THE CELLS IN THE WEIGHTED COXETER GROUP \((\tilde{C}_n, \tilde{\ell}_m)\)

JIAN-YI SHI
Department of Mathematics
East China Normal University
Shanghai, 200241, P.R.China

Abstract. The affine Weyl group \((\tilde{C}_n, S)\) can be realized as the fixed point set of the affine Weyl group \((\tilde{A}_m, \tilde{S}_m)\), \(m \in \{2n - 1, 2n, 2n + 1\}\), under a certain group automorphism \(\alpha_{n,m}\) with \(\alpha_{n,m}(\tilde{S}_m) = \tilde{S}_m\). Let \(\tilde{\ell}_m\) be the length function of \(\tilde{A}_m\). The present paper is to give some criterion for a left cell of \(\tilde{A}_m\) intersecting \(\tilde{C}_n\) and to use this criterion to deduce some formulae for the number of left cells of the weighted Coxeter group \((\tilde{C}_n, \tilde{\ell}_m)\) in the set \(E_\lambda\) of elements associated to any partition \(\lambda\) of \(m + 1\).

§0. Introduction.

0.1. The affine Weyl group \((\tilde{C}_n, S)\) can be realized as the fixed point set of the affine Weyl group \((\tilde{A}_m, \tilde{S}_m)\), \(m \in \{2n - 1, 2n, 2n + 1\}\), under a certain automorphism \(\alpha_{n,m}\) with \(\alpha_{n,m}(\tilde{S}_m) = \tilde{S}_m\), where \(\tilde{S}_m, S\) are the Coxeter generator sets of \(\tilde{A}_m, \tilde{C}_n\), respectively. The restriction to \(\tilde{C}_n\) of the length function \(\tilde{\ell}_m\) of \(\tilde{A}_m\) is a weight function of \(\tilde{C}_n\). It is known that there is a surjective map \(\psi\) from \(\tilde{A}_m\) to the set \(\Lambda_{m+1}\) of partitions of \(m + 1\) which induces a bijection from the set of two-sided cells of \(\tilde{A}_m\) to \(\Lambda_{m+1}\) (see [6], [3]). For any \(i \leq j\) in the set \(\mathbb{N} := \{0, 1, 2, \ldots\}\), denote \([i, j] := \{i, i + 1, \ldots, j\}\) and denote \([1, i]\) simply by \([i]\). Let \(E_\lambda := \psi^{-1}(\lambda) \cap \tilde{C}_n\) for \(\lambda \in \Lambda_{m+1}\). In the paper [7], we described all the cells of the weighted Coxeter group \((\tilde{C}_n, \tilde{\ell}_{2n-1})\) corresponding to the partitions \(k1^{2n-k}\) and \(h1^{2n-h-2}\) for all

Key words and phrases. Affine Weyl group; weighted Coxeter group; quasi-split case; cells; partitions.

Supported by the NSF of China, the SFUDP of China, Shanghai Leading Academic Discipline Project (B407) and Program of Shanghai Subject Chief Scientist (11xd1402200).

Typeset by AMSTeX
\( k \in [2n] \) and \( h \in [2, 2n - 2] \) and also all the cells of the weighted Coxeter group \((\tilde{C}_3, \tilde{\ell}_5)\).

**0.2.** Denote by \( \lambda^\vee \) the dual partition of \( \lambda \in \Lambda_{m+1} \) (see 1.8). Let \( \tilde{\Lambda}_{m+1} \) be the set of all compositions of \( m + 1 \) (see 2.1). There is a natural surjective map \( \zeta \) from the set \( \tilde{\Lambda}_{m+1} \) to \( \Lambda_{m+1} \) (see 2.1). Call \( a = (a_1, a_2, ..., a_r) \in \tilde{\Lambda}_{m+1} \) symmetric, if \( a_i = a_{r+1-i} \) for any \( i \in [r] \). Let \( C_{m+1} \) be the set of all tabloids of rank \( m + 1 \) (see 2.2). We can define an equivalence relation \( \approx \) on \( C_{m+1} \) (see 2.13). There exists a bijective map from the set \( \Pi^\gamma_{m} \) of left cells of \( \tilde{A}_m \) to the set of \( \approx \)-equivalence classes of \( C_{m+1} \) (see [6, Subsection 19.4]). There exists a natural surjective map \( \xi \) from \( C_{m+1} \) to \( \tilde{\Lambda}_{m+1} \) (see 2.2).

**0.3.** In the present paper, we prove that a left cell \( \Gamma \) of \( \tilde{A}_m \) has a non-empty intersection with \( \tilde{C}_n \) if and only if the \( \approx \)-equivalence class of \( C_{m+1} \) corresponding to \( \Gamma \) is \((m, n)\)-selfdual (see 2.13-2.14, Lemma 2.15 and Theorem 3.1). By this result, we can deduce some formulae for the number \( \gamma_{m+1-2n}(a) \) of left cells of \((\tilde{C}_n, \tilde{\ell}_m)\) in the set \( E_{\xi}(a) \) for any \( a \in \tilde{\Lambda}_{m+1} \). More precisely, we give a close formula for the number \( \gamma_{m+1-2n}(a) \) if \( a \in \tilde{\Lambda}_{m+1} \) is symmetric (Theorem 3.3). For an arbitrary \( a = (a_1, a_2, ..., a_r) \in \tilde{\Lambda}_{m+1} \), we have \( \gamma_{m+1-2n}(a) = \gamma_0(a_1)\gamma_{m+1-2n}(a_2)_{l}^{n} \) for some symmetric \( a_1 = (a_{i_1}, a_{i_2}, ..., a_{i_{2l}}) \in \tilde{\Lambda}_{2l} \), and some \( a_2 = (a_{j_1}, a_{j_2}, ..., a_{j_q}) \in \tilde{\Lambda}_{m+1-2l} \), \( a_{j_1} > a_{j_2} > \cdots > a_{j_q} \), with some \( l \in \mathbb{N} \), where \( (\frac{n}{l}) := \frac{n!}{(n-l)!} \) and \( \{i_h, j_l \ | \ h \in [0, 2p], l \in [0, q]\} = [r] \) and the notation \( \gamma_k(b) \), \( k \in \{0, 1, 2\} \), stands for the numbers of \((m, n)\)-selfdual tabloids \( T \) with \( \xi(T) = b \) over an \((m, n)\)-selfdual subset of \([m+1]\) containing exactly \( k \) \((m, n)\)-selfdual elements (see 3.6 and Theorem 3.7). Hence to calculate the number \( \gamma_{m+1-2n}(a) \), we are reduced to the case where \( a = (a_1, a_2, ..., a_r) \in \tilde{\Lambda}_{m+1-2l} \) satisfies \( a_1 > a_2 > \cdots > a_r \) and \( l \in \mathbb{N} \). We get a close formula for \( \gamma_{m+1-2n}(a) \) in the case of \( r = 2 \) (see Propositions 4.7-4.9 and Corollary 4.12). Then in the case of \( r = 3, 4 \), we describe the \((m, n)\)-selfdual tabloids in \( \xi^{-1}(a) \) (see Proposition 4.15 and Subsection 4.16).

**0.4.** The contents of the paper are organized as follows. In Section 1, we collect some concepts and known results concerning cells of the weighted Coxeter groups \((\tilde{A}_m, \tilde{\ell}_m)\) and \((\tilde{C}_n, \tilde{\ell}_m)\). Then we introduce the tabloids of rank \( m + 1 \) in Section 2. In Section 3, we characterize all the tabloids parameterizing the left cells of
The cells in the weighted Coxeter group \((\tilde{C}_n, \tilde{\ell}_m)\) and give some formulae for the number of left cells of \((\tilde{C}_n, \tilde{\ell}_m)\) in the set \(E_\lambda\) for any \(\lambda \in \Lambda_{m+1}\). Finally, we deduce some more formulae for those numbers and describe the \((m, n)\)-selfdual tabloids in some special cases in Section 4.

§1. The weighted Coxeter groups \((\tilde{A}_m, \tilde{S}_m)\) and \((\tilde{C}_n, \tilde{\ell}_m)\).

In this section, we collect some concepts and results concerning the weighted Coxeter groups \((\tilde{A}_m, \tilde{S}_m)\) and \((\tilde{C}_n, \tilde{\ell}_m)\).

1.1. Let \((W, S)\) be a Coxeter system with \(\ell\) its length function and \(\preceq\) the Bruhat-Chevalley ordering on \(W\). An expression \(w = s_1 s_2 \cdots s_r \in W\) with \(s_i \in S\) is called reduced if \(r = \ell(w)\). By a weight function on \(W\), we mean a map \(L\) from \(W\) to the integer set \(\mathbb{Z}\) satisfying that \(L(s) = L(t)\) for any \(s, t \in S\) conjugate in \(W\) and that \(L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)\) for any reduced expression \(w = s_1 s_2 \cdots s_r\) in \(W\). Call \((W, L)\) is a weighted Coxeter group.

A weighted Coxeter group \((W, L)\) is called in the split case if \(L = \ell\).

Suppose that there exists a group automorphism \(\alpha : W \rightarrow W\) with \(\alpha(S) = S\). Let \(W^\alpha = \{w \in W \mid \alpha(w) = w\}\). For any \(\alpha\)-orbit \(J\) on \(S\), let \(w_J \in W^\alpha\) be the longest element in the subgroup \(W_J\) of \(W\) generated by \(J\). Let \(S_\alpha\) be the set of elements \(w_J\) with \(J\) ranging over all \(\alpha\)-orbits on \(S\). Then \((W^\alpha, S_\alpha)\) is a Coxeter group and the restriction to \(W^\alpha\) of the length function \(\ell : W \rightarrow \mathbb{N}\) is a weight function on \(W^\alpha\). The weighted Coxeter group \((W^\alpha, \ell)\) is called in the quasi-split case.

1.2. Let \(\preceq_L\) (respectively, \(\preceq_R\), \(\preceq_{LR}\)) be the preorder on a weighted Coxeter group \((W, L)\) defined in [5]. The equivalence relation associated to this preorder is denoted by \(\sim_L\) (respectively, \(\sim_R\), \(\sim_{LR}\)). The corresponding equivalence classes in \(W\) are called left cells (respectively, right cells, two-sided cells) of \(W\).

1.3. Lusztig introduced a subset \(D\) of \(W\) consisting of certain involutive elements \(w\) (hence \(w^2 = 1\)) in a weighted Coxeter group \((W, L)\) (see [5, Chapter 14]). When \((W, L)\) is a Weyl or affine Weyl group which is either in the split case or in the quasi-split case, Lusztig proved that each left (respectively, right) cell of \(W\) contains exactly one element in \(D\) (see [5, Chapters 14–16]). Note that the elements of \(D\) were called distinguished involutions when \((W, L)\) is in the split case (see [4]).
1.4. The group $\tilde{A}_m$, $m \geq 1$, can be realized as the following permutation group on the set $\mathbb{Z}$ (see [2, Subsection 3.6] and [6, Subsection 4.1]):

$$
\tilde{A}_m = \left\{ w : \mathbb{Z} \rightarrow \mathbb{Z} \mid (i + m + 1)w = (i)w + m + 1, \sum_{i=1}^{m+1} (i)w = \sum_{i=1}^{m+1} i \right\}.
$$

The Coxeter generator set $\tilde{S}_m = \{s_i \mid i \in [0, m]\}$ of $\tilde{A}_m$ is given by

$$
(t)s_i = \begin{cases} 
  t, & \text{if } t \neq i, i + 1 \pmod{m + 1}, \\
  t + 1, & \text{if } t \equiv i \pmod{m + 1}, \\
  t - 1, & \text{if } t \equiv i + 1 \pmod{m + 1},
\end{cases}
$$

for any $t \in \mathbb{Z}$ and $i \in [0, m]$. Any $w \in \tilde{A}_m$ can be realized as a $\mathbb{Z} \times \mathbb{Z}$ monomial matrix $A_w = (a_{ij})_{i,j \in \mathbb{Z}}$, where $a_{ij}$ is 1 if $j = (i)w$ and 0 if otherwise. The row (respectively, column) indices of $A_w$ are increasing from top to bottom (respectively, from left to right).

1.5. For $m \in \{2n - 1, 2n, 2n + 1\}$, let $\alpha_{m,n} : \tilde{A}_m \rightarrow \tilde{A}_m$ be the group automorphism determined by $\alpha_{m,n}(s_i) = s_{2n-i}$ for $i \in [0, m]$ if $m = 2n - 1$ and by $\alpha_{m,n}(s_i) = s_{2n+1-i}$ for $i \in [0, m]$ if $m \in \{2n, 2n + 1\}$, where we stipulate $s_{i+m+1} = s_i$ for any $i \in \mathbb{Z}$. In terms of matrix form, for any $w \in \tilde{A}_m$, the matrix $A_{\alpha_{m,n}(w)}$ can be obtained from the matrix $A_w$ by rotating with the angle $\pi$ around the point $(n + \frac{1}{2}, n + \frac{1}{2})$ (respectively, $(n + 1, n + 1)$) if $m = 2n - 1$ (respectively, $m \in \{2n, 2n + 1\}$), where we identify $A_w$ with a plane and the positions $(i, j)$, $i, j \in \mathbb{Z}$, of $A_w$ are identified with the corresponding integer lattice points. Then $\alpha_{m,n}$ gives rise to a permutation on the set $\Pi_m^l$ (respectively, $\Pi_m^r$, $\Pi_m^t$) of left cells (respectively, right cells, two-sided cells) of $\tilde{A}_m$. Also, $\alpha_{m,n}(D) = D$ by the definition of the set $D$ in [5, Chapter 14] (see 1.3).

1.6. The affine Weyl group $\tilde{C}_n$ can be realized as the fixed point set of $\tilde{A}_m$, $m \in \{2n - 1, 2n, 2n + 1\}$, under the automorphism $\alpha_{m,n}$, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$
\tilde{C}_n = \{ w : \mathbb{Z} \rightarrow \mathbb{Z} \mid (i+m+1)w = (i)w+m+1, (i)w+(\epsilon_{m,n}-i)w = \epsilon_{m,n}, \forall i \in \mathbb{Z} \}.
$$
The cells in the weighted Coxeter group \((\widetilde{C}_n, \ell_m)\)

where \(e_{m,n} = 1\) if \(m \in \{2n-1, 2n\}\) and 0 if \(m = 2n+1\). The Coxeter generator set \(S = \{t_i \mid i \in [0,n]\}\) of \(\widetilde{C}_n\) is given by setting \(t_i = s_i s_{2n-i}\) for \(i \in [n-1]\), \(t_0 = s_0\) and \(t_n = s_n\) if \(m = 2n+1\); \(t_i = s_i s_{2n+1-i}\) for \(i \in [n-1]\), \(t_0 = s_0\) and \(t_n = s_n s_{n+1}s_n\) if \(m = 2n+1\).

In terms of matrix, an element \(w \in \tilde{A}_m\) is in \(\tilde{C}_n\) if and only if the matrix form \(A_w\) of \(w\) is centrally symmetric at the points \((qn + \frac{1}{2}, qn + \frac{1}{2})\) if \(m = 2n+1\) and, at the points \(((2n+1)q + \frac{1}{2}, (2n+1)q + \frac{1}{2})\) and \(((2n+1)q + (n+1), (2n+1)q + (n+1))\) if \(m = 2n+1\) and, at the points \(((n+1)q, (n+1)q)\) if \(m = 2n+1\), where \(q\) ranges over \(\mathbb{Z}\).

1.7. By a partition of a positive integer \(l\), we mean an \(r\)-tuple \(\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_r)\) of weakly decreasing positive integers \(\lambda_1 \geq \cdots \geq \lambda_r\) with \(\sum_{k=1}^r \lambda_k = l\) for some \(r \geq 1\). \(\lambda_i\) is called a part of \(\lambda\). We sometimes denote \(\lambda\) in the form \(j_1^{k_1} j_2^{k_2} \cdots j_m^{k_m}\) (boldfaced) with \(j_1 > j_2 > \cdots > j_m \geq 1\) if \(j_i\) is a part of \(\lambda\) with multiplicity \(k_i \geq 1\) for \(i \geq 1\). Let \(\Lambda_l\) be the set of all partitions of \(l\). For example, \(63^3 1^2\) stands for the partition \((6,3,3,3,1,1)\) of 17.

Fix \(w \in \tilde{A}_m\). For any \(i \neq j\) in \([m+1]\), we write \(i \prec_w j\), if there exist some \(p, q \in \mathbb{Z}\) such that both inequalities \(p(m+1)+i > q(m+1)+j\) and \((p(m+1)+i)w < (q(m+1)+j)w\) hold. This defines a partial order \(\preceq_w\) on the set \([m+1]\). \(i \neq j\) in \([m+1]\) are said \(w\)-comparable if either \(i \prec_w j\) or \(j \prec_w i\), and \(w\)-uncomparable if otherwise.

A sequence \(a_1, a_2, \ldots, a_r\) in \([m+1]\) is called a \(w\)-chain, if \(a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r\). Sometimes we identify a \(w\)-chain \(a_1, a_2, \ldots, a_r\) with the corresponding set \(\{a_1, a_2, \ldots, a_r\}\). For any \(k \geq 1\), a \(k\)-\(w\)-chain-family is by definition a union \(X = \bigcup_{i=1}^k X_i\) of \(k\) \(w\)-chains \(X_1, X_2, \ldots, X_k\) in \([m+1]\). Let \(d_k\) be the maximally possible cardinal of a \(k\)-\(w\)-chain-family for any \(k \geq 1\). Then there exists some \(r \geq 1\) such that \(d_1 < d_2 < \cdots < d_r = m+1\). Let \(\lambda_1 = d_1\) and \(\lambda_{k+1} = d_{k+1} - d_k\) for \(k \in [r-1]\). Then \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r\) by a result of Curtis Greene in [1].

Hence \(w \mapsto \psi(w) := (\lambda_1, \lambda_2, \ldots, \lambda_r)\) defines a map from the set \(\tilde{A}_m\) to \(\Lambda_m\).

1.8. Let \(\ell_m\) be the length function on the Coxeter systems \((\tilde{A}_m, \tilde{S}_m)\). By the definition in 1.1, we see that the weighted Coxeter group \((\tilde{A}_m, \ell_m)\) is in the split case, while \((\tilde{C}_n, \ell_m)\) is in the quasi-split case.
For any \( \lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{m+1} \), define \( \lambda^\vee = (\lambda_1^\vee, ..., \lambda_r^\vee) \in \Lambda_{m+1} \) by setting \( \lambda_j^\vee = \# \{ k \in [r] | \lambda_k \geq j \} \) for any \( j \geq 1 \), call \( \lambda^\vee \) the dual partition of \( \lambda \).

**Lemma 1.9.** (1) Regarding \( \tilde{C}_n \) as a subset of \( \tilde{A}_m \), \( m \in \{2n - 1, 2n, 2n + 1\} \). For any \( x, y \in \tilde{C}_n \), we have \( x \sim_L y \) in \( \tilde{C}_n \) if and only if \( x \sim_L y \) in \( \tilde{A}_m \) (see [5, Lemma 16.14]).

(2) The set \( \psi^{-1}(\lambda) \) forms a two-sided cell of \( \tilde{A}_m \) for any \( \lambda \in \Lambda_{m+1} \) (see [3, Theorem 6] and [6, Theorem 17.4]).

By Lemma 1.9 (1), we can just use the notation \( x \sim_L y \) for \( x, y \in \tilde{C}_n \) without indicating whether the relation refers to \( \tilde{A}_m \), \( m \in \{2n - 1, 2n, 2n + 1\} \), or \( \tilde{C}_n \).

For any \( \lambda \in \Lambda_{m+1} \), denote \( E_\lambda := \tilde{C}_n \cap \psi^{-1}(\lambda) \).

In the remaining part of the paper, when we mention the number \( m \), we always assume \( m \in \{2n - 1, 2n, 2n + 1\} \) unless otherwise specified.

**§2. Tabloids of rank \( m + 1 \).**

In the present section, we introduce the concept of tabloids of rank \( m + 1 \) which will be used to parametrize the left cells of \( \tilde{A}_m \) and of \( \tilde{C}_n \).

2.1. By a composition of \( m + 1 \), we mean an \( r \)-tuple \( a = (a_1, a_2, ..., a_r) \) of positive integers \( a_1, ..., a_r \) with some \( r \in \mathbb{N} \) such that \( \sum_{i=1}^{r} a_i = m + 1 \). Let \( \tilde{\Lambda}_{m+1} \) be the set of all compositions of \( m + 1 \). Clearly, \( \Lambda_{m+1} \subseteq \tilde{\Lambda}_{m+1} \). For any \( a = (a_1, a_2, ..., a_r) \in \tilde{\Lambda}_{m+1} \), let \( i_1, i_2, ..., i_r \) be a permutation of \( 1, 2, ..., r \) with \( a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_r} \). Denote \( \zeta(a) = (a_{i_1}, a_{i_2}, ..., a_{i_r}) \). This defines a surjective map \( \zeta : \tilde{\Lambda}_{m+1} \longrightarrow \Lambda_{m+1} \).

2.2. A (generalized) tabloid of rank \( m + 1 \) is, by definition, an \( r \)-tuple \( T = (T_1, T_2, ..., T_r) \) with some \( r \in \mathbb{N} \) such that \([m + 1]\) is a disjoint union of some non-empty subsets \( T_j, j \in [r] \). We have \( \xi(T) := ([T_1], [T_2], ..., [T_r]) \in \tilde{\Lambda}_{m+1} \), where \( [T_i] \) denotes the cardinal of the set \( T_i \). Two tabloids \( T = (T_1, ..., T_r) \) and \( T' = (T'_1, ..., T'_r) \) are said equal if \( r = t \) and \( T_i = T'_i \) for any \( i \in [r] \). Let \( \mathcal{C}_{m+1} \) be the set of all tabloids of rank \( m + 1 \). Then \( \xi : \mathcal{C}_{m+1} \longrightarrow \tilde{\Lambda}_{m+1} \) is a surjective map. Let \( \kappa = \zeta : \mathcal{C}_{m+1} \longrightarrow \Lambda_{m+1} \).

2.3. Let \( \Omega \) be the set of all \( w \in \tilde{A}_m \) such that for any \( w \in \Omega \), there is some \( T = (T_1, T_2, ..., T_r) \in \mathcal{C}_{m+1} \) satisfying that

(i) For any \( i < j \) in \([r]\), we have \( a \prec_w b \) for any \( a \in T_i \) and \( b \in T_j \);
The cells in the weighted Coxeter group \((\widetilde{C}_n, \ell_m)\)

(ii) For any \(i \in [r]\), \(a\) and \(b\) are \(w\)-uncomparable for any \(a \neq b\) in \(T_1\).

Clearly, the tabloid \(T\) is determined entirely by \(w \in \Omega\), denote \(T\) by \(T(w)\). The map \(T: \Omega \rightarrow C_{m+1}\) is surjective by [6, Proposition 19.1.2]. By a result of Curtis Greene in [1], we have \(\kappa(T(w)) = \psi(w)^Y\).

**Lemma 2.4.** (see [6, Proposition 19.1.2 and Lemma 19.4.6]) Suppose that \(y, w \in \widetilde{A}_m\) are two elements in \(\Omega\) with \(\xi(T(y)) = \xi(T(w))\). Then \(y \sim_L w\) if and only if \(T(y) = T(w)\).

**2.5.** By Lemma 2.4, it makes sense to write \(T \sim T'\) in \(C_{m+1}\) if there exist some \(x, y \in \Omega\) satisfying \(x \sim_L y\) and \(T(x) = T\) and \(T(y) = T'\). This defines an equivalence relation on \(C_{m+1}\).

Fix \(w \in \widetilde{A}_m\) and let \(\lambda = \psi(w)\). Take any \(a \in \zeta^{-1}(\lambda^\vee)\). There exists some \(y \in \Omega\) with \(y \sim_L w\) and \(\xi(T(y)) = a\). The tabloid \(T(y)\) is uniquely determined by the element \(w\) and the composition \(a\) of \(m+1\), denote it by \(T_a(w)\) (see [6, Propositions 19.1.2, 19.4.7 and 19.4.8]).

**Lemma 2.6.** (see [6, Propositions 19.4.7-19.4.8]) In the above setup, \(T_a\) gives rise to a surjective map from the set \(\psi^{-1}(\lambda)\) to \(\xi^{-1}(a)\), which induces a bijection (again denoted by \(T_a\)) from the set \(\Pi_\lambda^L\) of left cells of \(\widetilde{A}_m\) in \(\psi^{-1}(\lambda)\) to \(\xi^{-1}(a)\).

**2.7.** For further discussion on the left cells of \(\widetilde{A}_m\) and \(\widetilde{C}_n\), we need to recall some more concepts involving tabloids of rank \(m+1\) (see [6, Chapter 20]). Let \(k\) be a positive integer. Arrange the numbers \(1, 2, \ldots, k\) on a circle in the following way: in the clockwise direction, \(t+1\) is the successor of \(t\) for any \(t \in [k-1]\) and 1 is the successor of \(k\). We call such a circle the \(k\)-circle. For example, the following is the 8-circle.
For $x \neq y$ in $[k]$, we denote by $xy$ the arc of the $k$-circle which, starting with the number $x$ and moving clockwise, ends with the number $y$. For $Z \subseteq [k]$, let $Z_{xy}$ be the set of all numbers of $Z$ on $xy$. Take the 8-circle in Figure 1 as an example, let $Z = \{2, 3, 4, 6\}$, $x = 2$, $y = 5$. Then $Z_{xy} = \{2, 3, 4\}$ and $Z_{yx} = \{1, 2, 6\}$.

Let $X = \{a_j \mid j \in [t], a_1 < \cdots < a_t\}$ and $Y = \{b_j \mid j \in [r], b_1 < \cdots < b_r\}$ be two subsets of $[k]$ with $X \cap Y = \emptyset$ and $t \leq r$.

(i) We define a subset $H_Y(X) = \{c_1, \ldots, c_t\}$ of $Y$ as follows. Define $c_1 \in Y$ by the condition $|Y_{a_1c_1} \cap Y| = 1$. Define $c_2 \in Y$ by the condition $|Y_{a_2c_2} \cap (Y - \{c_1\})| = 1$. In general, suppose that $c_1, \ldots, c_{h-1}$ have been defined for $h \in [t]$. Then $c_h \in Y$ is defined by the condition $|Y_{a_hc_h} \cap (Y - \{c_1, \ldots, c_{h-1}\})| = 1$. Clearly, the set $H_Y(X)$ is well defined. In particular, $H_Y(X) = Y$ if $|X| = |Y|$.

(ii) We define a subset $L_Y(X) = \{d_1, \ldots, d_t\}$ of $Y$ as follows. Define $d_1 \in Y$ by the condition $|Y_{d_1a_1} \cap Y| = 1$. Define $d_2 \in Y$ by the condition $|Y_{d_2a_t} \cap (Y - \{d_1\})| = 1$. Recursively, define $d_h \in Y$ by the condition $|Y_{d_ha_{t+1-h}} \cap (Y - \{d_1, \ldots, d_{h-1}\})| = 1$ for any $h \in [t]$.

By the above definition, we see that the sets $H_Y(X)$ and $L_Y(X)$ depend only on the relative positions of the elements of $X \cup Y$ on the $k$-circle, but neither on the positive integer $k$ and nor on any of those integers in the set $[k] - X \cup Y$.

The following result can be checked directly from the above definition.

**Lemma 2.8.** Let $k$ be a positive integer. If $\eta$ is a permutation on $[k]$ such that $\eta(i + 1) \equiv \eta(i) - 1 \pmod{k}$ for any $i \in [k]$ (hence the order of the numbers $1, 2, \ldots, k$ on the $k$-circle are reversed by $\eta$) then $\eta(H_Y(X)) = L_{\eta(Y)}(\eta(X))$ and
The cells in the weighted Coxeter group \((\widetilde{C}_n, \tilde{e}_m)\) 9

\[ \eta(L_Y(X)) = H_{\eta(Y)}(\eta(X)) \text{ for any } X, Y \subseteq \{k\} \text{ with } |Y| \geq |X| \text{ and } Y \cap X = \emptyset. \]

Take the 8-circle in Figure 1 as an example. Let \(X = \{1, 4\}\) and \(Y = \{2, 6, 7\}\). Then \(H_Y(X) = \{2, 6\}\) and \(L_Y(X) = \{2, 7\}\). Define \(\eta: [8] \rightarrow [8]\) by setting \(\eta(i) = 9 - i\) for any \(i \in [8]\). Then \(\eta(H_Y(X)) = \{3, 7\} = L_{\{2,3,7\}}(\{5,8\}) = L_{\eta(Y)}(\eta(X))\).

The following results describe the sets \(H_Y(X)\) and \(L_Y(X)\) in more intrinsic way.

**Lemma 2.9.** (see [6, Lemmas 20.1.2-20.1.3]) For a positive integer \(k\), take \(X, Y \subseteq \{k\}\) satisfying \(X \cap Y = \emptyset\) and \(|X| \leq |Y|\). Then for any \(y \in Y\), we have

(a) \(y \in H_Y(X)\) if and only if there exists some \(x \in X\) satisfying \(|Y_{xy}| = |X_{xy}|\).

(b) \(y \in L_Y(X)\) if and only if there exists some \(x \in X\) satisfying \(|Y_{yx}| = |X_{yx}|\).

**2.10.** For \(i, j \in [m + 1]\), we say that \(j\) is the \((m, n)\)-dual of \(i\), denote \(j = \bar{i}\), if either \(m = 2n - 1\) and \(i + j = 2n + 1\), or \(m \in \{2n, 2n + 1\}\) and \(i + j = 2n + 2\) (mod \(2n + 2\)); in this case, we also have \(i = \bar{j}\), and call \(i, j\) an \((m, n)\)-dual pair. Denote \(\overline{E} = \{\bar{i} \mid i \in E\}\) for any \(E \subseteq [m + 1]\) (The notation \(\bar{i}, \overline{E}\) for \(i \in [m + 1]\) and \(E \subseteq [m + 1]\) will cause no confusion in the context since the pair \((m, n)\) is fixed in each case).

For any \(i \in [m + 1]\), we have \(i = \bar{i}\) if and only if either \(m = 2n\) and \(i = n + 1\), or \(m = 2n + 1\) and \(i \in \{n + 1, 2n + 2\}\). When the equivalent conditions hold, \(i\) with itself forms an \((m, n)\)-dual pair, call \(i\) an \((m, n)\)-selfdual element. Hence the number of \((m, n)\)-selfdual elements in \([m + 1]\) is \(m + 1 - 2n\).

Next result shows that for any \(Y \subseteq [m + 1]\), the operations \(H_Y\) and \(L_Y\) on \(X \subseteq [m + 1]\) with \(|X| \leq |Y|\) and \(X \cap Y = \emptyset\) are inverse to each other in some sense.

**Lemma 2.11.** Let \(X, Y \subseteq [m + 1]\) satisfy \(X \cap Y = \emptyset\) and \(|X| \leq |Y|\).

(a) Let \(Y' = H_Y(X)\) and \(X' = X \cup (Y - H_Y(X))\). Then \(X = L_{X'}(Y')\) and \(Y = Y' \cup (X' - L_{X'}(Y'))\).

(b) Let \(Y'' = L_Y(X)\) and \(X'' = X \cup (Y - L_Y(X))\). Then \(X = H_{X''}(Y'')\) and \(Y = Y'' \cup (X'' - H_{X''}(Y''))\).

(c) \(\overline{H_Y(X)} = L_{\overline{X}}(\overline{X})\) and \(\overline{L_Y(X)} = H_{\overline{X}}(\overline{X})\).

Proof. (a) and (b) are just the results in [6, Proposition 20.1.4]. Then (c) follows by Lemma 2.8. \(\Box\)
Recall the relation $\sim_L$ on $C_{m+1}$ defined in 2.5.

**Proposition 2.12.** (see [6, Proposition 20.2.2 and Corollary 20.2.3]) Let $T = (T_1, \ldots, T_t) \in C_{m+1}$ and $j \in [t - 1]$.

(a) If $|T_j| \leq |T_{j+1}|$, let

$$T' = (T_1, \ldots, T_{j-1}, T_j \cup (T_{j+1} - H_{T_{j+1}}(T_j)), H_{T_{j+1}}(T_j), T_{j+2}, \ldots, T_t)$$

then $T \sim_L T'$.

(b) If $|T_j| \geq |T_{j+1}|$, let

$$T'' = (T_1, \ldots, T_{j-1}, L_{T_j}(T_{j+1}), T_{j+1} \cup (T_j - L_{T_j}(T_{j+1})), T_{j+2}, \ldots, T_t).$$

Then $T \sim_L T''$.

**2.13.** Let $T, T', T'' \in C_{m+1}$ be given as in (2.12.1)-(2.12.2). We say that $T'$ (respectively, $T''$) is obtained from $T$ by a $\{j, j+1\}$-transformation. This definition does not cause any confusion since $T'$ (respectively, $T''$) is defined only when $|T_j| \leq |T_{j+1}|$ (respectively, $|T_j| \geq |T_{j+1}|$). Note that if $|T_j| = |T_{j+1}|$ then $T' = T'' = T$.

Fix $E$ with $\emptyset \neq E \subseteq [m+1]$. Let $C_E$ be the set of all tabloids $T = (T_1, T_2, \ldots, T_r)$ with $E = \cup_{i=1}^r T_i$ (hence $C_{m+1} = C_{[m+1]}$).

For any $T, T' \in C_E$, written $T \simeq T'$, if there exists a sequence $T_0 = T, T_1, \ldots, T_r = T'$ in $C_E$ such that for every $i \in [r]$, $T_i$ can be obtained from $T_{i-1}$ by an $\{h_i, h_i + 1\}$-transformation for some integer $h_i$. This defines an equivalence relation on the set $C_E$.

Let $l = |E|$ and $\xi_E(T) := (|T_1|, |T_2|, \ldots, |T_r|)$ for any $T = (T_1, T_2, \ldots, T_r) \in C_E$. Then $\xi_E : C_E \rightarrow \tilde{A}_l$ is a surjective map.

**2.14.** Take $E$ with $\emptyset \neq E \subseteq [m+1]$ and $\overline{E} = E$. Denote $\overline{T} = (\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_r)$ and $\overline{T}^\text{op} = (\overline{T}_r, \ldots, \overline{T}_2, \overline{T}_1)$ for any $T = (T_1, T_2, \ldots, T_r) \in C_E$. Then $T, T^\text{op} \in C_E$. We say that $T \in C_E$ is $(m, n)$-selfdual, if $\overline{T}^\text{op} \approx T$.

Denote $a^\text{op} = (a_r, \ldots, a_2, a_1)$ for $a = (a_1, a_2, \ldots, a_r) \in \tilde{A}_{m+1}$. Call $a$ symmetric, if $a^\text{op} = a$. 
Lemma 2.15. Let $T, T' \in C_{m+1}$.

(1) $T \sim L T'$ if and only if $T \approx T'$.

(2) When $a \in \Lambda_{m+1}$ is symmetric, $T \in \xi^{-1}(a)$ is $(m,n)$-selfdual if and only if $T^{\text{op}} = T$.

(3) If $T \approx T'$, then $T$ is $(m,n)$-selfdual if and only if so is $T'$.

Proof. (1) follows by Proposition 2.12, Lemmas 2.4 and 2.6. For (2), by the assumption of $a$ being symmetric, we have $\xi(T^{\text{op}}) = a$ for any $T \in \xi^{-1}(a)$. So $T^{\text{op}} \approx T$ if and only if $T^{\text{op}} = T$ by Lemma 2.4. This implies (2). For (3), let $T = (T_1, T_2, ..., T_r)$. We may assume without loss of generality that $T'$ is obtained from $T$ by an $\{i, i + 1\}$-transformation for some $i \in [r - 1]$. Then $T^{\text{op}}$ can be obtained from $T^{\text{op}}$ by an $\{r - i, r + 1 - i\}$-transformation by Lemma 2.11 (c). This implies that $T \approx T^{\text{op}}$ if and only if $T' \approx T^{\text{op}}$. Hence (3) follows. 

By Lemma 2.15 (3), we can call an $\approx$-equivalence class of $C_{m+1}$ $(m,n)$-selfdual if some (hence all) tabloids in this class are $(m,n)$-selfdual.

§3. A formula for the number of left cells of $\tilde{C}_n$ in the set $E_{\lambda}$, $\lambda \in \Lambda_{m+1}$.

In the present section, we first characterize all the tabloids of rank $m + 1$ which correspond to the left cells of $\tilde{C}_n$. Applying this result, we deduce a formula for the number of left cells of $\tilde{C}_n$ in the set $E_{\lambda}$ for any $\lambda \in \Lambda_{m+1}$.

Theorem 3.1. Let $\lambda \in \Lambda_{m+1}$ and $a \in \zeta^{-1}(\lambda')$. Then for any $\Gamma \in \Pi_{\lambda}^1$ (see Lemma 2.6), we have $\Gamma \cap \tilde{C}_n \neq \emptyset$ if and only if $T_a(\Gamma)$ is $(m,n)$-selfdual.

Proof. The automorphism $\alpha := \alpha_{m,n}$ of $\tilde{A}_m$ stabilizes the set $\Omega$ (see 2.3). We have $T(\alpha(w)) = \overline{T(w)^{\text{op}}}$ for any $w \in \Omega$ (see the matrix description for the action of $\alpha$ on $\tilde{A}_m$ in 1.5). This implies $T_a^{\text{op}}(\alpha(\Gamma)) = \overline{T_a(\Gamma)^{\text{op}}}$ for any $\Gamma \in \Pi_{\lambda}^1$. Hence by Lemmas 2.6, 2.15 and Proposition 2.12, we see that

(*) $\alpha(\Gamma) = \Gamma \iff T_a(\Gamma)$ is $(m,n)$-selfdual.

First assume $\Gamma \cap \tilde{C}_n \neq \emptyset$. Then $\alpha(\Gamma) \cap \Gamma \neq \emptyset$, hence $\alpha(\Gamma) = \Gamma$ since both $\Gamma$ and $\alpha(\Gamma)$ are left cells of $\tilde{A}_m$. This implies that $T_a(\Gamma)$ is $(m,n)$-selfdual by (*).

Next assume that $T_a(\Gamma)$ is $(m,n)$-selfdual. Then $\alpha(\Gamma) = \Gamma$ by (*). Recall the set $D$ mentioned in 1.3. The set $\Gamma \cap D$ consists of a single element (say $d$) by 1.3.
Then $\alpha(d) \in \alpha(\Gamma) \cap \mathcal{D}$ by the fact $\alpha(\mathcal{D}) = \mathcal{D}$ (see 1.5). This implies $d = \alpha(d)$ by the equation $\alpha(\Gamma) = \Gamma$ and the fact $|\Gamma \cap \mathcal{D}| = 1$ (by 1.3), i.e., $d \in \Gamma \cap \tilde{C}_n$. Hence $\Gamma \cap \tilde{C}_n \neq \emptyset$. □

3.2. Suppose that $\emptyset \neq E \subseteq [m + 1]$ and $\overline{E} = E$. For any $b \in \tilde{\Lambda}_{|E|}$, let $\gamma_E(b)$ be the number of all $(m,n)$-selfdual tabloids in $\xi_E^{-1}(b)$ (see 2.13). Under the conditions assumed on $E$, we see that the number $\gamma_E(b)$ depends only on $|E|$ and the number of $(m,n)$-selfdual elements contained in $E$, but not on a particular choice of a subset $E$ in $[m+1]$. Since $|E|$ is determined by $b$, we may write $\gamma_E(b)$ by $\gamma_k(b)$ if the number of $(m,n)$-selfdual elements contained in $E$ is $k$.

We have not yet found any efficient way to calculate the number $\gamma_k(a)$ in general. However, there is a simple formula for $\gamma_{m+1-2n}(a)$ when $a \in \tilde{\Lambda}_{m+1}$ is symmetric (see 2.14).

**Theorem 3.3.** Suppose that $a = (a_1, ..., a_r) \in \tilde{\Lambda}_{m+1}$ is symmetric. Then

\[
\gamma_{m+1-2n}(a) = \begin{cases} 
0, & \text{if } m = 2n + 1 \text{ and } r = 2l, \\
2^{a_1 + \cdots + a_i} \frac{n!}{\left(n - \sum_{k=1}^l a_k \right) \prod_{k=1}^l a_k!}, & \text{if otherwise},
\end{cases}
\]

where $l \in \mathbb{N}$.

**Proof.** Any $(m,n)$-selfdual tabloid $T = (T_1, T_2, ..., T_r) \in \xi^{-1}(a)$ is determined entirely by its first $l$ components if $r \in \{2l, 2l + 1\}$ with $l \in \mathbb{N}$ by the facts that $T_i = T_{r+1-i}$ for any $i \in [l]$ and that $T_{l+1} = [m+1] - \bigcup_{i=1}^l (T_i \cup T_i)$ is a union of some $(m,n)$-dual pairs (see 2.10) if $r = 2l + 1$ is odd. If $m = 2n + 1$ and $r = 2l$ then the $(m,n)$-selfdual elements $n+1, 2n+2$ can not be in $T_i$ for any $i \in [2l]$ and hence $\gamma_{m+1-2n}(a) = 0$. If $m = 2n$ then the number $r$ must be odd as $m+1$ is odd. If $r = 2l + 1$ is odd then any $(m,n)$-selfdual elements, whenever they exist, must be in $T_{l+1}$. Since the elements of $\bigcup_{i=1}^l T_i$ are pairwise not $(m,n)$-dual and none of them is $(m,n)$-selfdual, the number of the choices for $T_1$ is $2^{a_1} \binom{n}{a_1}$. Recurrently, when $T_1, T_2, ..., T_{h-1}$ have been chosen for $h \in [l]$, the number of the choices for
The cells in the weighted Coxeter group \((\widetilde{C}_n, \ell_m)\) 

\(T_h = 2^{a_h} \left( \frac{n-a_1 - \cdots - a_{h-1}}{a_h} \right) \). We have \(n = a_1 + \cdots + a_t\) if \(m = 2n - 1\) and \(r = 2l\). This proves the formula (3.3.1). \(\square\)

Next result gives a necessary and sufficient condition on \(\lambda \in \Lambda_{m+1}\) that there is some symmetric \(a\) in \(\zeta^{-1}(\lambda^\vee)\).

**Lemma 3.4.** Let \(\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda_{m+1}\).

1. There exists some symmetric \(a\) in \(\zeta^{-1}(\lambda^\vee)\) if and only if \(\lambda\) satisfies the condition (3.4.1) below.

2. \((3.4.1)\) \(\lambda_i\) is odd and \(\lambda_j\) is even for some \(k \in [0, r]\) and any \(i, j, 1 \leq i < k < j \leq r\).

3. When the condition (3.4.1) holds, the set \(E^\lambda\) is empty if and only if \(m = 2n + 1\) and \(k = 0\).

**Proof.** The proof for (1) is straightforward. Then (2) follows by Theorem 3.3. \(\square\)

**Example 3.5.** Let \(\lambda = 97642\). Then \(\lambda^\vee = 5^24^23^22^1\). The composition \(a = (5, 4, 3, 1, 2, 1, 3, 4, 5) \in \zeta^{-1}(\lambda^\vee)\) is symmetric.

3.6. Assume that \(\lambda \in \Lambda_{m+1}\) satisfies the condition (3.4.1). By Theorems 3.1, 3.3 and Lemmas 3.4, 2.6, we see that for any symmetric \(a \in \zeta^{-1}(\lambda^\vee)\), the number of left cells of \(\widetilde{C}_n\) in \(E^\lambda\) is equal to \(\gamma_{m+1-2n}(a)\), which can be computed by the formula (3.3.1).

Next we consider the number of left cells of \(\widetilde{C}_n\) in \(E^\lambda\) for an arbitrary \(\lambda \in \Lambda_{m+1}\).

For any \(\lambda \in \Lambda_{m+1}\), let \(\lambda^\vee = b_1^{k_1}b_2^{k_2} \cdots b_r^{k_r}\). Write \(k_i = 2l_i + p_i\) for any \(i \in [r]\), where \(l_i \in \mathbb{N}\) and \(p_i \in \{0, 1\}\). Define \(q_1 < q_2 < \cdots < q_u\) in \(\mathbb{N}\) by the condition \(\{q_j \mid j \in [u]\} = \{i \in [r] \mid p_i = 1\}\) for some \(u \in \mathbb{N}\). Take \(a \in \zeta^{-1}(\lambda^\vee)\) as follows.

\((3.6.1)\) \(a = (b_1^{q_1}b_2^{q_2} \cdots b_r^{q_r})\).

Define

\((3.6.2)\) \(a_1 = (b_1^{q_1}b_2^{q_2} \cdots b_r^{q_r})\).

\((3.6.3)\) \(a_2 = (b_1^{q_1}, b_2^{q_2}, \ldots, b^{q_u})\).

We have
Theorem 3.7. Let $\lambda \in \Lambda_{m+1}$ be given as in 3.6, and let $a \in \tilde{\Lambda}_{m+1}$, $a_1 \in \tilde{\Lambda}_{2l}$ and $a_2 \in \tilde{\Lambda}_{m+1-2l}$ be obtained from $\lambda$ as in (3.6.1)-(3.6.3), respectively, where $l = \sum_{i=1}^{r} l_i b_i$. Then

$$(3.7.1) \quad \gamma_{m+1-2n}(a) = \binom{n}{l} \gamma_0(a_1) \gamma_{m+1-2n}(a_2).$$

Proof. Let $p = \sum_{i=1}^{r} l_i$. For any

$$T = (T_1, T_2, \ldots, T_p, T_{p+1}, \ldots, T_{p+u}, T_{p+u+1}, \ldots, T_{2p+u}) \in \xi^{-1}(a),$$

let

$$T_1 = (T_1, T_2, \ldots, T_p, T_{p+u+1}, T_{p+u+2}, \ldots, T_{2p+u}) \quad \text{and} \quad T_2 = (T_{p+1}, T_{p+2}, \ldots, T_{p+u})$$

and $E = [m+1] - \bigcup_{i=p+1}^{p+u} T_i$. Then $|E| = 2l$ and $T_1 \in \xi^{-1}(a_1)$ and $T_2 \in \xi^{-1}_{[m+1]-E}(a_2)$. We see by Lemma 2.15 that $T$ is $(m, n)$-selfdual if and only if both $T_1$ and $T_2$ are $(m, n)$-selfdual. When the equivalent conditions hold, we have $\overline{E} = E$ again by Lemma 2.15. For any $k \in [n]$, denote by $[m+1]_{2k}$ the set of all $E \subseteq [m+1]$ with $|E| = 2k$ and $\overline{E} = E$ such that $E$ contains no $(m, n)$-selfdual element. For any $E \in [m+1]_{2l}$, let $C_E^a$ be the set of all $(m, n)$-selfdual tabloids

$$T' = (T'_1, T'_2, \ldots, T'_p, T'_{p+1}, \ldots, T'_{p+u}, T'_{p+u+1}, \ldots, T'_{2p+u}) \in \xi^{-1}(a)$$

with $E = [m+1] - \bigcup_{i=p+1}^{p+u} T'_i$. Then

$$\gamma_{m+1-2n}(a) = |[m+1]_{2l}| \cdot |C_E^a| = \binom{n}{l} |C_E^a| \quad \text{for any fixed } E \in [m+1]_{2l}.$$

$T \mapsto (T_1, T_2)$ is a bijective map from the set $C_E^a$ to the Cartesian product $C_E^{a_1} \times C_E^{a_2}_{[m+1]-E}$, where $C_E^{a_1}$, $C_E^{a_2}_{[m+1]-E}$ are the sets of all $(m, n)$-selfdual tabloids in $\xi^{-1}_{E}^{-1}(a_1)$, $\xi^{-1}_{[m+1]-E}(a_2)$, respectively. This proves the formula (3.7.1) by the facts

$\gamma_0(a_1) = |C_E^{a_1}|$ and $\gamma_{m+1-2n}(a_2) = |C_E^{a_2}_{[m+1]-E}|$ for any $E \in [m+1]_{2l}$. 

\section{4. Enumeration of some special tabloids in $\xi_{m+1}$.}

For any $a \in \tilde{\Lambda}_{m+1}$, let $C_{m+1}^a$ be the set of all $(m, n)$-selfdual tabloids $T$ in $\xi^{-1}(a)$. We want to formulate the number $\gamma_{m+1-2n}(a) := |C_{m+1}^a|$. By Theorems 3.3 and 3.7, it is enough to consider the case where $a = (a_1, a_2, \ldots, a_r) \in \tilde{\Lambda}_{m+1}$ satisfies $a_1 > a_2 > \cdots > a_r$ for some $r > 1$.

First consider the case of $r = 2$. 

Lemma 4.1. Let $T = (Y, X) \in C_{m+1}$ satisfy $|Y| \geq |X|$. Then $T$ is $(m,n)$-selfdual if and only if $L_Y(X) = \overline{X}$ and $Y - L_Y(X) = \overline{Y - L_Y(X)}$.

Proof. Let $X' = L_Y(X)$ and $Y' = X \cup (Y - L_Y(X))$. Then $(Y, X) \approx (X', Y')$. So

\[
T \text{ is (m,n)-selfdual} \iff X' = \overline{X} \text{ and } Y' = \overline{Y} \\
\iff L_Y(X) = \overline{X} \text{ and } X \cup (Y - L_Y(X)) = \overline{Y} \\
\iff L_Y(X) = \overline{X} \text{ and } Y - L_Y(X) = \overline{Y - L_Y(X)}.
\]

The last equivalence follows by the facts that

\[
Y = L_Y(X) \cup (Y - L_Y(X)) \quad \text{and} \quad \overline{X \cup (Y - L_Y(X))} = \overline{X} \cup \overline{Y - L_Y(X)}. \quad \square
\]

4.2. First assume $m = 2n - 1$. Hence $m + 1 - 2n = 0$ and $i := 2n + 1 - i$ for any $i \in [2n]$. Define an admissible subsequence $\alpha$ in each of the following sequences $\beta$ (note that $\beta$ has even number of terms in $[2n]$).

(a) Consider the sequence $\beta_{n,q} : \bar{n}, n - 1, \ldots, \bar{q}, q + 1, \ldots, n - 1, n$ for any $q \in [0, n - 1]$. A subsequence $\alpha : i_1, i_2, \ldots, i_r$ of $\beta_{n,q}$ is called admissible, if the following two conditions hold:

(a1) $r = n - q$ and $\bar{i}_h \neq i_k$ for any $h, k \in [n - q]$;

(a2) Let $\alpha' : j_1, j_2, \ldots, j_{n-q}$ be the subsequence of $\beta_{n,q}$ complement to $\alpha$ (i.e., $\{i_h, j_h \mid h \in [n-q]\} = \beta_{n,q}$ regarding the sequences as the corresponding sets).

Then the term $j_h$ occurs after $i_h$ in the sequence $\beta_{n,q}$ for every $h \in [n - q]$.

Let $\Delta_{n,q}$ be the set of all admissible subsequences of $\beta_{n,q}$ and let $\delta_{n,q} := |\Delta_{n,q}|$.

Denote $\beta_{n,0}$, $\Delta_{n,0}$, $\delta_{n,0}$ simply by $\beta_n$, $\Delta_n$, $\delta_n$, respectively. Clearly, the equation $\delta_{n,q} = \delta_{n-q}$ holds for any $q \in [0, n - 1]$.

(b) For any $i < j$ in $[n]$ with $j - i$ odd, denote by $\beta_{ij}$ (respectively, $\beta_{ji}$) the sequence $i + 1, i + 2, \ldots, j - 1$ (respectively, $\bar{j} - \bar{1}, \bar{j} - \bar{2}, \ldots, \bar{i} + \bar{1}$). A subsequence $\alpha : h_1, h_2, \ldots, h_r$ of $\beta_{ij}$ (respectively, $\beta_{ji}$) is called admissible, if $r = \frac{j-i-1}{2}$, and if, let $\alpha' : k_1, k_2, \ldots, k_{\frac{j-i-1}{2}}$ be the subsequence of $\beta_{ij}$ (respectively, $\beta_{ji}$) complement to $\alpha$, then $k_l$ occurs after $h_l$ in $\beta_{ij}$ (respectively, $\beta_{ji}$) for any $l \in \left[\frac{j-i-1}{2}\right]$.

It is well known that the number of admissible subsequences in $\beta_{ij}$ (respectively, $\beta_{ji}$) is $C_{\frac{j-i-1}{2}}$, where $C_l := \frac{1}{l+1} \binom{2l}{l}$ is the $l$-th Catalan number. The following is a formula for the number $\delta_n$ of admissible subsequences in $\beta_n$. 

Proposition 4.3. \( \delta_n := \left( \frac{n}{2} \right) \) for any \( n \geq 1 \), where \( \lfloor x \rfloor \) stands for the largest integer not greater than \( x \) for any \( x \in \mathbb{Q} \).

To show Proposition 4.3, we need some preparation. Let \( \alpha_{n,q} : i_1, i_2, \ldots, i_{n-q} \) be a subsequence of \( \beta_{n,q} \) satisfying the condition 4.2 (a1). Let \( p(\alpha_{n,q}) \) be the largest \( k \in [0, n - q] \) with \( i_1, i_2, \ldots, i_k \) a subsequence of \( \bar{n}, \bar{n} - 1, \ldots, q + 1 \). Denote \( i_1, i_2, \ldots, i_p(\alpha_{n,q}) \) by \( \alpha_{n,q}^0 \). Then \( \alpha_{n,q} \) is entirely determined by \( \alpha_{n,q}^0 \).

Let \( \alpha : i_1, i_2, \ldots, i_n \) be a subsequence of \( \beta_n \) satisfying the condition 4.2 (a1). For any \( q \in [0, n - 1] \), let \( \alpha_{n,q} \) be obtained from \( \alpha \) by removing all the terms in \( \beta_q \) and let \( \alpha'_{n,q} \) be the subsequence of \( \beta_{n,q} \) complement to \( \alpha_{n,q} \) (see 4.2 (a2)), where we stipulate \( \beta_0 \) to be the empty sequence. Then the following result can be checked easily:

Lemma 4.4. Let \( \alpha : i_1, i_2, \ldots, i_n \) be a subsequence of \( \beta_n \) satisfying the condition 4.2 (a1).

1. The following three conditions on \( \alpha \) are equivalent:
   (a) \( \alpha \) is admissible in \( \beta_n \);
   (b) \( \alpha_{n,q} \) is admissible in \( \beta_{n,q} \) for every \( q \in [0, n - 1] \);
   (c) \( p(\alpha) \geq \frac{n}{2} \) and the term \( j_h \) occurs after \( i_h \) in \( \beta_n \) for every \( h \in [p(\alpha)] \), where \( \alpha' : j_1, j_2, \ldots, j_n \) is the subsequence of \( \beta_n \) complement to \( \alpha \) (see 4.2 (a2)).

2. For \( q \in [0, n - 1] \), if \( \alpha_{n,q} \) is admissible in \( \beta_{n,q} \), then \( p(\alpha_{n,q}) \geq p(\alpha'_{n,q}) \), in particular, \( p(\alpha_{n,q}) \geq \frac{n-q}{2} \).

4.5. Proof of Proposition 4.3. Consider the set \( \Delta_n \). We may assume \( n > 1 \), for otherwise the result is obvious. By Lemma 4.4 (1), we see that \( \alpha_{n,1} \in \Delta_{n,1} \) for any \( \alpha \in \Delta_n \). On the other hand, for any \( \lambda : i_1, i_2, \ldots, i_{n-1} \in \Delta_{n,1} \), let \( \lambda_1 \) (respectively, \( \lambda_1 \)) be obtained from \( \lambda \) by inserting the term \( \bar{1} \) (respectively, 1) immediately after \( i_{p(\lambda)} \). Then \( \lambda_1 \) is always in \( \Delta_n \), while \( \lambda_1 \) is not in \( \Delta_n \) if and only if \( p(\lambda) < \frac{n}{2} \). Since \( p(\lambda) \geq \frac{n-1}{2} \) by the condition \( \lambda \in \Delta_{n,1} \) and Lemma 4.4 (2), this implies that \( \lambda_1 \) is not in \( \Delta_n \) if and only if \( n \) is odd (say \( n = 2l + 1 \)) and \( p(\lambda) = l \). When \( n = 2l + 1 \), let \( \Delta_{n,1} \) be the set of all such subsequences \( \lambda : i_1, i_2, \ldots, i_l \) of \( \bar{n}, \bar{n} - 1, \ldots, \bar{3}, \bar{2} \) that, if \( \lambda' : j_1, j_2, \ldots, j_l \) is the subsequence of \( \bar{n}, \bar{n} - 1, \ldots, \bar{3}, \bar{2} \) complement to \( \lambda \), then the term \( j_h \) occurs after the term \( i_h \) for every \( h \in [l] \). Then \( |\Delta_{n,1}| \) is equal to the
number of all \( \lambda \in \Delta_{n,1} \) with \( \lambda_1 \notin \Delta_n \). It is well known that \( |\Delta_{n,1}'| = C_l \) (the \( l \)-th Catalan number). So by applying induction on \( n \geq 1 \) and by the fact that \( \delta_{n,1} = \delta_{n-1} \), we have

\[
\delta_n = \begin{cases} 
\delta_{n-1} + (\delta_{n-1} - C_l) = 2\binom{2l}{l} - \frac{1}{l+1} \binom{2l}{l} = \binom{2l+1}{l+1}, & \text{if } n = 2l+1 \text{ is odd,} \\
2\delta_{n-1} = 2\binom{2l-1}{l-1} = \binom{2l}{l}, & \text{if } n = 2l \text{ is even.}
\end{cases}
\]

Our result is proved. \( \square \)

**Remark 4.6.** The result in Proposition 4.3 can be extended to a more general case. Let \( \beta : \overline{t}_1, \overline{i}_{t-1}, \ldots, \overline{i}_1, i_1, i_2, \ldots, i_t \) (respectively, \( \overline{\beta} : i_1, i_2, \ldots, i_t, \overline{i}_{t-1}, \ldots, \overline{i}_1 \)) satisfy \( 1 \leq i_1 < i_2 < \cdots < i_t \leq n \). A subsequence \( \alpha : j_1, j_2, \ldots, j_r \) of \( \beta \) (respectively, \( \overline{\beta} \)) is called admissible, if the following conditions are satisfied:

(i) \( r = t \) and \( \overline{j}_h \neq j_k \) for any \( h, k \in [t] \);

(ii) Let \( \alpha' : j'_1, j'_2, \ldots, j'_t \) be the subsequence of \( \beta \) (respectively, \( \overline{\beta} \)) complement to \( \alpha \). Then \( j'_{h_t} \) occurs after \( j_h \) in \( \beta \) (respectively, \( \overline{\beta} \)) for any \( h \in [t] \).

By the same way as that for Proposition 4.3, one can prove that the number of admissible subsequences of \( \beta \) (respectively, \( \overline{\beta} \)) is equal to \( \binom{t}{\frac{t}{2}} \).

The following is a formula for the number \( \gamma_0(\mathbf{a}) \) with \( \mathbf{a} = (a_1, a_2) \in \bar{\Lambda}_{2n} \) having just two parts \( a_1, a_2 \).

**Proposition 4.7.** For \( \mathbf{a} = (n+t, n-t), \) \( t \in [n-1], \) let \( C_{2n}^{n,t} \) be the set of all \((2n-1, n)\)-selfdual tabloids in \( \xi^{-1}(\mathbf{a}) \) and let \( q_{2n}^{n,t} = |C_{2n}^{n,t}| \). Then

\[
q_{2n}^{n,t} = \sum_{1 \leq h_1 < h_2 < \cdots < h_t \leq n \atop h_i+1-h_i \text{ odd } \forall i} \binom{n-h_t}{\lfloor n/2 \rfloor} \binom{h_1-1}{\lfloor h_1/2 \rfloor} \prod_{i=1}^{t-1} C_{h_{i+1}-h_i+1-1},
\]

where \( C_l \) is the \( l \)-th Catalan number for any \( l \in \mathbb{N} \).

**Proof.** Let \( \mathbf{T} = (Y, X) \in C_{2n}^{n,t} \). By the condition of \( \mathbf{T} \) being \((2n-1, n)\)-selfdual, we have \( L_Y(X) = \{ i \in Y \mid i \notin Y \} \) and \( Y - L_Y(X) = \{ h_1, h_2, \ldots, h_t, \overline{h}_{t+1}, \ldots, \overline{h}_1 \} \) with some \( 1 \leq h_1 < h_2 < \cdots < h_t \leq n \) by Lemma 4.1. According to the definition of the set \( L_Y(X) \) with respect to \( X, Y \), we see by Lemma 2.9 (b) that
(i) For any \( j \in \{t - 1\} \), let \( Y_{h_j+1, h_j+1-1} = \{h_{j1}, h_{j2}, \ldots, h_{jn_j}\} \) be with \( h_j < h_{j1} < h_{j2} < \cdots < h_{jn_j} < h_{j+1} \), then \( h_{j1}, h_{j2}, \ldots, h_{jn_j} \) is an admissible subsequence of \( \beta_{h_j, h_j+1} : h_j + 1, h_j + 2, \ldots, h_{j+1} - 1 \) (hence \( h_{j+1} - h_j \) is odd and \( n_j = \frac{h_{j+1} - h_j - 1}{2} \) by 4.2 (b)), and \( Y_{h_j+1-1, h_j+1} = [h_j + 1, h_{j+1} - 1] - Y_{h_j+1, h_j+1-1} \). Write \( Y_{h_j+1-1, h_j+1} = (h_{j1}', h_{j2}', \ldots, h_{jn_j}') \) with \( h_j < h_{j1}' < h_{j2}' < \cdots < h_{jn_j}' < h_{j+1} \). Then \( h_{j1}', h_{j2}', \ldots, h_{jn_j}' \) is an admissible subsequence of \( h_{j+1} - 1, \ldots, h_j + 2, h_j + 1 \).

(ii) Let \( Y_{h_t+1, h_t+1} = \{h_{t1}, h_{t2}, \ldots, h_{tn_t}\} \) be with \( \alpha : h_{t1}, h_{t2}, \ldots, h_{tn_t} \) a subsequence of \( \beta_{h_t, n} : h_t + 1, h_t + 2, \ldots, n, n, n - 1, \ldots, h_t + 1 \). Then \( \alpha \) is admissible in \( \beta_{h_t, n} \).

(iii) Let \( Y_{\bar{h}_1-1, \bar{h}_1} = \{h_{01}, h_{02}, \ldots, h_{0n_0}\} \) be with \( \alpha : h_{01}, h_{02}, \ldots, h_{0n_0} \) a subsequence of \( \beta_{1, \bar{h}_1} : \bar{h}_1 - 1, \bar{h}_1 - 2, \ldots, 1, 2, \ldots, h_1 - 1 \). Then \( \alpha \) is admissible in \( \beta_{1, \bar{h}_1} \).

(iv) \( L_Y(X) = \left( \bigcup_{Y \in \{t-1\}} \left( Y_{h_j+1, h_j+1-1} \cup Y_{h_{j+1-1}, h_{j+1}} \right) \right) \cup Y_{h_t+1, h_t+1} \cup Y_{\bar{h}_1-1, \bar{h}_1-1} \).

Now fix \( h_1, h_2, \ldots, h_t \in \mathbb{N} \) with \( t \in \mathbb{N} \) and \( 1 \leq h_1 < h_2 < \cdots < h_t \leq n \) and \( h_{i+1} - h_i \) odd for all \( i \in \{t - 1\} \). Take an admissible subsequence \( \alpha_j \) in \( \beta_{h_j, h_{j+1}} \) for any \( j \in \{t - 1\} \). Also, take an admissible subsequence \( \alpha_t \) (respectively, \( \alpha_0 \)) in \( \beta_{h_t, n} \) (respectively, \( \beta_{1, \bar{h}_1} \)). For \( j \in \{t - 1\} \), let \( \alpha_j' \) be the subsequence of \( \beta_{h_j, h_{j+1}} \) complement to \( \alpha_j \) and let \( \alpha_j' \) be the subsequence of \( \beta_{h_{j+1-1}, h_{j+1}} \) such that \( \alpha_j = \alpha_j' \) by regarding the sequences as the corresponding sets. Let \( Y' \) be the union of the sets \( \{h_l, \bar{h}_l \mid l \in \{t\} \} \), \( \alpha_t \), \( \alpha_0 \) and \( \alpha_j \), \( \alpha_j' \) with \( j \in \{t - 1\} \), regarding the sequences as the corresponding sets. Let \( X = [2n] - Y \). Then \( (Y, X) \in C_{2n}^{m, t} \).

By 4.2 (b) and Proposition 4.3, we see that the numbers of admissible subsequences in \( \beta_{h_j, h_{j+1}}, j \in \{t - 1\}, \beta_{h_t, n}, \beta_{1, \bar{h}_1} \) are \( C_{\frac{h_{j+1} - h_j - 1}{2}}, \left( \frac{n - h_t}{n - h_1} \right), \left( \frac{h_1 - 1}{h_1 - 2} \right) \), respectively. This implies the formula (4.7.1). \( \square \)

We can get the corresponding results in the case of \( m \in \{2n, 2n + 1\} \) similarly by noting that the number of \((m, n)\)-selfdual elements in \([m + 1]\) is \( m + 1 - 2n \).

**Proposition 4.8.** For \( a = (n + 1 + t, n - t), t \in \{n - 1\} \), let \( C_{2n+1}^{m, t} \) be the set of all \((2n, n)\)-selfdual tableaux in \( \xi^{-1}(a) \) and let \( q_{2n+1}^{n, t} = |C_{2n+1}^{m, t}|. \) Then...
The cells in the weighted Coxeter group \((\widetilde{C}_n, \ell_m)\) \(n, t\)

\[
q_{2n+1}^{n,t} = \sum_{h_1, h_2, \ldots, h_{t+1} \in \mathbb{N}} \left( \frac{h_1 - 1}{h_1 - 1} \right) \prod_{i=1}^{t} \frac{C_{h_{i+1} - h_i - 1}}{2}.
\]

**Proposition 4.9.** For \(a = (n + 1 + t, n + 1 - t)\), \(t \in [n]\), let \(C_{2n+2}^{n,t}\) be the set of all \((2n + 1, n)\)-selfdual tabloids in \(\xi^{-1}(a)\) and let \(q_{2n+2}^{n,t} = |C_{2n+2}^{n,t}|\). Then

\[
q_{2n+2}^{n,t} = \sum_{h_1, h_2, \ldots, h_{t+1} \in \mathbb{N}} \prod_{i=1}^{t} \frac{C_{h_{i+1} - h_i - 1}}{2}.
\]

From Theorem 3.3 and Propositions 4.7-4.9, we see that for \(k \in \mathbb{N}\) with \(2k \leq m + 1\), the set \(E_{2k+1m+1-2k}\) is empty if and only if \(m = 2n + 1\) and \(2k = m + 1\).

**Example 4.10.**

(1) In Proposition 4.7, take \(n = 5\) and \(t = 2\), then \(a = (7, 3)\) and \(\zeta(a)^\vee = 2^3 1^4\). The pairs \((h_1, h_2)\) occurring in the summation of (4.7.1) are \((1, 2), (2, 3), (3, 4), (4, 5), (1, 4), (2, 5)\). Then \(q_{10}^{5,2} = \binom{3}{1} + \binom{2}{1} + \binom{3}{1} + 1 + 1 = 12\), which is just the number of left cells of \(\widetilde{C}_5\) in the set \(E_{2^3 1^4}\). The set \(C_{10}^{5,2}\) consists of the following tabloids:

- \(T_1 = \{(3, 4, 5) \cup \{1, 2, 9, 10\}, \{6, 7, 8\}\}, \quad T_2 = \{(3, 4, 6) \cup \{1, 2, 9, 10\}, \{5, 7, 8\}\},\)
- \(T_3 = \{(3, 5, 7) \cup \{1, 2, 9, 10\}, \{4, 6, 8\}\}, \quad T_4 = \{(4, 5, 10) \cup \{2, 3, 8, 9\}, \{1, 6, 7\}\},\)
- \(T_5 = \{(4, 6, 10) \cup \{2, 3, 8, 9\}, \{1, 5, 7\}\}, \quad T_6 = \{(5, 9, 10) \cup \{3, 4, 7, 8\}, \{1, 2, 6\}\},\)
- \(T_7 = \{(1, 5, 9) \cup \{3, 4, 7, 8\}, \{2, 6, 10\}\}, \quad T_8 = \{(8, 9, 10) \cup \{4, 5, 6, 7\}, \{1, 2, 3\}\},\)
- \(T_9 = \{(1, 8, 9) \cup \{4, 5, 6, 7\}, \{2, 3, 10\}\}, \quad T_{10} = \{(2, 8, 10) \cup \{4, 5, 6, 7\}, \{1, 3, 9\}\},\)
- \(T_{11} = \{(2, 5, 8) \cup \{1, 4, 7, 10\}, \{3, 6, 9\}\}, \quad T_{12} = \{(3, 7, 10) \cup \{2, 5, 6, 9\}, \{1, 4, 8\}\}.
\)

(2) In Proposition 4.8, take \(n = 5\) and \(t = 2\), then \(a = (8, 3)\) and \(\zeta(a)^\vee = 2^3 1^5\). The triples \((h_1, h_2, h_3)\) occurring in the summation of (4.8.1) are \((4, 5, 6), (2, 5, 6), (2, 3, 6)\). Then \(q_{11}^{5,2} = \binom{3}{1} + 1 + 1 = 5\), which is just the number of left cells of \(\widetilde{C}_5\) in the set \(E_{2^3 1^5}\). The set \(C_{11}^{5,2}\) consists of the following tabloids:
\[ T_1 = (\{4, 7, 11\} \cup \{2, 3, 6, 9, 10\}, \{1, 5, 8\}), \]
\[ T_2 = (\{3, 8, 11\} \cup \{2, 5, 6, 7, 10\}, \{1, 4, 9\}), \]
\[ T_3 = (\{9, 10, 11\} \cup \{4, 5, 6, 7, 8\}, \{1, 2, 3\}), \]
\[ T_4 = (\{1, 9, 10\} \cup \{4, 5, 6, 7, 8\}, \{2, 3, 11\}), \]
\[ T_5 = (\{2, 9, 11\} \cup \{4, 5, 6, 7, 8\}, \{1, 3, 10\}). \]

(3) In Proposition 4.9, take \( n = 5 \) and \( t = 2 \), then \( a = (8, 4) \) and \( \zeta(a)^\gamma = 2^{414} \).

The triples \((h_1, h_2, h_3)\) occurring in the summation of (4.9.1) are \((0, 1, 6), (0, 3, 6), (0, 5, 6)\).

Then \( q_{12}^{5,2} = 2 + 1 + 2 = 5 \), which is just the number of left cells of \( C_5 \) in the set \( E_{2^{14}} \). The set \( C_{12}^{5,2} \) consists of the following tabloids:

\[ T_1 = (\{2, 3, 7, 8\} \cup \{1, 6, 11, 12\}, \{4, 5, 9, 10\}), \]
\[ T_2 = (\{2, 4, 7, 9\} \cup \{1, 6, 11, 12\}, \{3, 5, 8, 10\}), \]
\[ T_3 = (\{1, 4, 7, 10\} \cup \{3, 6, 9, 12\}, \{2, 5, 8, 11\}), \]
\[ T_4 = (\{1, 2, 8, 9\} \cup \{5, 6, 7, 12\}, \{3, 4, 10, 11\}), \]
\[ T_5 = (\{1, 3, 8, 10\} \cup \{5, 6, 7, 12\}, \{2, 4, 9, 11\}). \]

**Remark 4.11.** The results in Propositions 4.7-4.9 can be extended to a more general case. Let \( \lambda = (2l_1, 2l_2, \ldots, 2l_r, 2l_{r+1} + 1, \ldots, 2l_t + 1) \in \Lambda_{m+1} \) for some \( r, t, l_i \in \mathbb{N} \) with \( 1 \leq r < t \) and \( i \in [t] \) (Comparing with the partitions in Lemma 3.4). Then \( a = (a_1, a_2, \ldots, a_{l_i-1}, r, t, a_{l_i-1}, \ldots, a_2, a_1) \in \zeta^{-1}(\lambda^\gamma) \) for some \( 1 \leq a_1 \leq a_2 \leq \cdots \leq a_{l_i-1} \). Let \( C_{m+1}^a \) be the set of all \((m, n)\)-selfdual tabloids in \( \xi^{-1}(a) \) and let \( \gamma_{m+1-2n}(a) = |C_{m+1}^a| \).

**Corollary 4.12.** In the above setup, we have

\[(4.12.1)\]
\[ \gamma_{m+1-2n}(a) = 2^{a_1+\cdots+a_{l_i-1}-n} n! \frac{(n-a_1-\cdots-a_{l_i-1})!}{(n-a_1-\cdots-a_{l_i-1})! \prod_{l_i=1}^{l_i-1} a_l!} \cdot q^{\gamma(m, n)} \cdot q^{\frac{r}{2}}, \]

where \( \epsilon(m, n) = 0 \) if \( m \in \{2n-1, 2n\} \) and \(-1\) if \( m = 2n+1 \).

**Proof.** Let \( a_1 = (a_1, a_2, \ldots, a_{l_i-1}, a_{l_i-1}, \ldots, a_2, a_1) \) and \( a_2 = (t, r) \). Then \( \gamma_{m+1-2n}(a) = \gamma_0(a_1) \gamma_{m+1-2n}(a_2) \frac{n!}{(n-a_1-\cdots-a_{l_i-1})! \prod_{l_i=1}^{l_i-1} a_l!} \) by Theorem 3.7. We have \( \gamma_{m+1-2n}(a_2) = q^{\frac{r}{2}} \) by Propositions 4.7-4.9 and
The cells in the weighted Coxeter group \((\widetilde{C}_n, \widetilde{\ell}_m)\)

\[
\gamma_0(a_1)\left(a_1 + \cdots + a_{i-1}\right) = 2^{a_1+\cdots+a_{i-1}} \frac{n!}{(n-a_1-\cdots-a_{i-1})! \prod_{i=1}^{i-1} a_i!}
\]

by Theorem 3.3. This proves the formula (4.12.1).

4.13. Let \(T = (T_1, T_2, \ldots, T_r)\) and \(T' = (T'_1, T'_2, \ldots, T'_r)\) in \(\mathcal{C}_{m+1}\) satisfy \(|T_1| > |T_2| > \cdots > |T_r|\) and \(|T'_1| < |T'_2| < \cdots < |T'_r|\) and \(T' \approx T\). Then \(|T'_i| = |T_{r+1-i}|\) for any \(i \in [r]\). The tabloid \(T\) is \((m, n)\)-selfdual if and only if \(T'\) is \((m, n)\)-selfdual if and only if \(T'_i = \overline{T_{r+1-i}}\) for any \(i \in [r]\). When the equivalent conditions hold, define a partition \(T_j = T_{j1} \cup T_{j2} \cup \cdots \cup T_{j,r+1-j}\) for any \(j \in [r]\) such that the sets \(T^h_j := T_{j1} \cup T_{j2} \cup \cdots \cup T_{jh}\) for \(j \in [r]\) and \(h \in [r+1-j]\) satisfy the condition \(L_{T_j}(T^h_j) = T^h_j\) for any \(h \in [r-j]\).

4.14. Consider the case of \(r = 3\). Let \(T = (T_1, T_2, T_3)\) and \(T' = (T'_1, T'_2, T'_3)\) be \((m, n)\)-selfdual tabloids of rank \(m + 1\) with \(|T_1| > |T_2| > |T_3|\) and \(|T'_1| < |T'_2| < |T'_3|\) and \(T \approx T'\). We want to describe \(T'\) in terms of \(T\). Define the partitions \(T_1 = T_{11} \cup T_{12} \cup T_{13}\) and \(T_2 = T_{21} \cup T_{22}\) and \(T_3 = T_{31}\) as those in 4.13 with \(r = 3\).

Define

\[
X := (T_{11}, T_{21} \cup T_{12} \cup T_{13}, T_{31} \cup T_{22}) \quad \text{and} \quad Y := (T_{11} \cup T_{12}, T_{21} \cup T_{22} \cup T_{13}, T_{31}).
\]

Then \(X\) is obtained from \(T\) by a \(\{2, 3\}\)-transformation, while \(Y\) is obtained from \(T\) by a \(\{1, 2\}\)-transformation (see 2.13).

So \(X \approx T \approx Y\). We see by Lemma 2.15 that both \(X\) and \(Y\) are \((m, n)\)-selfdual and that \(Y = X^\text{op}\). This implies that \(T_{31} = \overline{T_{11}}\) and \(T_{22} = \overline{T_{12}}\) and \(T_{13} \cup T_{21} = \overline{T_{21}} \cup \overline{T_{13}}\). Denote \(E^0 = \{i \in E \mid \overline{i} \in E\}\) and \(E^1 = E - E^0\) for any \(E \subseteq [m+1]\). Then \(T^1_{13} = \overline{T^1_{21}}\) and \(T' = (T_{11}, T_{21} \cup T_{12} \cup T_{13}, T_{31} \cup T_{22} \cup T^1_{21} \cup T^1_{13})\).

Hence we have

**Proposition 4.15.** For any \(a = (a_1, a_2, a_3) \in \tilde{A}_{m+1}\) with \(a_1 > a_2 > a_3\), a tabloid \(T \in \xi^{-1}(a)\) is \((m, n)\)-selfdual if and only if \(T = (T_{11} \cup T_{12} \cup T_{13}, T_{21} \cup T_{12}, T_{11})\) for some \(T_{11}, T_{12}, T_{13}, T_{21} \subset [m+1]\) satisfying the following conditions:

(i) \(T_{11} = L_{T_{11} \cup T_{12} \cup T_{13}}(T_{21})\);

(ii) \(T_{11} \cup T_{12} = L_{T_{11} \cup T_{12} \cup T_{13}}(T_{21} \cup T_{12})\);
(iii) $T_{21} = L_{T_{21} \cup \overline{T_{12}}} (T_{11})$;
(iv) $T_{11}^0 = T_{12}^0 = \emptyset$ and $T_{13}^1 = \overline{T_{21}}$.

4.16. Next consider the case of $r = 4$. Let

$$T = (T_1, T_2, T_3, T_4) = (T_{11} \cup T_{12} \cup T_{13} \cup T_{14}, T_{21} \cup T_{22} \cup T_{23}, T_{31} \cup T_{32}, T_{41})$$

be defined as in 4.13. By the argument similar to that in 4.14 (of course, more complicated), one can shown that if $T$ is $(m, n)$-selfdual then the following conditions hold:

(i) $T_{41} = T_{11}$ and $T_{32} = T_{12}$.

(ii) There are some partitions $T_{31} = T_{31}' \cup T_{31}''$, $T_{23} = T_{23}' \cup T_{23}''$, $T_{14} = T_{14}' \cup T_{14}''$ and $T_{21} = T_{21}' \cup T_{21}''$ which satisfy (also are determined by) the following conditions:

(iia) $T_{31}' \cup T_{23}' = \overline{T_{13}} = T_{31}' \cup T_{22} \cup T_{23} = L_{T_{21} \cup T_{22} \cup T_{23}} (T_{41} \cup T_{32})$;

(iib) $T_{21}' \cup T_{14}' = \overline{T_{31}'} \cup T_{23}' = L_{T_{21} \cup T_{22} \cup T_{23} \cup T_{14}} (T_{31})$;

(iic) $T_{22}' \cup T_{21}' \cup T_{14}'' = \overline{T_{22}'} \cup T_{21}' \cup T_{14}''$;

(iid) $T_{11}' = T_{12}' = T_{13}' = T_{23}' = T_{31}' = T_{32}' = T_{41}' = \emptyset$.

REFERENCES