THE CELLS IN THE WEIGHTED COXETER GROUP $(\widetilde{C}_n, \widetilde{\ell}_m)$

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Abstract. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_m, \widetilde{S}_m)$, $m \in \{2n-1, 2n, 2n+1\}$, under a certain group automorphism $\alpha_{m,n}$. Let $\widetilde{\ell}_m$ be the length function of $\widetilde{A}_m$. The present paper is to give a combinatorial description for all the left cells of $\widetilde{A}_m$ which have non-empty intersection with $\widetilde{C}_n$. Then we use this description to deduce some formulae for the number of left cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set E_λ associated to any partition λ of $m+1$.

§0. Introduction.

0.1. Let $\mathbb{Z}$ (respectively, $\mathbb{N}$, $\mathbb{P}$) be the set of all integers (respectively, non-negative integers, positive integers). The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_m, \widetilde{S}_m)$, $m \in \{2n-1, 2n, 2n+1\}$, under a certain automorphism $\alpha_{m,n}$ with $\alpha_{m,n}(\widetilde{S}_m) = \widetilde{S}_m$, where $\widetilde{S}_m$, S are the Coxeter generator sets of $\widetilde{A}_m$, $\widetilde{C}_n$, respectively. The restriction to $\widetilde{C}_n$ of the length function $\widetilde{\ell}_m$ of $\widetilde{A}_m$ is a weight function of $\widetilde{C}_n$. It is known that there is a surjective map ψ from $\widetilde{A}_m$ to the set Λ_{m+1} of partitions of $m+1$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_m$ to Λ_{m+1} (see [6], [3]). For any $i \leqslant j$ in $\mathbb{N}$, denote $[i,j] := \{i, i+1, ..., j\}$ and denote $[1,i]$ simply by $[i]$. Let $E_\lambda := \psi^{-1}(\lambda) \cap \widetilde{C}_n$ for $\lambda \in \Lambda_{m+1}$. In the paper [7], we described all the cells of the weighted Coxeter

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group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ corresponding to the partitions $\mathbf{k1^{2n-k}}$ and $\mathbf{h21^{2n-h-2}}$ for all $k \in [2n]$ and $h \in [2, 2n-2]$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$.

0.2. Denote by $\lambda^\vee$ the dual partition of $\lambda \in \Lambda_{m+1}$ (see 1.8). Let $\widetilde{\Lambda}_{m+1}$ be the set of all compositions of $m+1$ (see 2.1). There is a natural surjective map ζ from the set $\widetilde{\Lambda}_{m+1}$ to Λ_{m+1} (see 2.1). Call $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$ *symmetric*, if $a_i = a_{r+1-i}$ for any $i \in [r]$. Let $\mathcal{C}_{m+1}$ be the set of all tabloids of rank $m+1$ (see 2.2). We can define an equivalence relation $\approx$ on $\mathcal{C}_{m+1}$ (see 2.13). There exists a bijective map from the set Π^l_m of left cells of $\widetilde{A}_m$ to the set of $\approx$-equivalence classes of $\mathcal{C}_{m+1}$ (see [6, Subsection 19.4]). There exists a natural surjective map ξ from $\mathcal{C}_{m+1}$ to $\widetilde{\Lambda}_{m+1}$ (see 2.2).

0.3. In the present paper, we prove that a left cell Γ of $\widetilde{A}_m$ has a non-empty intersection with $\widetilde{C}_n$ if and only if the $\approx$-equivalence class of $\mathcal{C}_{m+1}$ corresponding to Γ is (m,n)-selfdual (see 2.13-2.14, Lemma 2.15 and Theorem 3.1). By this result, we can deduce some formulae for the number $\gamma_{m+1-2n}(\mathbf{a})$ of left cells of $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set $E_{\zeta(\mathbf{a})^\vee}$ for any $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$. More precisely, we give a close formula for the number $\gamma_{m+1-2n}(\mathbf{a})$ if $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$ is symmetric (Theorem 3.3). For an arbitrary $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$, we have $\gamma_{m+1-2n}(\mathbf{a}) = \gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2)\binom{n}{l}$ for some symmetric $\mathbf{a}_1 = (a_{i_1}, a_{i_2}, ..., a_{i_{2p}}) \in \widetilde{\Lambda}_{2l}$, and some $\mathbf{a}_2 = (a_{j_1}, a_{j_2}, ..., a_{j_q}) \in \widetilde{\Lambda}_{m+1-2l}$, $a_{j_1} > a_{j_2} > \cdots > a_{j_q} > 0$, with some $l \in \mathbb{N}$, where $\binom{n}{l} := \frac{n!}{l!(n-l)!}$ and $\{i_h, j_l \mid h \in [0, 2p], l \in [0, q]\} = [r]$ and the notation $\gamma_k(\mathbf{b})$, $k \in \{0, 1, 2\}$, stands for the numbers of (m,n)-selfdual tabloids $\mathbf{T}$ with $\xi(\mathbf{T}) = \mathbf{b}$ over an (m,n)-selfdual subset of $[m+1]$ containing exactly k (m,n)-selfdual elements (see 3.6 and Theorem 3.7). Hence to calculate the number $\gamma_{m+1-2n}(\mathbf{a})$, we are reduced to the case where $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1-2l}$ satisfies $a_1 > a_2 > \cdots > a_r$ and $l \in \mathbb{N}$. We get a close formula for $\gamma_{m+1-2n}(\mathbf{a})$ in the case of $r = 2$ (see Propositions 4.7-4.9 and Corollary 4.12). Then in the case of $r = 3$, we describe the (m,n)-selfdual tabloids in $\xi^{-1}(\mathbf{a})$ (see Proposition 4.15).

0.4. The contents of the paper are organized as follows. In Section 1, we collect some concepts and known results concerning cells of the weighted Coxeter groups $(\widetilde{A}_m, \widetilde{\ell}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$. Then we introduce the tabloids of rank $m+1$ in Section

2. In Section 3, we characterize all the tabloids parameterizing the left cells of $(\widetilde{C}_n, \widetilde{\ell}_m)$ and give some formulae for the number of left cells of $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set E_λ for any $\lambda \in \Lambda_{m+1}$. Finally, we deduce some more formulae for those numbers and describe the (m, n)-selfdual tabloids in some special cases in Section 4.

§1. The weighted Coxeter groups $(\widetilde{A}_m, \widetilde{\ell}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$.

In this section, we collect some concepts and results concerning the weighted Coxeter groups $(\widetilde{A}_m, \widetilde{\ell}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$.

1.1. Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. Call $L : W \to \mathbb{Z}$ a *weight function* on W if $L(xy) = L(x) + L(y)$ for any $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$. Hence $L(s) = L(t)$ for any $s, t \in S$ conjugate in W. Call (W, L) is a *weighted Coxeter group.*

A weighted Coxeter group (W, L) is called in the *split* case if $L = \ell$.

Suppose that there exists a group automorphism $\alpha : W \longrightarrow W$ with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J in S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J whenever W_J is finite. Let S_α be the set of elements w_J with J ranging over all such α-orbits in S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function ℓ of W is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the *quasi-split* case.

1.2. Let $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on a weighted Coxeter group (W, L) defined in [5]. The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (respectively, $\underset{R}{\sim}$, $\underset{LR}{\sim}$). The corresponding equivalence classes in W are called *left cells* (respectively, *right cells, two-sided cells*) of W.

1.3. Lusztig introduced a subset $\mathcal{D}$ of a weighted Coxeter group (W, L) (see [5, Chapter 14]). When (W, L) is a Weyl or affine Weyl group which is either in the split case or in the quasi-split case, Lusztig proved that the set $\mathcal{D}$ consists of certain involutive elements w (hence $w^2 = 1$) and that each left (respectively, right) cell of W contains exactly one element in $\mathcal{D}$ (see [5, Chapters 14–16]). The elements of $\mathcal{D}$ were called *distinguished involutions* when (W, L) is in the split case (see [4]).

1.4. The group $\widetilde{A}_m$, $m \geqslant 1$, can be realized as the following permutation group on the set $\mathbb{Z}$ (see [2, Subsection 3.6] and [6, Subsection 4.1)]:

$$\widetilde{A}_m = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \,\middle|\, (i+m+1)w = (i)w + m + 1, \sum_{i=1}^{m+1} (i)w = \sum_{i=1}^{m+1} i \right\}.$$

The Coxeter generator set $\widetilde{S}_m = \{s_i \mid i \in [0, m]\}$ of $\widetilde{A}_m$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{m+1}, \\ t+1, & \text{if } t \equiv i \pmod{m+1}, \\ t-1, & \text{if } t \equiv i+1 \pmod{m+1}, \end{cases}$$

for $t \in \mathbb{Z}$ and $i \in [0, m]$. Any $w \in \widetilde{A}_m$ can be realized as a $\mathbb{Z} \times \mathbb{Z}$ monomial matrix $A_w = (a_{ij})_{i,j \in \mathbb{Z}}$, where a_{ij} is 1 if $j = (i)w$ and 0 otherwise. The row (respectively, column) indices of A_w increase from top to bottom (respectively, from left to right).

1.5. For $m \in \{2n-1, 2n, 2n+1\}$, let $\alpha_{m,n} : \widetilde{A}_m \longrightarrow \widetilde{A}_m$ be the group automorphism determined by $\alpha_{m,n}(s_i) = s_{2n-i}$ if $m = 2n-1$ and $\alpha_{m,n}(s_i) = s_{2n+1-i}$ if $m \in \{2n, 2n+1\}$ for $i \in [0, m]$, where we stipulate $s_{i+m+1} = s_i$ for any $i \in \mathbb{Z}$. In terms of matrix form, for any $w \in \widetilde{A}_m$, the matrix $A_{\alpha_{m,n}(w)}$ can be obtained from the matrix A_w by rotating with the angle π around the point $(n + \frac{1}{2}, n + \frac{1}{2})$ (respectively, $(n+1, n+1)$) if $m = 2n-1$ (respectively, $m \in \{2n, 2n+1\}$), where we identify A_w with a plane and identify the positions (i, j), $i, j \in \mathbb{Z}$, of A_w with the corresponding integer lattice points. Then $\alpha_{m,n}$ gives rise to a permutation on the set Π_m^l (respectively, Π_m^r, Π_m^t) of left cells (respectively, right cells, two-sided cells) of $\widetilde{A}_m$. Also, $\alpha_{m,n}(\mathcal{D}) = \mathcal{D}$ by the definition of the set $\mathcal{D}$ in [5, Chapter 14].

1.6. The affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$, under the automorphism $\alpha_{m,n}$, hence can be described as follows.

$$\widetilde{C}_n = \{w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+m+1)w = (i)w+m+1, (i)w+(\epsilon_{m,n}-i)w = \epsilon_{m,n}, \ \forall\, i \in \mathbb{Z}\},$$

where $\epsilon_{m,n}$ is 1 if $m \in \{2n-1, 2n\}$ and 0 if $m = 2n+1$. The Coxeter generator set $S = \{t_i \mid i \in [0, n]\}$ of $\widetilde{C}_n$ is given by setting $t_i = s_i s_{2n-i}$ for $i \in [n-1]$, $t_0 = s_0$ and

$t_n = s_n$ if $m = 2n-1$; $t_i = s_i s_{2n+1-i}$ for $i \in [n-1]$, $t_0 = s_0$ and $t_n = s_n s_{n+1} s_n$ if $m = 2n$; $t_i = s_i s_{2n+1-i}$ for $i \in [n-1]$, $t_0 = s_0 s_1 s_0$ and $t_n = s_n s_{n+1} s_n$ if $m = 2n+1$. In terms of matrix, an element $w \in \widetilde{A}_m$ is in $\widetilde{C}_n$ if and only if the matrix form A_w of w is centrally symmetric at the points $(qn+\frac{1}{2}, qn+\frac{1}{2})$ if $m = 2n-1$ and, at the points $((2n+1)q+\frac{1}{2}, (2n+1)q+\frac{1}{2})$ and $((2n+1)q+(n+1), (2n+1)q+(n+1))$ if $m = 2n$ and, at the points $((n+1)q, (n+1)q)$ if $m = 2n+1$, where q ranges over $\mathbb{Z}$

1.7. By a partition of $l \in \mathbb{P}$, we mean an r-tuple $\lambda := (\lambda_1, \lambda_2, ..., \lambda_r)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ in $\mathbb{P}$ and $\sum_{k=1}^r \lambda_k = l$ for some $r \in \mathbb{P}$. Call λ_i a *part* of λ. We sometimes denote λ by $\mathbf{j_1^{k_1} j_2^{k_2} \cdots j_m^{k_m}}$ (boldfaced) with $j_1 > j_2 > \cdots > j_m$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$. Let Λ_l be the set of all partitions of l.

Fix $w \in \widetilde{A}_m$. For any $i \neq j$ in $[m+1]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both $p(m+1)+i > q(m+1)+j$ and $(p(m+1)+i)w < (q(m+1)+j)w$ hold. This defines a partial order $\preceq_w$ on the set $[m+1]$. $i \neq j$ in $[m+1]$ is said *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* otherwise.

A sequence $a_1, a_2, ..., a_r$ in $[m+1]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. Sometimes we identify a w-chain $a_1, a_2, ..., a_r$ with the corresponding set $\{a_1, a_2, ..., a_r\}$. For any $k \geqslant 1$, a *k-w-chain-family* is by definition a union $X = \cup_{i=1}^k X_i$ of k w-chains $X_1, X_2, ..., X_k$ in $[m+1]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = m+1$. Let $\lambda_1 = d_1$ and $\lambda_{k+1} = d_{k+1} - d_k$ for $k \in [r-1]$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of C. Greene in [1]. Hence $w \mapsto \psi(w) := (\lambda_1, \lambda_2, ..., \lambda_r)$ defines a map from the set $\widetilde{A}_m$ to Λ_{m+1}.

A subset E of $[m+1]$ is called a *w-antichain*, if the elements of E are pairwise w-uncomparable.

1.8. Let $\widetilde{\ell}_m$ be the length function for the Coxeter group $(\widetilde{A}_m, \widetilde{S}_m)$. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_m, \widetilde{\ell}_m)$ is in the split case, while $(\widetilde{C}_n, \widetilde{\ell}_m)$ is in the quasi-split case.

For any $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{m+1}$, define $\lambda^\vee = (\lambda_1^\vee, ..., \lambda_t^\vee) \in \Lambda_{m+1}$ by setting $\lambda_j^\vee = {}^\#\{k \in [r] \mid \lambda_k \geqslant j\}$ for any $j \geqslant 1$, call $\lambda^\vee$ the *dual partition* of λ.

Lemma 1.9. *(1) Regarding $\widetilde{C}_n$ as a subset of $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$. For* *any* $x, y \in \widetilde{C}_n$, *we have* $x \underset{L}{\sim} y$ *in* $\widetilde{C}_n$ *if and only if* $x \underset{L}{\sim} y$ *in* $\widetilde{A}_m$ (see [5, Lemma 16.14]).

(2) The set $\psi^{-1}(\lambda)$ *forms a two-sided cell of* $\widetilde{A}_m$ *for any* $\lambda \in \Lambda_{m+1}$ (see [3, Theorem 6] and [6, Theorem 17.4]).

By Lemma 1.9 (1), we can just use the notation $x \underset{L}{\sim} y$ for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$, or $\widetilde{C}_n$.

For any $\lambda \in \Lambda_{m+1}$, denote $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

In the remaining part of the paper, when we mention the number m, we always assume $m \in \{2n-1, 2n, 2n+1\}$ unless otherwise specified.

§2. Tabloids of rank $m+1$.

In the present section, we introduce the concept of tabloids of rank $m+1$ which will be used to parametrize the left cells of $\widetilde{A}_m$ and of $\widetilde{C}_n$.

2.1. By a *composition* of $m+1$, we mean an r-tuple $\mathbf{a} = (a_1, a_2, ..., a_r)$ with $a_1, ..., a_r, r \in \mathbb{P}$ and $\sum_{i=1}^{r} a_i = m+1$. Let $\widetilde{\Lambda}_{m+1}$ be the set of all compositions of $m+1$. Clearly, $\Lambda_{m+1} \subseteq \widetilde{\Lambda}_{m+1}$. For any $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$, let $i_1, i_2, ..., i_r$ be a permutation of $1, 2, ..., r$ with $a_{i_1} \geqslant a_{i_2} \geqslant \cdots \geqslant a_{i_r}$. Denote $\zeta(\mathbf{a}) = (a_{i_1}, a_{i_2}, ..., a_{i_r})$. This defines a surjective map $\zeta : \widetilde{\Lambda}_{m+1} \longrightarrow \Lambda_{m+1}$.

2.2. A *(generalized) tabloid of rank* $m+1$ is, by definition, an r-tuple $\mathbf{T} = (T_1, T_2, ..., T_r)$ with some $r \in \mathbb{N}$ such that $[m+1]$ is a disjoint union of some non-empty subsets T_j, $j \in [r]$. We have $\xi(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|) \in \widetilde{\Lambda}_{m+1}$, where $|T_i|$ denotes the cardinal of the set T_i. Let $\mathcal{C}_{m+1}$ be the set of all tabloids of rank $m+1$. Then $\xi : \mathcal{C}_{m+1} \longrightarrow \widetilde{\Lambda}_{m+1}$ is a surjective map. Let $\kappa = \zeta\xi : \mathcal{C}_{m+1} \longrightarrow \Lambda_{m+1}$.

2.3. For any $i \in \mathbb{Z}$, define $\langle i \rangle \in [m+1]$ by the condition $\langle i \rangle \equiv i \pmod{m+1}$. Let Ω be the set of all $w \in \widetilde{A}_m$ such that there is some $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_{m+1}$ satisfying that

(i) If $i < j$ in $[r]$, then $\langle (a)w^{-1} \rangle \prec_w \langle (b)w^{-1} \rangle$ for any $a \in T_i$ and $b \in T_j$;

(ii) $\langle (T_i)w^{-1} \rangle$ is a maximal w-antichain in $[m+1]$ for any $i \in [r]$.

Clearly, the tabloid $\mathbf{T}$ is determined entirely by $w \in \Omega$, denote $\mathbf{T}$ by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{m+1}$ is surjective by [6, Proposition 19.1.2]. By a result of C. Greene in [1], we have $\kappa(T(w)) = \psi(w)^\vee$.

Lemma 2.4. (see [6, Proposition 19.1.2 and Lemma 19.4.6]) *Suppose that* $y, w \in \Omega$ *satisfy* $\xi(T(y)) = \xi(T(w))$. *Then* $y \underset{L}{\sim} w$ *if and only if* $T(y) = T(w)$.

2.5. By Lemma 2.4, it makes sense to write $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$ in $\mathcal{C}_{m+1}$ if there exist some $x, y \in \Omega$ satisfying $x \underset{L}{\sim} y$ and $T(x) = \mathbf{T}$ and $T(y) = \mathbf{T}'$. This defines an equivalence relation on $\mathcal{C}_{m+1}$.

Fix $w \in \widetilde{A}_m$ and let $\lambda = \psi(w)$. Take any $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$. There exists some $y \in \Omega$ with $y \underset{L}{\sim} w$ and $\xi(T(y)) = \mathbf{a}$. The tabloid $T(y)$ is uniquely determined by the element w and the composition $\mathbf{a}$ of $m+1$, denote it by $T_{\mathbf{a}}(w)$ (see [6, Propositions 19.1.2, 19.4.7 and 19.4.8]).

Lemma 2.6. *(see* [6, Propositions 19.4.7-19.4.8]*) In the above setup,* $T_{\mathbf{a}}$ *gives rise to a surjective map from the set* $\psi^{-1}(\lambda)$ *to* $\xi^{-1}(\mathbf{a})$*, which induces a bijection (again denoted by* $T_{\mathbf{a}}$*) from the set* Π^l_λ *of left cells of* $\widetilde{A}_m$ *in* $\psi^{-1}(\lambda)$ *to* $\xi^{-1}(\mathbf{a})$.

2.7. For further discussion on the left cells of $\widetilde{A}_m$ and $\widetilde{C}_n$, let us recall some more concepts involving tabloids of rank $m+1$ (see [6, Chapter 20]). Let $k \in \mathbb{P}$. Arrange the numbers $1, 2, ..., k$ on a circle in clockwise order, hence $t+1$ is the successor of t for any $t \in [k-1]$ and 1 is the successor of k. We call such a circle the *k-circle*. For example, the following is the 8-circle.

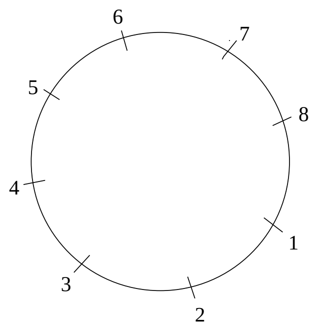

Figure 1

For $x \neq y$ in $[k]$, we denote by $\widehat{xy}$ the arc of the k-circle which, starting with the number x and moving clockwise, ends with the number y. For $Z \subseteq [k]$, let Z_{xy} be the set of all elements of Z on $\widehat{xy}$. Take the 8-circle in Figure 1 as an example, let $Z = \{1, 2, 3, 4, 6\}$, $x = 2$, $y = 5$. Then $Z_{xy} = \{2, 3, 4\}$ and $Z_{yx} = \{1, 2, 6\}$.

Let $X=\{a_j\mid j\in[t],a_1<\cdots<a_t\}$ and $Y=\{b_j\mid j\in[r],b_1<\cdots<b_r\}$ be two subsets of $[k]$ with $X\cap Y=\emptyset$ and $t\leqslant r$.

(i) Define a subset $H_Y(X)=\{c_1,...,c_t\}$ of Y such that $c_h\in Y$ is given recurrently by the condition $|(Y-\{c_1,...,c_{h-1}\})_{a_hc_h}|=1$ for any $h\in[t]$.

(ii) Define a subset $L_Y(X)=\{d_1,...,d_t\}$ of Y such that $d_h\in Y$ is given recurrently by the condition $|(Y-\{d_1,...,d_{h-1}\})_{d_ha_{t+1-h}}|=1$ for any $h\in[t]$.

By the definition, we see that the sets $H_Y(X)$ and $L_Y(X)$ depend only on the relative positions of the elements of $X\cup Y$ on the k-circle, but neither on $k\in\mathbb{P}$ nor on any element in $[k]-X\cup Y$. In particular, $H_Y(X)=L_Y(X)=Y$ if $|X|=|Y|$.

The following result can be checked directly from the above definition.

Lemma 2.8. *Fix $k\in\mathbb{P}$. If η is a permutation on $[k]$ such that $\eta(i+1)\equiv\eta(i)-1$ (* mod k*) for any $i\in[k]$ (hence the order of the numbers $1,2,...,k$ on the k-circle are reversed by η) then $\eta(H_Y(X))=L_{\eta(Y)}(\eta(X))$ and $\eta(L_Y(X))=H_{\eta(Y)}(\eta(X))$ for any $X,Y\subseteq[k]$ with $|Y|\geqslant|X|$ and $Y\cap X=\emptyset$.*

Take the 8-circle in Figure 1 as an example. Let $X=\{1,4\}$ and $Y=\{2,6,7\}$. Then $H_Y(X)=\{2,6\}$ and $L_Y(X)=\{2,7\}$. Define $\eta:[8]\longrightarrow[8]$ by setting $\eta(i)=9-i$ for any $i\in[8]$. Then $\eta(H_Y(X))=\{3,7\}=L_{\{2,3,7\}}(\{5,8\})=L_{\eta(Y)}(\eta(X))$.

The following results describe the sets $H_Y(X)$ and $L_Y(X)$ in more intrinsic way.

Lemma 2.9. (see [6, Lemmas 20.1.2-20.1.3]) *Fix $k\in\mathbb{P}$ and take $X,Y\subseteq[k]$ such that $X\cap Y=\emptyset$ and $|X|\leqslant|Y|$. Then for any $y\in Y$, we have*

(a) $y\in H_Y(X)$ if and only if there exists some $x\in X$ satisfying $|Y_{xy}|=|X_{xy}|$.

(b) $y\in L_Y(X)$ if and only if there exists some $x\in X$ satisfying $|Y_{yx}|=|X_{yx}|$.

2.10. For $i,j\in[m+1]$, we say that j is the (m,n)-*dual* of i, denote $j=\bar{i}$, if either $m=2n-1$ and $i+j=2n+1$, or $m\in\{2n,2n+1\}$ and $i+j\equiv 2n+2$ (mod $2n+2$); in this case, we also have $i=\bar{j}$, and call $i,\bar{i}$ *an* (m,n)-*dual pair*. Denote $\overline{E}=\{\bar{i}\mid i\in E\}$ for any $E\subseteq[m+1]$ (The notation $\bar{i}$, $\overline{E}$ for $i\in[m+1]$ and $E\subseteq[m+1]$ will cause no confusion in the context since the pair (m,n) is fixed in each case).

For any $i\in[m+1]$, we have $i=\bar{i}$ if and only if either $m=2n$ and $i=n+1$, or $m=2n+1$ and $i\in\{n+1,2n+2\}$. When the equivalent conditions hold, i

with itself forms an (m,n)-dual pair, call i an (m,n)-*selfdual element.* Hence the number of (m,n)-selfdual elements in $[m+1]$ is $m+1-2n$.

Next result shows that for any $Y \subseteq [m+1]$, the operations H_Y and L_Y on $X \subseteq [m+1]$ with $|X| \leqslant |Y|$ and $X \cap Y = \emptyset$ are inverse to each other in some sense.

Lemma 2.11. *Let $X, Y \subseteq [m+1]$ satisfy $|X| \leqslant |Y|$ and $X \cap Y = \emptyset$.*

(a) Let $Y' = H_Y(X)$ and $X' = X \cup (Y - H_Y(X))$. Then $X = L_{X'}(Y')$ and $Y = Y' \cup (X' - L_{X'}(Y'))$.

(b) Let $Y'' = L_Y(X)$ and $X'' = X \cup (Y - L_Y(X))$. Then $X = H_{X''}(Y'')$ and $Y = Y'' \cup (X'' - H_{X''}(Y''))$.

(c) $\overline{H_Y(X)} = L_{\overline{Y}}(\overline{X})$ and $\overline{L_Y(X)} = H_{\overline{Y}}(\overline{X})$.

Proof. (a) and (b) are just the results in [6, Proposition 20.1.4]. Then (c) follows by Lemma 2.8. □

Recall the relation $\underset{L}{\sim}$ on $\mathcal{C}_{m+1}$ defined in 2.5.

Proposition 2.12. (see [6, Proposition 20.2.2 and Corollary 20.2.3]) *Let $\mathbf{T} = (T_1, ..., T_t) \in \mathcal{C}_{m+1}$ and $j \in [t-1]$.*

(a) If $|T_j| \leqslant |T_{j+1}|$, let

$$\mathbf{T}' = (T_1, ..., T_{j-1}, T_j \cup (T_{j+1} - H_{T_{j+1}}(T_j)), H_{T_{j+1}}(T_j), T_{j+2}, ..., T_t) \tag{2.12.1}$$

then $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$.

(b) If $|T_j| \geqslant |T_{j+1}|$, let

$$\mathbf{T}'' = (T_1, ..., T_{j-1}, L_{T_j}(T_{j+1}), T_{j+1} \cup (T_j - L_{T_j}(T_{j+1})), T_{j+2}, ..., T_t). \tag{2.12.2}$$

Then $\mathbf{T} \underset{L}{\sim} \mathbf{T}''$.

2.13. Let $\mathbf{T}, \mathbf{T}', \mathbf{T}'' \in \mathcal{C}_{m+1}$ be given as in (2.12.1)-(2.12.2). We say that $\mathbf{T}'$ (respectively, $\mathbf{T}''$) is obtained from $\mathbf{T}$ by a $\{j, j+1\}$-*transformation.* This definition does not cause any confusion since $\mathbf{T}'$ (respectively, $\mathbf{T}''$) is defined only when $|T_j| \leqslant |T_{j+1}|$ (respectively, $|T_j| \geqslant |T_{j+1}|$). Note that if $|T_j| = |T_{j+1}|$ then $\mathbf{T}' = \mathbf{T}'' = \mathbf{T}$.

Fix E with $\emptyset \neq E \subseteq [m+1]$. Let $\mathcal{C}_E$ be the set of all tabloids $\mathbf{T} = (T_1, T_2, ..., T_r)$ with $E = \dot\cup_{i=1}^r T_i$ (hence $\mathcal{C}_{m+1} = \mathcal{C}_{[m+1]}$).

For any $\mathbf{T}, \mathbf{T}' \in \mathcal{C}_E$, written $\mathbf{T} \approx \mathbf{T}'$, if there exists a sequence $\mathbf{T}_0 = \mathbf{T}$, $\mathbf{T}_1$,..., $\mathbf{T}_r = \mathbf{T}'$ in $\mathcal{C}_E$ such that for every $i \in [r]$, $\mathbf{T}_i$ can be obtained from $\mathbf{T}_{i-1}$ by an $\{h_i, h_i + 1\}$-transformation for some integer h_i. This defines an equivalence relation on the set $\mathcal{C}_E$.

Let $l = |E|$ and $\xi_E(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|)$ for any $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_E$. Then $\xi_E : \mathcal{C}_E \longrightarrow \widetilde{\Lambda}_l$ is a surjective map.

2.14. Take E with $\emptyset \neq E \subseteq [m + 1]$ and $\overline{E} = E$. Denote $\overline{\mathbf{T}} = (\overline{T_1}, \overline{T_2}, ..., \overline{T_r})$ and $\mathbf{T}^{\rm op} = (T_r, ..., T_2, T_1)$ for any $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_E$. Then $\overline{\mathbf{T}}, \mathbf{T}^{\rm op} \in \mathcal{C}_E$. We say that $\mathbf{T} \in \mathcal{C}_E$ is (m, n)-*selfdual*, if $\overline{\mathbf{T}}^{\rm op} \approx \mathbf{T}$.

Denote $\mathbf{a}^{\rm op} = (a_r, ..., a_2, a_1)$ for $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$. Call $\mathbf{a}$ *symmetric*, if $\mathbf{a}^{\rm op} = \mathbf{a}$.

When $m \in \{2n - 1, 2n + 1\}$, define a map $\tau_{m+1} : [m + 1] \longrightarrow [m + 1]$ by

$$\tau_{m+1}(i) = \begin{cases} i + \frac{m+1}{2}, & \text{if } i \in [\frac{m+1}{2}], \\ i - \frac{m+1}{2}, & \text{if } i \in [\frac{m+1}{2} + 1, m + 1]. \end{cases}$$

Then for any $i, j \in [m+1]$, we have that $\tau_{m+1}(\bar{i}) = \overline{\tau_{m+1}(i)}$, that i is (m, n)-selfdual if and only if so is $\tau_{m+1}(i)$, and that on the $(m + 1)$-circle, j is the successor of i if and only if $\tau_{m+1}(j)$ is the successor of $\tau_{m+1}(i)$.

Define $\tau_{m+1}(\mathbf{T})$ to be the tabloid obtained from $\mathbf{T}$ by replacing each $i \in [m+1]$ by $\tau_{m+1}(i)$ for any $\mathbf{T} \in \mathcal{C}_{m+1}$.

Lemma 2.15. *Let* $\mathbf{T}, \mathbf{T}' \in \mathcal{C}_{m+1}$.

(1) $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$ *if and only if* $\mathbf{T} \approx \mathbf{T}'$.

(2) When $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$ *is symmetric,* $\mathbf{T} \in \xi^{-1}(\mathbf{a})$ *is* (m, n)*-selfdual if and only if* $\overline{\mathbf{T}}^{\rm op} = \mathbf{T}$.

(3) If $\mathbf{T} \approx \mathbf{T}'$, *then* $\mathbf{T}$ *is* (m, n)*-selfdual if and only if so is* $\mathbf{T}'$.

(4) When $(m, n) \in \{(2n - 1, n), (2n + 1, n)\}$, $\mathbf{T}$ *is* (m, n)*-selfdual if and only if so is* $\tau_{m+1}(\mathbf{T})$.

Proof. (1) follows by Proposition 2.12, Lemmas 2.4 and 2.6. For (2), by the assumption of $\mathbf{a}$ being symmetric, we have $\xi(\overline{\mathbf{T}}^{\rm op}) = \mathbf{a}$ for any $\mathbf{T} \in \xi^{-1}(\mathbf{a})$. So

$\overline{\mathbf{T}}^{\rm op} \approx \mathbf{T}$ if and only if $\overline{\mathbf{T}}^{\rm op} = \mathbf{T}$ by (1) and Lemma 2.4. This implies (2). For (3), let $\mathbf{T} = (T_1, T_2, ..., T_r)$. We may assume without loss of generality that $\mathbf{T}'$ is obtained from $\mathbf{T}$ by an $\{i, i+1\}$-transformation for some $i \in [r-1]$. Then $\overline{\mathbf{T}}^{\rm op}$ can be obtained from $\overline{\mathbf{T}'}^{\rm op}$ by an $\{r-i, r+1-i\}$-transformation by Lemma 2.11 (c). This implies that $\mathbf{T} \approx \overline{\mathbf{T}}^{\rm op}$ if and only if $\mathbf{T}' \approx \overline{\mathbf{T}'}^{\rm op}$. Hence (3) follows. Finally, (4) follows by the properties of the map τ_{m+1} mentioned proceeding the lemma. □

By Lemma 2.15 (3), we can call an $\approx$-equivalence class of $\mathcal{C}_{m+1}$ (m, n)*-selfdual* if some (hence all) tabloid in this class is (m, n)-selfdual.

§3. A formula for the number of left cells of $\widetilde{C}_n$ in the set E_λ, $\lambda \in \Lambda_{m+1}$.

In the present section, we first characterize all the tabloids of rank $m+1$ which correspond to the left cells of $\widetilde{C}_n$. Applying this result, we deduce a formula for the number of left cells of $\widetilde{C}_n$ in the set E_λ for any $\lambda \in \Lambda_{m+1}$.

Theorem 3.1. *Let $\lambda \in \Lambda_{m+1}$ and $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$. Then for any $\Gamma \in \Pi^l_\lambda$ (see Lemma 2.6), we have $\Gamma \cap \widetilde{C}_n \neq \emptyset$ if and only if $T_{\mathbf{a}}(\Gamma)$ is (m, n)-selfdual.*

Proof. The automorphism $\alpha := \alpha_{m,n}$ of $\widetilde{A}_m$ stabilizes the set Ω (see 1.5 and 2.3). We have $T(\alpha(w)) = \overline{T(w)}^{\rm op}$ for any $w \in \Omega$ (see the matrix description for the action of α on $\widetilde{A}_m$ in 1.5). This implies $T_{\mathbf{a}^{\rm op}}(\alpha(\Gamma)) = \overline{T_{\mathbf{a}}(\Gamma)}^{\rm op}$ for any $\Gamma \in \Pi^l_\lambda$. Hence by Lemmas 2.6, 2.15 and Proposition 2.12, we see that

$(*)$ $\alpha(\Gamma) = \Gamma \Longleftrightarrow T_{\mathbf{a}}(\Gamma)$ is (m, n)-selfdual.

First assume $\Gamma \cap \widetilde{C}_n \neq \emptyset$. Then $\alpha(\Gamma) \cap \Gamma \neq \emptyset$, hence $\alpha(\Gamma) = \Gamma$ since both Γ and $\alpha(\Gamma)$ are left cells of $\widetilde{A}_m$. This implies that $T_{\mathbf{a}}(\Gamma)$ is (m, n)-selfdual by $(*)$. Next assume that $T_{\mathbf{a}}(\Gamma)$ is (m, n)-selfdual. Then $\alpha(\Gamma) = \Gamma$ by $(*)$. Recall the set $\mathcal{D}$ mentioned in 1.3. The set $\Gamma \cap \mathcal{D}$ consists of a single element (say d) by 1.3. Then $\alpha(d) \in \alpha(\Gamma) \cap \mathcal{D}$ by the fact $\alpha(\mathcal{D}) = \mathcal{D}$ (see 1.5). This implies $d = \alpha(d)$ by the facts $\alpha(\Gamma) = \Gamma$ and $|\Gamma \cap \mathcal{D}| = 1$, i.e., $d \in \Gamma \cap \widetilde{C}_n$. Hence $\Gamma \cap \widetilde{C}_n \neq \emptyset$. □

3.2. Suppose that $\emptyset \neq E \subseteq [m+1]$ and $\overline{E} = E$. For any $\mathbf{b} \in \widetilde{\Lambda}_{|E|}$, let $\gamma_E(\mathbf{b})$ be the number of all (m, n)-selfdual tabloids in $\xi_E^{-1}(\mathbf{b})$ (see 2.13). Under the conditions assumed on E, we see that the number $\gamma_E(\mathbf{b})$ depends only on $|E|$ and the number of (m, n)-selfdual elements contained in E, but not on a particular choice of a subset E in $[m+1]$. Since $|E|$ is determined by $\mathbf{b}$, we may write $\gamma_E(\mathbf{b})$ by $\gamma_k(\mathbf{b})$ if the number of (m,n)-selfdual elements contained in E is k.

Note that the number of (m,n)-selfdual elements in $[m+1]$ is $m+1-2n$ (see 2.10).

We have not yet found any efficient way to calculate the number $\gamma_k(\mathbf{a})$ in general. However, there is a simple formula for $\gamma_{m+1-2n}(\mathbf{a})$ when $\mathbf{a}\in\widetilde{\Lambda}_{m+1}$ is symmetric (see 2.14).

Theorem 3.3. *Suppose that* $\mathbf{a}=(a_1,...,a_r)\in\widetilde{\Lambda}_{m+1}$ *is symmetric with* $r\in\{2l,2l+1\}$ *for some* $l\in\mathbb{N}$. *Then*

(3.3.1)
$$\gamma_{m+1-2n}(\mathbf{a})=\begin{cases}0, & \text{if } m=2n+1 \text{ and } r=2l,\\ 2^{a_1+\cdots+a_l}\dfrac{n!}{\left(n-\sum_{k=1}^{l}a_k\right)!\prod_{k=1}^{l}a_k!}, & \text{if otherwise,}\end{cases}$$

Proof. Any (m,n)-selfdual tabloid $\mathbf{T}=(T_1,T_2,...,T_r)\in\xi^{-1}(\mathbf{a})$ is determined entirely by its first l components if $r\in\{2l,2l+1\}$ with $l\in\mathbb{N}$ by the facts that $T_i=\overline{T_{r+1-i}}$ for any $i\in[l]$ and that $T_{l+1}=[m+1]-\bigcup_{i=1}^{l}(T_i\cup\overline{T_i})$ is a union of some (m,n)-dual pairs (see 2.10) if $r=2l+1$ is odd. If $m=2n+1$ and $r=2l$ then the (m,n)-selfdual elements $n+1,2n+2$ can not be in T_i for any $i\in[2l]$ and hence $\gamma_{m+1-2n}(\mathbf{a})=0$. If $m=2n$ then the number r must be odd as $m+1$ is odd. If $r=2l+1$ is odd then any (m,n)-selfdual elements, whenever they exist, must be in T_{l+1}. Since the elements of $\bigcup_{i=1}^{l}T_i$ are pairwise not (m,n)-dual and none of them is (m,n)-selfdual, the number of the choices for T_1 is $2^{a_1}\binom{n}{a_1}$. Recurrently, when $T_1,T_2,...,T_{h-1}$ have been chosen for $h\in[l]$, the number of the choices for T_h is $2^{a_h}\binom{n-a_1-\cdots-a_{h-1}}{a_h}$. This proves the formula (3.3.1). □

When $m=2n-1$ and $r=2l$, we have $n=a_1+\cdots+a_l$, hence (3.3.1) becomes

(3.3.2)
$$\gamma_0(\mathbf{a})=2^n\frac{n!}{\prod_{k=1}^{l}a_k!}.$$

Next result gives a necessary and sufficient condition on $\lambda\in\Lambda_{m+1}$ that there is some symmetric $\mathbf{a}$ in $\zeta^{-1}(\lambda^\vee)$.

Lemma 3.4. *Let* $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{m+1}$.

(1) There exists some symmetric $\mathbf{a}$ *in* $\zeta^{-1}(\lambda^\vee)$ *if and only if* λ *satisfies the condition (3.4.1) below.*

(3.4.1) λ_i *is odd and* λ_j *is even for some* $k \in [0, r]$ *and any* i, j, $1 \leqslant i \leqslant k < j \leqslant r$.

(2) When the condition (3.4.1) holds, the set E_λ *is empty if and only if* $m = 2n + 1$ *and* $k = 0$.

Proof. The proof for (1) is straightforward. Then (2) follows by (1) and Theorem 3.3. □

Example 3.5. Let $\lambda = \mathbf{97642}$. Then $\lambda^\vee = \mathbf{5^2 4^2 3^2 2 1^2}$. The composition $\mathbf{a} = (5, 4, 3, 1, 2, 1, 3, 4, 5) \in \zeta^{-1}(\lambda^\vee)$ is symmetric.

3.6. Assume that $\lambda \in \Lambda_{m+1}$ satisfies the condition (3.4.1). By Theorems 3.1, 3.3 and Lemmas 3.4, 2.6, we see that for any symmetric $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$, the number of left cells of $\widetilde{C}_n$ in E_λ is equal to $\gamma_{m+1-2n}(\mathbf{a})$, which can be computed by the formula (3.3.1).

Next we consider the number of left cells of $\widetilde{C}_n$ in E_λ for an arbitrary $\lambda \in \Lambda_{m+1}$.

For any $\lambda \in \Lambda_{m+1}$, let $\lambda^\vee = \mathbf{b_1^{k_1} b_2^{k_2}} \cdots \mathbf{b_r^{k_r}}$. Write $k_i = 2l_i + p_i$ for any $i \in [r]$, where $l_i \in \mathbb{N}$ and $p_i \in \{0, 1\}$. Define $q_1 < q_2 < \cdots < q_u$ in $\mathbb{N}$ by the condition $\{q_j \mid j \in [u]\} = \{i \in [r] \mid p_i = 1\}$ for some $u \in \mathbb{N}$. Take $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$ as follows.

(3.6.1)

$$\mathbf{a} = (\underbrace{b_1, ..., b_1}_{l_1}, \underbrace{b_2, ..., b_2}_{l_2}, ..., \underbrace{b_r, ..., b_r}_{l_r}, b_{q_1}, b_{q_2}, ..., b_{q_u}, \underbrace{b_r, ..., b_r}_{l_r}, ..., \underbrace{b_2, ..., b_2}_{l_2}, \underbrace{b_1, ..., b_1}_{l_1}).$$

Define

(3.6.2)

$$\mathbf{a}_1 = (\underbrace{b_1, ..., b_1}_{l_1}, \underbrace{b_2, ..., b_2}_{l_2}, ..., \underbrace{b_r, ..., b_r}_{l_r}, \underbrace{b_r, ..., b_r}_{l_r}, ..., \underbrace{b_2, ..., b_2}_{l_2}, \underbrace{b_1, ..., b_1}_{l_1}).$$

(3.6.3)

$$\mathbf{a}_2 = (b_{q_1}, b_{q_2}, ..., b_{q_u}).$$

We have

Theorem 3.7. *Let* $\lambda \in \Lambda_{m+1}$ *be given as in 3.6, and let* $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$, $\mathbf{a}_1 \in \widetilde{\Lambda}_{2l}$ *and* $\mathbf{a}_2 \in \widetilde{\Lambda}_{m+1-2l}$ *be obtained from* λ *as in (3.6.1)-(3.6.3), respectively, where*

$l=\sum_{i=1}^r l_ib_i$. *Then*

$$\text{(3.7.1)}\qquad \gamma_{m+1-2n}(\mathbf{a})=\binom{n}{l}\gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2).$$

Proof. Let $p=\sum_{i=1}^r l_i$. For any

$$\mathbf{T}=(T_1,T_2,...,T_p,T_{p+1},...,T_{p+u},T_{p+u+1},...,T_{2p+u})\in\xi^{-1}(\mathbf{a}),$$

let

$$\mathbf{T}_1=(T_1,T_2,...,T_p,T_{p+u+1},T_{p+u+2},...,T_{2p+u})\ \text{ and }\ \mathbf{T}_2=(T_{p+1},T_{p+2},...,T_{p+u})$$

and $E=[m+1]-\bigcup_{i=p+1}^{p+u}T_i$. Then $|E|=2l$ and $\mathbf{T}_1\in\xi_E^{-1}(\mathbf{a}_1)$ and $\mathbf{T}_2\in\xi_{[m+1]-E}^{-1}(\mathbf{a}_2)$. We see by Lemma 2.15 that $\mathbf{T}$ is (m,n)-selfdual if and only if both $\mathbf{T}_1$ and $\mathbf{T}_2$ are (m,n)-selfdual. When the equivalent conditions hold, we have $\overline{E}=E$ again by Lemma 2.15. For any $k\in[n]$, denote by $[m+1]_{2k}$ the set of all $E\subseteq[m+1]$ with $|E|=2k$ and $\overline{E}=E$ such that E contains no (m,n)-selfdual element. For any $E\in[m+1]_{2l}$, let $\mathcal{C}_E^{\mathbf{a}}$ be the set of all (m,n)-selfdual

$$\mathbf{T}'=(T_1',T_2',...,T_p',T_{p+1}',...T_{p+u}',T_{p+u+1}',...,T_{2p+u}')\in\xi^{-1}(\mathbf{a})$$

with $E=[m+1]-\bigcup_{i=p+1}^{p+u}T_i'$. Then

$$\gamma_{m+1-2n}(\mathbf{a})=|[m+1]_{2l}|\cdot|\mathcal{C}_E^{\mathbf{a}}|=\binom{n}{l}|\mathcal{C}_E^{\mathbf{a}}|\qquad\text{for any fixed } E\in[m+1]_{2l}.$$

$\mathbf{T}\mapsto(\mathbf{T}_1,\mathbf{T}_2)$ is a bijective map from the set $\mathcal{C}_E^{\mathbf{a}}$ to the Cartesian product $\mathcal{C}_E^{\mathbf{a}_1}\times\mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}$, where $\mathcal{C}_E^{\mathbf{a}_1}$, $\mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}$ are the sets of all (m,n)-selfdual tabloids in $\xi_E^{-1}(\mathbf{a}_1)$, $\xi_{[m+1]-E}^{-1}(\mathbf{a}_2)$, respectively. This proves the formula (3.7.1) by the facts $\gamma_0(\mathbf{a}_1)=|\mathcal{C}_E^{\mathbf{a}_1}|$ and $\gamma_{m+1-2n}(\mathbf{a}_2)=|\mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}|$ for any $E\in[m+1]_{2l}$. □

§4. Enumeration of some special tabloids in $\mathcal{C}_{m+1}$.

For any $\mathbf{a}\in\widetilde{\Lambda}_{m+1}$, let $\mathcal{C}_{m+1}^{\mathbf{a}}$ be the set of all (m,n)-selfdual tabloids $\mathbf{T}$ in $\xi^{-1}(\mathbf{a})$. We want to formulate the number $\gamma_{m+1-2n}(\mathbf{a}):=|\mathcal{C}_{m+1}^{\mathbf{a}}|$. By Theorems 3.3 and 3.7, it is enough to consider the case where $\mathbf{a}=(a_1,a_2,...,a_r)\in\widetilde{\Lambda}_{m+1}$ satisfies $a_1>a_2>\cdots>a_r$ for some $r>1$.

First consider the case of $r=2$.

Lemma 4.1. *Let* $\mathbf{T} = (Y, X) \in \mathcal{C}_{m+1}$ *satisfy* $|Y| \geqslant |X|$*. Then* $\mathbf{T}$ *is* (m,n)*-selfdual if and only if* $L_Y(X) = \overline{X}$ *and* $Y - L_Y(X) = \overline{Y - L_Y(X)}$*.*

Proof. Let $X' = L_Y(X)$ and $Y' = X \cup (Y - L_Y(X))$. Then $(Y, X) \approx (X', Y')$. So

$$\begin{aligned} \mathbf{T} \text{ is } (m,n)\text{-selfdual} &\iff X' = \overline{X} \text{ and } Y' = \overline{Y} \\ &\iff L_Y(X) = \overline{X} \text{ and } X \cup (Y - L_Y(X)) = \overline{Y} \\ &\iff L_Y(X) = \overline{X} \text{ and } Y - L_Y(X) = \overline{Y - L_Y(X)}. \end{aligned}$$

The last equivalence follows by the facts that

$$Y = L_Y(X) \cup (Y - L_Y(X)) \quad \text{and} \quad \overline{X \cup (Y - L_Y(X))} = \overline{X} \cup \overline{Y - L_Y(X)}. \qquad \square$$

4.2. First assume $m = 2n-1$. Hence $m+1-2n = 0$ and $\bar{i} := 2n+1-i$ for any $i \in [2n]$. Define an admissible subsequence β' in each of the following sequences β_{ij} (note that β_{ij} has even number of terms in $[2n]$).

(a) Consider the sequence $\beta_{n,q} : \bar{n}, \overline{n-1}, ..., \overline{q+1}, q+1, ..., n-1, n$ for any $q \in [0, n-1]$. A subsequence $\beta' : i_1, i_2, ..., i_r$ of $\beta_{n,q}$ is called *admissible*, if the following two conditions hold:

(a1) $r = n - q$ and $\overline{i_h} \neq i_k$ for any $h, k \in [n-q]$;

(a2) Let $\beta'' : j_1, j_2, ..., j_{n-q}$ be the subsequence of $\beta_{n,q}$ complement to β' (i.e., $\{i_h, j_h \mid h \in [n-q]\} = \beta_{n,q}$ identifying the sequences with the corresponding sets). Then the term j_h occurs after i_h in the sequence $\beta_{n,q}$ for any $h \in [n-q]$.

Let $\Delta_{n,q}$ be the set of all admissible subsequences of $\beta_{n,q}$ and let $\delta_{n,q} := |\Delta_{n,q}|$. Denote $\beta_{n,0}$, $\Delta_{n,0}$, $\delta_{n,0}$ simply by β_n, Δ_n, δ_n, respectively. Clearly, the equation $\delta_{n,q} = \delta_{n-q}$ holds for any $q \in [0, n-1]$.

(b) For any $i < j$ in $[n]$ with $j - i$ odd, denote by β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) the sequence $i+1, i+2, ..., j-1$ (respectively, $\overline{j-1}, \overline{j-2}, ..., \overline{i+1}$). A subsequence $\beta' : h_1, h_2, ..., h_r$ of β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) is called *admissible*, if $r = \frac{j-i-1}{2}$ and if, let $\beta'' : k_1, k_2, ..., k_{\frac{j-i-1}{2}}$ be the subsequence of β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) complement to β', then k_l occurs after h_l in β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) for any $l \in \left[\frac{j-i-1}{2}\right]$.

It is well known that the number of admissible subsequences in β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) is $C_{\frac{j-i-1}{2}}$, where $C_l := \frac{1}{l+1}\binom{2l}{l}$ is the l-th Catalan number. The following is a formula for the number δ_n of admissible subsequences in β_n.

Proposition 4.3. *$\delta_n := \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any $n \in \mathbb{P}$, where $\lfloor x \rfloor$ stands for the largest integer not greater than x for any $x \in \mathbb{Q}$.*

To show Proposition 4.3, we need some preparation. Let $\beta'_{n,q}$: $i_1, i_2, ..., i_{n-q}$ be a subsequence of $\beta_{n,q}$ satisfying the condition 4.2 (a1). Let $p(\beta'_{n,q})$ be the largest $k \in [0, n-q]$ with $i_1, i_2, ..., i_k$ a subsequence of $\bar{n}, \overline{n-1}, ..., \overline{q+1}$. Denote $i_1, i_2, ..., i_{p(\beta'_{n,q})}$ by $\beta'^0_{n,q}$. Then $\beta'_{n,q}$ is entirely determined by $\beta'^0_{n,q}$.

Let $\beta' : i_1, i_2, ..., i_n$ be a subsequence of β_n satisfying the condition 4.2 (a1). For any $q \in [0, n-1]$, let $\beta'_{n,q}$ be obtained from β' by removing all the terms in β_q and let $\beta''_{n,q}$ be the subsequence of $\beta_{n,q}$ complement to $\beta'_{n,q}$ (see 4.2 (a2)), where we stipulate β_0 to be the empty sequence. Then the following result can be checked easily:

Lemma 4.4. *Let $\beta' : i_1, i_2, ..., i_n$ be a subsequence of β_n satisfying the condition 4.2 (a1).*
(1) The following three conditions on β' are equivalent:
(a) β' is admissible in β_n;
(b) $\beta'_{n,q}$ is admissible in $\beta_{n,q}$ for every $q \in [0, n-1]$;
(c) $p(\beta') \geqslant \frac{n}{2}$ and the term j_h occurs after i_h in β_n for every $h \in [p(\beta')]$, where β'': $j_1, j_2, ..., j_n$ is the subsequence of β_n complement to β' (see 4.2 (a2)).
(2) For $q \in [0, n-1]$, if $\beta'_{n,q}$ is admissible in $\beta_{n,q}$, then $p(\beta'_{n,q}) \geqslant p(\beta''_{n,q})$, in particular, $p(\beta'_{n,q}) \geqslant \frac{n-q}{2}$.

4.5. *Proof of Proposition 4.3.* Consider the set Δ_n. We may assume $n > 1$, for otherwise the result is obvious. By Lemma 4.4 (1), we see that $\beta'_{n,1} \in \Delta_{n,1}$ for any $\beta' \in \Delta_n$. On the other hand, for any λ: $i_1, i_2, ..., i_{n-1}$ in $\Delta_{n,1}$, let $\lambda_{\bar{1}}$ (respectively, λ_1) be obtained from λ by inserting the term $\bar{1}$ (respectively, 1) immediately after $i_{p(\lambda)}$. Then $\lambda_{\bar{1}}$ is always in Δ_n, while λ_1 is not in Δ_n if and only if $p(\lambda) < \frac{n}{2}$. Since $p(\lambda) \geqslant \frac{n-1}{2}$ by the condition $\lambda \in \Delta_{n,1}$ and Lemma 4.4 (2), this implies that λ_1 is not in Δ_n if and only if n is odd (say $n = 2l+1$) and $p(\lambda) = l$. When $n = 2l+1$, let $\Delta'_{n,1}$ be the set of all such subsequences λ: $i_1, i_2, ..., i_l$ of $\bar{n}, \overline{n-1}, ..., \bar{3}, \bar{2}$ that, if λ': $j_1, j_2, ..., j_l$ is the subsequence of $\bar{n}, \overline{n-1}, ..., \bar{3}, \bar{2}$ complement to λ, then the term j_h occurs after the term i_h for every $h \in [l]$. Then $|\Delta'_{n,1}|$ is equal to the number of all $\lambda \in \Delta_{n,1}$ with $\lambda_1 \notin \Delta_n$. It is well known that $|\Delta'_{n,1}| = C_l$ (the l-th Catalan number). So by applying induction on $n \geqslant 1$ and by the fact that $\delta_{n,1} = \delta_{n-1}$, we have

$$\delta_n = \begin{cases} \delta_{n-1} + (\delta_{n-1} - C_l) = 2\binom{2l}{l} - \frac{1}{l+1}\binom{2l}{l} = \binom{2l+1}{l}, & \text{if } n = 2l+1 \text{ is odd,} \\ 2\delta_{n-1} = 2\binom{2l-1}{l-1} = \binom{2l}{l}, & \text{if } n = 2l \text{ is even.} \end{cases}$$

Our result is proved. □

Remark 4.6. The result in Proposition 4.3 can be extended to a more general case. Let $\beta : \overline{i_t}, \overline{i_{t-1}}, ..., \overline{i_1}, i_1, i_2, ..., i_t$ (respectively, $\overline{\beta} : i_1, i_2, ..., i_t, \overline{i_t}, \overline{i_{t-1}}, ..., \overline{i_1}$) satisfy $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n$. A subsequence $\beta' : j_1, j_2, ..., j_r$ of β (respectively, $\overline{\beta}$) is called *admissible*, if the following conditions are satisfied:

(i) $r = t$ and $\overline{j_h} \neq j_k$ for any $h, k \in [t]$;

(ii) Let $\beta'' : j'_1, j'_2, ..., j'_t$ be the subsequence of β (respectively, $\overline{\beta}$) complement to β'. Then j'_h occurs after j_h in β (respectively, $\overline{\beta}$) for any $h \in [t]$.

By the same way as that for Proposition 4.3, one can prove that the number of admissible subsequences of β (respectively, $\overline{\beta}$) is equal to $\binom{t}{\lfloor \frac{t}{2} \rfloor}$.

The following is a formula for the number $\gamma_0(\mathbf{a})$ with $\mathbf{a} \in \widetilde{\Lambda}_{2n}$ having exactly two different parts.

Proposition 4.7. *For $\mathbf{a} = (n+t, n-t)$ with $t \in [n-1]$, we have*

$$\gamma_0(\mathbf{a}) = \sum_{\substack{h_1, h_2, ..., h_t \in \mathbb{N} \\ 1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n \\ h_{i+1} - h_i \text{ odd } \forall i}} \binom{n - h_t}{\lfloor \frac{n-h_t}{2} \rfloor} \binom{h_1 - 1}{\lfloor \frac{h_1 - 1}{2} \rfloor} \prod_{i=1}^{t-1} C_{\frac{h_{i+1} - h_i - 1}{2}}, \tag{4.7.1}$$

where C_l is the l-th Catalan number for any $l \in \mathbb{N}$.

Proof. Let $\mathbf{T} = (Y, X) \in \mathcal{C}^{\mathbf{a}}_{2n}$. By the condition of $\mathbf{T}$ being $(2n-1, n)$-selfdual, we have $L_Y(X) = \{i \in Y \mid \bar{i} \notin Y\}$ and $Y - L_Y(X) = \{h_1, h_2, ..., h_t, \overline{h_t}, ..., \overline{h_2}, \overline{h_1}\}$ with some $1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n$ by Lemma 4.1. According to the definition of the set $L_Y(X)$, we get the following results by Lemma 2.9 (b).

(i) For any $j \in [t-1]$, let $Y_{h_j+1,h_{j+1}-1} = \{h_{j1}, h_{j2}, ..., h_{jn_j}\}$ be with $h_j < h_{j1} < h_{j2} < \cdots < h_{jn_j} < h_{j+1}$, then $h_{j1}, h_{j2}, ..., h_{jn_j}$ is an admissible subsequence of $\beta_{h_j,h_{j+1}} : h_j+1, h_j+2, ..., h_{j+1}-1$ (hence $h_{j+1}-h_j$ is odd and $n_j = \frac{h_{j+1}-h_j-1}{2}$ by 4.2 (b)), and $Y_{\overline{h_{j+1}-1},\overline{h_j+1}} = \overline{[h_j+1, h_{j+1}-1] - Y_{h_j+1,h_{j+1}-1}}$. Write $Y_{\overline{h_{j+1}-1},\overline{h_j+1}} = \{\overline{h'_{j1}}, \overline{h'_{j2}}, ..., \overline{h'_{jn_j}}\}$ with $h_j < h'_{j1} < h'_{j2} < \cdots < h'_{jn_j} < h_{j+1}$. Then $\overline{h'_{jn_j}}, ..., \overline{h'_{j2}}, \overline{h'_{j1}}$ is an admissible subsequence of $\overline{h_{j+1}-1}, ..., \overline{h_j+2}, \overline{h_j+1}$.

(ii) Let $Y_{h_t+1,\overline{h_t+1}} = \{h_{t1}, h_{t2}, ..., h_{tn_t}\}$ be with $\beta' : h_{t1}, h_{t2}, ..., h_{tn_t}$ a subsequence of $\beta_{h_t,n} : h_t+1, h_t+2, ..., n, \bar{n}, \overline{n-1}, ..., \overline{h_t+1}$. Then β' is admissible in $\beta_{h_t,n}$.

(iii) Let $Y_{\overline{h_1-1},h_1-1} = \{h_{01}, h_{02}, ..., h_{0n_0}\}$ be with $\beta' : h_{01}, h_{02}, ..., h_{0n_0}$ a subsequence of $\beta_{1,h_1} : \overline{h_1-1}, \overline{h_1-2}, ..., \bar{1}, 1, 2, ..., h_1-1$. Then β' is admissible in β_{1,h_1}.

(iv) $L_Y(X) = \left(\bigcup_{j\in[t-1]} \left(Y_{h_j+1,h_{j+1}-1} \cup Y_{\overline{h_{j+1}-1},\overline{h_j+1}}\right)\right) \cup Y_{h_t+1,\overline{h_t+1}} \cup Y_{\overline{h_1-1},h_1-1}$.

Conversely, fix $h_1, h_2, ..., h_t \in \mathbb{P}$ with $t \in \mathbb{P}$ and $1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n$ and $h_{i+1}-h_i$ odd for all $i \in [t-1]$. Take an admissible subsequence β'_j in $\beta_{h_j,h_{j+1}}$ for any $j \in [t-1]$. Also, take an admissible subsequence β'_t (respectively, β'_0) in $\beta_{h_t,n}$ (respectively, β_{1,h_1}). For $j \in [t-1]$, let β''_j be the subsequence of $\beta_{h_j,h_{j+1}}$ complement to β'_j and let $\beta'_{\bar{j}}$ be the subsequence of $\beta_{\overline{h_{j+1}},\overline{h_j}}$ such that $\beta'_{\bar{j}} = \overline{\beta''_j}$ by regarding the sequences as the corresponding sets. Let Y be the union of the sets $\{h_l, \overline{h_l} \mid l \in [t]\}$, β'_t, β'_0 and β'_j, $\beta'_{\bar{j}}$ with $j \in [t-1]$, regarding the sequences as the corresponding sets. Let $X = [2n] - Y$. Then $(Y, X) \in \mathcal{C}^{\mathbf{a}}_{2n}$ by Lemma 4.1.

By 4.2 (b) and Proposition 4.3, we see that the numbers of admissible subsequences in $\beta_{h_j,h_{j+1}}$, $j \in [t-1]$, $\beta_{h_t,n}$, β_{1,h_1} are $C_{\frac{h_{j+1}-h_j-1}{2}}$, $\binom{n-h_t}{\lfloor\frac{n-h_t}{2}\rfloor}$, $\binom{h_1-1}{\lfloor\frac{h_1-1}{2}\rfloor}$, respectively. This implies the formula (4.7.1). $\square$

We can get the corresponding results in the case of $m \in \{2n, 2n+1\}$ similarly by noting that the number of (m,n)-selfdual elements in $[m+1]$ is $m+1-2n$.

Proposition 4.8. *For* $\mathbf{a} = (n+1+t, n-t)$ *with* $t \in [0, n-1]$, *we have*

$$\gamma_1(\mathbf{a}) = \sum_{\substack{h_1,h_2,...,h_{t+1}\in\mathbb{N} \\ 1\leqslant h_1<h_2<\cdots<h_{t+1}=n+1 \\ h_{i+1}-h_i \text{ odd } \forall i}} \binom{h_1-1}{\lfloor\frac{h_1-1}{2}\rfloor} \prod_{i=1}^{t} C_{\frac{h_{i+1}-h_i-1}{2}}, \tag{4.8.1}$$

with the convention that $\gamma_1((n+1,n)) = \binom{n}{\lfloor\frac{n}{2}\rfloor}$.

Proposition 4.9. *For* $\mathbf{a} = (n+1+t, n+1-t)$ *with* $t \in [n]$, *we have*

$$\gamma_2(\mathbf{a}) = \sum_{\substack{h_1,h_2,\ldots,h_{t+1}\in\mathbb{N}\\ 0=h_1<h_2<\cdots<h_{t+1}=n+1\\ h_{i+1}-h_i \text{ odd } \forall i}} \prod_{i=1}^{t} C_{\frac{h_{i+1}-h_i-1}{2}}. \tag{4.9.1}$$

From Theorem 3.3 and Propositions 4.7-4.9, we see that for $k \in \mathbb{P}$ with $2k \leqslant m+1$, the set $E_{\mathbf{2^k1^{m+1-2k}}}$ is empty if and only if $m = 2n+1$ and $2k = m+1$.

Example 4.10. (1) In Proposition 4.7, take $n = 5$ and $t = 2$, then $\mathbf{a} = (7,3)$ and $\zeta(\mathbf{a})^\vee = \mathbf{2^3 1^4}$. The pairs (h_1, h_2) occurring in the summation of (4.7.1) are $(1,2),(2,3),(3,4),(4,5),(1,4),(2,5)$. Then $\gamma_0(\mathbf{a}) = \binom{3}{1}+\binom{2}{1}+\binom{2}{1}+\binom{3}{1}+1+1 = 12$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^31^4}}$. The set $\mathcal{C}^{\mathbf{a}}_{10}$ consists of the following tabloids:

$\mathbf{T}_1 = (\{3,4,5\} \cup \{1,2,9,10\}, \{6,7,8\})$, $\mathbf{T}_2 = (\{3,4,6\} \cup \{1,2,9,10\}, \{5,7,8\})$,

$\mathbf{T}_3 = (\{3,5,7\} \cup \{1,2,9,10\}, \{4,6,8\})$, $\mathbf{T}_4 = (\{4,5,10\} \cup \{2,3,8,9\}, \{1,6,7\})$,

$\mathbf{T}_5 = (\{4,6,10\} \cup \{2,3,8,9\}, \{1,5,7\})$, $\mathbf{T}_6 = (\{5,9,10\} \cup \{3,4,7,8\}, \{1,2,6\})$,

$\mathbf{T}_7 = (\{1,5,9\} \cup \{3,4,7,8\}, \{2,6,10\})$, $\mathbf{T}_8 = (\{8,9,10\} \cup \{4,5,6,7\}, \{1,2,3\})$,

$\mathbf{T}_9 = (\{1,8,9\} \cup \{4,5,6,7\}, \{2,3,10\})$, $\mathbf{T}_{10} = (\{2,8,10\} \cup \{4,5,6,7\}, \{1,3,9\})$,

$\mathbf{T}_{11} = (\{2,5,8\} \cup \{1,4,7,10\}, \{3,6,9\})$, $\mathbf{T}_{12} = (\{3,7,10\} \cup \{2,5,6,9\}, \{1,4,8\})$.

(2) In Proposition 4.8, take $n = 5$ and $t = 2$, then $\mathbf{a} = (8,3)$ and $\zeta(\mathbf{a})^\vee = \mathbf{2^3 1^5}$. The triples (h_1,h_2,h_3) occurring in the summation of (4.8.1) are $(4,5,6),(2,5,6),(2,3,6)$. Then $\gamma_1(\mathbf{a}) = \binom{3}{1} + 1 + 1 = 5$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^31^5}}$. The set $\mathcal{C}^{\mathbf{a}}_{11}$ consists of the following tabloids:

$$\begin{aligned}
\mathbf{T}_1 &= (\{4,7,11\} \cup \{2,3,6,9,10\}, \{1,5,8\}),\\
\mathbf{T}_2 &= (\{3,8,11\} \cup \{2,5,6,7,10\}, \{1,4,9\}),\\
\mathbf{T}_3 &= (\{9,10,11\} \cup \{4,5,6,7,8\}, \{1,2,3\}),\\
\mathbf{T}_4 &= (\{1,9,10\} \cup \{4,5,6,7,8\}, \{2,3,11\}),\\
\mathbf{T}_5 &= (\{2,9,11\} \cup \{4,5,6,7,8\}, \{1,3,10\}).
\end{aligned}$$

(3) In Proposition 4.9, take $n = 5$ and $t = 2$, then $\mathbf{a} = (8, 4)$ and $\zeta(\mathbf{a})^\vee = \mathbf{2^4 1^4}$. The triples (h_1, h_2, h_3) occurring in the summation of (4.9.1) are $(0, 1, 6), (0, 3, 6), (0, 5, 6)$. Then $\gamma_2(\mathbf{a}) = 2 + 1 + 2 = 5$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^4 1^4}}$. The set $\mathcal{C}_{12}^{\mathbf{a}}$ consists of the following tabloids:

$$\begin{aligned}
\mathbf{T}_1 &= (\{2,3,7,8\} \cup \{1,6,11,12\}, \{4,5,9,10\}),\\
\mathbf{T}_2 &= (\{2,4,7,9\} \cup \{1,6,11,12\}, \{3,5,8,10\}),\\
\mathbf{T}_3 &= (\{1,4,7,10\} \cup \{3,6,9,12\}, \{2,5,8,11\}),\\
\mathbf{T}_4 &= (\{1,2,8,9\} \cup \{5,6,7,12\}, \{3,4,10,11\}),\\
\mathbf{T}_5 &= (\{1,3,8,10\} \cup \{5,6,7,12\}, \{2,4,9,11\}).
\end{aligned}$$

Remark 4.11. (1) From Propositions 4.7-4.9, we can get a formula of the number $\gamma_{m+1-2n}(\mathbf{a})$ for any $\mathbf{a} = (r, s)$ with $r, s \in \mathbb{P}$ and $r+s \in [m+1]$ (see 3.2). Note that here we allow the case $r \leqslant s$. For, if $r = s$ then $\mathbf{a}$ is symmetric, hence $\gamma_{m+1-2n}(\mathbf{a})$ is known by Theorem 3.3; if $r < s$ then $\gamma_{m+1-2n}(\mathbf{a}) = \gamma_{m+1-2n}(\mathbf{a}^{\mathrm{op}})$.

We also allow the case $r+s < m+1$. When $m+1-2n \in \{0, 2\}$, we have $r+s = 2p$ and $r-s = 2q$ for some $p, q \in \mathbb{Z}$. If $r > s$, then the formula of the number $\gamma_0((r, s))$ (respectively, $\gamma_2((r, s))$) can be obtained from (4.7.1) (respectively, (4.9.1)) by replacing n, t by p, q (respectively, $p - 1$, q), respectively. When $m + 1 - 2n = 1$, we have $r + s = 2p + 1$ and $r - s = 2q + 1$ for some $p, q \in \mathbb{Z}$. If $r > s$, then the formula of the number $\gamma_1((r, s))$ can be obtained from (4.8.1) by replacing n, t by p, q, respectively.

(2) The results in Propositions 4.7-4.9 can be extended to a more general case. Let $\lambda = (2l_1, 2l_2, ..., 2l_r, 2l_{r+1} + 1, ..., 2l_t + 1) \in \Lambda_{m+1}$ for some $r, t, l_i \in \mathbb{N}$ with $r \in [t - 1]$ and $i \in [t]$ (Comparing with the partitions in Lemma 3.4). Then $\mathbf{a} = (a_1, a_2, ..., a_{l_1-1}, t, r, a_{l_1-1}, ..., a_2, a_1) \in \zeta^{-1}(\lambda^\vee)$ for some $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{l_1-1}$. Then the following is a formula of the number $\gamma_{m+1-2n}(\mathbf{a})$.

Corollary 4.12. *In the setup of 4.11 (2), we have*

(4.12.1)
$$\gamma_{m+1-2n}(\mathbf{a}) = 2^{a_1+\cdots+a_{l_1-1}} \frac{n!}{(n - a_1 - \cdots - a_{l_1-1})! \prod_{i=1}^{l_1-1} a_i!} \cdot \gamma_{m+1-2n}((t, r)).$$

Proof. Let $\mathbf{a}_1 = (a_1, a_2, ..., a_{l_1-1}, a_{l_1-1}, ..., a_2, a_1)$ and $\mathbf{a}_2 = (t, r)$. Then $\gamma_{m+1-2n}(\mathbf{a}) =$

$\gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2)\binom{n}{a_1+\cdots+a_{l_1-1}}$ by Theorem 3.7. Since

$$\gamma_0(\mathbf{a}_1)\binom{n}{a_1+\cdots+a_{l_1-1}} = 2^{a_1+\cdots+a_{l_1-1}}\frac{n!}{(n-a_1-\cdots-a_{l_1-1})!\prod_{i=1}^{l_1-1}a_i!}$$

by Theorem 3.3, this proves the formula (4.12.1). □

4.13. Let $\mathbf{T} = (T_1, T_2, ..., T_r)$ and $\mathbf{T}' = (T_1', T_2', ..., T_r')$ in $\mathcal{C}_{m+1}$ satisfy $|T_1| > |T_2| > \cdots > |T_r|$ and $|T_1'| < |T_2'| < \cdots < |T_r'|$ and $\mathbf{T}' \approx \mathbf{T}$. Then $|T_i'| = |T_{r+1-i}|$ for any $i \in [r]$. The tabloid $\mathbf{T}$ is (m, n)-selfdual if and only if $\mathbf{T}'$ is (m, n)-selfdual if and only if $T_i' = \overline{T_{r+1-i}}$ for any $i \in [r]$. When the equivalent conditions hold, define a partition $T_j = T_{j1}\dot\cup T_{j2}\dot\cup\cdots\dot\cup T_{j,r+1-j}$ for any $j \in [r]$ such that the sets $T_j^h := T_{j1}\dot\cup T_{j2}\dot\cup\cdots\dot\cup T_{jh}$ for $j \in [r]$ and $h \in [r+1-j]$ satisfy the condition $L_{T_j}(T_{j+1}^h) = T_j^h$ for any $h \in [r-j]$.

4.14. Let us describe (m, n)-selfdual $\mathbf{T} = (T_1, T_2, T_3) \in \mathcal{C}_{m+1}$ with $|T_1| > |T_2| > |T_3|$. Define the partitions $T_1 = T_{11}\dot\cup T_{12}\dot\cup T_{13}$ and $T_2 = T_{21}\dot\cup T_{22}$ and $T_3 = T_{31}$ as those in 4.13 with $r = 3$. Define

$$\mathbf{X} := (T_{11}, T_{21} \cup T_{12} \cup T_{13}, T_{31} \cup T_{22}) \quad \text{and} \quad \mathbf{Y} := (T_{11} \cup T_{12}, T_{21} \cup T_{22} \cup T_{13}, T_{31}).$$

Then $\mathbf{X}$ is obtained from $\mathbf{T}$ by a $\{2, 3\}$-transformation followed by a $\{1, 2\}$-transformation, while $\mathbf{Y}$ is obtained from $\mathbf{T}$ by a $\{1, 2\}$-transformation (see 2.13). So $\mathbf{X} \approx \mathbf{T} \approx \mathbf{Y}$. We see by Lemma 2.15 that both $\mathbf{X}$ and $\mathbf{Y}$ are (m, n)-selfdual and that $\mathbf{Y} = \overline{\mathbf{X}}^{\text{op}}$. This implies that $T_{31} = \overline{T_{11}}$ and $T_{22} = \overline{T_{12}}$ and $T_{13} \cup T_{21} = \overline{T_{21} \cup T_{13}}$. Denote $E^0 = \{i \in E \mid \bar{i} \in E\}$ and $E^1 = E - E^0$ for any $E \subseteq [m+1]$. Then $T_{13}^1 = \overline{T_{21}^1}$ and $\mathbf{T}' = (T_{11}, T_{21}^0 \cup T_{12} \cup T_{13}^1, T_{31} \cup T_{22} \cup T_{21}^1 \cup T_{13}^0)$.

Hence we have

Proposition 4.15. *For any $\mathbf{a} = (a_1, a_2, a_3) \in \widetilde{\Lambda}_{m+1}$ with $a_1 > a_2 > a_3$, a tabloid $\mathbf{T} \in \xi^{-1}(\mathbf{a})$ is (m, n)-selfdual if and only if $\mathbf{T} = (T_{11}\dot\cup T_{12}\dot\cup T_{13}, T_{21}\dot\cup\overline{T_{12}}, \overline{T_{11}})$ for some $T_{11}, T_{12}, T_{13}, T_{21} \subset [m+1]$ satisfying the following conditions:*

(i) $T_{11} = L_{T_{11}\dot\cup T_{12}\dot\cup T_{13}}(T_{21})$;

(ii) $T_{11}\dot\cup T_{12} = L_{T_{11}\dot\cup T_{12}\dot\cup T_{13}}(T_{21}\dot\cup\overline{T_{12}})$;

(iii) $T_{21} = L_{T_{21}\dot\cup\overline{T_{12}}}(\overline{T_{11}})$ and $T_{21}^0\dot\cup T_{13}^1 = L_{T_{21}\dot\cup\overline{T_{12}}\dot\cup T_{13}}(\overline{T_{11}})$;

(iv) $T_{11}^0 = T_{12}^0 = \emptyset$ and $T_{13}^1 = \overline{T_{21}^1}$.

Example 4.16. Let $(m,n)=(15,8)$ and $\mathbf{a}=(8,5,3)$. Then

$$\mathbf{T}_1=(\{12,13,14\}\cup\{7,8\}\cup\{2,6,11\},\{1,15,16\}\cup\{9,10\},\{3,4,5\}),$$
$$\mathbf{T}_2=(\{4,12,14\}\cup\{2,11\}\cup\{8,9,10\},\{1,7,16\}\cup\{6,15\},\{3,5,13\}),$$
$$\mathbf{T}'_1=(\{4,5,6\}\cup\{15,16\}\cup\{3,10,14\},\{7,8,9\}\cup\{1,2\},\{11,12,13\}),$$
$$\mathbf{T}'_2=(\{4,6,12\}\cup\{3,10\}\cup\{1,2,16\},\{8,9,15\}\cup\{7,14\},\{5,11,13\})$$

are four tabloids in $\mathcal{C}^{\mathbf{a}}_{16}$ with $\mathbf{T}'_i=\tau_{16}(\mathbf{T}_i)$ for $i=1,2$ (see 2.14).

Question 4.17. Can one find a close formula of the number $\gamma_{m+1-2n}(\mathbf{a})$ for any $\mathbf{a}=(a_1,...,a_r)\in\widetilde{\Lambda}_{m+1}$ with $a_1>\cdots>a_r$ and $r\geqslant 3$?

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