

LOWER BOUNDARY HYPERPLANES OF THE CANONICAL LEFT CELLS IN THE AFFINE WEYL GROUP $W_a(\tilde{A}_{n-1})$

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ABSTRACT. Let Γ be any canonical left cell of the affine Weyl group W_a of type $\tilde{A}_{n-1}$, $n > 1$. We describe the lower boundary hyperplanes for Γ , which answer two questions of Humphreys.

Let W_a be an affine Weyl group with Φ the root system of the corresponding Weyl group. Fix a positive root system Φ^+ of Φ , there is a bijection from W_a to the set of alcoves in the euclidean space E spanned by Φ . We identify the elements of W_a with the alcoves (also with the topological closure of the alcoves) of E . According to a result of Lusztig and Xi in [6], we know that the intersection of any two-sided cell of W_a with the dominant chamber of E is exactly a single left cell of W_a , called a canonical left cell. When W_a is of type $\tilde{A}_{n-1}$, $n > 1$, there is a bijection ϕ from the set of two sided cells of W_a to the set of partitions of n (see 2.4-2.6 and [7]). Recently, J. E. Humphreys raised the following

Questions ([2]): Let W_a be the affine Weyl group of type $\tilde{A}_{n-1}$, $n > 1$.

(1) Could one find the set $B(L)$ of all the lower boundary hyperplanes for any canonical left cell L of W_a ?

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(2) How does the partition $\phi(L)$ determine the set $B(L)$? and in which case is $\phi(L)$ also determined by the set $B(L)$?

In this paper, we shall answer the two questions.

From now on, we always assume that W_a is of type $\tilde{A}_{n-1}$ unless otherwise specified.

In the first two sections, we collect some concepts and known results for the later use. In Section 3, we give some criteria for a hyperplane to be lower boundary of a canonical left cell of W_a . Then we prove our main results in Section 4.

§1. Sign types.

1.1. Let $\mathbf{n} = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. An *$\mathbf{n}$ -sign type* (or just a sign type) is by definition a matrix $X = (X_{ij})_{i,j \in \mathbf{n}}$ over the symbol set $\{+, \circ, -\}$ with

$$(1.1.1) \quad \{X_{ij}, X_{ji}\} \in \{\{+, -\}, \{\circ, \circ\}\} \quad \text{for } i, j \in \mathbf{n}.$$

X is determined entirely by its “upper-unitriangular” part $X^\Delta = (X_{ij})_{i < j}$. We identify X with X^Δ . X is *dominant*, if $X_{ij} \in \{+, \circ\}$ for any $i < j$ in $\mathbf{n}$, and is *admissible*, if

$$(1.1.2) \quad - \in \{X_{ij}, X_{jk}\} \implies X_{ik} \leq \max\{X_{ij}, X_{jk}\},$$

$$(1.1.3) \quad - \notin \{X_{ij}, X_{jk}\} \implies X_{ik} \geq \max\{X_{ij}, X_{jk}\}$$

for any $i < j < k$ in $\mathbf{n}$, where we set a total ordering: $- < \circ < +$.

The following is an easy consequence of conditions (1.1.2)-(1.1.3).

Lemma 1.2. ([9, Lemma 3.1; 12, Corollary 2.8]) (1) A dominant sign type $X = (X_{ij})$ is admissible if and only if for any $i \leq h < k \leq j$, condition $X_{ij} = \circ$ implies $X_{hk} = \circ$.

(2) If an admissible sign type $X = (X_{ij})$ is not dominant, then there exists at least one k , $1 \leq k < n$, with $X_{k,k+1} = -$.

1.3. Let $E = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i = 0\}$. This is a euclidean space of dimension $n - 1$ with inner product $\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = \sum_{i=1}^n a_i b_i$. For $i \neq j$ in $\mathbf{n}$, let $\alpha_{ij} = (0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$, with 1, -1 the i th, j th components respectively. Then $\Phi = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}$ is the root system of type A_{n-1} , which spans E . $\Phi^+ = \{\alpha_{ij} \in \Phi \mid i < j\}$ is a positive root system of Φ with $\Pi = \{\alpha_{i,i+1} \mid 1 \leq i < n\}$ the corresponding simple root system. For any $\epsilon \in \mathbb{Z}$ and $i < j$ in $\mathbf{n}$, define a hyperplane

$$H_{ij;\epsilon} = \{(a_1, \dots, a_n) \in E \mid a_i - a_j = \epsilon\}.$$

Encode a connected component C of $E - \bigcup_{\substack{1 \leq i < j \leq n \\ \epsilon \in \{0,1\}}} H_{ij;\epsilon}$ (set difference) by a sign type $X = (X_{ij})_{i < j}$ as follows. Take any $v = (a_1, \dots, a_n) \in C$ and set

$$X_{ij} = \begin{cases} +, & \text{if } a_i - a_j > 1; \\ -, & \text{if } a_i - a_j < 0; \\ \circ, & \text{if } 0 < a_i - a_j < 1. \end{cases}$$

for $i < j$ in $\mathbf{n}$ (X only depends on C , but not on the choice of v , see [7, Chapter 5]). Note that not all the sign types can be obtained in this way.

Proposition 1.4. ([7, Proposition 7.1.1; 7, §2]) *A sign type $X = (X_{ij})$ can be obtained in the above way if and only if it is admissible.*

The following is an easy consequence of conditions (1.1.2)-(1.1.3).

Lemma 1.5. *Let $X = (X_{ij})$ be a dominant admissible sign type with $X_{p,p+1} = \circ$ for some p , $1 \leq p < n$. Let $X' = (X'_{ij})$ be a sign type given by*

$$X'_{ij} = \begin{cases} X_{ij} & \text{if } (i, j) \neq (p, p+1), \\ - & \text{if } (i, j) = (p, p+1) \end{cases}$$

for $i < j$ in $\mathbf{n}$. Then X' is admissible if and only if $X_{ph} = X_{p+1,h}$ for all $h \in \mathbf{n}$.

1.6. For $\alpha \in \Phi$, let s_α be the reflection in E : $v \mapsto v - 2 \frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$ and T_α the translation in E : $v \mapsto v + \alpha$. Define $s_i = s_{\alpha_{i,i+1}}$, $1 \leq i < n$, and $s_0 = T_{\alpha_{1n}} s_{\alpha_{1n}}$. Then $S = \{s_i \mid 0 \leq i < n\}$ forms a distinguished generator set of the affine Weyl group W_a of type $\tilde{A}_{n-1}$.

1.7. A connected component in $E - \bigcup_{\substack{1 \leq i < j \leq n \\ k \in \mathbb{Z}}} H_{ij;k}$ is called an alcove. The (right) action of W_a on E induces a simply transitive permutation on the set $\mathfrak{A}$ of alcoves in E . There exists a bijection $w \mapsto A_w$ from W_a to $\mathfrak{A}$ such that A_1 (1 the identity element of W_a) is the unique alcove in the dominant chamber of E whose closure containing the origin and that $(A_y)w = A_{yw}$ for $y, w \in W_a$ (see [8, Proposition 4.2]). To each $w \in W_a$, we associate an admissible sign type $X(w)$ which contains the alcove A_w . An admissible sign type X can be identified with the set $\{w \in W_a \mid X(w) = X\}$.

1.8. To each $w \in W_a$, we associate a set $\mathcal{R}(w) = \{s \in S \mid ws < w\}$, where $\leq$ is the Bruhat order in the Coxeter system (W_a, S) .

§2. Partitions and Kazhdan-Lusztig cells.

2.1. Let $(P, \preceq)$ be a finite poset. By a *chain* of P , we mean a totally ordered subset of P (allow to be an empty set). Also, a *cochain* of P is a subset K of P whose elements are pairwise incomparable. A *k-chain-family* (resp. a *k-cochain-family*) in P ($k \geq 1$) is a subset J of P which is a disjoint union of k chains (resp. cochains) J_i ($1 \leq i \leq k$). We usually write $J = J_1 \cup \dots \cup J_k$.

2.2. A *partition* of n ($n \in \mathbb{N}$) is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$. In particular, when $\lambda_1 = \dots = \lambda_r = a$, we also denote $\lambda = (a^r)$ (call it a *rectangular partition*). Let Λ_n be the set of all partitions of n .

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\mu = (\mu_1, \mu_2, \dots, \mu_t) \in \Lambda_n$. Write $\lambda \leq \mu$, if the inequalities $\sum_{j=1}^i \lambda_j \leq \sum_{j=1}^i \mu_j$ hold for $i \geq 1$. μ is *conjugate* to λ , if $\mu_i = |\{j \mid \lambda_j \geq i, 1 \leq j \leq r\}|$ for $1 \leq i \leq t$, where $|X|$ stands for the cardinality of the set X .

2.3. Let d_k be the maximal cardinality of a k -chain-family in P for $k \geq 1$. Then $d_1 < d_2 < \dots < d_r = n = |P|$ for some $r \geq 1$. Let $\lambda_1 = d_1$, $\lambda_i = d_i - d_{i-1}$ for $1 < i \leq r$. Then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ by [1, Theorem 1.6], a result of C. Greene. We get $\phi(P) = (\lambda_1, \dots, \lambda_r) \in \Lambda_n$, called the *partition associated to chains* in P . Replacing the word “ k -chain-family” by “ k -cochain-family” in the above, we can also define $\psi(P) = (\mu_1, \dots, \mu_t) \in \Lambda_n$ again by [1, Theorem 1.6], called the *partition associated to cochains* in P . Moreover, $\psi(P)$ is conjugate to $\phi(P)$.

2.4. Let $(P, \preceq)$ be a finite poset with $\psi(P) = (\mu_1, \dots, \mu_t)$. For $1 \leq k \leq t$, let $P^{(k)} = P_1 \cup \dots \cup P_k$ be a k -cochain-family of P with $|P^{(k)}| = \sum_{h=1}^k \mu_h$. Then $\mu_1 \geq |P_1| \geq \mu_k$ for $1 \leq i \leq k$. In particular, when $\psi(P) = (a^t)$ is rectangular, we have $|P_1| = \dots = |P_k| = a$. This fact will be used in the proof of Theorem 4.2.

2.5. To each admissible sign type $X = (X_{ij})$, we write $i \leq_X j$ in $\mathbf{n}$ if either $i = j$ or $X_{ij} = +$. It is well-known by [12, Lemma 2.2] that $\leq_X$ is a partial order on $\mathbf{n}$. We associate to X two partitions $\phi(X)$ and $\psi(X)$ of n , which are determined by the poset $(\mathbf{n}, \leq_X)$ as in 2.3.

2.6. Recall that in [4], Kazhdan and Lusztig defined some equivalence classes in a Coxeter system (W, S) , called a *left cell*, a *right cell* and a *two-sided cell*.

Let W_a be the affine Weyl group of type $\tilde{A}_{n-1}$, $n \geq 2$. Each element w of W_a

determines a sign type $X(w)$ (see 1.7), and hence by 2.5 it in turn determines two partitions $\phi(w) := \phi(X(w))$ and $\psi(w) := \psi(X(w))$. This defines two maps $\phi, \psi : W_a \longrightarrow \Lambda_n$, each induces a bijection from the set of two-sided cells of W_a to the set Λ_n (see [7, Theorem 17.4]). Let $Y_0 = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s_0\}\}$ (see 1.8). By a result of Lusztig-Xi in [6, Theorem 1.2], the intersection of Y_0 with any two-sided cell $\phi^{-1}(\lambda)$ ($\lambda \in \Lambda_n$) is a single left cell, written Γ_λ , of W_a , called a *canonical left cell*.

§3. Lower boundary of a canonical left cell.

In this section, we define a lower boundary hyperplane for any $F \subset W_a$. We give some criteria for a hyperplane of E to be lower boundary for a canonical left cell of W_a .

3.1. For $i < j$ in $\mathbf{n}$ and $k \in \mathbb{Z}$, the hyperplane $H_{ij;k}$ divides the space E into three parts: $H_{ij;k}^+ = \{v \in E \mid \langle v, \alpha_{ij} \rangle > k\}$, $H_{ij;k}^- = \{v \in E \mid \langle v, \alpha_{ij} \rangle < k\}$, and $H_{ij;k}$. For any set F of alcoves in E , call $H_{ij;k}$ a *lower boundary hyperplane* of F , if $\cup_{A \in F} A \subset H_{ij;k}^+$ and if there exists some alcove C in F such that $\overline{C} \cap H_{ij;k}$ is a facet of C of dimension $n - 2$, where $\overline{C}$ stands for the closure of C in E under the usual topology.

3.2. Let Γ be a canonical left cell of W_a . As a subset in W_a , Γ is a union of some dominant sign types (see [7, Proposition 18.2.2]), denote by $S(\Gamma)$ the set of these sign types. Regarded as a union of alcoves, the topological closure of Γ in E is connected (see [7, Theorem 18.2.1]) and is bounded by a certain set of hyperplanes in E of the form $H_{ij;\epsilon}$, $1 \leq i < j \leq n$, $\epsilon = 0, 1$. (see 1.3). Then a lower boundary hyperplane of Γ must be one of such hyperplanes. Let $B(\Gamma)$ be the set of all the lower boundary hyperplanes of Γ . Given a hyperplane $H_{ij;\epsilon}$ with $1 \leq i < j \leq n$ and $\epsilon = 0, 1$, we see that $H_{ij;\epsilon} \in B(\Gamma)$ if and only if one of the following conditions holds.

(1) $\epsilon = 1$, $X_{ij} = +$ for all $X = (X_{ab}) \in S(\Gamma)$, and there exists some $Y = (Y_{ab}) \in S(\Gamma)$ such that the sign type $Y' = (Y'_{ab})$ defined below is admissible:

$$Y'_{ab} = \begin{cases} Y_{ab}, & \text{if } (a, b) \neq (i, j), \\ \circ, & \text{if } (a, b) = (i, j). \end{cases}$$

(2) $\epsilon = 0$, and there exists some $X = (X_{ab}) \in S(\Gamma)$ with $X_{ij} = \circ$ such that the sign type $X' = (X'_{ab})$ defined below is admissible:

$$X'_{ab} = \begin{cases} X_{ab}, & \text{if } (a, b) \neq (i, j), \\ -, & \text{if } (a, b) = (i, j). \end{cases}$$

Notice that by Lemma 1.2 (2), the case (2) happens only if $j = i + 1$.

Proposition 3.3. (1) $H_{i,i+1;0} \in B(\Gamma)$ if and only if there exists some $X = (X_{ab}) \in S(\Gamma)$ such that $X_{i,h} = X_{i+1,h}$ for all $h \in \mathbf{n}$. In particular, when these conditions hold, we have $X_{i,i+1} = \circ$.

(2) If $H_{ij;1} \in B(\Gamma)$ and if either $i \leq k < l \leq j$ or $k \leq i < j \leq l$, then $H_{kl;1} \in B(\Gamma)$ if and only if $(i, j) = (k, l)$.

Proof. (1) follows from 3.2 (2) and Lemma 1.5. Then (2) is a direct consequence of 3.2 (1) and Lemma 1.2 (1). $\square$

§4. Description of the sets $B_0(\Gamma_\lambda)$ and $B_1(\Gamma_\lambda)$.

In this section, we shall answer the two questions of Humphreys.

Let Γ_λ be the canonical left cell of W_a corresponding to $\lambda \in \Lambda_n$. Let $B_\epsilon(\Gamma_\lambda) = \{H_{ij;\epsilon} \mid H_{ij;\epsilon} \in B(\Gamma_\lambda)\}$ for $\epsilon = 0, 1$.

Lemma 4.1. Suppose that $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda_n$ contains at least two different parts. Then $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leq i < n\}$.

Proof. Let $\mu = (\mu_1, \dots, \mu_t)$ be the conjugate partition of λ . Then μ also contains at least two different parts. Given any p , $1 \leq p < n$. There exists a permutation $a_1, a_2, \dots, a_t$ of $1, 2, \dots, t$ such that $m_s < p$ and $m_{s+1} > p$ for some s , $0 \leq s < t$, where $m_u := \sum_{k=1}^u \mu_{a_k}$ for $0 \leq u \leq t$ with the convention that $m_0 = 0$. Define a dominant sign type $X = (X_{ij})$ such that for any i, j , $1 \leq i < j \leq n$, $X_{ij} = \circ$ if and only if $m_h < i < j \leq m_{h+1}$ for some h , $0 \leq h < t$. Clearly, X is admissible with $\psi(X) = \mu$. Hence $X \in S(\Gamma_\lambda)$. We see also that $X_{ph} = X_{p+1,h}$ for all h , $1 \leq h \leq n$. So we conclude that $H_{p,p+1;0} \in B_0(\Gamma_\lambda)$ by Proposition 3.3 (1). Our result follows by the notice preceding Proposition 3.3. $\square$

Lemma 4.2. For a rectangular partition $(k^a) \in \Lambda$ with $a, k \in \mathbb{N}$. we have $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leq p < n, a \nmid p\}$.

Proof. Let $X = (X_{ij})$ be a dominant admissible sign type. Then a maximal cochain in $\mathbf{n}$ with respect to $\leq_X$ must consist of consecutive numbers. Now suppose $\psi(X) = (a^k)$. Then by 2.4, we can take a maximal k -cochain-family $\mathbf{n} = P_1 \cup \dots \cup P_k$ such that $P_h = \{a(h-1) + 1, a(h-1) + 2, \dots, ah\}$, $1 \leq h \leq k$, are the maximal cochains in $\mathbf{n}$ with respect to $\leq_X$. We have $X_{a(h-1)+1, ah} = \circ$ and $X_{a(h-1)+1, ah+1} = +$, which are different. So by the arbitrariness of X and by Proposition 3.3 (1), we see that

$$(4.2.1) \quad H_{ah, ah+1;0} \notin B_0(\Gamma_{(k^a)}) \quad \text{for} \quad 1 \leq h < k.$$

On the other hand, let $Y = (Y_{ij})$ be a sign type defined by

$$Y_{ij} = \begin{cases} \circ & \text{if } a(h-1) < i < j \leq ah \text{ for some } 1 \leq h \leq k, \\ + & \text{otherwise} \end{cases}$$

for $1 \leq i < j \leq n$. Then it is clear that Y is dominant admissible with $\psi(Y) = (a^k)$. Suppose $a(h-1) < p < ah$ for some h , $1 \leq h \leq k$. Then $Y_{p,p+1} = \circ$. We see also that $Y_{ph} = Y_{p+1,h}$ for all h , $1 \leq h \leq n$. So by Proposition 3.3 (1), we have

$$(4.2.2) \quad H_{p,p+1;0} \in B_0(\Gamma_{(k^a)}) \quad \text{for all } p \text{ with } 1 \leq p < n \text{ and } a \nmid p.$$

Therefore our result follows by (4.2.1), (4.2.2) and by the notice preceding Proposition 3.3. $\square$

Theorem 4.3. $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leq i < n\}$ for all $\lambda \in \Lambda_n$ unless λ is a rectangular partition. In the latter case, say $\lambda = (k^a)$, $k, a \in \mathbb{N}$, we have $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leq p < n, a \nmid p\}$.

Proof. We see that a partition is nonrectangular if and only if it contains at least two different parts. So our result follows immediately from Lemmas 4.1 and 4.2. $\square$

Theorem 4.4. $B_1(\Gamma_\lambda) = \{H_{i,i+r;1} \mid 1 \leq i \leq n-r\}$ for $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda_n$.

Proof. Let $\mu = (\mu_1, \dots, \mu_t) \in \Lambda_n$ be conjugate to λ . First we claim that

$$(4.3.1) \quad X_{i,i+r} = +, \quad \forall 1 \leq i \leq n-r$$

for any $X = (X_{ij}) \in S(\Gamma_\lambda)$. Since otherwise, there would exist some $X = (X_{ij}) \in S(\Gamma_\lambda)$ with $X_{i,i+r} = \circ$ for some i , $1 \leq i \leq n-r$. By Lemma 1.2 (1), we have $X_{hk} = \circ$ for all h, k , $i \leq h < k \leq i+r$. Then $\{i, i+1, \dots, i+r\}$ would be a cochain in $\mathbf{n}$ with respect to the partial order $\leq_X$, whose cardinality is $r+1 > \mu_1 = r$, contradicting the assumption $\psi(\Gamma_\lambda) = (\mu_1, \mu_2, \dots, \mu_t)$.

Next we want to find, for any p , $1 \leq p \leq n - r$, some $Y = (Y_{ij}) \in S(\Gamma_\lambda)$ such that the sign type $Y' = (Y'_{ij})$ defined by

$$(4.3.2) \quad Y'_{ij} = \begin{cases} Y_{ij}, & \text{if } (i, j) \neq (p, p+r), \\ \circ, & \text{if } (i, j) = (p, p+r). \end{cases}$$

for $1 \leq i < j \leq n$, is admissible (in this case, we automatically have $\psi(Y') \geq \mu$ by the proof of (4.3.1)).

Take a permutation $a_1, a_2, \dots, a_t$ of $1, 2, \dots, t$ satisfying the following conditions:

- (1) Let $m_u = \sum_{k=1}^u \mu_{a_k}$ for $0 \leq u \leq t$ with the convention that $m_0 = 0$. Then there exists some s , $0 \leq s < t$, such that $a_{s+1} = 1$, $m_s < p$ and $m_{s+1} \geq p$.
- (2) s is the largest possible number with the property (1) when $a_1, a_2, \dots, a_t$ ranges over all the permutations of $1, 2, \dots, t$.

Then we have $t - s \geq 2$, $p \leq m_{s+1} < p + r$ and $m_{s+2} \geq p + r$. Define a dominant sign type $Y = (Y_{ij})$ such that $Y_{ij} = \circ$ if and only if one of the following cases occurs:

- (1) $m_u < i < j \leq m_{u+1}$, $0 \leq u < t$.
- (2) $p \leq i < j \leq p + r$ with $(i, j) \neq (p, p + r)$.

Then we see by Lemma 1.2 (1) that Y is admissible with $\psi(Y) = \mu$, i.e., $Y \in S(\Gamma_\lambda)$. Clearly, the sign type Y' obtained from Y as in (4.3.2) is also dominant admissible by Lemma 1.2 (1). This implies by 3.2 (1) that $H_{p, p+r; 1} \in B_1(\Gamma_\lambda)$ for any p , $1 \leq p \leq n - r$. Therefore our result follows by Proposition 3.3 (2). $\square$

Remarks 4.5. (1) Theorems 4.3 and 4.4 answer the two questions of Humphreys. In particular, the canonical left cells of W_a associated to the rectangular partitions are determined entirely by the corresponding B_1 -set of hyperplanes. From the above description of B_0 -sets of hyperplanes, we see that comparing with the other canonical left cells of W_a , the positions of the canonical left cells associated to rectangular partitions are farther from the walls of the dominant chamber.

(2) When $\lambda = (n)$, we have $B_0(\Gamma_\lambda) = \emptyset$ and $B_1(\Gamma_\lambda) = \{H_{i, i+1; 1} \mid 1 \leq i < n\}$. Actually, this is the unique canonical left cell whose B_1 -set contains a hyperplane of the form $H_{i, i+1; 1}$. Also, this is the unique canonical left cell whose B_0 -set is empty. On the other hand, $B_0(\Gamma_{(1^n)}) = \{H_{i, i+1; 0} \mid 1 \leq i < n\}$ and $B_1(\Gamma_{(1^n)}) = \emptyset$. $\Gamma_{(1^n)}$ is the unique canonical left cell whose B_1 -set is empty.

(3) When $n \in \mathbb{N}$ is a prime number, the B_0 -sets of all the canonical left cells Γ_λ of W_a are $\{H_{i, i+1; 0} \mid 1 \leq i < n\}$, except for the case where $\lambda = (n)$.

(4) Now assume that (W_a, S) is an irreducible affine Weyl group of arbitrary type with Δ a choice of simple roots system of Φ . We are unable to describe the lower boundary hyperplanes for a canonical left cell L of W_a in general. This is because that L is not always a union of some sign types (e.g., in the case of type $\tilde{B}_2$). But we know that L is a single sign type when L is in the either lowest or highest two-sided cell of W_a (see [10, 11]) for which we can describe its lower boundary hyperplanes: if L is in the lowest two-sided cell of W_a , $B_1(L) = \{H_{\alpha;1} \mid \alpha \in \Delta\}$ and $B_0(L) = \emptyset$, where $H_{\alpha;1} := \{v \in E \mid \langle v, \alpha^\vee \rangle = 1\}$ and $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$; if L is in the highest two-sided cell of W_a , then $B_1(L) = \emptyset$ and $B_0(L) = \{H_{\alpha;0} \mid \alpha \in \Delta\}$. This extends the result (2). We conjecture that any canonical left cell of W_a is a union of some sign types whenever W_a has a simply-laced type, i.e. type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$. If this is true, then one would be able to describe the lower boundary hyperplanes for the canonical left cells of these groups.

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