Presentations for finite complex reflection groups

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Abstract. We survey our achievements on the classification of congruence classes of presentations for the finite complex reflection groups. The classification is described in terms of certain graphs for the imprimitive groups, and is with the help of the computer programmes for the primitive groups.

Shephard and Todd classified all the finite complex reflection groups in their paper [10]. A finite complex reflection group $G$ can be presented by generators and relations just as that for a Coxeter group. We already have one presentation for each irreducible finite complex reflection group (see [1]). However, such a presentation is not unique for $G$ in general. Different presentations of $G$ may reveal various properties of $G$ (see [5] for example). Then it is desirable to define an equivalent relation, called congruent relation, among the presentations of $G$ and then to classify all the presentations of $G$ into congruence classes.

Finite complex reflection groups are divided into two main classes: primitive and imprimitive. Any imprimitive complex reflection group has the form $G(m, p, n)$ for some positive integers $m, p, n$ with $p|m$ (reading “$p$ divides $m$”), $m > 2$, $n > 1$, and $(m, p, n) \neq (m, m, 2)$ (see [2]). The imprimitive complex reflection groups form an infinite series. There are 23 primitive complex reflection groups in total, 8 of them has exactly one congruence class of presentations since they can be generated by only two reflections (see [1, Appendix 2]).

I completed the classification of the presentations $(S, P)$ for the groups $G(m, p, n)$ according to their congruence (see [8] [9]). The classification was made separately in the cases of $p = 1$, $p = m$ and $1 < p < m$. We established a bijection between the set of congruence classes of presentations $(S, P)$ of the group $G(m, m, n)$ (resp., $G(m, 1, n)$, $G(m, p, n)$ with $1 < p < m$) and the set of isomorphism classes of certain graphs $\Gamma_S$ (resp., of certain rooted graphs $\Gamma_S^r$). The relation set $P$ was chosen to be the set $P_S$ of the basic relations on $S$, which was defined separately in the cases of $p = 1$, $p = m$ and $1 < p < m$. The latter can be treated with uniformly now (see Section 4).

I, together with my students L. Wang and P. Zeng, found the number of congruence classes of the presentations for all the primitive complex reflection groups.

Key words and phrases. Complex reflection groups, presentations, congruence classes.

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Let $G$ except for the group $G_{34}$. We also found a representative set for all the congruence classes of presentations for 10 primitive complex reflection groups: $G_7, G_{11}, G_{12}, G_{15}, G_{19}, G_{24}, G_{25}, G_{26}, G_{27}, G_{32}$ (the notations are due to Shephard-Todd, see [10]), each of which is generated by at least three reflections (see [6] [7] [11] [12]). We achieved these results with the help of the computer programmes.

In the present paper, we give a survey on the above achievements.

1. Preliminaries

1.1. Let $V$ be an $n$-dimensional complex vector space with a hermitian form $( , )$. A reflection $s$ on $V$ is by definition an invertible linear transformation on $V$ with $o(s) < \infty$ and $\dim V^s = n - 1$, where $o(s)$ denotes the order of $s$ and $V^s := \{v \in V \mid s \cdot v = v\}$. Any reflection has the form $s_{\alpha,\zeta}$ for some non-zero vector $\alpha \in V$ and some root $\zeta$ of unity, where $s_{\alpha,\zeta}$ is defined by

$$s_{\alpha,\zeta}(v) = v + (\zeta - 1)(v, \alpha)\alpha$$

for all $v \in V$.

We also write $s_{a,d}$ for $s_{\alpha,\zeta}$ if $\zeta = e^{2\pi i/d}$. A reflection group $G$ on $V$ is a finite group generated by reflections on $V$.

A reflection group $G$ on $V$ is called a real group or a Coxeter group if there is a $G$-invariant $\mathbb{R}$-subspace $V_0$ of $V$ such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, then $G$ is called complex. (Note that, according to this definition, a real reflection group is not complex.)

1.2. A reflection group $G$ in $V$ is imprimitive, if there exists a decomposition $V = V_1 \oplus \ldots \oplus V_r$ into a direct sum of proper subspaces $V_1, \ldots, V_r$ such that $G$ permutes $V := \{V_i \mid 1 \leq i \leq r\}$ ($V$ is called an imprimitive system of $G$ in $V$). $G$ is primitive if otherwise.

1.3. Let $S_n$ be the symmetric group over $n$ numbers $1, 2, \ldots, n$. For $\sigma \in S_n$, denote by $[(a_1, \ldots, a_n)]\sigma$ the $n \times n$ monomial matrix with non-zero entries $a_i$ in the $(i, (i)\sigma)$-positions. For $p|m$ in $\mathbb{N}$, set

$$G(m, p, n) = \left\{[(a_1, \ldots, a_n)]\sigma \mid a_i \in \mathbb{C}, a_i^m = 1, \sigma \in S_n, \left(\prod_j a_j\right)^{m/p} = 1\right\}.$$

Any imprimitive complex reflection group $G$ on $V$ has the matrix form $G(m, p, n)$ with respect to a basis $e_1, e_2, \ldots, e_n$ for some $m, p, n \in \mathbb{N}$ with $m > 2, n > 1, p|m$ and $(m, p, n) \neq (m, m, 2)$, in particular, the imprimitive system $\{V_1, \ldots, V_r\}$ of $G$ in $V$ consists of one-dimensional subspaces.

1.4. It is well known that to each finite complex reflection group $G$, we can associate a root system $(R, f)$, where $R$ is a finite $G$-invariant set of unit vectors in the vector space $V$ and $f : R \to \mathbb{N} \setminus \{1\}$ is a function, constant on any $G$-orbit, such that $G$ is generated by the reflection set $\{s_{\alpha, f(\alpha)} \mid \alpha \in R\}$. As a root system for $G$, $(R, f)$ is determined by $G$ up to scalar factors (see [3, Subsection 1.9]). One can define a simple root system $(B, w)$ in $(R, f)$, where $B \subset R$ has the minimal cardinality with the following properties: $G \cdot B = R$, $G$ is generated by $\{s_{\alpha, w(\alpha)} \mid \alpha \in B\}$, and $w = f|_B$. A simple root system $B$ for $R$ is uniquely determined by $G$. In [6], we introduce an equivalence relation on simple root systems: Two simple root systems $(B, w)$ and $(B', w')$ of $(R, f)$ are equivalent, written $B \sim B'$, if there exists a bijection $\phi : B \to B'$ such that for any $\alpha, \beta \in B$,

1. $w(\alpha) = w'(\phi(\alpha))$ and,
1.5. For a reflection group $G$, a presentation of $G$ by generators and relations (or a presentation in short) is by definition a pair $(S, P)$, where

1. $S$ is a finite set of reflection generators for $G$ with minimal possible cardinality.

2. $P$ is a finite relation set on $S$, and any other relation on $S$ is a consequence of the relations in $P$.

Clearly, for any simple system $(B, w)$ for $G$, $S = \{s_{\alpha, w(\alpha)} \mid \alpha \in B\}$ forms a generator set of a presentation of $G$. Call $(B, w)$ the associated simple system of the presentation.

1.6. Two presentations $(S, P)$ and $(S', P')$ for $G$ are congruent, if there exists a bijection $\eta : S \rightarrow S'$ such that for any $s, t \in S$, $(s, t) \equiv (\eta(s), \eta(t))$.

Note that when a generator set $S$ of the group $G$ is given, we assume that all the relations on $S$ are known. Thus by the definition, we see that the congruence of a presentation $(S, P)$ of $G$ is determined entirely by the generator set $S$, the relation set $P$ plays no role concerning it.

We see that two presentations of $G$ are congruent if and only if their associated simple root systems are equivalent.

1.7. For a given $G$, one way to calculate the number of congruence classes of presentations is to find the number of equivalence classes of simple root systems in the root system of $G$. The latter can be done by calculation of the groups $S, P$ given rise by the action of the reflections $s_{\alpha}$, $\alpha \in R$. These can be done by a computer in general. We did it in such a way when $G$ is primitive (see 5.1).

2. Graphs associated to reflection sets of $G(m, p, n)$

2.1. We have $G(m, m, n) \subseteq G(m, p, n) \subseteq G(m, 1, n)$ for any $1 \leq p \leq m$ with $p|m$. There exists two kinds of reflections in $G(m, 1, n)$ as follows. (i) $s(i, j; k) := [(1, \ldots, 1, \zeta_m^k, 1, \ldots, 1)](i, j)$, where $\zeta_m := e^{2\pi i/m}$, the numbers $\zeta_m^k$, $\zeta_m^k$ are the $i$th, resp. $j$th components of the $n$-tuple and $(i, j)$ is the transposition of $i$ and $j$ for some $i < j$. Call $s(i, j; k)$ a reflection of type I. Set $s(j, i; k) = s(i, j; -k)$. (ii) $s(i; k) := [(1, \ldots, 1, \zeta_m^k, 1, \ldots, 1)]$ for some $k \in \mathbb{Z}$ with $m \mid k$, where $\zeta_m^k$ is the $i$th component of the $n$-tuple. Call $s(i; k)$ a reflection of type II. $s(i; k)$ has the order $m/\gcd(m, k)$.

All the reflections of type I lie in the subgroup $G(m, m, n)$.

2.2. Let $X = \{s(i, j; k) \mid h \in J\}$ be a set of reflections of type I in $G(m, 1, n)$ for some finite index set $J$. We associate to $X$ a digraph $\Gamma_X = (V, E)$ as follows. Its node set $V$ is $[n] := \{1, 2, \ldots, n\}$, and its arrow set $E$ consists of all the ordered pairs $(i, j)$, $i < j$, with labels $k$ for any $s(i, j; k) \in X$ (hence, if $s(i, j; k) \in X$ and $i > j$, then $\Gamma_X = (V, E)$ contains an arrow $(j, i)$ with the label $-k$). Denote by $\Gamma_X$ the underlying graph of $\Gamma_X$, i.e., $\Gamma_X$ is obtained from $\Gamma_X$ by replacing each labelled arrow by an unlabelled edge.

Clearly, the graph $\Gamma_X$ has no loop but may have multi-edges between two nodes.
The above definition of a graph can be extended: to any set \( X \) of reflections of \( G(m,1,n) \), we define a graph \( \Gamma_X \) to be \( \Gamma_{X'} \), where \( X' \) is the subset of \( X \) consisting of all the reflections of type I. When \( X \) contains exactly one reflection of type II (say \( s(i;k) \)), we define another graph, denoted by \( \Gamma_X^r \), which is obtained from \( \Gamma_X \) by rooting the node \( i \), i.e., \( \Gamma_X^r \) is a rooted graph with the rooted node \( i \). Sometimes we denote \( \Gamma_X^r \) by \( ([n], E, i) \).

### 2.3. Example.

Let \( n = 6 \).

1. If \( X = \{s(1,2;4), s(3,4;2), s(4,6;0), s(3,4;3)\} \), then \( \Gamma_X \) is as in Fig. 1 (a).

![Fig. 1](attachment:image1)

and \( \Gamma_X \) is as in Fig. 1 (b).

2. Let \( Y = X \cup \{s(6;3)\} \) be with \( X \) as in (1). Then \( \Gamma_Y^r \) is as in Fig. 2.

![Fig. 2](attachment:image2)

where the node labelled by 6 is rooted.

### 2.4. Two graphs \((N,E)\) and \((N',E')\) are isomorphic, if there exists a bijection \( \eta : N \to N' \) such that for any \( v, w \in N \), \( \{v, w\} \) is in \( E \) if and only if \( \{\eta(v), \eta(w)\} \) is in \( E' \).

Two rooted graphs \((N,E,i)\) and \((N',E',i')\) are isomorphic, if there exists a bijection \( \eta : N \to N' \) with \( \eta(i) = i' \) such that for any \( v, w \in N \), \( \{v, w\} \) is in \( E \) if and only if \( \{\eta(v), \eta(w)\} \) is in \( E' \).

### 2.5. In [8, Lemma 2.1], we showed that the generator set \( S \) of a presentation \((S,P)\) of the group \( G(m,1,n) \) consists of \( n-1 \) reflections of type I and one reflection of order \( m \). Then we showed the following

### 2.6. Theorem. (see [8, Theorem 2.8]) Let \( S \) be a subset of \( G(m,1,n) \) consisting of \( n-1 \) reflections of type I and one reflection of order \( m \) (\( m > 2 \) as assumed). Then \( S \) is the generator set of a presentation for \( G(m,1,n) \) if and only if the graph \( \Gamma_S^r \) is a rooted tree.

### 2.7. Assume that \( X \) is a reflection set of \( G(m,1,n) \) with \( \Gamma_X \) connected and having exactly one circle. Then for some integer \( 2 \leq r \leq n \), \( X \) contains the reflections \( s(a_ha_{h+1}; k_h) \) with some integers \( k_h \) for any \( 1 \leq h \leq r \) (the subscripts are modulo \( r \)). Denote by \( \delta(X) \) the absolute value of \( \sum_{h=1}^{r} k_h \).

By [8, Lemma 2.7] and [1, Appendix 2], we see that the generator set \( S \) of a presentation \((S,P)\) of the group \( G(m,m,n) \) consists of \( n \) reflections of type I such that the graph \( \Gamma_S^r \) is connected (hence contains exactly one circle). Then we showed the following.
2.8. **Theorem.** (see [8, Theorem 2.19]) Let \( S \) be a subset of \( G(m, m, n) \) consisting of \( n \) reflections of type I with \( \Gamma_S \) connected. Then \( S \) is the generator set of a presentation of \( G(m, m, n) \) if and only if the value \( \delta(S) \) is coprime to \( m \).

2.9. Next we consider the group \( G(p, m, n) \) for any \( m, p, n \in \mathbb{N} \) with \( p | m \) and \( 1 < p < m \). By [9, Lemma 2.2], we know that for \( 1 < p < m \) with \( p | m \), the generator set \( S \) in a presentation \((S, P)\) of the group \( G(m, p, n) \) consists of \( n \) reflections of type I and one reflection of order \( m/p \) and type II such that the graph \( \Gamma_S \) is connected (hence containing exactly one circle). Then we get the following

2.10. **Theorem.** (see [9, Theorem 2.4]) Assume that \( S \) is a subset of \( G(m, p, n) \) consisting of \( n \) reflections of type I and one reflection of order \( m/p \) and type II such that \( \Gamma_S \) is connected. Then \( S \) is the generator set of a presentation of \( G(m, p, n) \) if and only if \( \gcd\{p, \delta(S)\} = 1 \).

3. **The classification of presentations of** \( G(m, p, n) \)

Let \( \Sigma(m, p, n) \) be the set of the presentations \((S, P)\) of \( G(m, p, n) \) and let \( \tilde{\Sigma}(m, p, n) \) be the set of congruence classes of \( \Sigma(m, p, n) \). In the present section, we shall describe the set \( \tilde{\Sigma}(m, p, n) \) in the cases of \( p = 1 \), \( p = m \) and \( 1 < p < m \), separately.

3.1. It is known by [8, Subsection 3.1] that \( S, S' \in \Sigma(m, 1, n) \) are congruent if and only if \( \Gamma_S \cong \Gamma_{S'} \). Also, it is known by [8, Subsection 3.3] that \( S, S' \in \Sigma(m, m, n) \) are congruent if and only if \( \Gamma_S \cong \Gamma_{S'} \). So we get the following two theorems concerning the classification of congruence classes of presentations for the groups \( G(m, 1, n) \) and \( G(m, m, n) \).

3.2. **Theorem.** (see [8, Theorem 3.2]) The map \((S, P) \rightarrow \Gamma_S^r \) induces a bijection from the set \( \tilde{\Sigma}(m, 1, n) \) to the set of isomorphism classes of rooted trees with \( n \) nodes.

3.3. **Theorem.** (see [8, Theorem 3.4]) The map \((S, P) \rightarrow \Gamma_S \) induces a bijection from the set \( \tilde{\Sigma}(m, m, n) \) to the set of isomorphism classes of connected graphs with \( n \) nodes and \( n \) edges (or equivalently with \( n \) nodes and exactly one circle).

3.4. **Example.** Let \( n = 4 \).

(1) There are 4 isomorphic classes of rooted trees of nodes 4 (see Fig. 3).

Hence \( G(m, 1, 4) \) has 4 congruence classes of presentations.

(2) There are 5 isomorphic classes of connected graphs with exactly one circle (see Fig 4).
Hence \( G(m, m, 4) \) has 5 congruence classes of presentations.
In the remaining part of the section, we always assume \( 1 < p < m \) and \( p|m \).

### 3.5
It is known by [9, Lemma 2.7] that \( S, S' \in \Sigma(m, p, n) \) are congruent if and only if one of the following conditions holds:
1. The circle of \( \Gamma^r_S \) contains more than two nodes and \( \Gamma^r_S \cong \Gamma^r_{S'} \) (see 2.4);
2. The circle of \( \Gamma^r_S \) contains exactly two nodes, \( \Gamma^r_S \cong \Gamma^r_{S'} \) and \( \gcd\{m, \delta(S)\} = \gcd\{m, \delta(S')\} \).

### 3.6
Denote by \( \Lambda(m, p) \) the set of all the numbers \( d \in \mathbb{N} \) such that \( d|m \) and \( \gcd\{d, p\} = 1 \). Let \( \Gamma(m, p, n) \) be the set of all the connected rooted graphs with \( n \) nodes and \( n \) edges. Let \( \Gamma_1(m, p, n) \) be the set consisting of all the rooted graphs in \( \Gamma(m, p, n) \) each of which contains a two-nodes circle. Let \( \Gamma_2(m, p, n) \) be the complement of \( \Gamma_1(m, p, n) \) in \( \Gamma(m, p, n) \). Denote by \( \tilde{\Gamma}(m, p, n) \), resp., \( \tilde{\Gamma}_i(m, p, n) \) the set of the isomorphism classes in the set \( \Gamma(m, p, n) \), resp., \( \Gamma_i(m, p, n) \) for \( i = 1, 2 \) (see 2.4).

The following result describes all the congruence classes of presentations for \( G(m, p, n) \) in terms of rooted graphs.

### 3.7 Theorem
(see [9, Theorem 2.9])
1. The map \( \psi : S \mapsto \Gamma^r_S \) from \( \Sigma(m, p, n) \) to \( \Gamma(m, p, n) \) induces a surjection \( \tilde{\psi} : \tilde{\Sigma}(m, p, n) \to \tilde{\Gamma}(m, p, n) \).
2. Let \( \tilde{\Sigma}_1(m, p, n) = \tilde{\psi}^{-1}(\tilde{\Gamma}_1(m, p, n)) \) for \( i = 1, 2 \). Then the map \( \psi \) gives rise to a bijection: \( \tilde{\Sigma}_2(m, p, n) \mapsto \tilde{\Gamma}_2(m, p, n) \); also, \( S \mapsto (\Gamma^r_S, \gcd\{m, \delta(S)\}) \) induces a bijection: \( \tilde{\Sigma}_1(m, p, n) \mapsto \tilde{\Gamma}_1(m, p, n) \times \Lambda(m, p) \).

### 3.8 Example
Let \( n = 4, m = 6 \) and \( p = 2 \). Then \( \Lambda(6, 2) = \{1, 3\} \). There exist 13 isomorphic classes of rooted connected graphs with exactly one circle, 9 of them contain a two-nodes circle (see Fig. 5). So \( G(6, 2, 4) \) has \( 22 = 9 \times 2 + 4 \) congruence classes of presentations.
4. The relation set of a presentation for $G(m, p, n)$

We always assume $1 \leq p \leq m$, $m > 2$, $n > 1$, $p|m$ and $(m, p, n) \neq (m, m, 2)$ in the section. In [8, Section 4] and [9, Section 4], we defined the basic relations on the generator set $S$ of a presentation $(S, P)$ for the groups $G(m, p, n)$ in the cases of $p = 1$, $p = m$ and $1 < p < m$ separately. In the present section, we shall give a uniform treatment for these relations.

4.1. Let $S \in \Sigma(m, p, n)$. By the results stated in the previous sections, we can write

\[ S = \{ s = s(a, k), t_h \mid h \in J \}, \]

where $\gcd\{m, k\} = \gcd\{m, p\}$ (Thus, if $p = m$ then $s = 1$, which can be removed from $S$), $J$ is an index set with $|J| = n - 1$ if $p = 1$ and $|J| = n$ if $1 < p \leq m$, and all the reflections $t_h$, $h \in J$, are of type I. The graph $\Gamma^r_S$ is always connected. When $1 \leq p < m$, we have the rooted graph $\Gamma^r_S$ with the node $a$ rooted; when $1 < p \leq m$, the graph $\Gamma_S$ contains exactly one circle.

4.2. Now assume $1 < p \leq m$. Take any node $x$ of $\Gamma_S$. Call a sequence of nodes $\xi_x : a_0 = x, a_1, ..., a_r = x$ in $\Gamma_S$ a generalized circle sequence of $\Gamma^r_S$ at the node $x$ if $S$ contains reflections $t_h = s(a_{h-1}, a_h; k_h)$ for $1 \leq h \leq r$ with some integers $k_h$, where $t_l \neq t_{l+1}$ for $1 \leq l < r$. Since the graph $\Gamma_S$ is connected and contains a unique circle, the sequence $\xi_x$ always exists. $\xi_x$ contains all the nodes on the circle of $\Gamma_S$ and is uniquely determined by the set $S$ and the node $x$ up to an orientation of the circle.

When $x = a$ is the rooted node of $\Gamma^r_S$ (this is the case only when $1 < p < m$), $\xi_x$ is also called a root-circle sequence of $\Gamma^r_S$.

Call $s_{h_j} := t_h t_{h+1} ... t_j t_{j-1} ... t_h$ a path reflection of $\Gamma_S$ in $\xi_x$ for any $1 \leq h < j \leq r$ (see Fig. 6).

\[ t_h \quad t_{h+1} \quad \ldots \quad t_j \]

Fig. 6

Let $c_x, c'_x$ be the smallest, resp., the largest integer with the node $a_{c_x}$, resp., $a_{c'_x}$ lying on the circle of $\Gamma_S$. Then $x$ is on the circle of $\Gamma_S$ if and only if $c_x = 0$ and $c'_x = r$.

4.3. Example.
Here the generalized circle sequence $\xi_x$ at the node $x$ is $a_0 = x, a_1, ..., a_{10} = x$, which is also a root-circle sequence. Hence $r = 10$, $c_x = 2$, $c'_x = 8$, $a_1 = a_9$, $a_2 = a_8$ and $s_1, a = t_1t_2t_3t_4t_5t_2t_1$. We have $s_{1,2} = s_{9,10}$ since $t_1 = t_{10}$, $t_2 = t_9$ and $t_1t_2t_1 = t_2t_1t_2$.

4.4. Let $S$ be as in (4.1). By [8, Theorems 4.17 and 4.20] and [9, Theorem 6.2], we see that $(S, P)$ is a presentation of $G(m, p, n)$ if $P$ consists of relations (A)-(M) as follow. Here the path reflections $s_{ij}, s_{j+1,r}$ are with respect to the generalized circle sequence $\xi_x$ in 4.2 and satisfy $c_x < j < c'_x$ whenever it is applicable. $\{x, a_j\}$ is called an admissible node pair of $\Gamma_S$, at which we are allowed to talk about relations (J)-(L) on $S$, where $x$ is required to be the rooted node $a$ of $\Gamma_S^x$ for relations (K)-(L). $u, v \in S$ in (M) satisfy that the edges $e(u), e(v)$ are incident to $\xi_x$ at the nodes $x, a_j$ respectively.

(A) $s^{m/p} = 1$.
(B) $t_i^2 = 1$ for $i \in J$.
(C) $t_it_j = t_jt_i$ if the edges $e(t_i)$ and $e(t_j)$ have no common end node.
(D) $t_it_jt_i = t_jt_it_j$ if the edges $e(t_i)$ and $e(t_j)$ have exactly one common end node.

(E) $st_is = t_is^t_i$ if $a$ is an end node of $e(t_i)$.
(F) $st_is = t_is$ if $a$ is not an end node of $e(t_i)$.
(G) $(t_it_j)^{m/d} = 1$ if $t_i \neq t_j$ with $e(t_i)$ and $e(t_j)$ having two common end nodes, where $d = \gcd\{m, \delta(S)\}$.

(H) $t_i \cdot t_jt_i = t_jt_it_i \cdot t_i$ for any triple $X = \{t_i, t_j, t_k\} \subseteq S$ with $\Gamma_X$ having a branching node.

(I) $s \cdot t_it_jt_i = t_it_jt_i \cdot s$, if $e(t_i)$ and $e(t_j)$ have exactly one common end node $a$.

(J) $(s_j s_{j+1,r})^{m/p} = 1$.

(K) $s_j s_{j+1,r} = s_j s_{j+1,r} s$.

(L) $(s_j s_{j+1,r})^{p-1} = s^{-\delta(S)} s_j s^\delta(S) s_{j+1,r}$.

(M) $a us_j u \cdot vs_j v u = vs_j v u \cdot us_j u$.

Here any of the above relations involving the reflection $s$ (i.e., any of the relations (A), (E), (F), (I), (K) and (L)) is applicable only in the case of $1 \leq p < m$; while any of the above relations involving a circle (i.e., any of the relations (G), (J), (K), (L) and (M)) is applicable only in the case of $1 < p < m$.

4.5. In 4.4, call (A)-(B) the order relations, (C)-(G) the braid relations on $S$. Call (A)-(F) the order-braid relations (or o.b. relations in short) on $S$ (note that relation (G) is not included in the o.b. relations on $S$). Call (H) the branching relations, (I) the root-braid relations, (H) the branching relations, (I) the root-braid relations, (G), (J), (K) the circle relations, (L) the circle-root relations, and (M) the branching-circle relations on $S$.

Call all the relations (A)-(M) above the basic relations on $S$. We have the following

4.6. Theorem. (see [8, Theorems 4.17 and 4.20] and [9, Theorem 6.2]) Let $S \in \Sigma(m, p, n)$ and let $P_S$ be the set of all the basic relations on $S$. Then $(S, P_S)$ forms a presentation of $G(m, p, n)$.
4.7. A presentation \((S, P)\) of a reflection group \(G\) is essential if \((S, P_0)\) is not a presentation of \(G\) for any proper subset \(P_0\) of \(P\).

Let \(S \in \Sigma(m, p, n)\) be as in (4.1) with \(1 \leq p \leq m\) and let \(P_S\) be the set of all the basic relations \((A)\)-(M) on \(S\). Then the presentation \((S, P_S)\) of \(G(m, p, n)\) is not essential in general.

Let us take the case of \(1 < p < m\) as an example,

(a) Let \((B')\) be any relation in \((B)\). Then \((B')\) is equivalent to \((B)\) under the assumption of \((D)\).

(b) Let \((K')\) (resp., \((L')\)) be a relation in \((K)\) (resp., \((L)\)) at any one admissible node pair. Then \((K')\) (resp., \((L')\)) is equivalent to \((K)\) (resp., \((L)\)) under the assumption of the o.b. relations on \(S\).

(c) For any branching node \(v\) of \(\Gamma_S\), fix some \(t_v \in S\) of type I with \(e(t_v)\) incident to \(v\). Set

\[(H')\quad \text{The relation } t_v \cdot tt't = tt't \cdot t_v \text{ holds for any } t \neq t' \text{ in } S \setminus \{t_v\} \text{ of type I with } e(t), e(t_a) \text{ having just one common end node } a.

Then \((H')\) is a subset of and is equivalent to \((H)\) under the assumption of the o.b. relations on \(S\).

(d) Let \(a\) be the rooted node in \(\Gamma_S^r\). Fix some \(t_a \in S\) of type I with \(e(t_a)\) incident to \(a\). Set \((\Gamma')\quad s \cdot t_a t_a = t_a t_a a \cdot s\) for any \(t \in \Gamma_S \setminus \{t_a\}\) of type I with \(e(t), e(t_a)\) having just one common end node \(a\).

Then \((\Gamma')\) is a subset of and is equivalent to \((\Gamma)\) under the assumption of the o.b. relations and the branching relations on \(S\). (e) \((G)\) is a special case of \((J)\), while \((J)\) is a consequence of the o.b. relations on \(S\).

(f) Let \((M')\) be the relations \((M)\) if \(\Gamma_S^r\) has a two-nodes circle and be the empty set of relations if otherwise.

(g) Assume that \(\Gamma_S^r\) has a two-nodes circle with the rooted node on the circle and not adjacent to any node outside the circle and that \(\gcd\delta(S), m\} = p\). Then relation \((E)\) is a consequence of \((L)\) and the other o.b. relations on \(S\). Let \((E')\) be the empty set of relations in this case and be the relation \((E)\) in any other case.

Let \(P'_S\) be the collection of relations \((A), (B'), (C), (D), (E'), (F), (H'), (\Gamma'), (L'), (M')\). Then \((S, P'_S)\) is again a presentation of \(G(m, p, n)\) (see [9, Remark 6.9 (1)]).

One may ask if the presentation \((S, P'_S)\) is always essential. The answer is still negative. Recently, Liu and Shi have showed that each of the relations \((H'), (\Gamma'), (M')\) could be further reduced (see [4]).

4.8. Among all the presentation \((S, P_S)\) of \(G(m, 1, n)\), the relation set \(P_S\) has a simpler form when \(\Gamma_S\) is a string with the rooted node at one end (see [1, Appendix 2]). Among all the presentations \((S, P_S)\) of \(G(m, p, n)\), \(1 < p \leq m\), we single out two kinds of presentations whose relation sets have simpler forms:

(i) One is when \(\Gamma_S\) is a string with a two-nodes circle at one end, and with the rooted node on the circle, not incident to any node outside the circle if \(1 < p < m\) (see [1, Appendix 2]);

(ii) The other is when \(\Gamma_S\) is a circle. In this case, if \(p = m\), then the relation set \(P_S\) can only consist of some o.b. relations and one circle relation (see [5, Proposition 3.3]); if \(1 < p < m\), then \(P_S\) can only consist of some o.b. relations, one root-braid relation, one root-circle relation and one circle-root relation (see [9, Remark 6.9 (2)]).
5. Presentations for the primitive complex reflection groups

5.1. In Table 1, we record results of L. Wang, P. Zeng and J. Y. Shi on the numbers $N(G)$ of congruence classes of presentations for the primitive complex reflection groups $G$ (see [6]), where the numbers $N(G_i)$ for $i = 12, 24, 25, 26$ were got by Shi (see [6]), for $i = 13, 22, 27, 29, 31, 33$ by Wang (see [11]), and for $i = 7, 11, 15, 19, 32$ by Zeng (see [12]).

<table>
<thead>
<tr>
<th>$G$</th>
<th>$N(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_7$</td>
<td>2</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>4</td>
</tr>
<tr>
<td>$G_{15}$</td>
<td>18</td>
</tr>
<tr>
<td>$G_{22}$</td>
<td>2</td>
</tr>
<tr>
<td>$G_{25}$</td>
<td>6</td>
</tr>
<tr>
<td>$G_{29}$</td>
<td>61</td>
</tr>
<tr>
<td>$G_{33}$</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 1

Since any $G$ generated by $\leq 2$ reflections satisfies $N(G) = 1$, $G_{34}$ is the only primitive complex reflection group with $N(G_{34})$ unknown.

5.2. We list a representative set for the congruence classes of presentations (or r.c.p. for brevity) of the primitive complex reflection groups $G_{12}, G_{24}, G_{27}, G_{25}, G_{26}, G_{7}, G_{15}, G_{19}, G_{11}, G_{32}$ (see 5.3–5.12), where the groups $G_{12}, G_{24}, G_{25}, G_{26}, G_{19}, G_{11}, G_{32}$ were done by Shi (see [6] [7]), $G_{27}$ by Wang (see [11]), and $G_{7}, G_{15}$ by Zeng (see [12]). The first presentation for each group was given in [1, Appendix 2]. According to the Shephard-Todd’s classification (see [10]), the groups $G_{25}, G_{26}, G_{7}, G_{15}, G_{19}, G_{11}$ and $G_{32}$ form the full set of such primitive complex reflection groups each of which is generated by more than two reflections and contains some reflections of order $> 2$.

5.3. r.c.p. for the group $G_{12}$. (see [6, Propositions 3.3–3.7]):

1. $G_{12} = \langle s, u, t \mid s^2 = u^2 = t^2 = 1, susu = ussu, ussu = uusu = wsu, swus = \text{utut} \rangle$.
2. $G_{12} = \langle s, u, w \mid s^2 = u^2 = w^2 = 1, sususu = ususu, ususu = wusu = uswu \rangle$.
3. $G_{12} = \langle s, u, x \mid s^2 = u^2 = x^2 = 1, wusu = wusu, wsusu = uswu \rangle$.
4. $G_{12} = \langle s, u, y \mid s^2 = u^2 = y^2 = 1, susuy = susuy, yusu = yusu \rangle$.
5. $G_{12} = \langle s, x, y \mid s^2 = x^2 = y^2 = 1, = xys, yx = yxy, xysx = yxsx \rangle$.

Here $w = utut, x = utu, y = tst, t = ususu, u = uyus = yxsx$ and $u = sxsx$.

5.4. r.c.p. for the group $G_{24}$. (see [6, Propositions 4.3–4.5]):

1. $G_{24} = \langle s, u, t \mid s^2 = u^2 = t^2 = 1, stst = stst, usu = usu, utut = tusutus \rangle$.
2. $G_{24} = \langle s, t, x \mid s^2 = t^2 = x^2 = 1, stst = stst, sxs = stsx, txt = xtx, stsxstxs = stsxstxs \rangle$.
3. $G_{24} = \langle s, t, y \mid s^2 = t^2 = y^2 = 1, stst = stst, tyt = ytyt, tstyts = ytysty \rangle$.

Here $x = sus, y = tut$ and $u = sxs = tyt$. 

5.5. r.c.p. for the group $G_{27}$. (see [11, Propositions 3.1.1–3.1.6]):

1. $G_{27} = \langle s, u, t \mid s^2 = u^2 = t^2 = 1, utu = tut, stst = tsts, susus = ususu, usutsus = tusutu \rangle$.

2. $G_{27} = \langle s, u, w \mid s^2 = u^2 = w^2 = 1, susus = ususu, uwu = uwu, sws = wsw, swsuwsuw = wuwusuwus \rangle$.

3. $G_{27} = \langle s, t, w \mid s^2 = t^2 = w^2 = 1, stst = tsts, wt = twt, sws = sws, swtswtsw = wntswtswt \rangle$.

4. $G_{27} = \langle s, u, x \mid s^2 = u^2 = x^2 = 1, susus = ususu, uxsxux = xxsux, uxexux = xesux \rangle$.

5. $G_{27} = \langle s, t, y \mid s^2 = t^2 = y^2 = 1, stst = tsts, sysys = ysysy, ysysys = ysysy, ysysys = ysysys \rangle$.

Here $w = utu, x = sts, y = susus, t = uwu = sxs = ysysys$ and $u = ysy$.

5.6. r.c.p. for the group $G_{25}$. (see [6, Propositions 5.3–5.4]):

1. $G_{25} = \langle t, u, v \mid t^3 = u^3 = v^3 = 1, tut = utu, uwu = vuy, tv = vt \rangle$.

2. $G_{25} = \langle t, u, x \mid t^3 = u^3 = x^3 = 1, tut = utu, wx = xux, xtu = xtu \rangle$.

Here $x = u^2v$ and $v = uxu^2$.

Each of the above presentations for $G_{12}, G_{24}$, $G_{27}$ and $G_{25}$ becomes essential after removing any two of the order relations.

5.7. r.c.p. for the group $G_{26}$. (see [6, Propositions 6.3–6.4]):

1. $G_{26} = \langle t, u, v \mid t^2 = u^3 = v^3 = 1, tutu = utut, uwu = vuv, tv = vt \rangle$.

2. $G_{26} = \langle t, u, x \mid t^2 = u^3 = x^3 = 1, tutu = utut, wx = xux, uxt = uxtu \rangle$.

Here $x = u^2v$ and $v = uxu^2$.

Each presentation becomes essential after removing any one of the order 3 relations.

5.8. r.c.p. for the group $G_{7}$. (see [12, Propositions 4.1.1–4.1.2]):

1. $G_{7} = \langle t, u, s \mid t^2 = u^3 = s^3 = 1, tus = ust = stu \rangle$.

2. $G_{7} = \langle t, x, s \mid t^2 = x^3 = s^3 = 1,uxt = xsx, txtsts = stxxt \rangle$.

Here $x = stus^2t$ and $u = stxts$.

5.9. r.c.p. for the group $G_{15}$. (see [12, Propositions 4.2.1–4.2.4]):

1. $G_{15} = \langle t, u, s \mid t^2 = u^3 = s^3 = 1, tus = ust = stu \rangle$.

2. $G_{15} = \langle t, x, s \mid t^2 = x^2 = s^3 = 1, txxs^2txs = xsxts^2tx = xsxtsxs \rangle$.

3. $G_{15} = \langle t, y, s \mid t^2 = y^2 = s^3 = 1, tyytysys = tsysy \rangle$.

4. $G_{15} = \langle t, z, s \mid t^2 = z^2 = s^3 = 1, tzst = zstz = zsztst \rangle$.

Here $x = ststxs, y = s^2as$ and $z = ttt$. Hence $u = tssxt^2 = syt^2 = tzt$.

5.10. r.c.p. for the group $G_{19}$. (see [7, (2.1.1) and Propositions 2.4–2.8]) :

1. $G_{19} = \langle t, u, s \mid t^2 = u^3 = s^5 = 1, tus = ust = stu \rangle$.

2. $G_{19} = \langle t, u, w \mid t^2 = u^3 = w^5 = 1, wuww = uwuw, u^2t \cdot w^2 = w^2 \cdot utu \rangle$.

3. $G_{19} = \langle t, v, w \mid t^2 = v^3 = w^5 = 1, vv^2wv = vuv^2w, vv^3wv = vtw^2wv \rangle$.

4. $G_{19} = \langle t, x, z \mid t^2 = x^2 = z^3 = 1, xxzx = xxx, xxz^2z \cdot ztt = tz \cdot xxz^2z \rangle$.

5. $G_{19} = \langle t, y, w \mid t^2 = y^2 = w^3 = 1, ywv = wyw, yw^2y \cdot wt = w^2 \cdot yw^2y \rangle$.

Here $v = s^2u^3s^2, y = ustus^4tu^2, y = x^2 = ustu^4tu^2, w = su^3s^2us^4$ and $z = s^2$. Hence $s = w^2tu^2uwv = tvu^3v^2t = tvy^3y^2t = z^3$ and $u = s^2v^2s^3 = w^3v^2w^2 = w^2yw^3 = z^2t^2xztx$. 

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5.11. r.c.p. for the group $G_{11}$. (see [7, (3.1.1) and Propositions 3.3–3.5]):

(1) $G_{11} = \langle t, u, s \mid t^2 = u^3 = s^4 = 1, tus = ust = stu \rangle$

(2) $G_{11} = \langle t, u, z \mid t^2 = u^3 = z^4 = 1, tuz = zut, tuzu = uzt \rangle$

(3) $G_{11} = \langle r, u, s \mid r^2 = u^3 = s^4 = 1, usus = susu, usur = rs \cdot su_2 \rangle$

(4) $G_{11} = \langle r, u, w \mid r^2 = w^3 = u^4 = 1, wuwu = uvw, w \cdot w_2 r \cdot w = u \cdot w_2 r \cdot w \rangle$

Here $w = w^3 su, z = u^2 s^3 u$ and $r = us^4 tsu_2$. Hence $s = uwu^2 = uz^3 u^2$ and $t = su^2 rus^3 = w^3 rw = rz^3$.

All the above presentations for the groups $G_7, G_{15}, G_{19}$ and $G_{11}$ are essential.

5.12. r.c.p. for the group $G_{32}$. (see [7, (4.1.1) and Propositions 4.3–4.6]):

(1) $G_{32} = \langle t, u, v, w \mid t^3 = u^3 = v^3 = w^3 = 1, tut = utu, wvu = vuw, vuv = \rangle$

(2) $G_{32} = \langle t, u, v, x \mid t^3 = u^3 = v^3 = x^3 = 1, tut = utu, tv = vt, tx = xt, vux =\rangle$

(3) $G_{32} = \langle r, u, v, x \mid r^3 = u^3 = v^3 = x^3 = 1, uuw = uuw, wur = wur, vuv = \rangle$

(4) $G_{32} = \langle t, s, w, y \mid t^3 = s^3 = w^3 = y^3 = 1, tw = wt, tvt = ytv, ywy = \rangle$

(5) $G_{32} = \langle t, u, v, m \mid t^2 = u^3 = z^3 = m^3 = 1, uzu^2 m^2 tm = m^2 tzu_2 u^2, tzm = zt, um = mu, zmz = zmz, tut = utu,uzu = zuu, tmt = tmt \rangle$

Here $x = vwu^2, r = utu^2, s = uw^2, y = u^2 v, z = uw^2$ and $m = t^2 u^2 v u t$. Hence $t = u^2 ru, u = sys^2, v = s^2 y = utmt^2 u^2$ and $w = v^2 xv = utmt^2 u^2 z t u t^2 u^2$.

Any of the presentations (1)–(4) of the group $G_{32}$ becomes essential after removing any three of the four order relations. However, in order to make presentation (5) of $G_{32}$ essential, one need remove any one of the relations $z zm = zm, tut = utu, uzu = zuu$ and $tmt = tmt$ in addition.

References


[8] J. Y. Shi, Congruence classes of presentations for the complex reflection groups $G(m, 1, n)$ and $G(m, m, n)$, to appear in Indag. Math.


