THE CELLS IN THE WEIGHTED COXETER GROUP $(\widetilde{C}_n, \widetilde{\ell}_m)$

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Abstract. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_m, \widetilde{S}_m)$, $m \in \{2n-1, 2n, 2n+1\}$, under a certain group automorphism $\alpha_{n,m}$ with $\alpha_{n,m}(\widetilde{S}_m) = \widetilde{S}_m$. Let $\widetilde{\ell}_m$ be the length function of $\widetilde{A}_m$. The present paper is to give some criterion for a left cell of $\widetilde{A}_m$ intersecting $\widetilde{C}_n$ and to use this criterion to deduce some formulae for the number of left cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set E_λ of elements associated to any partition λ of $m+1$.

§0. Introduction.

0.1. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_m, \widetilde{S}_m)$, $m \in \{2n-1, 2n, 2n+1\}$, under a certain automorphism $\alpha_{n,m}$ with $\alpha_{m,n}(\widetilde{S}_m) = \widetilde{S}_m$, where $\widetilde{S}_m$, S are the Coxeter generator sets of $\widetilde{A}_m$, $\widetilde{C}_n$, respectively. The restriction to $\widetilde{C}_n$ of the length function $\widetilde{\ell}_m$ of $\widetilde{A}_m$ is a weight function of $\widetilde{C}_n$. It is known that there is a surjective map ψ from $\widetilde{A}_m$ to the set Λ_{m+1} of partitions of $m+1$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_m$ to Λ_{m+1} (see [6], [3]). For any $i \leqslant j$ in the set $\mathbb{N} := \{0, 1, 2, ...\}$, denote $[i, j] := \{i, i+1, ..., j\}$ and denote $[1, i]$ simply by $[i]$. Let $E_\lambda := \psi^{-1}(\lambda) \cap \widetilde{C}_n$ for $\lambda \in \Lambda_{m+1}$. In the paper [7], we described all the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ corresponding to the partitions $\mathbf{k1^{2n-k}}$ and $\mathbf{h21^{2n-h-2}}$ for all

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$k \in [2n]$ and $h \in [2, 2n-2]$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$.

0.2. Denote by $\lambda^\vee$ the dual partition of $\lambda \in \Lambda_{m+1}$ (see 1.8). Let $\widetilde{\Lambda}_{m+1}$ be the set of all compositions of $m+1$ (see 2.1). There is a natural surjective map ζ from the set $\widetilde{\Lambda}_{m+1}$ to Λ_{m+1} (see 2.1). Call $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$ *symmetric*, if $a_i = a_{r+1-i}$ for any $i \in [r]$. Let $\mathcal{C}_{m+1}$ be the set of all tabloids of rank $m+1$ (see 2.2). We can define an equivalence relation $\approx$ on $\mathcal{C}_{m+1}$ (see 2.13). There exists a bijective map from the set Π^l_m of left cells of $\widetilde{A}_m$ to the set of $\approx$-equivalence classes of $\mathcal{C}_{m+1}$ (see [6, Subsection 19.4]). There exists a natural surjective map ξ from $\mathcal{C}_{m+1}$ to $\widetilde{\Lambda}_{m+1}$ (see 2.2).

0.3. In the present paper, we prove that a left cell Γ of $\widetilde{A}_m$ has a non-empty intersection with $\widetilde{C}_n$ if and only if the $\approx$-equivalence class of $\mathcal{C}_{m+1}$ corresponding to Γ is (m,n)-selfdual (see 2.13-2.14, Lemma 2.15 and Theorem 3.1). By this result, we can deduce some formulae for the number $\gamma_{m+1-2n}(\mathbf{a})$ of left cells of $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set $E_{\zeta(\mathbf{a})^\vee}$ for any $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$. More precisely, we give a close formula for the number $\gamma_{m+1-2n}(\mathbf{a})$ if $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$ is symmetric (Theorem 3.3). For an arbitrary $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$, we have $\gamma_{m+1-2n}(\mathbf{a}) = \gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2)\binom{n}{l}$ for some symmetric $\mathbf{a}_1 = (a_{i_1}, a_{i_2}, ..., a_{i_{2p}}) \in \widetilde{\Lambda}_{2l}$, and some $\mathbf{a}_2 = (a_{j_1}, a_{j_2}, ..., a_{j_q}) \in \widetilde{\Lambda}_{m+1-2l}$, $a_{j_1} > a_{j_2} > \cdots > a_{j_q} > 0$, with some $l \in \mathbb{N}$, where $\binom{n}{l} := \frac{n!}{l!(n-l)!}$ and $\{i_h, j_l \mid h \in [0, 2p], l \in [0, q]\} = [r]$ and the notation $\gamma_k(\mathbf{b})$, $k \in \{0, 1, 2\}$, stands for the numbers of (m,n)-selfdual tabloids $\mathbf{T}$ with $\xi(\mathbf{T}) = \mathbf{b}$ over an (m,n)-selfdual subset of $[m+1]$ containing exactly k (m,n)-selfdual elements (see 3.6 and Theorem 3.7). Hence to calculate the number $\gamma_{m+1-2n}(\mathbf{a})$, we are reduced to the case where $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1-2l}$ satisfies $a_1 > a_2 > \cdots > a_r$ and $l \in \mathbb{N}$. We get a close formula for $\gamma_{m+1-2n}(\mathbf{a})$ in the case of $r = 2$ (see Propositions 4.7-4.9 and Corollary 4.12). Then in the case of $r = 3, 4$, we describe the (m,n)-selfdual tabloids in $\xi^{-1}(\mathbf{a})$ (see Proposition 4.15 and Subsection 4.16).

0.4. The contents of the paper are organized as follows. In Section 1, we collect some concepts and known results concerning cells of the weighted Coxeter groups $(\widetilde{A}_m, \widetilde{\ell}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$. Then we introduce the tabloids of rank $m+1$ in Section 2. In Section 3, we characterize all the tabloids parameterizing the left cells of

$(\widetilde{C}_n, \widetilde{\ell}_m)$ and give some formulae for the number of left cells of $(\widetilde{C}_n, \widetilde{\ell}_m)$ in the set E_λ for any $\lambda \in \Lambda_{m+1}$. Finally, we deduce some more formulae for those numbers and describe the (m, n)-selfdual tabloids in some special cases in Section 4.

§1. The weighted Coxeter groups $(\widetilde{A}_m, \widetilde{S}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$.

In this section, we collect some concepts and results concerning the weighted Coxeter groups $(\widetilde{A}_m, \widetilde{\ell}_m)$ and $(\widetilde{C}_n, \widetilde{\ell}_m)$.

1.1. Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. By a *weight function* on W, we mean a map L from W to the integer set $\mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ in W. Call (W, L) is a *weighted Coxeter group.*

A weighted Coxeter group (W, L) is called in the *split* case if $L = \ell$.

Suppose that there exists a group automorphism $\alpha : W \longrightarrow W$ with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J on S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function $\ell : W \longrightarrow \mathbb{N}$ is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the *quasi-split* case.

1.2. Let $\underset{L}{\leqslant}$ (respectively, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on a weighted Coxeter group (W, L) defined in [5]. The equivalence relation associated to this preorder is denoted by $\underset{L}{\sim}$ (respectively, $\underset{R}{\sim}$, $\underset{LR}{\sim}$). The corresponding equivalence classes in W are called *left cells* (respectively, *right cells, two-sided cells*) of W.

1.3. Lusztig introduced a subset $\mathcal{D}$ of W consisting of certain involutive elements w (hence $w^2 = 1$) in a weighted Coxeter group (W, L) (see [5, Chapter 14]). When (W, L) is a Weyl or affine Weyl group which is either in the split case or in the quasi-split case, Lusztig proved that each left (respectively, right) cell of W contains exactly one element in $\mathcal{D}$ (see [5, Chapters 14–16]). Note that the elements of $\mathcal{D}$ were called *distinguished involutions* when (W, L) is in the split case (see [4]).

1.4. The group $\widetilde{A}_m$, $m \geqslant 1$, can be realized as the following permutation group on the set $\mathbb{Z}$ (see [2, Subsection 3.6] and [6, Subsection 4.1)]:

$$\widetilde{A}_m = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \,\middle|\, (i+m+1)w = (i)w + m + 1, \sum_{i=1}^{m+1} (i)w = \sum_{i=1}^{m+1} i \right\}.$$

The Coxeter generator set $\widetilde{S}_m = \{s_i \mid i \in [0, m]\}$ of $\widetilde{A}_m$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{m+1}, \\ t+1, & \text{if } t \equiv i \pmod{m+1}, \\ t-1, & \text{if } t \equiv i+1 \pmod{m+1}, \end{cases}$$

for any $t \in \mathbb{Z}$ and $i \in [0, m]$. Any $w \in \widetilde{A}_m$ can be realized as a $\mathbb{Z} \times \mathbb{Z}$ monomial matrix $A_w = (a_{ij})_{i,j \in \mathbb{Z}}$, where a_{ij} is 1 if $j = (i)w$ and 0 if otherwise. The row (respectively, column) indices of A_w are increasing from top to bottom (respectively, from left to right).

1.5. For $m \in \{2n-1, 2n, 2n+1\}$, let $\alpha_{m,n} : \widetilde{A}_m \longrightarrow \widetilde{A}_m$ be the group automorphism determined by $\alpha_{m,n}(s_i) = s_{2n-i}$ for $i \in [0, m]$ if $m = 2n-1$ and by $\alpha_{m,n}(s_i) = s_{2n+1-i}$ for $i \in [0, m]$ if $m \in \{2n, 2n+1\}$, where we stipulate $s_{i+m+1} = s_i$ for any $i \in \mathbb{Z}$. In terms of matrix form, for any $w \in \widetilde{A}_m$, the matrix $A_{\alpha_{m,n}(w)}$ can be obtained from the matrix A_w by rotating with the angle π around the point $(n+\frac{1}{2}, n+\frac{1}{2})$ (respectively, $(n+1, n+1)$) if $m = 2n-1$ (respectively, $m \in \{2n, 2n+1\}$), where we identify A_w with a plane and the positions (i, j), $i, j \in \mathbb{Z}$, of A_w are identified with the corresponding integer lattice points. Then $\alpha_{m,n}$ gives rise to a permutation on the set Π^l_m (respectively, Π^r_m, Π^t_m) of left cells (respectively, right cells, two-sided cells) of $\widetilde{A}_m$. Also, $\alpha_{m,n}(\mathcal{D}) = \mathcal{D}$ by the definition of the set $\mathcal{D}$ in [5, Chapter 14] (see 1.3).

1.6. The affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$, under the automorphism $\alpha_{m,n}$, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$\widetilde{C}_n = \{w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i+m+1)w = (i)w+m+1, (i)w+(\epsilon_{m,n}-i)w = \epsilon_{m,n}, \forall\, i \in \mathbb{Z}\},$$

where $\epsilon_{m,n}$ is 1 if $m \in \{2n-1, 2n\}$ and 0 if $m = 2n+1$. The Coxeter generator set $S = \{t_i \mid i \in [0,n]\}$ of $\widetilde{C}_n$ is given by setting $t_i = s_i s_{2n-i}$ for $i \in [n-1]$, $t_0 = s_0$ and $t_n = s_n$ if $m = 2n-1$; $t_i = s_i s_{2n+1-i}$ for $i \in [n-1]$, $t_0 = s_0$ and $t_n = s_n s_{n+1} s_n$ if $m = 2n$; $t_i = s_i s_{2n+1-i}$ for $i \in [n-1]$, $t_0 = s_0 s_1 s_0$ and $t_n = s_n s_{n+1} s_n$ if $m = 2n+1$. In terms of matrix, an element $w \in \widetilde{A}_m$ is in $\widetilde{C}_n$ if and only if the matrix form A_w of w is centrally symmetric at the points $(qn + \frac{1}{2}, qn + \frac{1}{2})$ if $m = 2n-1$ and, at the points $((2n+1)q + \frac{1}{2}, (2n+1)q + \frac{1}{2})$ and $((2n+1)q + (n+1), (2n+1)q + (n+1))$ if $m = 2n$ and, at the points $((n+1)q, (n+1)q)$ if $m = 2n+1$, where q ranges over $\mathbb{Z}$

1.7. By a partition of a positive integer l, we mean an r-tuple $\lambda := (\lambda_1, \lambda_2, ..., \lambda_r)$ of weakly decreasing positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ with $\sum_{k=1}^{r} \lambda_k = l$ for some $r \geqslant 1$. λ_i is called a *part* of λ. We sometimes denote λ in the form $\mathbf{j_1^{k_1} j_2^{k_2} \cdots j_m^{k_m}}$ (boldfaced) with $j_1 > j_2 > \cdots > j_m \geqslant 1$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$ for $i \geqslant 1$. Let Λ_l be the set of all partitions of l. For example, $\mathbf{63^3 1^2}$ stands for the partition $(6,3,3,3,1,1)$ of 17.

Fix $w \in \widetilde{A}_m$. For any $i \neq j$ in $[m+1]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both inequalities $p(m+1)+i > q(m+1)+j$ and $(p(m+1)+i)w < (q(m+1)+j)w$ hold. This defines a partial order $\preceq_w$ on the set $[m+1]$. $i \neq j$ in $[m+1]$ are said *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* if otherwise.

A sequence $a_1, a_2, ..., a_r$ in $[m+1]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. Sometimes we identify a w-chain $a_1, a_2, ..., a_r$ with the corresponding set $\{a_1, a_2, ..., a_r\}$. For any $k \geqslant 1$, a k-w-chain-family is by definition a union $X = \cup_{i=1}^{k} X_i$ of k w-chains $X_1, X_2, ..., X_k$ in $[m+1]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = m+1$. Let $\lambda_1 = d_1$ and $\lambda_{k+1} = d_{k+1} - d_k$ for $k \in [r-1]$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of Curtis Greene in [1]. Hence $w \mapsto \psi(w) := (\lambda_1, \lambda_2, ..., \lambda_r)$ defines a map from the set $\widetilde{A}_m$ to Λ_{m+1}.

1.8. Let $\widetilde{\ell}_m$ be the length function on the Coxeter systems $(\widetilde{A}_m, \widetilde{S}_m)$. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_m, \widetilde{\ell}_m)$ is in the split case, while $(\widetilde{C}_n, \widetilde{\ell}_m)$ is in the quasi-split case.

For any $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{m+1}$, define $\lambda^\vee = (\lambda_1^\vee, ..., \lambda_t^\vee) \in \Lambda_{m+1}$ by setting $\lambda_j^\vee = {}^\#\{k \in [r] \mid \lambda_k \geqslant j\}$ for any $j \geqslant 1$, call $\lambda^\vee$ the *dual partition* of λ.

Lemma 1.9. *(1) Regarding $\widetilde{C}_n$ as a subset of $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$. For any $x, y \in \widetilde{C}_n$, we have $x \underset{L}{\sim} y$ in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ in $\widetilde{A}_m$* (see [5, Lemma 16.14]).

(2) The set $\psi^{-1}(\lambda)$ forms a two-sided cell of $\widetilde{A}_m$ for any $\lambda \in \Lambda_{m+1}$ (see [3, Theorem 6] and [6, Theorem 17.4]).

By Lemma 1.9 (1), we can just use the notation $x \underset{L}{\sim} y$ for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to $\widetilde{A}_m$, $m \in \{2n-1, 2n, 2n+1\}$, or $\widetilde{C}_n$.

For any $\lambda \in \Lambda_{m+1}$, denote $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

In the remaining part of the paper, when we mention the number m, we always assume $m \in \{2n-1, 2n, 2n+1\}$ unless otherwise specified.

§2. Tabloids of rank $m+1$.

In the present section, we introduce the concept of tabloids of rank $m+1$ which will be used to parametrize the left cells of $\widetilde{A}_m$ and of $\widetilde{C}_n$.

2.1. By a *composition* of $m+1$, we mean an r-tuple $\mathbf{a} = (a_1, a_2, ..., a_r)$ of positive integers $a_1, ..., a_r$ with some $r \in \mathbb{N}$ such that $\sum_{i=1}^r a_i = m+1$. Let $\widetilde{\Lambda}_{m+1}$ be the set of all compositions of $m+1$. Clearly, $\Lambda_{m+1} \subseteq \widetilde{\Lambda}_{m+1}$. For any $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$, let $i_1, i_2, ..., i_r$ be a permutation of $1, 2, ..., r$ with $a_{i_1} \geqslant a_{i_2} \geqslant \cdots \geqslant a_{i_r}$. Denote $\zeta(\mathbf{a}) = (a_{i_1}, a_{i_2}, ..., a_{i_r})$. This defines a surjective map $\zeta : \widetilde{\Lambda}_{m+1} \longrightarrow \Lambda_{m+1}$.

2.2. A (generalized) tabloid of rank $m+1$ is, by definition, an r-tuple $\mathbf{T} = (T_1, T_2, ..., T_r)$ with some $r \in \mathbb{N}$ such that $[m+1]$ is a disjoint union of some non-empty subsets T_j, $j \in [r]$. We have $\xi(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|) \in \widetilde{\Lambda}_{m+1}$, where $|T_i|$ denotes the cardinal of the set T_i. Two tabloids $\mathbf{T} = (T_1, ..., T_r)$ and $\mathbf{T}' = (T_1', ..., T_t')$ are said *equal* if $r = t$ and $T_i = T_i'$ for any $i \in [r]$. Let $\mathcal{C}_{m+1}$ be the set of all tabloids of rank $m+1$. Then $\xi : \mathcal{C}_{m+1} \longrightarrow \widetilde{\Lambda}_{m+1}$ is a surjective map. Let $\kappa = \zeta\xi : \mathcal{C}_{m+1} \longrightarrow \Lambda_{m+1}$.

2.3. Let Ω be the set of all $w \in \widetilde{A}_m$ such that for any $w \in \Omega$, there is some $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_{m+1}$ satisfying that

(i) For any $i < j$ in $[r]$, we have $a \prec_w b$ for any $a \in T_i$ and $b \in T_j$;

(ii) For any $i \in [r]$, a and b are w-uncomparable for any $a \neq b$ in T_i.

Clearly, the tabloid $\mathbf{T}$ is determined entirely by $w \in \Omega$, denote $\mathbf{T}$ by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{m+1}$ is surjective by [6, Proposition 19.1.2]. By a result of Curtis Greene in [1], we have $\kappa(T(w)) = \psi(w)^\vee$.

Lemma 2.4. (see [6, Proposition 19.1.2 and Lemma 19.4.6]) *Suppose that $y, w \in \widetilde{A}_m$ are two elements in Ω with $\xi(T(y)) = \xi(T(w))$. Then $y \underset{L}{\sim} w$ if and only if $T(y) = T(w)$.*

2.5. By Lemma 2.4, it makes sense to write $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$ in $\mathcal{C}_{m+1}$ if there exist some $x, y \in \Omega$ satisfying $x \underset{L}{\sim} y$ and $T(x) = \mathbf{T}$ and $T(y) = \mathbf{T}'$. This defines an equivalence relation on $\mathcal{C}_{m+1}$.

Fix $w \in \widetilde{A}_m$ and let $\lambda = \psi(w)$. Take any $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$. There exists some $y \in \Omega$ with $y \underset{L}{\sim} w$ and $\xi(T(y)) = \mathbf{a}$. The tabloid $T(y)$ is uniquely determined by the element w and the composition $\mathbf{a}$ of $m+1$, denote it by $T_{\mathbf{a}}(w)$ (see [6, Propositions 19.1.2, 19.4.7 and 19.4.8]).

Lemma 2.6. *(see* [6, Propositions 19.4.7-19.4.8]*) In the above setup, $T_{\mathbf{a}}$ gives rise to a surjective map from the set $\psi^{-1}(\lambda)$ to $\xi^{-1}(\mathbf{a})$, which induces a bijection (again denoted by $T_{\mathbf{a}}$) from the set Π^l_λ of left cells of $\widetilde{A}_m$ in $\psi^{-1}(\lambda)$ to $\xi^{-1}(\mathbf{a})$.*

2.7. For further discussion on the left cells of $\widetilde{A}_m$ and $\widetilde{C}_n$, we need to recall some more concepts involving tabloids of rank $m + 1$ (see [6, Chapter 20]). Let k be a positive integer. Arrange the numbers $1, 2, ..., k$ on a circle in the following way: in the clockwise direction, $t + 1$ is the successor of t for any $t \in [k - 1]$ and 1 is the successor of k. We call such a circle the *k-circle*. For example, the following is the 8-circle.

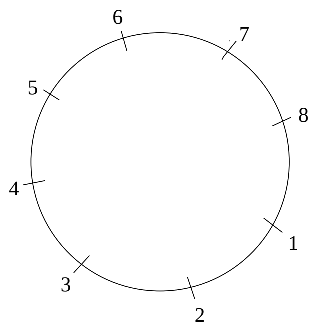

Figure 1

For $x \neq y$ in $[k]$, we denote by $\widehat{xy}$ the arc of the k-circle which, starting with the number x and moving clockwise, ends with the number y. For $Z \subseteq [k]$, let Z_{xy} be the set of all numbers of Z on $\widehat{xy}$. Take the 8-circle in Figure 1 as an example, let $Z = \{1, 2, 3, 4, 6\}$, $x = 2$, $y = 5$. Then $Z_{xy} = \{2, 3, 4\}$ and $Z_{yx} = \{1, 2, 6\}$.

Let $X = \{a_j \mid j \in [t], a_1 < \cdots < a_t\}$ and $Y = \{b_j \mid j \in [r], b_1 < \cdots < b_r\}$ be two subsets of $[k]$ with $X \cap Y = \emptyset$ and $t \leqslant r$.

(i) We define a subset $H_Y(X) = \{c_1, ..., c_t\}$ of Y as follows. Define $c_1 \in Y$ by the condition $|Y_{a_1c_1} \cap Y| = 1$. Define $c_2 \in Y$ by the condition $|Y_{a_2c_2} \cap (Y - \{c_1\})| = 1$. In general, suppose that $c_1, ..., c_{h-1}$ have been defined for $h \in [t]$. Then $c_h \in Y$ is defined by the condition $|Y_{a_hc_h} \cap (Y - \{c_1, ..., c_{h-1}\})| = 1$. Clearly, the set $H_Y(X)$ is well defined. In particular, $H_Y(X) = Y$ if $|X| = |Y|$.

(ii) We define a subset $L_Y(X) = \{d_1, ..., d_t\}$ of Y as follows. Define $d_1 \in Y$ by the condition $|Y_{d_1a_t} \cap Y| = 1$. Define $d_2 \in Y$ by the condition $|Y_{d_2a_{t-1}} \cap (Y - \{d_1\})| = 1$. Recurrently, define $d_h \in Y$ by the condition $|Y_{d_ha_{t+1-h}} \cap (Y - \{d_1, ..., d_{h-1}\})| = 1$ for any $h \in [t]$.

By the above definition, we see that the sets $H_Y(X)$ and $L_Y(X)$ depend only on the relative positions of the elements of $X \cup Y$ on the k-circle, but neither on the positive integer k and nor on any of those integers in the set $[k] - X \cup Y$.

The following result can be checked directly from the above definition.

Lemma 2.8. *Let k be a positive integer. If η is a permutation on $[k]$ such that $\eta(i+1) \equiv \eta(i) - 1 \pmod k$ for any $i \in [k]$ (hence the order of the numbers $1, 2, ..., k$ on the k-circle are reversed by η) then $\eta(H_Y(X)) = L_{\eta(Y)}(\eta(X))$ and*

$\eta(L_Y(X)) = H_{\eta(Y)}(\eta(X))$ *for any* $X, Y \subseteq [k]$ *with* $|Y| \geqslant |X|$ *and* $Y \cap X = \emptyset$.

Take the 8-circle in Figure 1 as an example. Let $X = \{1, 4\}$ and $Y = \{2, 6, 7\}$. Then $H_Y(X) = \{2, 6\}$ and $L_Y(X) = \{2, 7\}$. Define $\eta : [8] \longrightarrow [8]$ by setting $\eta(i) = 9 - i$ for any $i \in [8]$. Then $\eta(H_Y(X)) = \{3, 7\} = L_{\{2,3,7\}}(\{5, 8\}) = L_{\eta(Y)}(\eta(X))$.

The following results describe the sets $H_Y(X)$ and $L_Y(X)$ in more intrinsic way.

Lemma 2.9. (see [6, Lemmas 20.1.2-20.1.3]) *For a positive integer* k, *take* $X, Y \subseteq [k]$ *satisfying* $X \cap Y = \emptyset$ *and* $|X| \leqslant |Y|$. *Then for any* $y \in Y$, *we have*

(a) $y \in H_Y(X)$ *if and only if there exists some* $x \in X$ *satisfying* $|Y_{xy}| = |X_{xy}|$.

(b) $y \in L_Y(X)$ *if and only if there exists some* $x \in X$ *satisfying* $|Y_{yx}| = |X_{yx}|$.

2.10. For $i, j \in [m + 1]$, we say that j is the (m, n)-*dual* of i, denote $j = \bar{i}$, if either $m = 2n - 1$ and $i + j = 2n + 1$, or $m \in \{2n, 2n + 1\}$ and $i + j \equiv 2n + 2$ (mod $2n + 2$); in this case, we also have $i = \bar{j}$, and call $i, \bar{i}$ *an* (m, n)-*dual pair*. Denote $\overline{E} = \{\bar{i} \mid i \in E\}$ for any $E \subseteq [m+1]$ (The notation $\bar{i}$, $\overline{E}$ for $i \in [m+1]$ and $E \subseteq [m + 1]$ will cause no confusion in the context since the pair (m, n) is fixed in each case).

For any $i \in [m + 1]$, we have $i = \bar{i}$ if and only if either $m = 2n$ and $i = n + 1$, or $m = 2n + 1$ and $i \in \{n + 1, 2n + 2\}$. When the equivalent conditions hold, i with itself forms an (m, n)-dual pair, call i an (m, n)-*selfdual element*. Hence the number of (m, n)-selfdual elements in $[m + 1]$ is $m + 1 - 2n$.

Next result shows that for any $Y \subseteq [m + 1]$, the operations H_Y and L_Y on $X \subseteq [m + 1]$ with $|X| \leqslant |Y|$ and $X \cap Y = \emptyset$ are inverse to each other in some sense.

Lemma 2.11. *Let* $X, Y \subseteq [m + 1]$ *satisfy* $X \cap Y = \emptyset$ *and* $|X| \leqslant |Y|$.

(a) Let $Y' = H_Y(X)$ *and* $X' = X \cup (Y - H_Y(X))$. *Then* $X = L_{X'}(Y')$ *and* $Y = Y' \cup (X' - L_{X'}(Y'))$.

(b) Let $Y'' = L_Y(X)$ *and* $X'' = X \cup (Y - L_Y(X))$. *Then* $X = H_{X''}(Y'')$ *and* $Y = Y'' \cup (X'' - H_{X''}(Y''))$.

(c) $\overline{H_Y(X)} = L_{\overline{Y}}(\overline{X})$ *and* $\overline{L_Y(X)} = H_{\overline{Y}}(\overline{X})$.

Proof. (a) and (b) are just the results in [6, Proposition 20.1.4]. Then (c) follows by Lemma 2.8. □

Recall the relation $\underset{L}{\sim}$ on $\mathcal{C}_{m+1}$ defined in 2.5.

Proposition 2.12. (see [6, Proposition 20.2.2 and Corollary 20.2.3]) *Let* $\mathbf{T} = (T_1, ..., T_t) \in \mathcal{C}_{m+1}$ *and* $j \in [t-1]$.

(a) If $|T_j| \leqslant |T_{j+1}|$, *let*

$$\mathbf{T}' = (T_1, ..., T_{j-1}, T_j \cup (T_{j+1} - H_{T_{j+1}}(T_j)), H_{T_{j+1}}(T_j), T_{j+2}, ..., T_t) \tag{2.12.1}$$

then $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$.

(b) If $|T_j| \geqslant |T_{j+1}|$, *let*

$$\mathbf{T}'' = (T_1, ..., T_{j-1}, L_{T_j}(T_{j+1}), T_{j+1} \cup (T_j - L_{T_j}(T_{j+1})), T_{j+2}, ..., T_t). \tag{2.12.2}$$

Then $\mathbf{T} \underset{L}{\sim} \mathbf{T}''$.

2.13. Let $\mathbf{T}, \mathbf{T}', \mathbf{T}'' \in \mathcal{C}_{m+1}$ be given as in (2.12.1)-(2.12.2). We say that $\mathbf{T}'$ (respectively, $\mathbf{T}''$) is obtained from $\mathbf{T}$ by a $\{j, j+1\}$-*transformation*. This definition does not cause any confusion since $\mathbf{T}'$ (respectively, $\mathbf{T}''$) is defined only when $|T_j| \leqslant |T_{j+1}|$ (respectively, $|T_j| \geqslant |T_{j+1}|$). Note that if $|T_j| = |T_{j+1}|$ then $\mathbf{T}' = \mathbf{T}'' = \mathbf{T}$.

Fix E with $\emptyset \neq E \subseteq [m+1]$. Let $\mathcal{C}_E$ be the set of all tabloids $\mathbf{T} = (T_1, T_2, ..., T_r)$ with $E = \dot{\cup}_{i=1}^r T_i$ (hence $\mathcal{C}_{m+1} = \mathcal{C}_{[m+1]}$).

For any $\mathbf{T}, \mathbf{T}' \in \mathcal{C}_E$, written $\mathbf{T} \approx \mathbf{T}'$, if there exists a sequence $\mathbf{T}_0 = \mathbf{T}$, $\mathbf{T}_1$,..., $\mathbf{T}_r = \mathbf{T}'$ in $\mathcal{C}_E$ such that for every $i \in [r]$, $\mathbf{T}_i$ can be obtained from $\mathbf{T}_{i-1}$ by an $\{h_i, h_i + 1\}$-transformation for some integer h_i. This defines an equivalence relation on the set $\mathcal{C}_E$.

Let $l = |E|$ and $\xi_E(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|)$ for any $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_E$. Then $\xi_E : \mathcal{C}_E \longrightarrow \widetilde{\Lambda}_l$ is a surjective map.

2.14. Take E with $\emptyset \neq E \subseteq [m+1]$ and $\overline{E} = E$. Denote $\overline{\mathbf{T}} = (\overline{T_1}, \overline{T_2}, ..., \overline{T_r})$ and $\mathbf{T}^{\mathrm{op}} = (T_r, ..., T_2, T_1)$ for any $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_E$. Then $\overline{\mathbf{T}}, \mathbf{T}^{\mathrm{op}} \in \mathcal{C}_E$. We say that $\mathbf{T} \in \mathcal{C}_E$ is (m, n)-*selfdual*, if $\overline{\mathbf{T}}^{\mathrm{op}} \approx \mathbf{T}$.

Denote $\mathbf{a}^{\mathrm{op}} = (a_r, ..., a_2, a_1)$ for $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$. Call $\mathbf{a}$ *symmetric*, if $\mathbf{a}^{\mathrm{op}} = \mathbf{a}$.

Lemma 2.15. *Let $\mathbf{T}, \mathbf{T}' \in \mathcal{C}_{m+1}$.*

(1) $\mathbf{T} \underset{L}{\sim} \mathbf{T}'$ if and only if $\mathbf{T} \approx \mathbf{T}'$.

(2) When $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$ is symmetric, $\mathbf{T} \in \xi^{-1}(\mathbf{a})$ is (m,n)-selfdual if and only if $\overline{\mathbf{T}}^{\rm op} = \mathbf{T}$.

(3) If $\mathbf{T} \approx \mathbf{T}'$, then $\mathbf{T}$ is (m,n)-selfdual if and only if so is $\mathbf{T}'$.

Proof. (1) follows by Proposition 2.12, Lemmas 2.4 and 2.6. For (2), by the assumption of $\mathbf{a}$ being symmetric, we have $\xi(\overline{\mathbf{T}}^{\rm op}) = \mathbf{a}$ for any $\mathbf{T} \in \xi^{-1}(\mathbf{a})$. So $\overline{\mathbf{T}}^{\rm op} \approx \mathbf{T}$ if and only if $\overline{\mathbf{T}}^{\rm op} = \mathbf{T}$ by Lemma 2.4. This implies (2). For (3), let $\mathbf{T} = (T_1, T_2, ..., T_r)$. We may assume without loss of generality that $\mathbf{T}'$ is obtained from $\mathbf{T}$ by an $\{i, i+1\}$-transformation for some $i \in [r-1]$. Then $\overline{\mathbf{T}}^{\rm op}$ can be obtained from $\overline{\mathbf{T}'}^{\rm op}$ by an $\{r-i, r+1-i\}$-transformation by Lemma 2.11 (c). This implies that $\mathbf{T} \approx \overline{\mathbf{T}}^{\rm op}$ if and only if $\mathbf{T}' \approx \overline{\mathbf{T}'}^{\rm op}$. Hence (3) follows. □

By Lemma 2.15 (3), we can call an $\approx$-equivalence class of $\mathcal{C}_{m+1}$ *(m,n)-selfdual* if some (hence all) tabloids in this class are (m,n)-selfdual.

§3. A formula for the number of left cells of $\widetilde{C}_n$ in the set E_λ, $\lambda \in \Lambda_{m+1}$.

In the present section, we first characterize all the tabloids of rank $m+1$ which correspond to the left cells of $\widetilde{C}_n$. Applying this result, we deduce a formula for the number of left cells of $\widetilde{C}_n$ in the set E_λ for any $\lambda \in \Lambda_{m+1}$.

Theorem 3.1. *Let $\lambda \in \Lambda_{m+1}$ and $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$. Then for any $\Gamma \in \Pi_\lambda^l$ (see Lemma 2.6), we have $\Gamma \cap \widetilde{C}_n \neq \emptyset$ if and only if $T_{\mathbf{a}}(\Gamma)$ is (m,n)-selfdual.*

Proof. The automorphism $\alpha := \alpha_{m,n}$ of $\widetilde{A}_m$ stabilizes the set Ω (see 2.3). We have $T(\alpha(w)) = \overline{T(w)}^{\rm op}$ for any $w \in \Omega$ (see the matrix description for the action of α on $\widetilde{A}_m$ in 1.5). This implies $T_{\mathbf{a}^{\rm op}}(\alpha(\Gamma)) = \overline{T_{\mathbf{a}}(\Gamma)}^{\rm op}$ for any $\Gamma \in \Pi_\lambda^l$. Hence by Lemmas 2.6, 2.15 and Proposition 2.12, we see that

$(*)$ $\alpha(\Gamma) = \Gamma \Longleftrightarrow T_{\mathbf{a}}(\Gamma)$ is (m,n)-selfdual.

First assume $\Gamma \cap \widetilde{C}_n \neq \emptyset$. Then $\alpha(\Gamma) \cap \Gamma \neq \emptyset$, hence $\alpha(\Gamma) = \Gamma$ since both Γ and $\alpha(\Gamma)$ are left cells of $\widetilde{A}_m$. This implies that $T_{\mathbf{a}}(\Gamma)$ is (m,n)-selfdual by $(*)$. Next assume that $T_{\mathbf{a}}(\Gamma)$ is (m,n)-selfdual. Then $\alpha(\Gamma) = \Gamma$ by $(*)$. Recall the set $\mathcal{D}$ mentioned in 1.3. The set $\Gamma \cap \mathcal{D}$ consists of a single element (say d) by 1.3.

Then $\alpha(d) \in \alpha(\Gamma) \cap \mathcal{D}$ by the fact $\alpha(\mathcal{D}) = \mathcal{D}$ (see 1.5). This implies $d = \alpha(d)$ by the equation $\alpha(\Gamma) = \Gamma$ and the fact $|\Gamma \cap \mathcal{D}| = 1$ (by 1.3), i.e., $d \in \Gamma \cap \widetilde{C}_n$. Hence $\Gamma \cap \widetilde{C}_n \neq \emptyset$. $\square$

3.2. Suppose that $\emptyset \neq E \subseteq [m+1]$ and $\overline{E} = E$. For any $\mathbf{b} \in \widetilde{\Lambda}_{|E|}$, let $\gamma_E(\mathbf{b})$ be the number of all (m,n)-selfdual tabloids in $\xi_E^{-1}(\mathbf{b})$ (see 2.13). Under the conditions assumed on E, we see that the number $\gamma_E(\mathbf{b})$ depends only on $|E|$ and the number of (m,n)-selfdual elements contained in E, but not on a particular choice of a subset E in $[m+1]$. Since $|E|$ is determined by $\mathbf{b}$, we may write $\gamma_E(\mathbf{b})$ by $\gamma_k(\mathbf{b})$ if the number of (m,n)-selfdual elements contained in E is k.

Note that the number of (m,n)-selfdual elements in $[m+1]$ is $m+1-2n$ (2.10).

We have not yet found any efficient way to calculate the number $\gamma_k(\mathbf{a})$ in general. However, there is a simple formula for $\gamma_{m+1-2n}(\mathbf{a})$ when $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$ is symmetric (see 2.14).

Theorem 3.3. *Suppose that $\mathbf{a} = (a_1, ..., a_r) \in \widetilde{\Lambda}_{m+1}$ is symmetric. Then*

(3.3.1)
$$\gamma_{m+1-2n}(\mathbf{a}) = \begin{cases} 0, & \text{if } m = 2n+1 \text{ and } r = 2l, \\ 2^{a_1+\cdots+a_l} \dfrac{n!}{\left(n - \sum_{k=1}^{l} a_k\right)! \prod_{k=1}^{l} a_k!}, & \text{if otherwise,} \end{cases}$$

where $l \in \mathbb{N}$.

Proof. Any (m,n)-selfdual tabloid $\mathbf{T} = (T_1, T_2, ..., T_r) \in \xi^{-1}(\mathbf{a})$ is determined entirely by its first l components if $r \in \{2l, 2l+1\}$ with $l \in \mathbb{N}$ by the facts that $T_i = \overline{T_{r+1-i}}$ for any $i \in [l]$ and that $T_{l+1} = [m+1] - \bigcup_{i=1}^{l}(T_i \cup \overline{T_i})$ is a union of some (m,n)-dual pairs (see 2.10) if $r = 2l+1$ is odd. If $m = 2n+1$ and $r = 2l$ then the (m,n)-selfdual elements $n+1, 2n+2$ can not be in T_i for any $i \in [2l]$ and hence $\gamma_{m+1-2n}(\mathbf{a}) = 0$. If $m = 2n$ then the number r must be odd as $m+1$ is odd. If $r = 2l+1$ is odd then any (m,n)-selfdual elements, whenever they exist, must be in T_{l+1}. Since the elements of $\cup_{i=1}^{l} T_i$ are pairwise not (m,n)-dual and none of them is (m,n)-selfdual, the number of the choices for T_1 is $2^{a_1}\binom{n}{a_1}$. Recurrently, when $T_1, T_2, ..., T_{h-1}$ have been chosen for $h \in [l]$, the number of the choices for

T_h is $2^{a_h}\binom{n-a_1-\cdots-a_{h-1}}{a_h}$. We have $n = a_1 + \cdots + a_l$ if $m = 2n-1$ and $r = 2l$. This proves the formula (3.3.1). □

Next result gives a necessary and sufficient condition on $\lambda \in \Lambda_{m+1}$ that there is some symmetric $\mathbf{a}$ in $\zeta^{-1}(\lambda^\vee)$.

Lemma 3.4. *Let $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{m+1}$.*

(1) There exists some symmetric $\mathbf{a}$ in $\zeta^{-1}(\lambda^\vee)$ if and only if λ satisfies the condition (3.4.1) below.

(3.4.1) λ_i is odd and λ_j is even for some $k \in [0, r]$ and any i, j, $1 \leqslant i \leqslant k < j \leqslant r$.

(2) When the condition (3.4.1) holds, the set E_λ is empty if and only if $m = 2n+1$ and $k = 0$.

Proof. The proof for (1) is straightforward. Then (2) follows by Theorem 3.3. □

Example 3.5. Let $\lambda = \mathbf{97642}$. Then $\lambda^\vee = \mathbf{5^2 4^2 3^2 2 1^2}$. The composition $\mathbf{a} = (5, 4, 3, 1, 2, 1, 3, 4, 5) \in \zeta^{-1}(\lambda^\vee)$ is symmetric.

3.6. Assume that $\lambda \in \Lambda_{m+1}$ satisfies the condition (3.4.1). By Theorems 3.1, 3.3 and Lemmas 3.4, 2.6, we see that for any symmetric $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$, the number of left cells of $\widetilde{C}_n$ in E_λ is equal to $\gamma_{m+1-2n}(\mathbf{a})$, which can be computed by the formula (3.3.1).

Next we consider the number of left cells of $\widetilde{C}_n$ in E_λ for an arbitrary $\lambda \in \Lambda_{m+1}$.

For any $\lambda \in \Lambda_{m+1}$, let $\lambda^\vee = \mathbf{b_1^{k_1} b_2^{k_2} \cdots b_r^{k_r}}$. Write $k_i = 2l_i + p_i$ for any $i \in [r]$, where $l_i \in \mathbb{N}$ and $p_i \in \{0, 1\}$. Define $q_1 < q_2 < \cdots < q_u$ in $\mathbb{N}$ by the condition $\{q_j \mid j \in [u]\} = \{i \in [r] \mid p_i = 1\}$ for some $u \in \mathbb{N}$. Take $\mathbf{a} \in \zeta^{-1}(\lambda^\vee)$ as follows.

(3.6.1)

$$\mathbf{a} = (\underbrace{b_1, ..., b_1}_{l_1}, \underbrace{b_2, ..., b_2}_{l_2}, ..., \underbrace{b_r, ..., b_r}_{l_r}, b_{q_1}, b_{q_2}, ..., b_{q_u}, \underbrace{b_r, ..., b_r}_{l_r}, ..., \underbrace{b_2, ..., b_2}_{l_2}, \underbrace{b_1, ..., b_1}_{l_1}).$$

Define

$$(3.6.2)\quad \mathbf{a}_1 = (\underbrace{b_1, ..., b_1}_{l_1}, \underbrace{b_2, ..., b_2}_{l_2}, ..., \underbrace{b_r, ..., b_r}_{l_r}, \underbrace{b_r, ..., b_r}_{l_r}, ..., \underbrace{b_2, ..., b_2}_{l_2}, \underbrace{b_1, ..., b_1}_{l_1}),$$

$$(3.6.3)\quad \mathbf{a}_2 = (b_{q_1}, b_{q_2}, ..., b_{q_u}).$$

We have

Theorem 3.7. *Let $\lambda \in \Lambda_{m+1}$ be given as in 3.6, and let $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$, $\mathbf{a}_1 \in \widetilde{\Lambda}_{2l}$ and $\mathbf{a}_2 \in \widetilde{\Lambda}_{m+1-2l}$ be obtained from λ as in (3.6.1)-(3.6.3), respectively, where $l = \sum_{i=1}^{r} l_i b_i$. Then*

$$\gamma_{m+1-2n}(\mathbf{a}) = \binom{n}{l}\gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2). \tag{3.7.1}$$

Proof. Let $p = \sum_{i=1}^{r} l_i$. For any

$$\mathbf{T} = (T_1, T_2, ..., T_p, T_{p+1}, ..., T_{p+u}, T_{p+u+1}, ..., T_{2p+u}) \in \xi^{-1}(\mathbf{a}),$$

let

$$\mathbf{T}_1 = (T_1, T_2, ..., T_p, T_{p+u+1}, T_{p+u+2}, ..., T_{2p+u}) \text{ and } \mathbf{T}_2 = (T_{p+1}, T_{p+2}, ..., T_{p+u})$$

and $E = [m+1] - \bigcup_{i=p+1}^{p+u} T_i$. Then $|E| = 2l$ and $\mathbf{T}_1 \in \xi_E^{-1}(\mathbf{a}_1)$ and $\mathbf{T}_2 \in \xi_{[m+1]-E}^{-1}(\mathbf{a}_2)$. We see by Lemma 2.15 that $\mathbf{T}$ is (m,n)-selfdual if and only if both $\mathbf{T}_1$ and $\mathbf{T}_2$ are (m,n)-selfdual. When the equivalent conditions hold, we have $\overline{E} = E$ again by Lemma 2.15. For any $k \in [n]$, denote by $[m+1]_{2k}$ the set of all $E \subseteq [m+1]$ with $|E| = 2k$ and $\overline{E} = E$ such that E contains no (m,n)-selfdual element. For any $E \in [m+1]_{2l}$, let $\mathcal{C}_E^{\mathbf{a}}$ be the set of all (m,n)-selfdual

$$\mathbf{T}' = (T'_1, T'_2, ..., T'_p, T'_{p+1}, ...T'_{p+u}, T'_{p+u+1}, ..., T'_{2p+u}) \in \xi^{-1}(\mathbf{a})$$

with $E = [m+1] - \bigcup_{i=p+1}^{p+u} T'_i$. Then

$$\gamma_{m+1-2n}(\mathbf{a}) = |[m+1]_{2l}| \cdot |\mathcal{C}_E^{\mathbf{a}}| = \binom{n}{l}|\mathcal{C}_E^{\mathbf{a}}| \quad \text{for any fixed } E \in [m+1]_{2l}.$$

$\mathbf{T} \mapsto (\mathbf{T}_1, \mathbf{T}_2)$ is a bijective map from the set $\mathcal{C}_E^{\mathbf{a}}$ to the Cartesian product $\mathcal{C}_E^{\mathbf{a}_1} \times \mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}$, where $\mathcal{C}_E^{\mathbf{a}_1}$, $\mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}$ are the sets of all (m,n)-selfdual tabloids in $\xi_E^{-1}(\mathbf{a}_1)$, $\xi_{[m+1]-E}^{-1}(\mathbf{a}_2)$, respectively. This proves the formula (3.7.1) by the facts $\gamma_0(\mathbf{a}_1) = |\mathcal{C}_E^{\mathbf{a}_1}|$ and $\gamma_{m+1-2n}(\mathbf{a}_2) = |\mathcal{C}_{[m+1]-E}^{\mathbf{a}_2}|$ for any $E \in [m+1]_{2l}$. □

§4. Enumeration of some special tabloids in $\mathcal{C}_{m+1}$.

For any $\mathbf{a} \in \widetilde{\Lambda}_{m+1}$, let $\mathcal{C}_{m+1}^{\mathbf{a}}$ be the set of all (m,n)-selfdual tabloids $\mathbf{T}$ in $\xi^{-1}(\mathbf{a})$. We want to formulate the number $\gamma_{m+1-2n}(\mathbf{a}) := |\mathcal{C}_{m+1}^{\mathbf{a}}|$. By Theorems 3.3 and 3.7, it is enough to consider the case where $\mathbf{a} = (a_1, a_2, ..., a_r) \in \widetilde{\Lambda}_{m+1}$ satisfies $a_1 > a_2 > \cdots > a_r$ for some $r > 1$.

First consider the case of $r = 2$.

Lemma 4.1. *Let* $\mathbf{T} = (Y, X) \in \mathcal{C}_{m+1}$ *satisfy* $|Y| \geqslant |X|$. *Then* $\mathbf{T}$ *is* (m, n)*-selfdual if and only if* $L_Y(X) = \overline{X}$ *and* $Y - L_Y(X) = \overline{Y - L_Y(X)}$.

Proof. Let $X' = L_Y(X)$ and $Y' = X \cup (Y - L_Y(X))$. Then $(Y, X) \approx (X', Y')$. So

$$\begin{aligned}\mathbf{T} \text{ is } (m,n)\text{-selfdual} &\iff X' = \overline{X} \text{ and } Y' = \overline{Y} \\ &\iff L_Y(X) = \overline{X} \text{ and } X \cup (Y - L_Y(X)) = \overline{Y} \\ &\iff L_Y(X) = \overline{X} \text{ and } Y - L_Y(X) = \overline{Y - L_Y(X)}.\end{aligned}$$

The last equivalence follows by the facts that

$$Y = L_Y(X) \cup (Y - L_Y(X)) \quad \text{and} \quad \overline{X \cup (Y - L_Y(X))} = \overline{X} \cup \overline{Y - L_Y(X)}. \qquad \square$$

4.2. First assume $m = 2n - 1$. Hence $m + 1 - 2n = 0$ and $\overline{i} := 2n + 1 - i$ for any $i \in [2n]$. Define an admissible subsequence α in each of the following sequences β (note that β has even number of terms in $[2n]$).

(a) Consider the sequence $\beta_{n,q} : \bar{n}, \overline{n-1}, ..., \overline{q+1}, q+1, ..., n-1, n$ for any $q \in [0, n-1]$. A subsequence $\alpha : i_1, i_2, ..., i_r$ of $\beta_{n,q}$ is called *admissible*, if the following two conditions hold:

(a1) $r = n - q$ and $\overline{i_h} \neq i_k$ for any $h, k \in [n-q]$;

(a2) Let $\alpha' : j_1, j_2, ..., j_{n-q}$ be the subsequence of $\beta_{n,q}$ complement to α (i.e., $\{i_h, j_h \mid h \in [n-q]\} = \beta_{n,q}$ regarding the sequences as the corresponding sets). Then the term j_h occurs after i_h in the sequence $\beta_{n,q}$ for every $h \in [n-q]$.

Let $\Delta_{n,q}$ be the set of all admissible subsequences of $\beta_{n,q}$ and let $\delta_{n,q} := |\Delta_{n,q}|$. Denote $\beta_{n,0}$, $\Delta_{n,0}$, $\delta_{n,0}$ simply by β_n, Δ_n, δ_n, respectively. Clearly, the equation $\delta_{n,q} = \delta_{n-q}$ holds for any $q \in [0, n-1]$.

(b) For any $i < j$ in $[n]$ with $j - i$ odd, denote by β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) the sequence $i+1, i+2, ..., j-1$ (respectively, $\overline{j-1}, \overline{j-2}, ..., \overline{i+1}$). A subsequence $\alpha : h_1, h_2, ..., h_r$ of β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) is called *admissible*, if, $r = \frac{j-i-1}{2}$, and if, let $\alpha' : k_1, k_2, ..., k_{\frac{j-i-1}{2}}$ be the subsequence of β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) complement to α, then k_l occurs after h_l in β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) for any $l \in \left[\frac{j-i-1}{2}\right]$.

It is well known that the number of admissible subsequences in β_{ij} (respectively, $\beta_{\bar{j}\bar{i}}$) is $C_{\frac{j-i-1}{2}}$, where $C_l := \frac{1}{l+1}\binom{2l}{l}$ is the l-th Catalan number. The following is a formula for the number δ_n of admissible subsequences in β_n.

Proposition 4.3. *$\delta_n := \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any $n \geqslant 1$, where $\lfloor x \rfloor$ stands for the largest integer not greater than x for any $x \in \mathbb{Q}$.*

To show Proposition 4.3, we need some preparation. Let $\alpha_{n,q}$: $i_1, i_2, ..., i_{n-q}$ be a subsequence of $\beta_{n,q}$ satisfying the condition 4.2 (a1). Let $p(\alpha_{n,q})$ be the largest $k \in [0, n-q]$ with $i_1, i_2, ..., i_k$ a subsequence of $\bar{n}, \overline{n-1}, ..., \overline{q+1}$. Denote $i_1, i_2, ..., i_{p(\alpha_{n,q})}$ by $\alpha^0_{n,q}$. Then $\alpha_{n,q}$ is entirely determined by $\alpha^0_{n,q}$.

Let $\alpha : i_1, i_2, ..., i_n$ be a subsequence of β_n satisfying the condition 4.2 (a1). For any $q \in [0, n-1]$, let $\alpha_{n,q}$ be obtained from α by removing all the terms in β_q and let $\alpha'_{n,q}$ be the subsequence of $\beta_{n,q}$ complement to $\alpha_{n,q}$ (see 4.2 (a2)), where we stipulate β_0 to be the empty sequence. Then the following result can be checked easily:

Lemma 4.4. *Let $\alpha : i_1, i_2, ..., i_n$ be a subsequence of β_n satisfying the condition 4.2 (a1).*

(1) The following three conditions on α are equivalent:

(a) α is admissible in β_n;

(b) $\alpha_{n,q}$ is admissible in $\beta_{n,q}$ for every $q \in [0, n-1]$;

(c) $p(\alpha) \geqslant \frac{n}{2}$ and the term j_h occurs after i_h in β_n for every $h \in [p(\alpha)]$, where α': $j_1, j_2, ..., j_n$ is the subsequence of β_n complement to α (see 4.2 (a2)).

(2) For $q \in [0, n-1]$, if $\alpha_{n,q}$ is admissible in $\beta_{n,q}$, then $p(\alpha_{n,q}) \geqslant p(\alpha'_{n,q})$, in particular, $p(\alpha_{n,q}) \geqslant \frac{n-q}{2}$.

4.5. *Proof of Proposition 4.3.* Consider the set Δ_n. We may assume $n > 1$, for otherwise the result is obvious. By Lemma 4.4 (1), we see that $\alpha_{n,1} \in \Delta_{n,1}$ for any $\alpha \in \Delta_n$. On the other hand, for any λ: $i_1, i_2, ..., i_{n-1}$ in $\Delta_{n,1}$, let $\lambda_{\bar{1}}$ (respectively, λ_1) be obtained from λ by inserting the term $\bar{1}$ (respectively, 1) immediately after $i_{p(\lambda)}$. Then $\lambda_{\bar{1}}$ is always in Δ_n, while λ_1 is not in Δ_n if and only if $p(\lambda) < \frac{n}{2}$. Since $p(\lambda) \geqslant \frac{n-1}{2}$ by the condition $\lambda \in \Delta_{n,1}$ and Lemma 4.4 (2), this implies that λ_1 is not in Δ_n if and only if n is odd (say $n = 2l+1$) and $p(\lambda) = l$. When $n = 2l+1$, let $\Delta'_{n,1}$ be the set of all such subsequences λ: $i_1, i_2, ..., i_l$ of $\bar{n}, \overline{n-1}, ..., \bar{3}, \bar{2}$ that, if λ': $j_1, j_2, ..., j_l$ is the subsequence of $\bar{n}, \overline{n-1}, ..., \bar{3}, \bar{2}$ complement to λ, then the term j_h occurs after the term i_h for every $h \in [l]$. Then $|\Delta'_{n,1}|$ is equal to the number of all $\lambda \in \Delta_{n,1}$ with $\lambda_1 \notin \Delta_n$. It is well known that $|\Delta'_{n,1}| = C_l$ (the l-th Catalan number). So by applying induction on $n \geqslant 1$ and by the fact that $\delta_{n,1} = \delta_{n-1}$, we have

$$\delta_n = \begin{cases} \delta_{n-1} + (\delta_{n-1} - C_l) = 2\binom{2l}{l} - \frac{1}{l+1}\binom{2l}{l} = \binom{2l+1}{l}, & \text{if } n = 2l+1 \text{ is odd,} \\ 2\delta_{n-1} = 2\binom{2l-1}{l-1} = \binom{2l}{l}, & \text{if } n = 2l \text{ is even.} \end{cases}$$

Our result is proved. □

Remark 4.6. The result in Proposition 4.3 can be extended to a more general case. Let $\beta : \overline{i_t}, \overline{i_{t-1}}, ..., \overline{i_1}, i_1, i_2, ..., i_t$ (respectively, $\overline{\beta} : i_1, i_2, ..., i_t, \overline{i_t}, \overline{i_{t-1}}, ..., \overline{i_1}$) satisfy $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant n$. A subsequence $\alpha : j_1, j_2, ..., j_r$ of β (respectively, $\overline{\beta}$) is called *admissible*, if the following conditions are satisfied:

(i) $r = t$ and $\overline{j_h} \neq j_k$ for any $h, k \in [t]$;

(ii) Let $\alpha' : j'_1, j'_2, ..., j'_t$ be the subsequence of β (respectively, $\overline{\beta}$) complement to α. Then j'_h occurs after j_h in β (respectively, $\overline{\beta}$) for any $h \in [t]$.

By the same way as that for Proposition 4.3, one can prove that the number of admissible subsequences of β (respectively, $\overline{\beta}$) is equal to $\binom{t}{\lfloor \frac{t}{2} \rfloor}$.

The following is a formula for the number $\gamma_0(\mathbf{a})$ with $\mathbf{a} = (a_1, a_2) \in \widetilde{\Lambda}_{2n}$ having just two parts a_1, a_2.

Proposition 4.7. *For $\mathbf{a} = (n+t, n-t)$, $t \in [n-1]$, let $\mathcal{C}^{n,t}_{2n}$ be the set of all $(2n-1, n)$-selfdual tabloids in $\xi^{-1}(\mathbf{a})$ and let $q^{n,t}_{2n} = |\mathcal{C}^{n,t}_{2n}|$. Then*

$$q^{n,t}_{2n} = \sum_{\substack{h_1, h_2, ..., h_t \in \mathbb{N} \\ 1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n \\ h_{i+1} - h_i \text{ odd } \forall i}} \binom{n - h_t}{\lfloor \frac{n-h_t}{2} \rfloor} \binom{h_1 - 1}{\lfloor \frac{h_1-1}{2} \rfloor} \prod_{i=1}^{t-1} C_{\frac{h_{i+1}-h_i-1}{2}}, \tag{4.7.1}$$

where C_l is the l-th Catalan number for any $l \in \mathbb{N}$.

Proof. Let $\mathbf{T} = (Y, X) \in \mathcal{C}^{n,t}_{2n}$. By the condition of $\mathbf{T}$ being $(2n-1, n)$-selfdual, we have $L_Y(X) = \{i \in Y \mid \bar{i} \notin Y\}$ and $Y - L_Y(X) = \{h_1, h_2, ..., h_t, \overline{h_t}, ..., \overline{h_2}, \overline{h_1}\}$ with some $1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n$ by Lemma 4.1. According to the definition of the set $L_Y(X)$ with respect to X, Y, we see by Lemma 2.9 (b) that

(i) For any $j \in [t-1]$, let $Y_{h_j+1,h_{j+1}-1} = \{h_{j1}, h_{j2}, ..., h_{jn_j}\}$ be with $h_j < h_{j1} < h_{j2} < \cdots < h_{jn_j} < h_{j+1}$, then $h_{j1}, h_{j2}, ..., h_{jn_j}$ is an admissible subsequence of $\beta_{h_j,h_{j+1}} : h_j+1, h_j+2, ..., h_{j+1}-1$ (hence $h_{j+1}-h_j$ is odd and $n_j = \frac{h_{j+1}-h_j-1}{2}$ by 4.2 (b)), and $Y_{\overline{h_{j+1}-1},\overline{h_j+1}} = \overline{[h_j+1, h_{j+1}-1] - Y_{h_j+1,h_{j+1}-1}}$. Write $Y_{\overline{h_{j+1}-1},\overline{h_j+1}} = \{\overline{h'_{j1}}, \overline{h'_{j2}}, ..., \overline{h'_{jn_j}}\}$ with $h_j < h'_{j1} < h'_{j2} < \cdots < h'_{jn_j} < h_{j+1}$. Then $\overline{h'_{jn_j}}, ..., \overline{h'_{j2}}, \overline{h'_{j1}}$ is an admissible subsequence of $\overline{h_{j+1}-1}, ..., \overline{h_j+2}, \overline{h_j+1}$.

(ii) Let $Y_{h_t+1,\overline{h_t+1}} = \{h_{t1}, h_{t2}, ..., h_{tn_t}\}$ be with $\alpha : h_{t1}, h_{t2}, ..., h_{tn_t}$ a subsequence of $\beta_{h_t,n} : h_t+1, h_t+2, ..., n, \bar{n}, \overline{n-1}, ..., \overline{h_t+1}$. Then α is admissible in $\beta_{h_t,n}$.

(iii) Let $Y_{\overline{h_1-1},h_1-1} = \{h_{01}, h_{02}, ..., h_{0n_0}\}$ be with $\alpha : h_{01}, h_{02}, ..., h_{0n_0}$ a subsequence of $\beta_{1,h_1} : \overline{h_1-1}, \overline{h_1-2}, ..., \bar{1}, 1, 2, ..., h_1-1$. Then α is admissible in β_{1,h_1}.

(iv) $L_Y(X) = \left(\bigcup_{j\in[t-1]} \left(Y_{h_j+1,h_{j+1}-1} \cup Y_{\overline{h_{j+1}-1},\overline{h_j+1}}\right)\right) \cup Y_{h_t+1,\overline{h_t+1}} \cup Y_{\overline{h_1-1},h_1-1}$.

Now fix $h_1, h_2, ..., h_t \in \mathbb{N}$ with $t \in \mathbb{N}$ and $1 \leqslant h_1 < h_2 < \cdots < h_t \leqslant n$ and $h_{i+1} - h_i$ odd for all $i \in [t-1]$. Take an admissible subsequence α_j in $\beta_{h_j,h_{j+1}}$ for any $j \in [t-1]$. Also, take an admissible subsequence α_t (respectively, α_0) in $\beta_{h_t,n}$ (respectively, β_{1,h_1}). For $j \in [t-1]$, let α'_j be the subsequence of $\beta_{h_j,h_{j+1}}$ complement to α_j and let $\alpha_{\bar{j}}$ be the subsequence of $\beta_{\overline{h_{j+1}},\overline{h_j}}$ such that $\alpha_{\bar{j}} = \overline{\alpha'_j}$ by regarding the sequences as the corresponding sets. Let Y be the union of the sets $\{h_l, \overline{h_l} \mid l \in [t]\}$, α_t, α_0 and α_j, $\alpha_{\bar{j}}$ with $j \in [t-1]$, regarding the sequences as the corresponding sets. Let $X = [2n] - Y$. Then $(Y, X) \in \mathcal{C}_{2n}^{n,t}$.

By 4.2 (b) and Proposition 4.3, we see that the numbers of admissible subsequences in $\beta_{h_j,h_{j+1}}$, $j \in [t-1]$, $\beta_{h_t,n}$, β_{1,h_1} are $C_{\frac{h_{j+1}-h_j-1}{2}}$, $\binom{n-h_t}{\lfloor\frac{n-h_t}{2}\rfloor}$, $\binom{h_1-1}{\lfloor\frac{h_1-1}{2}\rfloor}$, respectively. This implies the formula (4.7.1). $\square$

We can get the corresponding results in the case of $m \in \{2n, 2n+1\}$ similarly by noting that the number of (m,n)-selfdual elements in $[m+1]$ is $m+1-2n$.

Proposition 4.8. *For $\mathbf{a} = (n+1+t, n-t)$, $t \in [n-1]$, let $\mathcal{C}_{2n+1}^{n,t}$ be the set of all $(2n,n)$-selfdual tabloids in $\xi^{-1}(\mathbf{a})$ and let $q_{2n+1}^{n,t} = |\mathcal{C}_{2n+1}^{n,t}|$. Then*

(4.8.1)
$$q^{n,t}_{2n+1} = \sum_{\substack{h_1,h_2,\ldots,h_{t+1}\in\mathbb{N}\\ 1\leqslant h_1<h_2<\cdots<h_{t+1}=n+1\\ h_{i+1}-h_i \text{ odd } \forall i}} \binom{h_1-1}{\lfloor\frac{h_1-1}{2}\rfloor}\prod_{i=1}^{t} C_{\frac{h_{i+1}-h_i-1}{2}}.$$

Proposition 4.9. *For* $\mathbf{a} = (n+1+t, n+1-t)$, $t \in [n]$, *let* $\mathcal{C}^{n,t}_{2n+2}$ *be the set of all* $(2n+1, n)$*-selfdual tabloids in* $\xi^{-1}(\mathbf{a})$ *and let* $q^{n,t}_{2n+2} = |\mathcal{C}^{n,t}_{2n+2}|$. *Then*

(4.9.1)
$$q^{n,t}_{2n+2} = \sum_{\substack{h_1,h_2,\ldots,h_{t+1}\in\mathbb{N}\\ 0=h_1<h_2<\cdots<h_{t+1}=n+1\\ h_{i+1}-h_i \text{ odd } \forall i}} \prod_{i=1}^{t} C_{\frac{h_{i+1}-h_i-1}{2}}.$$

From Theorem 3.3 and Propositions 4.7-4.9, we see that for $k \in \mathbb{N}$ with $2k \leqslant m+1$, the set $E_{\mathbf{2^k1^{m+1-2k}}}$ is empty if and only if $m = 2n+1$ and $2k = m+1$.

Example 4.10. (1) In Proposition 4.7, take $n = 5$ and $t = 2$, then $\mathbf{a} = (7, 3)$ and $\zeta(\mathbf{a})^\vee = \mathbf{2^3 1^4}$. The pairs (h_1, h_2) occurring in the summation of (4.7.1) are $(1,2), (2,3), (3,4), (4,5), (1,4), (2,5)$. Then $q^{5,2}_{10} = \binom{3}{1} + \binom{2}{1} + \binom{2}{1} + \binom{3}{1} + 1 + 1 = 12$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^3 1^4}}$. The set $\mathcal{C}^{5,2}_{10}$ consists of the following tabloids:

$$\begin{aligned}
&\mathbf{T}_1 = (\{3,4,5\}\cup\{1,2,9,10\}, \{6,7,8\}), && \mathbf{T}_2 = (\{3,4,6\}\cup\{1,2,9,10\}, \{5,7,8\}),\\
&\mathbf{T}_3 = (\{3,5,7\}\cup\{1,2,9,10\}, \{4,6,8\}), && \mathbf{T}_4 = (\{4,5,10\}\cup\{2,3,8,9\}, \{1,6,7\}),\\
&\mathbf{T}_5 = (\{4,6,10\}\cup\{2,3,8,9\}, \{1,5,7\}), && \mathbf{T}_6 = (\{5,9,10\}\cup\{3,4,7,8\}, \{1,2,6\}),\\
&\mathbf{T}_7 = (\{1,5,9\}\cup\{3,4,7,8\}, \{2,6,10\}), && \mathbf{T}_8 = (\{8,9,10\}\cup\{4,5,6,7\}, \{1,2,3\}),\\
&\mathbf{T}_9 = (\{1,8,9\}\cup\{4,5,6,7\}, \{2,3,10\}), && \mathbf{T}_{10} = (\{2,8,10\}\cup\{4,5,6,7\}, \{1,3,9\}),\\
&\mathbf{T}_{11} = (\{2,5,8\}\cup\{1,4,7,10\}, \{3,6,9\}), && \mathbf{T}_{12} = (\{3,7,10\}\cup\{2,5,6,9\}, \{1,4,8\}).
\end{aligned}$$

(2) In Proposition 4.8, take $n = 5$ and $t = 2$, then $\mathbf{a} = (8, 3)$ and $\zeta(\mathbf{a})^\vee = \mathbf{2^3 1^5}$. The triples (h_1, h_2, h_3) occurring in the summation of (4.8.1) are $(4,5,6), (2,5,6), (2,3,6)$. Then $q^{5,2}_{11} = \binom{3}{1} + 1 + 1 = 5$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^3 1^5}}$. The set $\mathcal{C}^{5,2}_{11}$ consists of the following tabloids:

$$\begin{aligned}
\mathbf{T}_1 &= (\{4,7,11\}\cup\{2,3,6,9,10\},\{1,5,8\}),\\
\mathbf{T}_2 &= (\{3,8,11\}\cup\{2,5,6,7,10\},\{1,4,9\}),\\
\mathbf{T}_3 &= (\{9,10,11\}\cup\{4,5,6,7,8\},\{1,2,3\}),.\\
\mathbf{T}_4 &= (\{1,9,10\}\cup\{4,5,6,7,8\},\{2,3,11\}),\\
\mathbf{T}_5 &= (\{2,9,11\}\cup\{4,5,6,7,8\},\{1,3,10\}).
\end{aligned}$$

(3) In Proposition 4.9, take $n=5$ and $t=2$, then $\mathbf{a}=(8,4)$ and $\zeta(\mathbf{a})^\vee=\mathbf{2^41^4}$. The triples (h_1,h_2,h_3) occurring in the summation of (4.9.1) are $(0,1,6),(0,3,6),(0,5,6)$. Then $q_{12}^{5,2}=2+1+2=5$, which is just the number of left cells of $\widetilde{C}_5$ in the set $E_{\mathbf{2^41^4}}$. The set $\mathcal{C}_{12}^{5,2}$ consists of the following tabloids:

$$\begin{aligned}
\mathbf{T}_1 &= (\{2,3,7,8\}\cup\{1,6,11,12\},\{4,5,9,10\}),\\
\mathbf{T}_2 &= (\{2,4,7,9\}\cup\{1,6,11,12\},\{3,5,8,10\}),\\
\mathbf{T}_3 &= (\{1,4,7,10\}\cup\{3,6,9,12\},\{2,5,8,11\}),.\\
\mathbf{T}_4 &= (\{1,2,8,9\}\cup\{5,6,7,12\},\{3,4,10,11\}),\\
\mathbf{T}_5 &= (\{1,3,8,10\}\cup\{5,6,7,12\},\{2,4,9,11\}).
\end{aligned}$$

Remark 4.11. The results in Propositions 4.7-4.9 can be extended to a more general case. Let $\lambda=(2l_1,2l_2,...,2l_r,2l_{r+1}+1,...,2l_t+1)\in\Lambda_{m+1}$ for some $r,t,l_i\in\mathbb{N}$ with $1\leqslant r<t$ and $i\in[t]$ (Comparing with the partitions in Lemma 3.4). Then $\mathbf{a}=(a_1,a_2,...,a_{l_1-1},t,r,a_{l_1-1},...,a_2,a_1)\in\zeta^{-1}(\lambda^\vee)$ for some $1\leqslant a_1\leqslant a_2\leqslant\cdots\leqslant a_{l_1-1}$. Let $\mathcal{C}_{m+1}^{\mathbf{a}}$ be the set of all (m,n)-selfdual tabloids in $\xi^{-1}(\mathbf{a})$ and let $\gamma_{m+1-2n}(\mathbf{a})=|\mathcal{C}_{m+1}^{\mathbf{a}}|$.

Corollary 4.12. *In the above setup, we have*

(4.12.1)
$$\gamma_{m+1-2n}(\mathbf{a})=2^{a_1+\cdots+a_{l_1-1}}\frac{n!}{(n-a_1-\cdots-a_{l_1-1})!\prod_{i=1}^{l_1-1}a_i!}\cdot q_{t+r}^{\lfloor\frac{t+r}{2}\rfloor+\epsilon_{m,n},\lfloor\frac{t-r}{2}\rfloor}.$$

where $\epsilon_{m,n}$ is 0 if $m\in\{2n-1,2n\}$ and -1 if $m=2n+1$.

Proof. Let $\mathbf{a}_1=(a_1,a_2,...,a_{l_1-1},a_{l_1-1},...,a_2,a_1)$ and $\mathbf{a}_2=(t,r)$. Then $\gamma_{m+1-2n}(\mathbf{a})=\gamma_0(\mathbf{a}_1)\gamma_{m+1-2n}(\mathbf{a}_2)\binom{n}{a_1+\cdots+a_{l_1-1}}$ by Theorem 3.7. We have $\gamma_{m+1-2n}(\mathbf{a}_2)=q_{t+r}^{\lfloor\frac{t+r}{2}\rfloor+\epsilon_{m,n},\lfloor\frac{t-r}{2}\rfloor}$ by Propositions 4.7-4.9 and

$$\gamma_0(\mathbf{a}_1)\binom{n}{a_1+\cdots+a_{l_1-1}} = 2^{a_1+\cdots+a_{l_1-1}}\frac{n!}{(n-a_1-\cdots-a_{l_1-1})!\prod_{i=1}^{l_1-1}a_i!}$$

by Theorem 3.3. This proves the formula (4.12.1). □

4.13. Let $\mathbf{T} = (T_1, T_2, ..., T_r)$ and $\mathbf{T}' = (T'_1, T'_2, ..., T'_r)$ in $\mathcal{C}_{m+1}$ satisfy $|T_1| > |T_2| > \cdots > |T_r|$ and $|T'_1| < |T'_2| < \cdots < |T'_r|$ and $\mathbf{T}' \approx \mathbf{T}$. Then $|T'_i| = |T_{r+1-i}|$ for any $i \in [r]$. The tabloid $\mathbf{T}$ is (m,n)-selfdual if and only if $\mathbf{T}'$ is (m,n)-selfdual if and only if $T'_i = \overline{T_{r+1-i}}$ for any $i \in [r]$. When the equivalent conditions hold, define a partition $T_j = T_{j1}\dot{\cup}T_{j2}\dot{\cup}\cdots\dot{\cup}T_{j,r+1-j}$ for any $j \in [r]$ such that the sets $T_j^h := T_{j1}\dot{\cup}T_{j2}\dot{\cup}\cdots\dot{\cup}T_{jh}$ for $j \in [r]$ and $h \in [r+1-j]$ satisfy the condition $L_{T_j}(T_{j+1}^h) = T_j^h$ for any $h \in [r-j]$.

4.14. Consider the case of $r = 3$. Let $\mathbf{T} = (T_1, T_2, T_3)$ and $\mathbf{T}' = (T'_1, T'_2, T'_3)$ be (m,n)-selfdual tabloids of rank $m+1$ with $|T_1| > |T_2| > |T_3|$ and $|T'_1| < |T'_2| < |T'_3|$ and $\mathbf{T} \approx \mathbf{T}'$. We want to describe $\mathbf{T}'$ in terms of $\mathbf{T}$. Define the partitions $T_1 = T_{11}\dot{\cup}T_{12}\dot{\cup}T_{13}$ and $T_2 = T_{21}\dot{\cup}T_{22}$ and $T_3 = T_{31}$ as those in 4.13 with $r = 3$. Define

$$\mathbf{X} := (T_{11}, T_{21}\cup T_{12}\cup T_{13}, T_{31}\cup T_{22}) \quad \text{and} \quad \mathbf{Y} := (T_{11}\cup T_{12}, T_{21}\cup T_{22}\cup T_{13}, T_{31}).$$

Then $\mathbf{X}$ is obtained from $\mathbf{T}$ by a $\{2,3\}$-transformation followed by a $\{1,2\}$-transformation, while $\mathbf{Y}$ is obtained from $\mathbf{T}$ by a $\{1,2\}$-transformation (see 2.13). So $\mathbf{X} \approx \mathbf{T} \approx \mathbf{Y}$. We see by Lemma 2.15 that both $\mathbf{X}$ and $\mathbf{Y}$ are (m,n)-selfdual and that $\mathbf{Y} = \overline{\mathbf{X}}^{\mathrm{op}}$. This implies that $T_{31} = \overline{T_{11}}$ and $T_{22} = \overline{T_{12}}$ and $T_{13}\cup T_{21} = \overline{T_{21}\cup T_{13}}$. Denote $E^0 = \{i \in E \mid \bar{i} \in E\}$ and $E^1 = E - E^0$ for any $E \subseteq [m+1]$. Then $T_{13}^1 = \overline{T_{21}^1}$ and $\mathbf{T}' = (T_{11}, T_{21}^0\cup T_{12}\cup T_{13}^1, T_{31}\cup T_{22}\cup T_{21}^1\cup T_{13}^0)$.

Hence we have

Proposition 4.15. *For any* $\mathbf{a} = (a_1, a_2, a_3) \in \widetilde{\Lambda}_{m+1}$ *with* $a_1 > a_2 > a_3$*, a tabloid* $\mathbf{T} \in \xi^{-1}(\mathbf{a})$ *is* (m,n)*-selfdual if and only if* $\mathbf{T} = (T_{11}\dot{\cup}T_{12}\dot{\cup}T_{13}, T_{21}\dot{\cup}\overline{T_{12}}, \overline{T_{11}})$ *for some* $T_{11}, T_{12}, T_{13}, T_{21} \subset [m+1]$ *satisfying the following conditions:*

(i) $T_{11} = L_{T_{11}\dot{\cup}T_{12}\dot{\cup}T_{13}}(T_{21})$*;*

(ii) $T_{11}\dot{\cup}T_{12} = L_{T_{11}\dot{\cup}T_{12}\dot{\cup}T_{13}}(T_{21}\dot{\cup}\overline{T_{12}})$*;*

(iii) $T_{21} = L_{T_{21}\dot{\cup}\overline{T_{12}}}(\overline{T_{11}})$*;*

(iv) $T_{11}^0 = T_{12}^0 = \emptyset$ *and* $T_{13}^1 = \overline{T_{21}^1}$.

4.16. Next consider the case of $r = 4$. Let

$$\mathbf{T} = (T_1, T_2, T_3, T_4) = (T_{11}\dot{\cup}T_{12}\dot{\cup}T_{13}\dot{\cup}T_{14}, T_{21}\dot{\cup}T_{22}\dot{\cup}T_{23}, T_{31}\dot{\cup}T_{32}, T_{41})$$

be defined as in 4.13. By the argument similar to that in 4.14 (of course, more complicated), one can shown that if $\mathbf{T}$ is (m, n)-selfdual then the following conditions hold:

(i) $T_{41} = \overline{T_{11}}$ and $T_{32} = \overline{T_{12}}$.

(ii) There are some partitions $T_{31} = T'_{31}\dot{\cup}T''_{31}$, $T_{23} = T'_{23}\dot{\cup}T''_{23}$, $T_{14}^1 = T'_{14}\dot{\cup}T''_{14}$ and $T_{21}^1 = T'_{21}\dot{\cup}T''_{21}$ which satisfy (also are determined by) the following conditions:

(iia) $T''_{31}\dot{\cup}T''_{23} = \overline{T_{13}} = T_{31}\dot{\cup}T_{22}\dot{\cup}T_{23} - L_{T_{31}\dot{\cup}T_{22}\dot{\cup}T_{23}}(T_{41}\dot{\cup}T_{32})$;

(iib) $T'_{21}\dot{\cup}T'_{14} = \overline{T'_{31}\dot{\cup}T'_{23}} = L_{T_{21}\dot{\cup}T_{22}\dot{\cup}T_{23}\dot{\cup}T_{14}}(T_{31})$;

(iic) $T_{22}^1\dot{\cup}T''_{21}\dot{\cup}T''_{14} = \overline{T_{22}^1\dot{\cup}T''_{21}\dot{\cup}T''_{14}}$.

(iid) $T_{11}^0 = T_{12}^0 = T_{13}^0 = T_{23}^0 = T_{31}^0 = T_{32}^0 = T_{41}^0 = \emptyset$.

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