) that for any $w \in E_{\mathbf{3^2}}$, there exists some $w' \in F'_{\mathbf{3^2}}$ such that w,w' are in the same left-connected component of $E_{\mathbf{3^2}}$.

6.4. Let

$$\begin{aligned} F_1 = \{&[4,2,0],[4,1,-1],[5,3,0],[5,1,-2],[5,1,-3],\\ &[6,3,-1],[7,3,-1],[4,2,6],[5,3,6]\},\\ F_2 = \{&[3,2,0],[3,1,-1],[4,2,1],[5,3,1]\}, \qquad F_3 = \{[3,2,1]\}. \end{aligned}$$

Then $w' \in F'_{\mathbf{3^2}}$ satisfies the condition (a) in (6.3.1) if and only if there exists some $w \in F_1$ such that w,w' are in the same left-connected component of $E_{\mathbf{3^2}}$. $w' \in F'_{\mathbf{3^2}}$ satisfies the condition (b) (respectively, (c)) in (6.3.1) if and only if w' is in the set F_2 (respectively, F_3).

6.5. Let $x_1 = [4,2,6]$, $x_2 = [4,2,1]$, $y_1 = [5,3,6]$ and $y_2 = [5,3,1]$. Then $x_1,y_1 \in F_1$, $x_2,y_2 \in F_2$, $x_2 = t_3x_1$ and $y_2 = t_3y_1$. So the elements x_1,x_2 (respectively, y_1,y_2) are in the same left-connected component of $E_{\mathbf{3^2}}$. Let

$$F_{\mathbf{3^2}} = (F_1 \cup F_2 \cup F_3) \setminus \{[4,2,6],[5,3,6]\}.$$

We see from Figure 12 that the elements of $F_{\mathbf{3^2}}$ are in the same right-connected component of $E_{\mathbf{3^2}}$ and have pairwise different generalized τ-invariants (see 2.9 and 2.11).

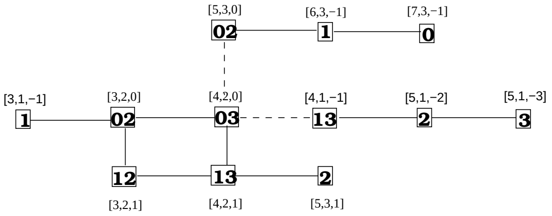


Figure 12

So by Lemmas 2.15-2.16 and 2.18, we see that

(6.5.1) $E_{\mathbf{3^2}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_3$ with $n(\mathbf{3^2}) = |F_{\mathbf{3^2}}| = 12$, each left cell of $\widetilde{C}_3$ in $E_{\mathbf{3^2}}$ is left-connected.

6.6. Next consider the set $E_{\mathbf{2^3}}$. Let $F'_{\mathbf{2^3}}$ be the set of all $w' \in \widetilde{C}_3$ satisfying the condition (6.6.1) below.

(6.6.1) There exists some pairwise not 6-dual $i, j, k \in [6]$ satisfying one of the conditions (a)-(c) below:

(a) i, j, k are all w'-wild heads satisfying (i) $i < j < k$ and $4 \leqslant (i)w' < (j)w' < (k)w' \leqslant 9$; (ii) $i \in [3]$ unless $(i)w' > 6$; (iii) $k = 6$ unless $(k)w' \leqslant 6$;

(b) i is a w'-tame head and j, k are both w'-wild heads such that (i) $j < k$ and $(j)w' < (k)w' < (j)w' + 6$; (ii) assume $j < \bar{i}$. if $k < i$ then $(k)w' < (i)w'$; if $k > i$ then $(j)w' < (i)w' < (k)w'$; (iii) assume $\bar{i} < j < i$. if $k < i$ then $(i)w' < (k)w' < (\bar{i})w' + 6$; if $k > i$ then either $(i)w' < (j)w' < (k)w' < (\bar{i})w' + 6$, or $(\bar{i})w' < (j)w' < (i)w'$ and $(\bar{i})w' < (k)w' - 6 < (i)w'$; (iv) if $i < j$ then $(\bar{i})w' < (k)w' - 6 < (i)w'$;

(c) k is a w'-wild head and i, j are both w'-tame heads with $i < j$ and $(j)w' < (i)w'$ such that (i) $\bar{j} < k$ and $3 < (k)w' < (\bar{j})w' + 6$; (ii) either $i < k$, or $k < i$ and $3 < (k)w' < (i)w'$.

By 6.2 (2) and 3.3, we see that for any $w \in E_{\mathbf{2^3}}$, there exists some $w' \in F'_{\mathbf{2^3}}$ such that w, w' are in the same left-connected component of $E_{\mathbf{2^3}}$.

6.7. Let

$$\begin{aligned} F_1' =&\{[4,5,6],[0,4,5],[-1,4,6],[-2,5,6],\\ &[-2,-1,0],[-1,0,4],[-2,0,5],[-2,-1,6]\},\\ F_2' =&\{[4,1,5],[0,4,2],[2,4,6],[3,5,6],[2,0,4],[0,3,5],[-1,3,6],\\ &[-1,1,4],[-2,1,5],[-1,3,0],[-2,-1,1],[-2,0,2]\}\\ F_3' =&\{[2,1,4],[3,1,5],[0,3,2],[-1,3,1]\}. \end{aligned}$$

Then we see by 3.3 that any $x \in F'_{\mathbf{2}^3}$ satisfying the condition (a) (respectively, (b), (c)) in (6.6.1) is in a left-connected component of $E_{\mathbf{2}^3}$ containing some element of F_1' (respectively, F_2', F_3').

Let $F_{\mathbf{2}^3} = F_1 \cup F_2$, where

$$\begin{aligned} F_1 =&\{[0,4,2],[0,3,2],[-1,3,1],[-1,3,0]\},\\ F_2 =&\{[2,0,4],[2,1,4],[3,1,5],[4,1,5]\}. \end{aligned}$$

Then it is easy to check that any $x \in \bigcup_{k=1}^3 F_k'$ is in a left-connected component of $E_{\mathbf{2}^3}$ containing some element of $F_{\mathbf{2}^3}$.

6.8. We see from Figure 13 that no two elements of $F_{\mathbf{2}^3}$ have the same generalized τ-invariants (see 2.9 and 2.11).

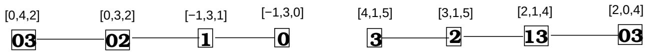


Figure 13

On the other hand, we have

$$[0,4,2] = t_2t_0t_3 \underset{R}{\sim} t_0t_2t_3t_1 \underset{L}{\sim} t_1t_3 \underset{R}{\sim} t_1t_0t_3 = [2,0,4].$$

This implies that $F_{\mathbf{2}^3}$ is contained in a two-sided-connected component of $E_{\mathbf{2}^3}$. By Lemmas 2.15-2.16 and 2.18, we see that

(6.8.1) $E_{\mathbf{2}^3}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_3$ with $n(\mathbf{2}^3) = |F_{\mathbf{2}^3}| = 8$, each left cell of $\widetilde{C}_3$ in $E_{\mathbf{2}^3}$ is left-connected.

So we complete the proof of Theorem 6.1 by Theorems 4.9, 5.1 and the results (6,2,1), (6.5.1), (6.8.1).

REFERENCES

1. T. Asai et al., *Open problems in algebraic groups*, Problems from the conference on algebraic groups and representations held at Katata (Ryoshi Hotta (ed.)), August 29–September 3, 1983.
2. K. Bremke, *On generalized cells in affine Weyl groups*, J. Algebra **191** (1997), 149-173.
3. C. Greene, *Some partitions associated with a partially ordered set*, J. Comb. Theory (A) **20** (1976), 69–79.
4. J. Guilhot, *Kazhdan-Lusztig cells in the affine Weyl groups of rank 2*, International Mathematics Research Notices **17** (2010), 3422-3462.
5. G. Lusztig, *Some examples in square integrable representations of semisimple p-adic groups*, Trans. of the AMS **277** (1983), 623–653.
6. G. Lusztig, *Left cells in Weyl groups*, In "Lie Group Representation I ", LNM 1024, Springer-Verlag, Berlin, 1984, pp. 99-111.
7. G. Lusztig, *The two-sided cells of the affine Weyl group of type* $\widetilde{A}_n$, in " Infinite Dimensional Groups with Applications ", (V. Kac, ed.), MSRI. Publications 4, Springer-Verlag,(1985), 275–283.
8. G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Series, vol. 18, AMS, USA, 2003.
9. J. Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Math. vol. 1179, Springer-Verlag, Germany, 1986.
10. J. Y. Shi, *A survey on the cell theory of affine Weyl groups*, Advances in Science of China, Math. **3** (1990), 79–98.
11. J. Y. Shi, *The partial order on two-sided cells of certain affine Weyl groups*, J. Algebra. **179(2)** (1996), 607–621.