Presentations for finite complex reflection groups

Jian-yi Shi

Abstract. We survey our achievements on the classification of congruence classes of presentations for the finite complex reflection groups. The classification is described in terms of certain graphs for the imprimitive groups, and is with the help of the computer programmes for the primitive groups.

Shephard and Todd classified all the finite complex reflection groups in their paper [**10**]. A finite complex reflection group G can be presented by generators and relations just as that for a Coxeter group. We already have one presentation for each irreducible finite complex reflection group (see [**1**]). However, such a presentation is not unique for G in general. Different presentations of G may reveal various properties of G (see [**5**] for example). Then it is desirable to define an equivalent relation, called *congruent relation*, among the presentations of G and then to classify all the presentations of G into congruence classes.

Finite complex reflection groups are divided into two main classes: primitive and imprimitive. Any imprimitive complex reflection group has the form $G(m, p, n)$ for some positive integers m, p, n with $p|m$ (reading " p divides m "), $m > 2$, $n > 1$, and $(m, p, n) \neq (m, m, 2)$ (see [**2**]). The imprimitive complex reflection groups form an infinite series. There are 23 primitive complex reflection groups in total, 8 of them has exactly one congruence class of presentations since they can be generated by only two reflections (see [**1**, Appendix 2]).

I completed the classification of the presentations (S, P) for the groups $G(m, p, n)$ according to their congruence (see [**8**] [**9**]). The classification was made separately in the cases of $p = 1$, $p = m$ and $1 < p < m$. We established a bijection between the set of congruence classes of presentations (S, P) of the group $G(m, m, n)$ (resp., $G(m, 1, n)$, $G(m, p, n)$ with $1 < p < m$) and the set of isomorphism classes of certain graphs Γ_S (resp., of certain rooted graphs Γ_S^r). The relation set P was chosen to be the set P_S of the basic relations on S, which was defined separately in the cases of $p = 1$, $p = m$ and $1 < p < m$. The latter can be treated with uniformly now (see Section 4).

I, together with my students L. Wang and P. Zeng, found the number of congruence classes of the presentations for all the primitive complex reflection groups

Key words and phrases. Complex reflection groups, presentations, congruence classes.

Supported by Nankai Univ., the 973 Project of MST of China, the NSF of China, the SF of the Univ. Doctoral Program of ME of China, the Shanghai Priority Academic Discipline, and the CST of Shanghai.

G except for the group G_{34}. We also found a representative set for all the congruence classes of presentations for 10 primitive complex reflection groups: G_7, G_{11}, G_{12}, G_{15}, G_{19}, G_{24}, G_{25}, G_{26}, G_{27}, G_{32} (the notations are due to Shephard-Todd, see [**10**]), each of which is generated by at least three reflections (see [**6**] [**7**] [**11**] [**12**]). We achieved these results with the help of the computer programmes.

In the present paper, we give a survey on the above achievements.

1. Preliminaries

1.1. Let V be an n-dimensional complex vector space with a hermitian form $(\ ,\ )$. A *reflection* s on V is by definition an invertible linear transformation on V with $o(s) < \infty$ and $\dim V^s = n-1$, where $o(s)$ denotes the order of s and $V^s := \{v \in V \mid s\cdot v = v\}$. Any reflection has the form $s_{\alpha,\zeta}$ for some non-zero vector $\alpha \in V$ and some root ζ of unity, where $s_{\alpha,\zeta}$ is defined by

$$s_{\alpha,\zeta}(v) = v + (\zeta - 1)(v,\alpha)\alpha \qquad \text{for all } v \in V.$$

We also write $s_{a,d}$ for $s_{a,\zeta}$ if $\zeta = e^{2\pi i/d}$. A *reflection group* G on V is a finite group generated by reflections on V.

A reflection group G on V is called a *real* group or a *Coxeter group* if there is a G-invariant $\mathbb{R}$-subspace V_0 of V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, then G is called *complex.* (Note that, according to this definition, a real reflection group is not complex.)

1.2. A reflection group G in V is *imprimitive*, if there exists a decomposition $V = V_1 \oplus \ldots\ldots \oplus V_r$ into a direct sum of proper subspaces $V_1, ..., V_r$ such that G permutes $\mathbf{V} := \{V_i \mid 1 \le i \le r\}$ ($\mathbf{V}$ is called an *imprimitive system* of G in V). G is *primitve* if otherwise.

1.3. Let S_n be the symmetric group over n numbers $1, 2, ..., n$. For $\sigma \in S_n$, denote by $[(a_1, ..., a_n)|\sigma]$ the $n \times n$ monomial matrix with non-zero entries a_i in the $(i, (i)\sigma)$-positions. For $p|m$ in $\mathbb{N}$, set

$$G(m,p,n) = \left\{ [(a_1,...,a_n)|\sigma] \,\middle|\, a_i \in \mathbb{C}, a_i^m = 1, \sigma \in S_n, \left(\prod\nolimits_j a_j\right)^{m/p} = 1 \right\}.$$

Any imprimitive complex reflection group G on V has the matrix form $G(m,p,n)$ with respect to a basis $e_1, e_2, ..., e_n$ for some $m, p, n \in \mathbb{N}$ with $m > 2$, $n > 1$, $p|m$ and $(m,p,n) \neq (m,m,2)$, in particular, the imprimitive system $\{V_1, ..., V_r\}$ of G in V consists of one-dimensional subspaces.

1.4. It is well known that to each finite complex reflection group G, we can associate a root system (R, f), where R is a finite G-invariant set of unit vectors in the vector space V and $f : R \to \mathbb{N} \setminus \{1\}$ is a function, constant on any G-orbit, such that G is generated by the reflection set $\{s_{\alpha,f(\alpha)} \mid \alpha \in R\}$. As a root system for G, (R, f) is determined by G up to scalar factors (see [**3**, Subsection 1.9]). One can define a simple root system (B, w) in (R, f), where $B \subset R$ has the minimal cardinality with the following properties: $G \cdot B = R$, G is generated by $\{s_{\alpha,w(\alpha)} \mid \alpha \in B\}$, and $w = f|_B$. A simple root system B for R is not uniquely determined by G. In [**6**], we introduce an equivalence relation on simple root systems: Two simple root systems (B, w) and (B', w') of (R, f) are *equivalent*, written $B \sim B'$, if there exists a bijection $\phi : B \longrightarrow B'$ such that for any $\alpha, \beta \in B$,

(1) $w(\alpha) = w'(\phi(\alpha))$ and,

(2) $\langle s_{\alpha,w(\alpha)}, s_{\beta,w(\beta)}\rangle \cong \langle s_{\phi(\alpha),w'(\phi(\alpha))}, s_{\phi(\beta),w'(\phi(\beta))}\rangle$, where the notation $\langle x, y\rangle$ stands for the subgroup of G generated by the elements $x, y \in G$.

1.5. For a reflection group G, *a presentation of* G *by generators and relations* (or *a presentation* in short) is by definition a pair (S, P), where

(1) S is a finite set of reflection generators for G with minimal possible cardinality.

(2) P is a finite relation set on S, and any other relation on S is a consequence of the relations in P.

Clearly, for any simple system (B, w) for G, $S = \{s_{\alpha,w(\alpha)} \mid \alpha \in B\}$ forms a generator set of a presentation of G. Call (B, w) *the associated simple system* of the presentation.

1.6. Two presentations (S, P) and (S', P') for G are *congruent*, if there exists a bijection $\eta : S \longrightarrow S'$ such that for any $s, t \in S$,$(*)$ $\quad \langle s, t\rangle \cong \langle\eta(s), \eta(t)\rangle$,

Note that when a generator set S of the group G is given, we assume that all the relations on S are known. Thus by the definition, we see that the congruence of a presentation (S, P) of G is determined entirely by the generator set S, the relation set P plays no role concerning it.

We see that two presentations of G are congruent if and only if their associated simple root systems are equivalent.

1.7. For a given G, one way to calculate the number of congruence classes of presentations is to find the number of equivalence classes of simple root systems in the root system of G. The latter can be done by calculation of the groups $\langle s_\alpha, s_\beta\rangle$ for all pairs of reflections $s_\alpha \neq s_\beta$ in G with respect to $\alpha, \beta \in R$, and also by calculation of all the permutations of R given rise by the action of the reflections s_α, $\alpha \in R$. These can be done by a computer in general. We did it in such a way when G is primitive (see 5.1).

2. Graphs associated to reflection sets of $G(m, p, n)$

2.1. We have $G(m, m, n) \subseteq G(m, p, n) \subseteq G(m, 1, n)$ for any $1 \leq p \leq m$ with $p|m$. There exists two kinds of reflections in $G(m, 1, n)$ as follows.(i) $s(i, j; k) := [(1, ..., 1, \zeta_m^{-k}, 1, ..., 1, \zeta_m^k, 1, ..., 1)|(i, j)]$, where $\zeta_m := e^{2\pi i/m}$, the numbers ζ_m^{-k}, ζ_m^k are the ith, resp. jth components of the n-tuple and (i, j) is the transposition of i and j for some $i < j$. Call $s(i, j; k)$ *a reflection of type I.* Set $s(j, i; k) = s(i, j; -k)$.(ii) $s(i; k) := [(1, ..., 1, \zeta_m^k, 1, ..., 1)|1]$ for some $k \in \mathbb{Z}$ with $m \mid k$, where ζ_m^k is the ith component of the n-tuple. Call $s(i; k)$ *a reflection of type II.* $s(i; k)$ has the order $m/\gcd\{m, k\}$.

All the reflections of type I lie in the subgroup $G(m, m, n)$.

2.2. Let $X = \{s(i_h, j_h; k_h) \mid h \in J\}$ be a set of reflections of type I in $G(m, 1, n)$ for some finite index set J. We associate to X a digraph $\overline{\Gamma}_X = (V, E)$ as follows. Its node set V is $[n] := \{1, 2, ..., n\}$, and its arrow set E consists of all the ordered pairs (i, j), $i < j$, with labels k for any $s(i, j; k) \in X$ (hence, if $s(i, j; k) \in X$ and $i > j$, then $\overline{\Gamma}_X = (V, E)$ contains an arrow (j, i) with the label $-k$). Denote by Γ_X the underlying graph of $\overline{\Gamma}_X$, i.e., Γ_X is obtained from $\overline{\Gamma}_X$ by replacing each labelled arrow by an unlabelled edge.

Clearly, the graph Γ_X has no loop but may have multi-edges between two nodes.

The above definition of a graph can be extended: to any set X of reflections of $G(m,1,n)$, we define a graph Γ_X to be $\Gamma_{X'}$, where X' is the subset of X consisting of all the reflections of type I. When X contains exactly one reflection of type II (say $s(i;k)$), we define another graph, denoted by Γ_X^r, which is obtained from Γ_X by rooting the node i, i.e., Γ_X^r is a rooted graph with the rooted node i. Sometimes we denote Γ_X^r by $([n],E,i)$.

2.3. Example. Let $n=6$.

(1) If $X=\{s(1,2;4),s(3,4;2),s(4,6;0),s(3,4;3)\}$, then $\overline{\Gamma}_X$ is as in Fig. 1 (a).

(a) (b)

Fig. 1

and Γ_X is as in Fig. 1 (b).

(2) Let $Y=X\cup\{s(6;3)\}$ be with X as in (1). Then Γ_Y^r is as in Fig. 2.

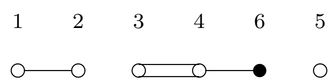

Fig. 2

where the node labelled by 6 is rooted.

2.4. Two graphs (N,E) and (N',E') are *isomorphic*, if there exists a bijection $\eta: N\rightarrow N'$ such that for any $v,w\in N$, $\{v,w\}$ is in E if and only if $\{\eta(v),\eta(w)\}$ is in E'.

Two rooted graphs (N,E,i) and (N',E',i') are *isomorphic*, if there exists a bijection $\eta: N\rightarrow N'$ with $\eta(i)=i'$ such that for any $v,w\in N$, $\{v,w\}$ is in E if and only if $\{\eta(v),\eta(w)\}$ is in E'.

2.5. In [**8**, Lemma 2.1], we showed that the generator set S of a presentation (S,P) of the group $G(m,1,n)$ consists of $n-1$ reflections of type I and one reflection of order m. Then we showed the following

2.6. Theorem. (see [**8**, Theorem 2.8]) *Let S be a subset of $G(m,1,n)$ consisting of $n-1$ reflections of type I and one reflection of order m ($m>2$ as assumed). Then S is the generator set of a presentation for $G(m,1,n)$ if and only if the graph Γ_S^r is a rooted tree.*

2.7. Assume that X is a reflection set of $G(m,1,n)$ with Γ_X connected and having exactly one circle. Then for some integer $2\le r\le n$, X contains the reflections $s(a_h,a_{h+1};k_h)$ with some integers k_h for any $1\le h\le r$ (the subscripts are modulo r). Denote by $\delta(X)$ the absolute value of $\sum_{h=1}^r k_h$.

By [**8**, Lemma 2.7] and [**1**, Appendix 2], we see that the generator set S of a presentation (S,P) of the group $G(m,m,n)$ consists of n reflections of type I such that the graph Γ_S is connected (hence contains exactly one circle). Then we showed the following

2.8. Theorem. (see [**8**, Theorem 2.19]) *Let S be a subset of $G(m,m,n)$ consisting of n reflections of type I with Γ_S connected. Then S is the generator set of a presentation of $G(m,m,n)$ if and only if the value $\delta(S)$ is coprime to m.*

2.9. Next we consider the group $G(m,p,n)$ for any $m,p,n \in \mathbb{N}$ with $p|m$ and $1 < p < m$. By [**9**, Lemma 2.2], we know that for $1 < p < m$ with $p|m$, the generator set S in a presentation (S,P) of the group $G(m,p,n)$ consists of n reflections of type I and one reflection of order m/p and type II such that the graph Γ_S is connected (hence containing exactly one circle). Then we get the following

2.10. Theorem. (see [**9**, Theorem 2.4]) *Assume that S is a subset of $G(m,p,n)$ consisting of n reflections of type I and one reflection of order m/p and type II such that Γ_S is connected. Then S is the generator set of a presentation of $G(m,p,n)$ if and only if* $\gcd\{p, \delta(S)\} = 1$.

3. The classification of presentations of $G(m,p,n)$

Let $\Sigma(m,p,n)$ be the set of the presentations (S,P) of $G(m,p,n)$ and let $\widetilde{\Sigma}(m,p,n)$ be the set of congruence classes of $\Sigma(m,p,n)$. In the present section, we shall describe the set $\widetilde{\Sigma}(m,p,n)$ in the cases of $p = 1$, $p = m$ and $1 < p < m$, separately.

3.1. It is known by [**8**, Subsection 3.1] that $S, S' \in \Sigma(m,1,n)$ are congruent if and only if $\Gamma_S^r \cong \Gamma_{S'}^r$. Also, it is known by [**8**, Subsection 3.3] that $S, S' \in \Sigma(m,m,n)$ are congruent if and only if $\Gamma_S \cong \Gamma_{S'}$. So we get the following two theorems concerning the classification of congruence classes of presentations for the groups $G(m,1,n)$ and $G(m,m,n)$.

3.2. Theorem. (see [**8**, Theorem 3.2]) *The map $(S,P) \to \Gamma_S^r$ induces a bijection from the set $\widetilde{\Sigma}(m,1,n)$ to the set of isomorphism classes of rooted trees with n nodes.*

3.3. Theorem. (see [**8**, Theorem 3.4]) *The map $(S,P) \to \Gamma_S$ induces a bijection from the set $\widetilde{\Sigma}(m,m,n)$ to the set of isomorphism classes of connected graphs with n nodes and n edges (or equivalently with n nodes and exactly one circle).*

3.4. Example. Let $n = 4$.
(1) There are 4 isomorphic classes of rooted trees of nodes 4 (see Fig. 3).

Fig. 3

Hence $G(m,1,4)$ has 4 congruence classes of presentations.
(2) There are 5 isomorphic classes of connected graphs with exactly one circle (see Fig 4).

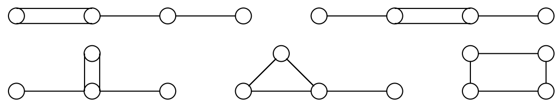

Fig. 4

Hence $G(m, m, 4)$ has 5 congruence classes of presentations.

In the remaining part of the section, we always assume $1 < p < m$ and $p|m$.

3.5. It is known by [**9**, Lemma 2.7] that $S, S' \in \Sigma(m, p, n)$ are congruent if and only if one of the following conditions holds:

(1) the circle of Γ_S^r contains more than two nodes and $\Gamma_S^r \cong \Gamma_{S'}^r$ (see 2.4);

(2) the circle of Γ_S^r contains exactly two nodes, $\Gamma_S^r \cong \Gamma_{S'}^r$ and $\gcd\{m, \delta(S)\} = \gcd\{m, \delta(S')\}$.

3.6. Denote by $\Lambda(m, p)$ the set of all the numbers $d \in \mathbb{N}$ such that $d|m$ and $\gcd\{d, p\} = 1$. Let $\Gamma(m, p, n)$ be the set of all the connected rooted graphs with n nodes and n edges. Let $\Gamma_1(m, p, n)$ be the set consisting of all the rooted graphs in $\Gamma(m, p, n)$ each of which contains a two-nodes circle. Let $\Gamma_2(m, p, n)$ be the complement of $\Gamma_1(m, p, n)$ in $\Gamma(m, p, n)$. Denote by $\widetilde{\Gamma}(m, p, n)$, resp., $\widetilde{\Gamma}_i(m, p, n)$ the set of the isomorphism classes in the set $\Gamma(m, p, n)$, resp., $\Gamma_i(m, p, n)$ for $i = 1, 2$ (see 2.4).

The following result describes all the congruence classes of presentations for $G(m, p, n)$ in terms of rooted graphs.

3.7. Theorem. (see [**9**, Theorem 2.9]) *(1) The map* $\psi : S \mapsto \Gamma_S^r$ *from* $\Sigma(m, p, n)$ *to* $\Gamma(m, p, n)$ *induces a surjection* $\widetilde{\psi}$*:* $\widetilde{\Sigma}(m, p, n) \twoheadrightarrow \widetilde{\Gamma}(m, p, n)$*. (2) Let* $\widetilde{\Sigma}_i(m, p, n) = \widetilde{\psi}^{-1}(\widetilde{\Gamma}_i(m, p, n))$ *for* $i = 1, 2$*. Then the map* $\widetilde{\psi}$ *gives rise to a bijection:* $\widetilde{\Sigma}_2(m, p, n) \rightarrowtail\!\!\!\twoheadrightarrow \widetilde{\Gamma}_2(m, p, n)$*; also,* $S \mapsto (\Gamma_S^r, \gcd\{m, \delta(S)\})$ *induces a bijection:* $\widetilde{\Sigma}_1(m, p, n) \rightarrowtail\!\!\!\twoheadrightarrow \widetilde{\Gamma}_1(m, p, n) \times \Lambda(m, p)$*.*

3.8. Example. Let $n = 4$, $m = 6$ and $p = 2$. Then $\Lambda(6, 2) = \{1, 3\}$. There exist 13 isomorphic classes of rooted connected graphs with exactly one circle, 9 of them contain a two-nodes circle (see Fig. 5). So $G(6, 2, 4)$ has $22 = 9 \times 2 + 4$ congruence classes of presentations.

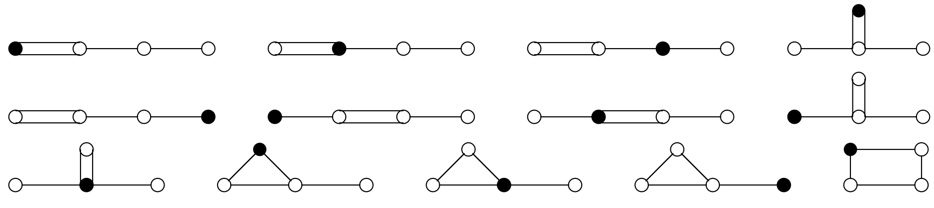

Fig. 5

4. The relation set of a presentation for $G(m,p,n)$

We always assume $1 \leq p \leq m$, $m > 2$, $n > 1$, $p|m$ and $(m,p,n) \neq (m,m,2)$ in the section. In [**8**, Section 4] and [**9**, Section 4], we defined the basic relations on the generator set S of a presentation (S,P) for the groups $G(m,p,n)$ in the cases of $p=1$, $p=m$ and $1<p<m$ separately. In the present section, we shall give a uniform treatment for these relations.

4.1. Let $S \in \Sigma(m,p,n)$. By the results stated in the previous sections, we can write

$$S = \{s = s(a,k), t_h \mid h \in J\}, \tag{4.1}$$

where $\gcd\{m,k\} = \gcd\{m,p\}$ (Thus, if $p=m$ then $s=1$, which can be removed from S), J is an index set with $|J| = n-1$ if $p=1$ and $|J| = n$ if $1 < p \leq m$, and all the reflections t_h, $h \in J$, are of type I. The graph Γ_S is always connected. When $1 \leq p < m$, we have the rooted graph Γ_S^r with the node a rooted; when $1 < p \leq m$, the graph Γ_S contains exactly one circle.

4.2. Now assume $1 < p \leq m$. Take any node x of Γ_S. Call a sequence of nodes $\xi_x : a_0 = x, a_1, ..., a_r = x$ in Γ_S *a generalized circle sequence of* Γ_S^r *at the node* x if S contains reflections $t_h = s(a_{h-1}, a_h; k_h)$ for $1 \leq h \leq r$ with some integers k_h, where $t_l \neq t_{l+1}$ for $1 \leq l < r$. Since the graph Γ_S is connected and contains a unique circle, the sequence ξ_x always exists. ξ_x contains all the nodes on the circle of Γ_S and is uniquely determined by the set S and the node x up to an orientation of the circle.

When $x = a$ is the rooted node of Γ_S^r (this is the case only when $1 < p < m$), ξ_x is also called *a root-circle sequence of* Γ_S^r.

Call $s_{hj} := t_h t_{h+1} ... t_{j-1} t_j t_{j-1} ... t_h$ *a path reflection of* Γ_S *in* ξ_x for any $1 \leq h < j \leq r$ (see Fig. 6).

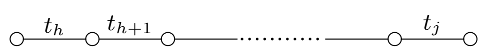

Fig. 6

Let c_x, c'_x be the smallest, resp., the largest integer with the node a_{c_x}, resp., $a_{c'_x}$ lying on the circle of Γ_S. Then x is on the circle of Γ_S if and only if $c_x = 0$ and $c'_x = r$.

4.3. Example.

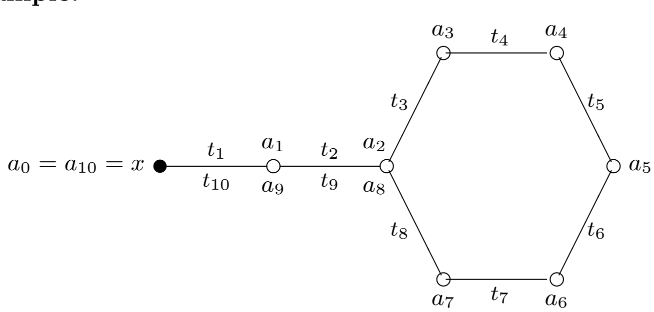

Fig. 7

Here the generalized circle sequence ξ_x at the node x is $a_0 = x, a_1, ..., a_{10} = x$, which is also a root-circle sequence. Hence $r = 10$, $c_x = 2$, $c'_x = 8$, $a_1 = a_9$, $a_2 = a_8$ and $s_{1,4} = t_1t_2t_3t_4t_3t_2t_1$. We have $s_{1,2} = s_{9,10}$ since $t_1 = t_{10}$, $t_2 = t_9$ and $t_1t_2t_1 = t_2t_1t_2$.

4.4. Let S be as in (4.1). By [**8**, Theorems 4.17 and 4.20] and [**9**, Theorem 6.2], we see that (S, P) is a presentation of $G(m, p, n)$ if P consists of relations (A)–(M) as follow. Here the path reflections s_{1j}, $s_{j+1,r}$ are with respect to the generalized circle sequence ξ_x in 4.2 and satisfy $c_x < j < c'_x$ whenever it is applicable. $\{x, a_j\}$ is called *an admissible node pair* of Γ_S, at which we are allowed to talk about relations (J)-(L) on S, where x is required to be the rooted node a of Γ_S^r for relations (K)-(L). $u, v \in S$ in (M) satisfy that the edges $e(u)$, $e(v)$ are incident to ξ_x at the nodes x, a_j respectively.

(A) $s^{m/p} = 1$.

(B) $t_i^2 = 1$ for $i \in J$.

(C) $t_it_j = t_jt_i$ if the edges $e(t_i)$ and $e(t_j)$ have no common end node.

(D) $t_it_jt_i = t_jt_it_j$ if the edges $e(t_i)$ and $e(t_j)$ have exactly one common end node.

(E) $st_ist_i = t_ist_is$ if a is an end node of $e(t_i)$.

(F) $st_i = t_is$ if a is not an end node of $e(t_i)$.

(G) $(t_it_j)^{m/d} = 1$ if $t_i \neq t_j$ with $e(t_i)$ and $e(t_j)$ having two common end nodes, where $d = \gcd\{m, \delta(S)\}$.

(H) $t_i \cdot t_jt_lt_j = t_jt_lt_j \cdot t_i$ for any triple $X = \{t_i, t_j, t_l\} \subseteq S$ with Γ_X having a branching node.

(I) $s \cdot t_it_jt_i = t_it_jt_i \cdot s$, if $e(t_i)$ and $e(t_j)$ have exactly one common end node a.

(J) $(s_{1j}s_{j+1,r})^{\frac{m}{\gcd\{m,\delta(S)\}}} = 1$.

(K) $ss_{1j}s_{j+1,r} = s_{1j}s_{j+1,r}s$.

(L) $(s_{j+1,r}s_{1j})^{p-1} = s^{-\delta(S)}s_{1j}s^{\delta(S)}s_{j+1,r}$.

(M) (a) $us_{1j}u \cdot vs_{j+1,r}v = vs_{j+1,r}v \cdot us_{1j}u$.

(b) $us_{1j}s_{j+1,r}us_{1j}s_{j+1,r} = s_{1j}s_{j+1,r}us_{1j}s_{j+1,r}u$.

(c) $vs_{1j}s_{j+1,r}vs_{1j}s_{j+1,r} = s_{1j}s_{j+1,r}vs_{1j}s_{j+1,r}v$.

Here any of the above relations involving the reflection s (i.e., any of the relations (A), (E), (F), (I), (K) and (L)) is applicable only in the case of $1 \leq p < m$; while any of the above relations involving a circle (i.e., any of the relations (G), (J), (K), (L) and (M)) is applicable only in the case of $1 < p \leq m$.

4.5. In 4.4, call (A)-(B) the *order relations*, (C)-(G) the *braid relations* on S. Call (A)-(F) the *order-braid relations* (or *o.b. relations* in short) on S (note that relation (G) is not included in the o.b. relations on S). Call (H) the *branching relations*, (I) the *root-braid relations*, (H) the *branching relations*, (I) the *root-braid relations*, (G), (J) the *circle relations*, (K) the *root-circle relations*, (L) the *circle-root relations*, and (M) the *branching-circle relations* on S.

Call all the relations (A)-(M) above the basic relations on S. We have the following

4.6. Theorem. (see [**8**, Theorems 4.17 and 4.20] and [**9**, Theorem 6.2]) *Let $S \in \Sigma(m, p, n)$ and let P_S be the set of all the basic relations on S. Then (S, P_S) forms a presentation of $G(m, p, n)$.*

4.7. A presentation (S, P) of a reflection group G is *essential* if (S, P_0) is not a presentation of G for any proper subset P_0 of P.

Let $S \in \Sigma(m, p, n)$ be as in (4.1) with $1 \leq p \leq m$ and let P_S be the set of all the basic relations (A)-(M) on S. Then the presentation (S, P_S) of $G(m, p, n)$ is not essential in general.

Let us take the case of $1 < p < m$ as an example,
(a) Let (B′) be any relation in (B). Then (B′) is equivalent to (B) under the assumption of (D).
(b) Let (K′) (resp., (L′)) be a relation in (K) (resp., (L)) at any one admissible node pair. Then (K′) (resp., (L′)) is equivalent to (K) (resp., (L)) under the assumption of the o.b. relations on S.
(c) For any branching node v of Γ_S, fix some $t_v \in S$ of type I with $e(t_v)$ incident to v. Set
(H′) The relation $t_v \cdot tt't = tt't \cdot t_v$ holds for any $t \neq t'$ in $S \setminus \{t_v\}$ of type I with $\Gamma_{\{t_v,t,t'\}}$ having v as a branching node.

Then (H′) is a subset of and is equivalent to (H) under the assumption of the o.b. relations on S.
(d) Let a be the rooted node in Γ_S^r. Fix some $t_a \in S$ of type I with $e(t_a)$ incident to a. Set (I′) $s \cdot t_a t t_a = t_a t t_a \cdot s$ for any $t \in \Gamma_S \setminus \{t_a\}$ of type I with $e(t)$, $e(t_a)$ having just one common end node a.

Then (I′) is a subset of and is equivalent to (I) under the assumption of the o.b. relations and the branching relations on S. (e) (G) is a special case of (J), while (J) is a consequence of the o.b. relations and the relations (K)-(L) on S.
(f) Let (M′) be the relations (M) if Γ_S^r has a two-nodes circle and be the empty set of relations if otherwise.
(g) Assume that Γ_S^r has a two-nodes circle with the rooted node on the circle and not adjacent to any node outside the circle and that $\gcd\{\delta(S), m\} = p$. Then relation (E) is a consequence of (L) and the other o.b. relations on S. Let (E′) be the empty set of relations in this case and be the relation (E) in any other case.

Let P'_S be the collection of relations (A), (B′), (C), (D), (E′), (F), (H′), (I′), (K′), (L′), (M′). Then (S, P'_S) is again a presentation of $G(m, p, n)$ (see [**9**, Remark 6.9 (1)]).

One may ask if the presentation (S, P'_S) is always essential. The answer is still negative. Recently, Liu and Shi have showed that each of the relations (H′), (I′), (M′) could be further reduced (see [**4**]).

4.8. Among all the presentation (S, P_S) of $G(m, 1, n)$, the relation set P_S has a simpler form when Γ_S^r is a string with the rooted node at one end (see [**1**, Appendix 2]). Among all the presentations (S, P_S) of $G(m, p, n)$, $1 < p \leq m$, we single out two kinds of presentations whose relation sets have simpler forms:

(i) One is when Γ_S is a string with a two-nodes circle at one end, and with the rooted node on the circle, not incident to any node outside the circle if $1 < p < m$ (see [**1**, Appendix 2]);

(ii) The other is when Γ_S is a circle. In this case, if $p = m$, then the relation set P_S can only consist of some o.b. relations and one circle relation (see [**5**, Proposition 3.3]); if $1 < p < m$, then P_S can only consist of some o.b. relations, one root-braid relation, one root-circle relation and one circle-root relation (see [**9**, Remark 6.9 (2)]).

5. Presentations for the primitive complex reflection groups

5.1. In Table 1, we record results of L. Wang, P. Zeng and J. Y. Shi on the numbers $N(G)$ of congruence classes of presentations for the primitive complex reflection groups G (see 1.7), where the numbers $N(G_i)$ for $i = 12, 24, 25, 26$ were got by Shi (see [**6**]), for $i = 13, 22, 27, 29, 31, 33$ by Wang (see [**11**]), and for $i = 7, 11, 15, 19, 32$ by Zeng (see [**12**]).

G	$N(G)$	G	$N(G)$
G_7	2	G_{11}	4
G_{12}	5	G_{13}	4
G_{15}	4	G_{19}	6
G_{22}	18	G_{24}	3
G_{25}	2	G_{26}	2
G_{27}	6	G_{29}	9
G_{31}	61	G_{32}	5
G_{33}	14		

Table 1

Since any G generated by ≤ 2 reflections satisfies $N(G) = 1$, G_{34} is the only primitive complex reflection group with $N(G_{34})$ unknown.

5.2. We list *a representative set for the congruence classes of presentations* (or *r.c.p.* for brevity) of the primitive complex reflection groups G_{12}, G_{24}, G_{27}, G_{25}, G_{26}, G_7, G_{15}, G_{19}, G_{11}, G_{32} (see 5.3–5.12), where the groups G_{12}, G_{24}, G_{25}, G_{26}, G_{19}, G_{11}, G_{32} were done by Shi (see [**6**] [**7**]), G_{27} by Wang (see [**11**]), and G_7, G_{15} by Zeng (see [**12**]). The first presentation for each group was given in [**1**, Appendix 2]). According to the Shephard-Todd's classification (see [**10**]), the groups G_{25}, G_{26}, G_7, G_{15}, G_{19}, G_{11} and G_{32} form the full set of such primitive complex reflection groups each of which is generated by more than two reflections and contains some reflections of order > 2.

5.3. r.c.p. for the group G_{12}. (see [**6**, Propositions 3.3–3.7]):

(1) $G_{12} = \langle s, u, t \mid s^2 = u^2 = t^2 = 1, suts = utsu = tsut\rangle$.

(2) $G_{12} = \langle s, u, w \mid s^2 = u^2 = w^2 = 1, sususu = ususus, sws = wsw, swus = uswu\rangle$.

(3) $G_{12} = \langle s, u, x \mid s^2 = u^2 = x^2 = 1, suxs = usux, xusx = uxus\rangle$.

(4) $G_{12} = \langle s, u, y \mid s^2 = u^2 = y^2 = 1, usuyu = susuy, ysuyu = suyus\rangle$.

(5) $G_{12} = \langle s, x, y \mid s^2 = x^2 = y^2 = 1, sxsx = xsxs, sys = ysy, xyx = yxy, sxsxy = ysxsx\rangle$.

Here $w = ututu$, $x = utu$, $y = tst$, $t = susu suw = uxu = uyusy = yxsxy$ and $u = sxyxs$.

5.4. r.c.p. for the group G_{24}. (see [**6**, Propositions 4.3–4.5]):

(1) $G_{24} = \langle s, u, t \mid s^2 = u^2 = t^2 = 1, stst = tsts, sus = usu, utut = tutu, tusutu = usutus\rangle$.

(2) $G_{24} = \langle s, t, x \mid s^2 = t^2 = x^2 = 1, stst = tsts, sxs = xsx, txt = xtx, stsxstsx = xstsxsts\rangle$.

(3) $G_{24} = \langle s, t, y \mid s^2 = t^2 = y^2 = 1, stst = tsts, tyty = ytyt, tstyts = ytstyt\rangle$.

Here $x = sus$, $y = tut$ and $u = sxs = tyt$.

5.5. r.c.p. for the group G_{27}. (see [**11**, Propositions 3.1.1–3.1.6]):

(1) $G_{27} = \langle s,u,t \mid s^2 = u^2 = t^2 = 1, utu = tut, stst = tsts, susus = ususu, usutus = tusutu\rangle$.

(2) $G_{27} = \langle s,u,w \mid s^2 = u^2 = w^2 = 1, susus = ususu, uwu = wuw, sws = wsw, suwusuwu = uwusuwus\rangle$.

(3) $G_{27} = \langle s,t,w \mid s^2 = t^2 = w^2 = 1, stst = tsts, wtw = twt, sws = wsw, swtwswtwsw = wtwswtwswt\rangle$.

(4) $G_{27} = \langle s,u,x \mid s^2 = u^2 = x^2 = 1, susus = ususu, sxsx = xsxs, usxsus = sxsusx, uxsxux = xsxuxs\rangle$.

(5) $G_{27} = \langle s,t,y \mid s^2 = t^2 = y^2 = 1, stst = tsts, sysys = ysysy, ysytys = tysyty, tsysts = systsy\rangle$.

(6) $G_{27} = \langle s,w,y \mid s^2 = w^2 = y^2 = 1, sws = wsw, sysys = ysysy, wysywy = ysywys, wsyswsys = syswsysw\rangle$.

Here $w = utu$, $x = sts$, $y = susus$, $t = uwu = sxs = ysywysy$ and $u = ysy$.

5.6. r.c.p. for the group G_{25}. (see [**6**, Propositions 5.3–5.4]):

(1) $G_{25} = \langle t,u,v \mid t^3 = u^3 = v^3 = 1, tut = utu, uvu = vuv, tv = vt\rangle$.

(2) $G_{25} = \langle t,u,x \mid t^3 = u^3 = x^3 = 1, tut = utu, uxu = xux, xtux = uxtu\rangle$.

Here $x = u^2vu$ and $v = uxu^2$.

Each of the above presentations for G_{12}, G_{24}, G_{27} and G_{25} becomes essential after removing any two of the order relations.

5.7. r.c.p. for the group G_{26}. (see [**6**, Propositions 6.3–6.4]):

(1) $G_{26} = \langle t,u,v \mid t^2 = u^3 = v^3 = 1, tutu = utut, uvu = vuv, tv = vt\rangle$.

(2) $G_{26} = \langle t,u,x \mid t^2 = u^3 = x^3 = 1, tutu = utut, uxu = xux, uxtu = xtux\rangle$.

Here $x = u^2vu$ and $v = uxu^2$.

Each presentation becomes essential after removing any one of the order 3 relations.

5.8. r.c.p. for the group G_7. (see [**12**, Propositions 4.1.1–4.1.2]):

(1) $G_7 = \langle t,u,s \mid t^2 = u^3 = s^3 = 1, tus = ust = stu\rangle$.

(2) $G_7 = \langle t,x,s \mid t^2 = x^3 = s^3 = 1, sxt = txs, txtsts = ststxt\rangle$.

Here $x = tsus^2t$ and $u = s^2txts$.

5.9. r.c.p. for the group G_{15}. (see [**12**, Propositions 4.2.1–4.2.4]):

(1) $G_{15} = \langle t,u,s \mid t^2 = u^2 = s^3 = 1, tus = ust, stusu = tusus\rangle$.

(2) $G_{15} = \langle t,x,s \mid t^2 = x^2 = s^3 = 1, tsxs^2ts = sxs^2tst, xstsxs^2t = stsxstsx\rangle$.

(3) $G_{15} = \langle t,y,s \mid t^2 = y^2 = s^3 = 1, tsy = syt, ytsys = tsysy\rangle$.

(4) $G_{15} = \langle t,z,s \mid t^2 = z^2 = s^3 = 1, tzts = ztst, sztstz = ztszts\rangle$.

Here $x = s^2tuts$, $y = s^2us$ and $z = tut$. Hence $u = tsxs^2t = sys^2 = tzt$.

5.10. r.c.p. for the group G_{19}. (see [**7**, (2.1.1) and Propositions 2.4–2.8]) :

(1) $G_{19} = \langle t,u,s \mid t^2 = u^3 = s^5 = 1, tus = ust = stu\rangle$

(2) $G_{19} = \langle t,u,w \mid t^2 = u^3 = w^5 = 1, wuwu = uwuw, u^2t \cdot w^2 = w^2 \cdot utu\rangle$

(3) $G_{19} = \langle t,v,w \mid t^2 = v^3 = w^5 = 1, wv^2wv = vwv^2w, vw \cdot wvt = wvt \cdot vw\rangle$

(4) $G_{19} = \langle t,x,z \mid t^2 = x^3 = z^5 = 1, xzxz = zxzx, zxz^2 \cdot zt = tz \cdot zxz^2\rangle$

(5) $G_{19} = \langle t,y,w \mid t^2 = y^3 = w^5 = 1, ywyw = wywy, yw^2y \cdot wt = wt \cdot w^2y^2\rangle$

Here $v = s^3u^2s^2$, $x = utsu^2s^4tu^2$, $y = x^2 = utsus^4tu^2$, $w = su^2s^2us^4$ and $z = s^2$. Hence $s = w^2tu^2tuw = tvw^3v^2t = tyw^3y^2t = z^3$ and $u = s^2v^2s^3 = w^3v^2w^2 = w^2yw^3 = z^2tz^2xztx$.

5.11. r.c.p. for the group G_{11}. (see [**7**, (3.1.1) and Propositions 3.3–3.5]):

(1) $G_{11} = \langle t, u, s \mid t^2 = u^3 = s^4 = 1, tus = ust = stu\rangle$

(2) $G_{11} = \langle t, u, z \mid t^2 = u^3 = z^4 = 1, tuz = zut, tuzu = uzut\rangle$

(3) $G_{11} = \langle r, u, s \mid r^2 = u^3 = s^4 = 1, usus = susu, usu \cdot rs = rs \cdot su^2\rangle$

(4) $G_{11} = \langle r, u, w \mid r^2 = u^3 = w^4 = 1, wuwu = uwuw, w \cdot w^2r \cdot u = u \cdot w^2r \cdot w\rangle$

Here $w = u^2su$, $z = u^2s^3u$ and $r = us^3tsu^2$. Hence $s = uwu^2 = uz^3u^2$ and $t = su^2rus^3 = w^3rw = zrz^3$.

All the above presentations for the groups G_7, G_{15}, G_{19} and G_{11} are essential.

5.12. r.c.p. for the group G_{32}. (see [**7**, (4.1.1) and Propositions 4.3–4.6]):

(1) $G_{32} = \langle t, u, v, w \mid t^3 = u^3 = v^3 = w^3 = 1, tut = utu, uvu = vuv, vwv = wvw, tv = vt, tw = wt, uw = wu\rangle$

(2) $G_{32} = \langle t, u, v, x \mid t^3 = u^3 = v^3 = x^3 = 1, tut = utu, tv = vt, tx = xt, vuxv = xvux = uxvu\rangle$

(3) $G_{32} = \langle r, u, v, x \mid r^3 = u^3 = v^3 = x^3 = 1, uvu = vuv, rur = uru, vxv = xvx, uvru = vruv, uxvu = vuxv, vrxv = xvrx\rangle$

(4) $G_{32} = \langle t, s, w, y \mid t^3 = s^3 = w^3 = y^3 = 1, tw = wt, tyt = yty, ywy = wyw, sws = wsw, stys = ysty, syws = wsyw\rangle$

(5) $G_{32} = \langle t, u, z, m \mid t^3 = u^3 = z^3 = m^3 = 1, uzu^2m^2tm = m^2tmuzu^2, tz = zt, um = mu, zmz = mzm, tut = utu, uzu = zuz, tmt = mtm\rangle$

Here $x = vwv^2$, $r = utu^2$, $s = uvu^2$, $y = u^2vu$, $z = wvw^2$ and $m = t^2u^2vut$. Hence $t = u^2ru$, $u = sys^2$, $v = s^2ys = utmt^2u^2$ and $w = v^2xv = utmt^2u^2zutm^2t^2u^2$.

Any of the presentations (1)–(4) of the group G_{32} becomes essential after removing any three of the four order relations. However, in order to make presentation (5) of G_{32} essential, one need remove any one of the relations $zmz = mzm$, $tut = utu$, $uzu = zuz$ and $tmt = mtm$ in addition.

References

[1] M. Broué, G. Malle and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math., **500** (1998), 127–190.

[2] A. M. Cohen, *Finite complex reflection groups*, Ann. scient. Éc. Norm. Sup. **4e** série t. **9** (1976), 379-436.

[3] R. B. Howlett and J. Y. Shi, *On regularity of finite reflection groups*, manuscripta mathematica, **102(3)** (2000), 325–333.

[4] X. L. Liu and J. Y. Shi, in preparation.

[5] J. Y. Shi, *Certain imprimitive reflection groups and their generic versions*, Trans. Amer. Math. Soc., **354** (2002), 2115–2129.

[6] J. Y. Shi, *Simple root systems and presentations for certain complex reflection groups*, to appear in Comm. in Algebra.

[7] J. Y. Shi, *Presentations for three complex reflection groups*, preprint.

[8] J. Y. Shi, *Congruence classes of presentations for the complex reflection groups $G(m, 1, n)$ and $G(m, m, n)$*, to appear in Indag. Math.

[9] J. Y. Shi, *Congruence classes of presentations for the complex reflection groups $G(m, p, n)$*, J. Algebra, **284**(1) (2005), 392–414.

[10] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math., **6** (1954), 274–304.

[11] L. Wang, *Simple root systems and presentations for the primitive complex reflection groups generated by involutive reflections*, Master Dissertation in ECNU, 2003.

[12] P. Zeng, *Simple root systems and presentations for the primitive complex reflection groups containing reflections of order* > 2, Master Dissertation in ECNU, 2003.

Department of Mathematics, East China Normal University, Shanghai, 200062, China and Center for Combinatorics, Nankai University, Tianjin, 300071, China
E-mail address: jyshi@math.ecnu.edu.cn