Some cells in the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$

Qian Huang and Jian-yi Shi

Abstract. The fixed point set of the affine Weyl group $(\widetilde{A}_{2n+1}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$ can be considered as the affine Weyl group $(\widetilde{C}_n, S)$. Then we study the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ with $\widetilde{\ell}_{2n+1}$ the length function of $\widetilde{A}_{2n+1}$. We give an explicit description for all the cells of $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ corresponding to the partitions $\mathbf{k1^{2n+2-k}}$ and $(h, 2n+2-h)$ for any $1 \leqslant k \leqslant 2n+2$ and $n+1 \leqslant h \leqslant 2n$, and also for all the cells of $(\widetilde{C}_3, \widetilde{\ell}_7)$.

Let $\mathbb{Z}$ (respectively, $\mathbb{P}$, $\mathbb{N}$) be the set of integers (respectively, positive integers, non-negative integers). For any $i \leqslant j$ in the set $\mathbb{Z}$, denote $[i,j] := \{i, i+1, ..., j\}$ and denote $[1,i]$ simply by $[i]$.

By a *Coxeter system* (W, S), we mean a Coxeter group W together with a Coxeter generator set S. Lusztig defined a weight function L on any Coxeter system (W, S), called (W, L) a *weighted Coxeter group* and also introduced the concepts of left, right and two-sided cells in a weighted Coxeter group in [**4**]. The affine Coxeter group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Coxeter group $(\widetilde{A}_m, \widetilde{S}_m)$, $m \in \{2n-1, 2n, 2n+1\}$, under a certain automorphism $\alpha_{m,n}$ with $\alpha_{m,n}(\widetilde{S}_m) = \widetilde{S}_m$, where $\widetilde{S}_m$, S are the Coxeter generator sets of $\widetilde{A}_m$, $\widetilde{C}_n$, respectively. The restriction to $\widetilde{C}_n$ of the length function $\widetilde{\ell}_m$ of $\widetilde{A}_m$ is a weight function of $\widetilde{C}_n$. It is known that there is a surjective map ψ from $\widetilde{A}_m$ to the set Λ_{m+1} of partitions of $m+1$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_m$ to Λ_{m+1} (see [**5**], [**3**]). Let $E_\lambda := \psi^{-1}(\lambda) \cap \widetilde{C}_n$ for $\lambda \in \Lambda_{m+1}$. In his papers [**7**] and [**8**], Shi described all the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$ in the set E_λ with $\lambda \in \{\mathbf{k1^{2n-k}}, \mathbf{h21^{2n-h-2}}, (j, 2n-j) \mid k \in [2n], h \in [2, 2n-2], j \in [n, 2n-1]\}$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$. In the present paper, we study left cells and two-sided cells in the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ and describe all the cells of $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ in the set E_λ with $\lambda \in \{\mathbf{k1^{2n+2-k}}, (j, 2n+2-j) \mid k \in [2n+2], j \in [n+1, 2n+1]\}$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_7)$.

Key words and phrases. Affine Weyl groups, left cells, two-sided cells, quasi-split case, weighted Coxeter group.

Supported by the NSF of China, the SFUDP of China, Shanghai Leading Academic Discipline Project (B407) and Program of Shanghai Subject Chief Scientist (11xd1402200).

Comparing with the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$, the set E_λ might be empty in the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ for some $\lambda \in \Lambda_{2n+2}$. In our considered cases, we prove that the sets $E_{\mathbf{k1^{2n+2-k}}}$ and $E_{(k,2n+2-k)}$ and E_λ, $\lambda \in \{\mathbf{431}, \mathbf{42^2}, \mathbf{2^4}, \mathbf{2^3 1^2}\}$, are empty, where k is even.

The connectedness is an important structural property for the cells. We prove in E_λ with $\lambda \in \{\mathbf{k1^{2n+2-k}}, (h, 2n+2-h) \mid k \in [2n+2], h \in [n+1, 2n+1]\} \cup \Lambda_8$ that all the left cells are left-connected and that all the two-sided cells are two-sided-connected.

The generalized tabloid and the generalized τ-invariants are two complete invariants for the left cells of $\widetilde{A}_{2n+1}$. Those invariants will be very useful in determination of left cells in our considered cases.

The contents of the paper are organized as follows. We collect some concepts and known results concerning cells of the weighted Coxeter groups $(\widetilde{A}_{2n+1}, \widetilde{\ell}_{2n+1})$ and $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ in Sections 1-2. We give some criteria for the vanishing of the set E_λ in Section 3. In Sections 4-5, we give an explicit description for all the cells of $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ corresponding to the partitions $\mathbf{k1^{2n+2\text{-}k}}$ and $(h, 2n+2-h)$ for all $k \in [2n+2]$ and $h \in [n+1, 2n+1]$ respectively. Finally, we describe all the cells of $(\widetilde{C}_3, \widetilde{\ell}_7)$ in Section 6.

1. The weighted Coxeter groups $(\widetilde{A}_{2n+1}, \widetilde{\ell}_{2n+1})$ and $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$.

In this section, we assemble some concepts and known results concerning cells of a weighted Coxeter group (W, L), in particular, in the cases where (W, L) is either $(\widetilde{A}_{2n+1}, \widetilde{\ell}_{2n+1})$ or $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$.

1.1. Let (W, S) be a Coxeter system with ℓ the length function and $\leq$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r$ of $w \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. By a *weight function* on W, we mean a map $L : W \longrightarrow \mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots + L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ of $w \in W$. Call (W, L) is a *weighted Coxeter group.* In particular, the length function ℓ is a weight function on W and the weighted Coxeter group (W, ℓ) is called in a *split* case.

Suppose that there is a group automorphism α of W with $\alpha(S) = S$. Let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J on S, let $w_J \in W^\alpha$ be the longest element in the subgroup of W generated by J. Let S_α be the set of all w_J with J ranging over the α-orbits in S. Then (W^α, S_α) is a Coxeter system. The restriction of ℓ to W^α is a weight function on (W^α, S_α). The weighted Coxeter group (W^α, ℓ) is called in a *quasi-split* case.

1.2. In [**4**], Lusztig introduced the preorders $\underset{L}{\leqslant}$, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$ and the associated equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ on a weighted Coxeter group (W, L), the corresponding equivalence classes of (W, L) are called *left cells*, *right cells* and *two-sided cells*.

For $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (resp., $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (resp., $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (resp., $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (resp., $\mathcal{L}(y) = \mathcal{L}(w)$) (see [**4**, Lemma 8.6]).

In [**4**, Chapter 13], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ in terms of structural coefficients of the Hecke algebra associated to W. Then in [**4**, Chapters 15-16], Lusztig proved that the function a is constant on any cell of W when W is either a finite or an affine Coxeter group and when (W, L) is either in a split case or in a quasi-split case.

For any $X \subset W$, write $X^{-1} := \{x^{-1} \mid x \in X\}$.

1.3. Lemma. (see [**7**, Lemma 1.7]) *Suppose that W is either a finite or an affine Coxeter group and that (W, L) is either in a split case or in a quasi-split case.*

Let E be a non-empty subset of W satisfying the following conditions:

(a) There exists some $k \in \mathbb{N}$ with $a(x) = k$ for any $x \in E$;

(b) E is a union of some left cells of W;

(c) $E^{-1} = E$.

Then E is a union of some two-sided cells of W.

From now on, we concentrate ourselves to the weighted Coxeter groups $(\widetilde{A}_{2n+1}, \widetilde{\ell}_{2n+1})$ and $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$, where $\widetilde{\ell}_{2n+1}$ is the length function of the affine Weyl group $\widetilde{A}_{2n+1}$.

1.4. The affine Weyl group $\widetilde{A}_{2n+1}$ can be realized as the following permutation group on the integer set $\mathbb{Z}$ (see [**2**, Subsection 3.6] and [**5**, Subsection 4.1]):

$$\widetilde{A}_{2n+1} = \left\{ w : \mathbb{Z} \longrightarrow \mathbb{Z} \,\middle|\, (i + 2n + 2)w = (i)w + 2n + 2, \sum_{i=1}^{2n+2} (i)w = \sum_{i=1}^{2n+2} i \right\}.$$

The Coxeter generator set $\widetilde{S} = \{s_i \mid i \in [0, 2n + 1]\}$ of $\widetilde{A}_{2n+1}$ is given by

$$(t)s_i = \begin{cases} t, & \text{if } t \not\equiv i, i+1 \pmod{2n+2}, \\ t+1, & \text{if } t \equiv i \pmod{2n+2}, \\ t-1, & \text{if } t \equiv i+1 \pmod{2n+2}, \end{cases}$$

for any $t \in \mathbb{Z}$.

Let $\alpha := \alpha_{2n+1,n} : \widetilde{A}_{2n+1} \longrightarrow \widetilde{A}_{2n+1}$ be the group automorphism determined by $\alpha(s_i) = s_{2n+1-i}$ for $i \in [0, 2n + 1]$. Then the affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n+1}$ under α, which can also be described as a permutation group on $\mathbb{Z}$ as follows.

$$\widetilde{C}_n = \{w : \mathbb{Z} \longrightarrow \mathbb{Z} \mid (i + 2n + 2)w = (i)w + 2n + 2, (-i)w = -(i)w, \forall i \in \mathbb{Z}\}$$

with the Coxeter generator set $S = \{t_i \mid i \in [0,n]\}$, where $t_i = s_i s_{2n+1-i}$ for $i \in [n-1]$, $t_0 = s_0 s_{2n+1} s_0$ and $t_n = s_n s_{n+1} s_n$. For any $w \in \widetilde{C}_n$, we can see that $(k(n+1))w = k(n+1)$, for any $k \in \mathbb{Z}$. For the sake of convenience, we define s_i, t_j for any $i, j \in \mathbb{Z}$ by setting $s_{(2n+2)q+b}$ to be s_b and setting both $t_{(2n+2)q+a}$ and $t_{(2n+2)q+(2n+1-a)}$ to be t_a for any $q \in \mathbb{Z}$ and $b \in [0, 2n+1]$ and $a \in [0, n]$.

1.5. By a partition of $n \in \mathbb{P}$, we mean an r-tuple $\lambda := (\lambda_1, \lambda_2, \ldots, ...\lambda_r)$ of weakly decreasing $\lambda_1, \lambda_2, ..., \lambda_r$ in $\mathbb{P}$ with $\sum_{k=1}^{r} \lambda_k = n$ for some $r \geqslant 1$. λ_i is called a *part* of λ. We sometimes denote λ in the form $\mathbf{j_1^{k_1} j_2^{k_2} \ldots j_m^{k_m}}$ (boldfaced) with $j_1 > j_2 > \cdots > j_m \geqslant 1$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$ for $i \geqslant 1$. For example, $\mathbf{63^2 2^3 1}$ stands for the partition $(6, 3, 3, 2, 2, 2, 1)$ of 19.

Fix $w \in \widetilde{A}_{2n+1}$. For any $i \neq j$ in $[2n+2]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that both inequalities $2pn + 2p + i > 2qn + 2q + j$ and $(2pn + 2p + i)w < (2qn + 2q + j)w$ hold. In terms of matrix entries of w, this means that the entry 1 at the position $(2qn + 2q + j, (2qn + 2q + j)w)$ is located at the northeastern of the entry 1 at the position $(2pn + 2p + i, (2pn + 2p + i)w)$. This defines a partial order $\prec_w$ on the set $[2n+2]$.

A sequence $a_1, a_2, ..., a_r$ in $[2n+2]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. Sometimes we identify a w-chain $a_1, a_2, \ldots, a_r$ with the corresponding set $\{a_1, a_2, \ldots, a_r\}$. For any $k \geqslant 1$, a k-w-chain-family is by definition a disjoint union $X = \bigcup_{i=1}^{k} X_i$ of k w-chains $X_1, ..., X_k$ in $[2n+2]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = 2n+2$. Let $\lambda_1 = d_1$ and $\lambda_k = d_k - d_{k-1}$ for any $k \in [2, r]$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of Curtis Greene in [**1**]. Let Λ_{2n+2} be the set of all partitions of $2n+2$. Hence $w \mapsto \psi(w) = (\lambda_1, \ldots, \lambda_r)$ defines a map from the set $\widetilde{A}_{2n+1}$ to Λ_{2n+2}.

$i \neq j$ in $[2n+2]$ are said *w-comparable* if either $i \prec_w j$ or $j \prec_w i$, and *w-uncomparable* if otherwise. A subset E in $[2n+2]$ is called a *w-antichain*, if the elements of E are pairwise w-uncomparable.

1.6. Let $\widetilde{\ell}_{2n+1}$, ℓ be the length functions on the Coxeter systems $(\widetilde{A}_{2n+1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. By the definition in 1.1, we see that the weighted Coxeter group $(\widetilde{A}_{2n+1}, \widetilde{\ell}_{2n+1})$ is in a split case, while $(\widetilde{C}_n, \widetilde{\ell}_{2n+1})$ is in a quasi-split case (see [**4**, Lemma 16.2]).

For any $x \in \widetilde{A}_{2n+1}$ and $k \in \mathbb{Z}$, let $m_k(x) = {}^{\#}\{i \in \mathbb{Z} \mid i < k \text{ and } (i)x > (k)x\}$. Then the formulae for the functions $\widetilde{\ell}_{2n+1}$ and ℓ are as follows.

1.7. Proposition. (comparing with [**7**, Proposition 2.4]) *For any $w \in \widetilde{A}_{2n+1}$ and $x \in \widetilde{C}_n$, we have*

(1) $\widetilde{\ell}_{2n+1}(w) = \sum_{1\leqslant i<j\leqslant 2n+2}\left|\left\lfloor \frac{(j)w-(i)w}{2n+2}\right\rfloor\right| = \sum_{k=1}^{2n+2} m_k(w)$*;*

(2) $\ell(x) = \frac{1}{2}(\widetilde{\ell}_{2n+1}(x) - m_0(x) - m_n(x))$*,*

where $\lfloor x\rfloor$ *stands for the largest integer not larger than* x *and* $|x|$ *for the absolute value of* x *for any* $x \in \mathbb{Q}$*.*

1.8. Let $\leqslant, \leqslant_C$ be the Bruhat-Chevalley orders on the Coxeter systems $(\widetilde{A}_{2n+1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. Since the condition $x \leqslant_C y$ is equivalent to $x \leqslant y$ for any $x, y \in \widetilde{C}_n$, it will cause no confusion if we use the notation $\leqslant$ in the place of $\leqslant_C$. Hence from now on we shall use $\leqslant$ for both $\leqslant$ and $\leqslant_C$.

Let $\widetilde{\mathcal{L}}(x) = \{s \in \widetilde{S} \mid sx < x\}$ and $\widetilde{\mathcal{R}}(x) = \{s \in \widetilde{S} \mid xs < x\}$ for $x \in \widetilde{A}_{2n+1}$ and let $\mathcal{L}(y) = \{t \in S \mid ty < y\}$ and $\mathcal{R}(y) = \{t \in S \mid yt < y\}$ for $y \in \widetilde{C}_n$.

1.9. Corollary. (comparing with [**7**, Corollary 2.6]) *For any* $x \in \widetilde{C}_n$ *and* $i \in [0, n]$*,*

$s_i \in \widetilde{\mathcal{L}}(x) \Longleftrightarrow s_{2n+1-i} \in \widetilde{\mathcal{L}}(x) \Longleftrightarrow t_i \in \mathcal{L}(x) \Longleftrightarrow (i)x > (i+1)x \Longleftrightarrow (2n+2-i)x < (2n+1-i)x$,

$s_i \in \widetilde{\mathcal{R}}(x) \Longleftrightarrow s_{2n+1-i} \in \widetilde{\mathcal{R}}(x) \Longleftrightarrow t_i \in \mathcal{R}(x) \Longleftrightarrow (i)x^{-1} > (i+1)x^{-1} \Longleftrightarrow (2n+2-i)x^{-1} < (2n+1-i)x^{-1}$.

1.10. For any $a \in \mathbb{Z}$, denote by $\langle a\rangle$ the unique integer in $[2n+2]$ satisfying $a \equiv \langle a\rangle \pmod{2n+2}$. It is known that any $w \in \widetilde{C}_n$ is determined uniquely by the n-tuple $((1)w, (2)w, \ldots, (n)w)$. Hence we shall identify w with the n-tuple $((1)w, (2)w, \ldots, (n)w)$ and denote the latter by $[(1)w, (2)w, \ldots, (n)w]$ in such a sense. Let $w = [a_1, a_2, \ldots, a_n]$ and $w' = t_i w = [a'_1, a'_2, \ldots, a'_n]$ and $w'' = wt_i = [a''_1, a''_2, \ldots, a''_n]$ be in $\widetilde{C}_n$. When $i \in [n-1]$, we have $a'_j = a_j$ for $j \in [n] - \{i, i+1\}$ (set difference) and $(a'_i, a'_{i+1}) = (a_{i+1}, a_i)$; when $i = 0$, we have $a'_j = a_j$ for $j \in [2, n]$ and $a'_1 = -a_1$; when $i = n$, we have $a'_j = a_j$ for $j \in [n-1]$ and $a'_n = 2n+2-a_n$. On the other hand, when $i \in [n-1]$, we have $a''_j = a_j$ if $\langle a_j\rangle \notin \{i, i+1, 2n+1-i, 2n+2-i\}$, $a''_j = a_j + 1$ if $\langle a_j\rangle \in \{i, 2n+1-i\}$ and $a''_j = a_j - 1$ if $\langle a_j\rangle \in \{i+1, 2n+2-i\}$; when $i = 0$, we have $a''_j = a_j$ if $\langle a_j\rangle \notin \{1, 2n+1\}$, $a''_j = a_j + 2$ if $\langle a_j\rangle = 2n+1$ and $a''_j = a_j - 2$ if $\langle a_j\rangle = 1$; when $i = n$, we have $a''_j = a_j$ if $\langle a_j\rangle \notin \{n, n+2\}$, $a''_j = a_j + 2$ if $\langle a_j\rangle = n$ and $a''_j = a_j - 2$ if $\langle a_j\rangle = n+2$.

Let η be the group automorphism of $\widetilde{C}_n$ determined by the condition $\eta(t_i) = t_{n-i}$ for any $i \in [0, n]$.

1.11. For any $i \in [0, 2n+1]$, let $\widetilde{D}_R(i)$ be the set of all $w \in \widetilde{A}_{2n+1}$ satisfying $|\{s_i, s_{i+1}\} \cap \widetilde{\mathcal{R}}(w)| = 1$. When $w \in \widetilde{D}_R(i)$, exactly one of ws_i and ws_{i+1} is in $\widetilde{D}_R(i)$, denote it by w^*, call the transformation from w to w^* a *right* $\{s_i, s_{i+1}\}$*-star operation* (or a *right star operation* in short) on w. Clearly, $(w^*)^* = w$ in this case., For any $w \in \widetilde{A}_{2n+1}$, let $\widetilde{M}(w)$ be the set of all

$y \in \widetilde{A}_{2n+1}$ satisfying the following condition: there exists a sequence $x_0 = w, x_1, \dots, x_r = y$ in $\widetilde{A}_{2n+1}$ with some $r \geqslant 0$ such that for every $i \in [r]$, x_i is obtained from x_{i-1} by a right $\{s_{k_i}, s_{k_i+1}\}$-star operation with some $k_i \in \mathbb{Z}$. Define a graph $\widetilde{\mathcal{M}}(w)$ as follows. Its vertex set is $\widetilde{M}(w)$, each $x \in \widetilde{M}(w)$ is labeled by $\widetilde{\mathcal{R}}(x)$. Two vertices $x, y \in \widetilde{M}(w)$ are joined by a solid edge if y can be obtained from x by a right star operation. By a *path* in $\widetilde{\mathcal{M}}(w)$, we mean a sequence $x_0, x_1, \dots, x_r$ in $\widetilde{M}(w)$ with some $r \geqslant 0$ such that x_{i-1} and x_i are joined by a solid edge for every $i \in [r]$. Two elements $w, y \in \widetilde{A}_{2n+1}$ are said to have *the same generalized τ-invariants*, if for any path $w_1 = w, w_2, \dots, w_r$ in $\widetilde{\mathcal{M}}(w)$, there exists a path $y_1 = y, y_2, \dots, y_r$ in $\widetilde{\mathcal{M}}(y)$ such that $\widetilde{\mathcal{R}}(w_i) = \widetilde{\mathcal{R}}(y_i)$ for every $i \in [r]$ and if the same condition holds when the roles of w and y are interchanged.

For any $i \in [0, n-1]$, let $D_R(i)$ be the set of all $w \in \widetilde{C}_n$ such that $|\{t_i, t_{i+1}\} \cap \mathcal{R}(w)| = 1$. Regarding $\widetilde{C}_n$ as a subset of $\widetilde{A}_{2n+1}$, an element $w \in \widetilde{C}_n$ is in $D_R(i)$ if and only if w is in $\widetilde{D}_R(i)$ if and only if w is in $\widetilde{D}_R(2n-i)$. When $w \in D_R(i)$, exactly one of wt_i and wt_{i+1} is in $D_R(i)$ unless that $i \in \{0, n-1\}$ and $w = xy$ with $x, y \in \widetilde{C}_n$ satisfying $\{t_i, t_{i+1}\} \cap \mathcal{R}(x) = \emptyset$ and $y \in \{t_i t_{i+1}, t_{i+1} t_i\}$. In this excepted case, both wt_i and wt_{i+1} are in $D_R(i)$. When $|\{wt_i, wt_{i+1}\} \cap D_R(i)| = 1$, denote by w^* the unique element in $\{wt_i, wt_{i+1}\} \cap D_R(i)$, then w^* can be obtained from w by a pair of right star operations in $\widetilde{A}_{2n+1}$ if and only if either $i \in [n-2]$ or $w^* = wt_m$ with $(i, m) \in \{(0,1), (n-1, n-1)\}$. When $\{wt_i, wt_{i+1}\} \subset D_R(i)$, define w_1^*, w_2^* by the conditions $\{w_1^*, w_2^*\} = \{wt_i, wt_{i+1}\}$ and $w_1^* < w_2^*$, then $x \in \{w_1^*, w_2^*\}$ can be obtained from w by a pair of right star operations in $\widetilde{A}_{2n+1}$ if and only if $x = wt_m$ with $m \in \{1, n-1\}$.

In the remaining part of the paper, when we mention a right star operation and the generalized τ-invariants on $w \in \widetilde{C}_n$, we always mean that w is regard as an element of $\widetilde{A}_{2n+1}$. We make such a convention once and forever.

1.12. Examples. (1) The elements $x = t_1, y = t_1 t_0$ and $z = t_1 t_0 t_1$ are in $D_R(0)$. We have $x^* = z^* = y$ and $y_1^* = x$ and $y_2^* = z$. The elements y, z can be obtained from one to another by a right $\{s_0, s_1\}$-star operation followed by a right $\{s_{2n}, s_{2n+1}\}$-star operation, but x, y can't be obtained from one to another by a pair of right star operations.

(2) Assume $n > 2$. The elements $x = t_2$ and $y = t_2 t_1$ are in $D_R(1)$. We have $x^* = y$ and $y^* = x$. The elements x, y can be obtained from one to another by a right $\{s_1, s_2\}$-star operation followed by a right $\{s_{2n-1}, s_{2n}\}$-star operation.

1.13. For any $w \in \widetilde{C}_n$, define $M(w)$ to be the set of all $y \in \widetilde{C}_n$ satisfying the following conditions: there exists a sequence $x_0 = w, x_1, \dots, x_r = y$ with some $r \geqslant 0$ such that for every $i \in [r]$, $x_i^{-1} x_{i-1} \in S$ and x_i can be obtained from x_{i-1} by a pair of right star operations. Define a graph $\mathcal{M}(w)$ as follows. Its vertex set is $M(w)$. Each vertex $x \in M(w)$ is labeled by $\mathcal{R}(x)$. Two elements $x, y \in M(w)$ are joined by a solid edge if $x^{-1}y \in S$ and x can be obtained from y by a pair of right star operations.

One can define a path in $\mathcal{M}(w)$ in the same way as that in $\widetilde{\mathcal{M}}(w)$. It is easily seen that if $y, w \in \widetilde{C}_n$ have the same generalized τ-invariants, then for any path $w_1 = w, w_2, \ldots, w_r$ in $\mathcal{M}(w)$, there exists a path $y_1 = y, y_2, \ldots, y_r$ in $\mathcal{M}(y)$ such that $\mathcal{R}(w_i) = \mathcal{R}(y_i)$ for every $i \in [r]$ and the same condition holds when the roles of w and y are interchanged. In Section 7, the graphs $\mathcal{M}(w)$ with $w \in \widetilde{C}_3$ will be used to confirm that two elements of $\widetilde{C}_3$ have different generalized τ-invariants.

Sometimes we join two vertices $x, y \in \widetilde{C}_n$ in a graph by a dashed edge to indicate the facts that $x^{-1}y \in S$, $\mathcal{R}(x) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{R}(y)$ and that y can't be obtained from x by a pair of right star operations in $\widetilde{A}_{2n+1}$.

1.14. Example. In Figure 4 (see 6.4), the elements $x = [-2, -3, -1]$, $y = [-2, -5, -1]$ and $z = [-3, -6, -1]$ in $\widetilde{C}_3$ have labels $\mathcal{R}(x) = \mathcal{R}(z) = \{t_0, t_2\}$ and $\mathcal{R}(y) = \{t_0, t_3\}$, where we use a boldfaced letter **i** to denote the generator t_i, hence, for example, the notation $\boxed{\mathbf{02}}$ stands for the set $\{t_0, t_2\}$. y and z are joined by a solid edge since $y^{-1}z = t_2 \in S$ and z can be obtained from y by a right $\{s_2, s_3\}$-star operation followed by a right $\{s_4, s_5\}$-star operation in $\widetilde{A}_7$. However, x and y are joined only by a dashed edge since $x^{-1}y = t_3$ and x can't be obtained from y by a pair of right star operations. We see from Figure 4 that x, y, z have pairwise different generalized τ-invariants.

1.15. For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ in Λ_{2n+2}, we write $\lambda \leqslant \mu$ if $\lambda_1 + \cdots + \lambda_k \leqslant \mu_1 + \cdots + \mu_k$ for any $1 \leqslant k \leqslant \min\{r, t\}$. This defines a partial order on Λ_{2n+2}. It is well known that if $x \in \widetilde{A}_{2n+1}$ and $s \in \widetilde{\mathcal{L}}(x)$ and $t \in \widetilde{\mathcal{R}}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$ (see [**5**, Lemma 5.5 and Corollary 5.6]). This implies by Corollary 1.9 that if $x \in \widetilde{C}_n$ and $s \in \mathcal{L}(x)$ and $t \in \mathcal{R}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$.

1.16. Lemma. (see [**4**, Lemma 16.14]) *Let $x, y \in \widetilde{C}_n$. Then $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) in $\widetilde{A}_{2n+1}$.*

By Lemma 1.16, we can just use the notation $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to the group $\widetilde{A}_{2n+1}$ or $\widetilde{C}_n$.

For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda_{2n+2}$, define $\lambda^\vee = (\mu_1, \mu_2, \ldots, \mu_t) \in \Lambda_{2n+2}$ by setting $\mu_j = {}^\#\{k \geqslant 1 \mid \lambda_k \geqslant j\}$, for any $j \geqslant 1$, and call $\lambda^\vee$ the *dual partition* of λ.

1.17. Lemma. *Let $x, y \in \widetilde{A}_{2n+1}$.*

(1) $x \underset{L}{\sim} y$ if and only if x, y have the same generalized τ-invariants (see [**5**, Theorem 16.1.2]*).*

(2) $x \underset{LR}{\leqslant} y$ if and only if $\psi(y) \leqslant \psi(x)$. In particular, $x \underset{LR}{\sim} y$ if and only if $\psi(x) = \psi(y)$ (see [**3**, Theorem 6] *and* [**5**, Theorem 17.4] *and* [**6**, Theorem B]*).*

1.18. A non-empty subset E of a Coxeter group $W = (W, S)$ is said *left-connected*, (resp., *right-connected*) if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, \ldots, x_r = y$ in E such that $x_{i-1}x_i^{-1} \in S$ (resp., $x_i^{-1}x_{i-1} \in S$) for every $i \in [r]$. E is said *two-sided-connected* if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, \ldots, x_r = y$ in E such that either $x_{i-1}x_i^{-1}$ or $x_i^{-1}x_{i-1}$ is in S for every $i \in [r]$.

Let $F \subseteq E$ in W. Call F a *left-connected component* of E, if F is a maximal left-connected subset of E. One can define a right-connected component and a two-sided-connected component of E similarly.

For any $\lambda \in \Lambda_{2n+2}$, define $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

1.19. Lemma. (comparing with [**7**, Lemma 2.18]) *Let $\lambda \in \Lambda_{2n+2}$.*

(1) Any left-connected (resp., right-connected, two-sided-connected) set of $\psi^{-1}(\lambda)$ is contained in some left (resp., right, two-sided) cell of $\widetilde{A}_{2n+1}$.

(2) Any left-connected (resp., right-connected, two-sided-connected) set of E_λ is contained in some left (resp., right, two-sided) cell of $\widetilde{C}_n$.

(3) The set E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$.

1.20. Corollary. (comparing with [**7**, Corollary 2.19]) *Let $x, y \in \widetilde{A}_{2n+1}$ be in $\psi^{-1}(\lambda)$ for some $\lambda \in \Lambda_{2n+2}$.*

(1) If $\widetilde{\ell}_{2n+1}(y) = \widetilde{\ell}_{2n+1}(x) + \widetilde{\ell}_{2n+1}(yx^{-1})$ then x, y are in the same left-connected component of $\psi^{-1}(\lambda)$ and hence $x \underset{L}{\sim} y$.

(2) If $\widetilde{\ell}_{2n+1}(y) = \widetilde{\ell}_{2n+1}(x) + \widetilde{\ell}_{2n+1}(x^{-1}y)$ then x, y are in the same right-connected component of $\psi^{-1}(\lambda)$ and hence $x \underset{R}{\sim} y$.

Let $x, y \in E_\lambda$ for some $\lambda \in \Lambda_{2n+2}$.

(3) If $\ell(y) = \ell(x) + \ell(yx^{-1})$ then x, y are in the same left-connected component of E_λ and hence $x \underset{L}{\sim} y$.

(4) If $\ell(y) = \ell(x) + \ell(x^{-1}y)$ then x, y are in the same right-connected component of E_λ and hence $x \underset{R}{\sim} y$.

2. Partial order $\preceq_w$ on $[2n+2]$ determined by some $w \in \widetilde{C}_n$.

In this section, we introduce two technical tools following Shi in [**7**, Section 3]. One is a transformation on an element in 2.3, which is a crucial step in proving the left-connectedness of a left cell and in finding a representative set for the left cells of $\widetilde{C}_n$ in the set E_λ, $\lambda \in \Lambda_{2n+2}$. The other is the generalized tabloids defined in 2.5, by which we can check whether two elements of $\widetilde{C}_n$ are in the same left cell.

2.1. Call $i, j \in [2n+2]$ to be $(2n+2)$-*dual*, if $i + j = 2n+2 \pmod{2n+2}$; in this case, we denote $j = \bar{i}$ (hence $i = \bar{j}$ also). In particular, $\overline{2n+2} = 2n+2$ and $\overline{n+1} = n+1$. Call each of $n+1$ and $2n+2$ to be $(2n+2)$-*selfdual.* Recall the partial order $\preceq_w$ on $[2n+2]$ defined in 1.5 for any $w \in \widetilde{A}_{2n+1}$ and that $\widetilde{C}_n$ can be regarded as a subset of $\widetilde{A}_{2n+1}$ (see 1.4). Denote $[2n+1]_{n+1} := [2n+1] - \{n+1\}$. Fix $w \in \widetilde{C}_n$. Call $i \in [2n+1]_{n+1}$ a *w-wild* if $i \neq \bar{i}$ are w-comparable and a *w-tame* if $i \neq \bar{i}$ are w-uncomparable. Call $i \in [2n+1]_{n+1}$ a *w-wild head* (resp., a *w-tame head*), if i is a w-wild (resp., a w-tame) with $(\bar{i})w < (i)w$; in this case, we call $\bar{i}$ a *w-wild tail* (resp., a *w-tame tail*).

It is easily seen that $i < j$ in $[2n+2]$ are w-uncomparable if and only if $(i)w < (j)w < (i)w + 2n + 2$.

2.2. Lemma. *Fix $w \in \widetilde{C}_n$.*

(i) For any $j \neq k$ in $[2n+2]$, $j \prec_w k$ if and only if $\bar{k} \prec_w \bar{j}$;

Now suppose that $j \neq k$ in $[2n+1]_{n+1}$ are w-wild heads and that $i \in [2n+1]_{n+1}$ is a w-tame head.

(ii) $\bar{j} \prec_w k$ if and only if $\bar{j}, k$ are w-comparable.

(iii) If $\bar{j}, k$ are w-uncomparable then so are j, k;

(iv) i and k are w-comparable if and only if $i \prec_w k$.

(v) $\{j, i, \bar{j}\}$ is a w-chain if and only if j is w-comparable with both i and $\bar{i}$;

(vi) $\{j, k, \bar{j}, \bar{k}\}$ is a w-chain if and only if j, k are w-comparable.

(vii) if $j < n+1$ then $\bar{j} \prec_w n+1 \prec_w j$; if $j > n+1$ then $\bar{j} \prec_w 2n+2 \prec_w j$;

(viii) $i, \bar{i}$ are w-uncomparable with $n+1, 2n+2$.

Proof. The results (i)-(vi) follow by [**7**, Lemma 3.2] and (vii)-(viii) can be checked directly.

2.3. Define

$$t'_k = \begin{cases} t_{\langle k \rangle}, & \text{if } \langle k \rangle \in [n], \\ t_{\overline{\langle k \rangle}-1}, & \text{if } \langle k \rangle \in [n+2, 2n+1], \\ 1, & \text{if } \langle k \rangle \in \{n+1, 2n+2\}. \end{cases}$$

and

$$t_{i,j} = t'_{i+j-1}t'_{i+j-2}\dots t'_{i+1}t'_i \tag{2.3.1}$$

for any $i, k \in \mathbb{Z}$ and $j \in \mathbb{P}$. Suppose that $x \in \widetilde{C}_n$ and $i \in \mathbb{Z}$ satisfy $i \not\equiv n+1, 2n+2$ (mod $2n+2$). If $(i)x - (j)x > 2n+2$ for any $j \in [i+1, i+a]$ with some $a \in [2n+1]$, let $x' = t_{i,a}x$, then $\ell(x') = \ell(x) - \ell(t_{i,a})$ and $\psi(x) = \psi(x')$. Moreover, if $(i)x - (j)x > 2n+2$ for any $j \in [i+1, i+2n+1]$, let $x'' = t_{i,2n+2}x$, then

$$(m)x'' = \begin{cases} (m)x - 2n - 2, & \text{if } \langle m\rangle = \langle i\rangle, \\ (m)x + 2n + 2, & \text{if } \langle m\rangle = \langle 2n+2-i\rangle, \\ (m)x, & \text{if otherwise.} \end{cases}$$

for any $m \in \mathbb{Z}$, where x'' satisfies $\ell(x'') = \ell(x) - 2n - 2$ and $\psi(x) = \psi(x'')$ by 1.5 and Proposition 1.7.

Fix $w \in \widetilde{C}_n$. Suppose that $E_1 = \{i_1, i_2, \dots, i_a\}$ and $E_2 = \{j_1, j_2, \dots, j_b\}$ are two subsets of $[2n+1]_{n+1}$ satisfying that

(i) $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$ with $a > 0$ and $b \geqslant 0$ and $a + b = n$;

(ii) the elements of $E_1 \cup E_2$ are pairwise not $(2n+2)$-dual;

(iii) $(\overline{k})w < (k)w$ for any $k \in E_1 \cup E_2$;

(iv) $(i)w - (j)w > l(2n+2)$ for any $i \in E_1$ and $j \in E_2 \cup \{2n+2\}$, where $l \in \mathbb{N}$.

Suppose $b > 0$. Then by repeatedly left-multiplying the elements $t_{i,j}$ with some $i \in \mathbb{Z}$, $j \in \mathbb{P}$, on w, we can obtain some $w' \in \widetilde{C}_n$ satisfying that

(1) $\ell(w') = \ell(w) - \ell(w'w^{-1})$;

(2) There exists a unique order-preserving bijective map $\phi : E_2 \cup \overline{E_2} \cup \{n+1\} \longrightarrow [a+1, n+b+1]$ such that $(p)w' = (i_p)w - l'(2n+2)$ and $(\phi(q))w' = (q)w$ for any $p \in [a]$ and $q \in E_2 \cup \overline{E_2} \cup \{n+1\}$, where $l' \leqslant l$ in $\mathbb{N}$;

(3) $(\overline{c})w' < (c)w'$ for any $c \in [a] \cup \{\phi(m) \mid m \in E_2\}$;

(4) $0 < \min\{(c)w' - (q)w' \mid c \in [a], q \in \{\phi(m) \mid m \in E_2\}\} \leqslant 2n+2$.

We have $\psi(w') = \psi(w)$ (denote it by λ) by Lemma 2.2 and that w, w' are in the same left-connected component of E_λ by Corollary 1.20.

2.4. Example. Let $w = [-1, -11, -10] \in \widetilde{C}_3$. Then $E_1 = \{5, 6\}$ and $E_2 = \{7\}$ satisfy the conditions (i)-(iv) in 2.3 with $n = 3$ and $(a, b, l) = (2, 1, 1)$. Let $w' = t_{5,3}t_{6,2}w$. Then $w' = [10, 11, -1] \in \widetilde{C}_3$ satisfies the conditions (1)-(4) in 2.3 and $\psi(w) = \psi(w') = \mathbf{521}$.

Let $w'' = t_{1,3}t_{2,2}w'$. Then $w'' = [-1, -3, -2] \in \widetilde{C}_3$ ($w'' \in F'_{\mathbf{521}}$ in 6.4). We see that $\psi(w'') = \psi(w') = \mathbf{521}$ and $\ell(w'') = \ell(w') - \ell(t_{1,3}t_{2,2})$. Hence by Corollary 1.20. w, w', w'' are in the same left-connected component of $E_{\mathbf{521}}$.

2.5. By a *composition* of $2n+2$, we mean an r-tuple $\mathbf{a} = (a_1, a_2, \dots, a_r)$ of positive integers $a_1, a_2, \dots, a_r$ with some $r \geqslant 1$ such that $\sum_{i=1}^r a_i = 2n+2$. Let $\widetilde{\Lambda}_{2n+2}$ be the set of all compositions of $2n+2$. Let $i_1, i_2, ..., i_r$ be some permutation of $1, 2, ..., r$ such that $a_{i_1} \geqslant a_{i_2} \geqslant \cdots \geqslant a_{i_r}$. Then $\zeta(\mathbf{a}) := (a_{i_1}, a_{i_2}, ..., a_{i_r}) \in \Lambda_{2n}$. Clearly, $\Lambda_{2n+2} \subseteq \widetilde{\Lambda}_{2n+2}$ and $\zeta : \widetilde{\Lambda}_{2n} \longrightarrow \Lambda_{2n}$ is a surjective map.

A generalized tabloid of rank $2n+2$ is, by definition, an r-tuple $T = (T_1, T_2, \dots, T_r)$ with some $r \in \mathbb{P}$ such that $[2n+2]$ is a disjoint union of non-empty subsets $T_j, j \in [r]$. We have $\xi(T) := (|T_1|, |T_2|, \dots, |T_r|) \in \widetilde{\Lambda}_{2n+2}$, where $|T_i|$ denotes the cardinal of the set T_i. Let $\mathcal{C}_{2n+2}$ be the set of all generalized tabloids of rank $2n+2$. Then $\xi : \mathcal{C}_{2n+2} \longrightarrow \widetilde{\Lambda}_{2n+2}$ is a surjective map.

Let Ω be the set of all $w \in \widetilde{A}_{2n+1}$ such that there is a generalized tabloid $T = (T_1, T_2, \dots, T_r) \in \mathcal{C}_{2n+2}$ satisfying:

(i) For any $i < j$ in $[r]$, we have $\langle (a)w^{-1} \rangle \prec_w \langle (b)w^{-1} \rangle$ for any $a \in T_i$ and $b \in T_j$;

(ii) $\langle (T_i)w^{-1} \rangle$ is a maximal w-antichain in $[2n+2]$ for any $i \in [r]$ (see 1.5).

Clearly, T is determined entirely by $w \in \Omega$, denote T by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{2n+2}$ is surjective by [**5**, Proposition 19.1.2]. By a result of Curtis Greene in [**1**], we have $\zeta\xi(T(w)) = \psi(w)^\vee$ for any $w \in \Omega$.

The following known result will be crucial in the proof of Lemmas 4.5 and 5.6.

2.6. Lemma. (see [**5**, Lemma 19.4.6]) *Suppose that $y, w \in \widetilde{A}_{2n+1}$ are two elements in Ω with $\xi(T(y)) = \xi(T(w))$. Then $y \underset{L}{\sim} w$ if and only if $T(y) = T(w)$.*

3. Some criteria for the set E_λ, $\lambda \in \Lambda_{2n+2}$, being empty.

Recall that in 1.18 we defined the set E_λ for any $\lambda \in \Lambda_{2n+2}$. We have $E_\lambda^{-1} = E_\lambda$. In the present section, we shall give some criteria for the vanishing of the set E_λ.

Fix $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \Lambda_{2n+2}$ in 3.1-3.3.

3.1. Lemma. *Let $w \in E_\lambda$. If $X = \{a_1, a_2, \dots, a_{\lambda_1}\}$ is a w-chain such that a_i is either a w-wild or in $\{n+1, 2n+2\}$ for any $i \in [\lambda_1]$, then λ_1 is odd.*

Proof: We may assume $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_{\lambda_1}$. By Lemma 2.2 (i) and (vii), $a_{i+1}, ..., a_{\lambda_1}$ are all w-wild heads in X with $i = \lceil \frac{\lambda_1}{2} \rceil$, where $\lceil x \rceil$ is the smallest integer not smaller than $x \in \mathbb{Q}$. Assume that λ_1 is even. Then $E := \{\overline{a}_{\lambda_1}, ..., \overline{a}_{i+1}, a_{i+1}, ..., a_{\lambda_1}\}$ is a w-chain by Lemma 2.2 (i). This would imply that either $E \cup \{n+1\}$ or $E \cup \{2n+2\}$ is a w-chain of cardinal $\lambda_1 + 1$ by Lemma 2.2 (viii), contradicting the assumption $w \in E_\lambda$. $\square$

3.2. Proposition. *$E_\lambda = \emptyset$ if λ_1 is even and if either $r \leqslant 3$ or $\lambda_2 = 1$.*

Proof. We argue by contrary. Suppose that there is some w in E_λ.

First assume $r \leqslant 3$. Then there is no any w-tame in $[2n+2]$ by Lemma 2.2 (viii). Hence any w-chain X in $[2n+2]$ with $|X| = \lambda_1$ consists of some w-wilds and some elements in $\{n+1, 2n+2\}$. But this would imply that λ_1 is odd by Lemma 3.1, contradicting our assumption.

Next assume $\lambda_2 = 1$. Then all w-tames in $[2n+2]$ must be pairwise w-uncomparable by Lemma 2.2 (i). Since any w-tame is w-uncomparable with each of $n+1, 2n+2$, any w-chain in $[2n+2]$ contains at most one element which is either a w-tame or one of $n+1, 2n+2$. So any w-chain X in $[2n+2]$ with $|X| = \lambda_1$ contains a subset X' with $|X'| \geqslant \lambda_1 - 1$ such that X' consists of some w-wilds. Write $X' = X'_1 \cup X'_2$ with X'_1 the set of all w-wild heads in X' and $X'_2 = X' - X'_1$. We see that at least one of $X'_1 \cup \overline{X'_1} \cup \{n+1\}$, $X'_1 \cup \overline{X'_1} \cup \{2n+2\}$ is a w-chain and also that at least one of $X'_2 \cup \overline{X'_2} \cup \{n+1\}$, $X'_2 \cup \overline{X'_2} \cup \{2n+2\}$ is a w-chain, where $\overline{E} := \{\bar{i} \mid i \in E\}$ for any $E \subset \mathbb{Z}$. We must have $|X'_1| = |X'_2| = \frac{\lambda_1 - 1}{2}$ with $\lambda_1 - 1$ even by the assumption of $w \in E_\lambda$. This would imply λ_1 odd, contradicting our assumption also.

This proves $E_\lambda = \emptyset$ in either case. □

3.3. Proposition. *Suppose that $m \in [r]$ and that $\lambda_i = 2$, $\lambda_j = 1$ for any $i \in [m]$ and any $j \in [m+1, r]$. If either m is odd or $m \geqslant r - 1$, then $E_\lambda = \emptyset$.*

Proof. Suppose that there is some $w \in E_\lambda$. We claim that there is no any w-wild in $[2n+2]$. For otherwise, we would have $\lambda_1 \geqslant 3$ by Lemma 2.2 (vii). Hence all elements of $[2n+1]_{n+1}$ are w-tames. So m is even by Lemma 2.2 (i). We see by Lemma 2.2 (viii) that each of $n+1$, $2n+2$ is w-uncomparable with any w-tame, hence $\lambda_r = \lambda_{r-1} = 1$. This completes our proof. □

4. The set $E_{\mathbf{k1^{2n+2-k}}}$.

In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{\mathbf{k1^{2n+2-k}}}$ for all $k \in [2n+2]$. The set $E_{\mathbf{1^{2n+2}}}$ consists of the identity element of $\widetilde{C}_n$ and $E_{\mathbf{k1^{2n+2-k}}} = \emptyset$ for any even $k \in [2n+2]$ by Proposition 3.2. In the subsequent discussion of the section, we shall always assume $k = 2m+1$ with $m \in [n]$.

4.1. Let $l = n - m$. Then $2n + 2 - k = 2l + 1$. By Lemma 2.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{\mathbf{k1^{2n+2-k}}}$ if and only if the conditions (4.1.1) (i), (ii) on w hold.

(4.1.1) There exist n disitnct $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m$ in $[2n+1]_{n+1}$ such that

(i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$;

(ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$;

Let $F_{\mathbf{k1^{2n+2-k}}}$ be the set of all $w \in E_{\mathbf{k1^{2n+2-k}}}$ satisfying one additional condition (iii).

(iii) $(\bar{i}_l, \bar{i}_{l-1}, \cdots, \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ and $(j_1)w \in [n+2, 3n+2]$ and $(j_{a+1})w - (j_a)w \in [2n+1]$ for any $a \in [m-1]$;

By 2.3 and 4.1, it is easily seen that

4.2. Lemma. *For any $w \in E_{\mathbf{k1^{2n+2-k}}}$, there exists some $w' \in F_{\mathbf{k1^{2n+2-k}}}$ such that w', w are in the same left-connected component of $E_{\mathbf{k1^{2n+2-k}}}$.*

4.3. Lemma. *The set $F_{\mathbf{k1^{2n+2-k}}}$ is contained in a right-connected component of $E_{\mathbf{k1^{2n+2-k}}}$.*

Proof. Let $J = \{t_{l+1}, t_{l+2}, ..., t_n\}$ and let w_J be the longest element in the subgroup W_J of $\widetilde{C}_n$ generated by J. Then $w_J = [1, 2, ..., l, 2n+1-l, 2n-l, ..., n+2]$ is in $F_{\mathbf{k1^{2n+2-k}}}$. By Corollary 1.9, we see that any $w \in F_{\mathbf{k1^{2n+2-k}}}$ satisfies $\mathcal{L}(w) = J$ and has an expression $w = w_J x$ with $\ell(w) = \ell(w_J) + \ell(x)$ for some $x \in \widetilde{C}_n$. So the set $F_{\mathbf{k1^{2n+2-k}}}$ is contained in the right-connected component of $E_{\mathbf{k1^{2n+2-k}}}$ containing w_J. □

4.4. Lemma. $|F_{\mathbf{k1^{2n+2-k}}}| = n!2^m/(n-m)!$.

Proof. The elements w of $F_{\mathbf{k1^{2n+2-k}}}$ are in one-to-one correspondence with the m-tuples

$$\langle A_w \rangle := (\langle (j_m)w \rangle, \langle (j_{m-1})w \rangle, ..., \langle (j_1)w \rangle). \tag{4.4.1}$$

with its components being in $[2n+1]_{n+1}$, pairwise distinct and pairwise not $(2n+2)$-dual. This is because that w is determined uniquely by the n-tuple

$$A_w := ((\bar{i}_l)w, (\bar{i}_{l-1})w, ..., (\bar{i}_1)w, (j_m)w, (j_{m-1})w, ..., (j_1)w)$$

subject to the conditions that $1 \leqslant (\bar{i}_l)w < (\bar{i}_{l-1})w < \cdots < (\bar{i}_1)w \leqslant n$, that $(j_1)w \in [n+2, 3n+2] - \{2n+2\}$, that $(j_{a+1})w - (j_a)w \in [2n+1]$ for any $a \in [m-1]$, and that the components of A_w are pairwise distinct and pairwise not $(2n+2)$-dual modulo $2n+2$. Now the number of choices for $\langle (j_1)w \rangle$ is $2n$. Recurrently, suppose that $h \in [2, m]$ and that all the $\langle (j_a)w \rangle$, $a \in [h-1]$, have been chosen, then the number of choices for $\langle (j_h)w \rangle$ should be $2(n+1-h)$. This implies our result. □

4.5. Lemma. *No two elements of $F_{\mathbf{k1^{2n+2-k}}}$ are in the same left cell of $\widetilde{C}_n$.*

Proof. For any $w \in F_{\mathbf{k1^{2n+2-k}}}$, let $w' = zw$ with $z = (t_n \cdots t_1 t_0 t_1 \cdots t_l)^m$. Then $\ell(w') = \ell(z) + \ell(w)$ by Corollary 1.9. We have $(i_t)w' = (i_t)w$ and $(j_r)w' = (j_r)w + 2n + 2$ for any $t \in [l]$ and $r \in [m]$. We also have $\psi(w') = \psi(w)$. This implies that w, w' are in the same left-connected component of $E_{\mathbf{k1^{2n+2-k}}}$ and further that $w \underset{L}{\sim} w'$ by Corollary 1.20.

We have $w' \in \Omega \cap \widetilde{C}_n$ (see 2.5). Write $T(w') = (T_1(w'), T_2(w'), ..., T_{2m+1}(w'))$. Then $T_b(w') = \{\langle (\bar{j}_{m+1-b})w \rangle\}$ and $T_{m+1}(w') = \{n+1, 2n+2, \langle (\bar{i}_a)w \rangle, \langle (i_a)w \rangle \mid a \in [l]\}$ and $T_c(w') = \{\langle (j_{c-m-1})w \rangle\}$ for any $b \in [m]$ and $c \in [m+2, 2m+1]$. Hence $T(w')$ is uniquely determined by $\langle A_w\rangle$ (see (4.4.1)). We see from the proof of Lemma 4.4 that $w_1 \neq w_2$ in $F_{\mathbf{k1^{2n+2-k}}}$ implies $\langle A_{w_1}\rangle \neq \langle A_{w_2}\rangle$and further $T(w_1') \neq T(w_2')$. This implies $w_1' \underset{L}{\not\sim} w_2'$ and hence $w_1 \underset{L}{\not\sim} w_2$ in $\widetilde{C}_n$ by Lemmas 2.6, 1.16 and by the fact $\xi(T(w_1')) = \xi(T(w_2'))$. $\Box$

4.6. Theorem. (1) $E_{\mathbf{k1^{2n+2-k}}} = \emptyset$ for any even $k \in [2n+2]$.

Now assume $k = 2m+1 \in [2n+2]$ odd.

(2) The set $E_{\mathbf{k1^{2n+2-k}}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$.

(3) The set $E_{\mathbf{k1^{2n+2-k}}}$ is infinite if $k > 1$ and $E_{\mathbf{1^{2n+2}}} = \{1\}$.

(4) The set $E_{\mathbf{k1^{2n+2-k}}}$ contains $n!2^m/(n-m)!$ left cells of $\widetilde{C}_n$ each of which is left-connected.

Proof. The assertion (1) follows by Proposition 3.2. Now assume $k = 2m+1 \in [2n+2]$ odd. By Lemma 1.19, we see that $E_{\mathbf{k1^{2n+2-k}}}$ is a union of some two-sided cells of $\widetilde{C}_n$. Hence the assertions (2) and (4) follow by Lemmas 4.2-4.5. For the assertion (3), we see that for $k = 2m+1 > 1$, the number of the choices for the integer $(j_m)w$ in the condition (4.1.1) (ii) is infinite. On the other hand, we have $E_{\mathbf{1^{2n+2}}} = \{1\}$. This proves (3). $\Box$

5. The set $E_{(k,2n+2-k)}$.

In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{(k,2n+2-k)}$ for all $k \in [n+1, 2n]$. Since $E_{(k,2n+2-k)} = \emptyset$ for all even $k \in [n+1, 2n]$ by Proposition 3.2, we shall always assume $k = 2m+1 \in [n+1, 2n]$ odd in the subsequent discussion of the section.

5.1. Let $l = n - m$. Then $2n+2-k = 2l+1$ and $m \geqslant l \geqslant 1$. By Lemma 2.2, we see that $w \in \widetilde{C}_n$ is in the set $E_{(k,2n+2-k)}$ if and only if there are n distinct w-wild heads $i_1, i_2, \cdots, i_l, j_1, j_2, ..., j_m$ in $[2n+1]_{n+1}$ satisfying the following conditions (i)-(iii).

(i) $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$;

(ii) $E := \{n+1, 2n+2, i_c, j_d \mid c \in [l], d \in [m]\}$ is a union of exactly two w-chains (or equivalently, the maximal size of a w-antichain in E is 2 by a result of C. Greene in [**1**]);

(iii) Any w-chain in $E' := E - \{n+1, 2n+2\}$ has cardinal $\leqslant m$.

For any $w \in \widetilde{C}_n$ satisfying the conditions (i)-(ii), define

$$Y_q(w) = \{r \in [m] \mid j_r \text{ is } w\text{-uncomparable with } i_q\}$$

for any $q \in [l]$.

Under the assumptions of (i)-(ii), we state the condition (iii$'$) on w below.

(iii$'$) There exists some $u_1 < u_2 < \cdots < u_l$ in $[m]$ such that $u_q \in Y_q$ for any $q \in [l]$.

5.2. Lemma. In 5.1, under the assumptions of (i)-(ii), the conditions (iii) and (iii$'$) on $w \in \widetilde{C}_n$ are equivalent.

Proof: Keep the notation in 5.1. First assume that $w \in \widetilde{C}_n$ satisfies the conditions 5.1 (i)-(iii). We claim that

(5.2.1) $\quad Y_q(w) \neq \emptyset$ for any $q \in [l]$.

For otherwise, there would exist some $q \in [l]$ with $Y_q(w) = \emptyset$. But this would imply that $\{j_1, j_2, ..., j_m, i_q\}$ is a w-chain of length $m+1$, contradicting the assumption 5.1 (iii).

For any $q \in [l]$, let r'_q, r_q be the largest and the smallest integers of Y_q respectively. Then $Y_q = [r_q, r'_q]$ by the definition of the set Y_q and the condition 5.1 (i). If $Y_q \cap Y_{q'} \neq \emptyset$ for some $q \neq q'$ in $[l]$, then i_q and $i_{q'}$ must be w-comparable by the condition 5.1 (ii). In other words, we have $Y_q \cap Y_{q'} = \emptyset$ for any w-uncomparable pair $i_q, i_{q'}$. This implies the following

(5.2.2) $\quad$ For any $q \in [l-1]$, we have either that $i_q \prec_w i_{q+1}$ or that i_q, i_{q+1} are w-uncomparable and $t < r$ for any $t \in Y_q$ and $r \in Y_{q+1}$.

Now we want to find a required sequence $u_1 < u_2 < \cdots < u_l$ in 5.1 (iii$'$) recurrently.

We can take u_1 to be the smallest integer in Y_1 by (5.2.1). If $l = 1$, then we are done. Now assume $l > 1$. Suppose that we have got all the integers $u_1 < u_2 < \cdots < u_p$ in $[m]$ for some $p \in [l-1]$ such that $u_q = \min\{i \in Y_q \mid i > u_{q-1}\}$ for any $q \in [2, p]$. Now we want to find u_{p+1}. If $E_{p+1} := \{i \in Y_{p+1} \mid i > u_p\} \neq \emptyset$, then we take $u_{p+1} = \min(E_{p+1})$. Hence the condition (iii$'$) holds by induction on $p \in [l]$.

It remains to show that there always exists some integer in Y_{p+1} larger than u_p. Suppose not. There should exist some $b \in [p]$ such that u_b is the smallest integer in Y_b but u_c is not the smallest integer in Y_c for any $c \in [b+1, p]$ and that u_p is the largest integer in Y_{p+1}. This would imply by the choice of the u_a's, $a \in [p]$, that $u_a \in Y_a \cap Y_{a+1}$ for any $a \in [b, p]$. By (5.2.1)-(5.2.2), we see that $u_{b+c} = u_b + c$ for any $c \in [p-b]$ and that $j_{u_b - 1} \prec_w i_b \prec_w i_{b+1} \prec_w \cdots \prec_w i_p \prec_w i_{p+1} \prec_w j_{u_p+1}$. But this would imply that $X = \{j_1, ..., j_{u_b-1}, i_b, i_{b+1}, ..., i_{p+1}, j_{u_p+1}, ..., j_m\}$ is a w-chain with $|X| = m+1$, contradicting the condition 5.1 (iii).

Next assume the conditions 5.1 (i), (ii), (iii$'$) on w. For any w-chain $X \subseteq \{i_c, j_d \mid c \in [l], d \in [m]\}$, we have $|X \cap \{i_q, j_{u_q}\}| \leqslant 1$ for any $q \in [l]$ by the condition 5.1 (iii$'$) on w. Hence $|X| \leqslant m$, the condition 5.1 (iii) on w holds. $\square$

5.3. Let $F'_{(k,2n+2-k)}$ be the set of all $w \in \widetilde{C}_n$ satisfying the condition (5.3.1) below.

(5.3.1) $\quad$ There exist n distinct w-wild heads $i_1, i_2, \cdots, i_l, j_1, j_2, ..., j_m$ in $[2n+1]_{n+1}$ such that $(j_m, i_l, j_{m-1}, i_{l-1}, ..., j_{m-l+1}, i_1, j_{m-l}, ..., j_2, j_1) = (1, 2, ..., n)$, where $(j_1)w \in [3n+4, 5n+4]$

and, the integers $(j_{e+1})w-(j_e)w$ and $(i_f)w-(j_{m-l+f})w$ and $(j_{m-l+q+1})w-(i_q)w$ are in $[2n+1]$ for any $e\in[m-l]$ and $f\in[l]$ and $q\in[l-1]$.

For any $w\in\widetilde{C}_n$ satisfying the condition (5.3.1), we have the w-uncomparable pairs i_q,j_{m-l+q} for any $q\in[l]$ and the relations $a\prec_w j_1\prec_w j_2\prec_w\cdots\prec_w j_{m-l}\prec_w h_1\prec_w h_2\prec_w\cdots\prec_w h_l$ for some $a\in\{n+1,2n+2\}$ and any $h_q\in\{i_q,j_{m-l+q}\}$. This implies that any $w\in F'_{(k,2n+2-k)}$ satisfies the conditions 5.1 (i), (ii), (iii′). So $F'_{(k,2n+2-k)}\subseteq E_{(k,2n+2-k)}\cap\Omega$ by 2.5 and Lemma 5.2.

5.4. Lemma. *For any $w\in E_{(k,2n+2-k)}$, there exists some $w'\in F'_{(k,2n+2-k)}$ such that w',w are in the same left-connected component of $E_{(k,2n+2-k)}$.*

Proof. We see by Lemma 5.2 that $w\in E_{(k,2n+2-k)}$ if and only if w satisfies the conditions 5.1 (i), (ii), (iii′).

Now assume $w\in E_{(k,2n+2-k)}$. Suppose that $i,i+1\in[2n+1]_{n+1}$ (resp., $n,n+2$) are a w-wild tail and a w-wild head, respectively. Then t'_iw (see 2.3) (resp., t_nw) satisfies the conditions 5.1 (i), (ii), (iii′) (meaning that the conditions 5.1 (i), (ii), (iii′) hold with t'_iw (resp., t_nw) in the place of w). Hence the elements w and t'_iw (resp., t_nw) are in the same left-connected component of $E_{(k,2n+2-k)}$. Keeping this fact in mind, by replacing w by some element in the same left-connected component of $E_{(k,2n+2-k)}$ if necessary and by symmetry between the intervals $[n]$ and $[n+2,2n+1]$, we may assume without loss of generality that the integers i_p,j_q in the conditions 5.1 (i)-(ii) on w are in $[n]$ for any $p\in[l]$ and $q\in[m]$.

Define the sets $X_1=\{j_1,j_2,...,j_{u_1-1}\}$ and $X_{2q+1}=\{j_{u_q+1},j_{u_q+2},...,j_{u_{q+1}-1}\}$ and $X_{2l+1}=\{j_{u_l+1},j_{u_l+2},...,j_m\}$ and $X_{2p}=\{i_p,j_{u_p}\}$ for any $q\in[l-1]$ and $p\in[l]$. Then we have the partition $\{i_p,j_q\mid p\in[l],q\in[m]\}=\dot{\bigcup}_{i=1}^{2l+1}X_i$, where any $i\in X_q$ and any $j\in X_{q+1}$ with $q\in[2l]$ either have the relation $i\prec_w j$ or are w-uncomparable (i.e., $j\not\prec_w i$ in either case). If $j\in X_p$ and $j+1\in X_q$ for some $p<q$ in $[2l+1]$, then the element t_jw satisfies the condition 5.1 (i),(ii), (iii′) as w does so, hence t_jw and w are in the same left-connected component of $E_{(k,2n+2-k)}$ by Lemma 5.2. Again, keep this fact in mind, by replacing w by some element in the same left-connected component of $E_{(k,2n+2-k)}$ if necessary, we may assume that $(j_m,...,j_{u_l+1},h_l,g_l,j_{u_l-1},\ ...,j_{u_q+1},h_q,g_q,j_{u_q-1},...,j_{u_1+1},h_1,g_1,j_{u_1-1},...,j_1)=(1,2,...,n)$, where for $q\in[l]$, we assign (h_q,g_q) to be (j_{u_q},i_q) or (i_q,j_{u_q}) according to $(i_q)w>(j_{u_q})w$ or $(i_q)w<(j_{u_q})w$.

Denote $p_i=|X_{2l+1}|+|X_{2l}|+\cdots+|X_i|$ and $x_p=(t_p\cdots t_{n-1}t_nt_{n-1}\cdots t_1t_0)^p$ for any $i\in[2l+1]$ and $p\in[n]$. Let $w^{(1)}=x_{p_2}\cdots x_{p_{2l}}x_{p_{2l+1}}w$. Then $\ell(w^{(1)})=\ell(w^{(1)}w^{-1})+\ell(w)$ and $(j')w^{(1)}=(j')w+(q-1)(2n+2)$ for any $j'\in X_q$ with $q\in[2l+1]$. We see that

$$j_1 \prec_{w^{(1)}} \cdots \prec_{w^{(1)}} j_{u_1-1} \prec_{w^{(1)}} h_1 \prec_{w^{(1)}} j_{u_1+1} \prec_{w^{(1)}} \cdots \prec_{w^{(1)}} j_{u_q-1} \prec_{w^{(1)}} h_q$$

$$\prec_{w^{(1)}} j_{u_q+1} \prec_{w^{(1)}} \cdots \prec_{w^{(1)}} j_{u_l-1} \prec_{w^{(1)}} h_l \prec_{w^{(1)}} j_{u_l+1} \prec_{w^{(1)}} \cdots \prec_{w^{(1)}} j_m$$

and that i_q, j_{u_q} are $w^{(1)}$-uncomparable for any $q \in [l]$ and any $h_q \in \{i_q, j_{u_q}\}$. This implies that $w^{(1)}$ satisfies the conditions 5.1 (i), (ii), (iii′). So $w^{(1)}$ and w are in the same left-connected component of $E_{(k,2n+2-k)}$ by Corollary 1.20.

Note that the set $F(w^{(1)}) := \{p \in [n] \mid p+1, p+2 \prec_{w^{(1)}} p \prec_{w^{(1)}} p-1\}$ is not empty, where we stipulate $1 \prec_{w^{(1)}} 0$ temporary. Define j to be the smallest integer in $F(w^{(1)})$.

Suppose $p(2n+2) < (j)w^{(1)} - (j+2)w^{(1)} < (p+1)(2n+2)$ for some $p \in \mathbb{N}$. Then $\ell(x_j^{-p}w^{(1)}) = \ell(w^{(1)}) - \ell(x_j^{-p})$ and $(i)x_j^{-p}w^{(1)} = (i)w^{(1)} - p(2n+2)$ and $(i')x_j^{-p}w^{(1)} = (i')w^{(1)}$ for any $i \in [j]$ and $i' \in [n] - [j]$. This implies that $0 < (j)x_j^{-p}w^{(1)} - (j+2)x_j^{-p}w^{(1)} < 2n+2$ and that $x_j^{-p}w^{(1)}$ satisfies the conditions 5.1 (i), (ii), (iii′). So by Lemma 5.2 and Corollary 1.20, the elements $w^{(1)}$ and $x_j^{-p}w^{(1)}$ are in the same left-connected component of $E_{(k,2n+2-k)}$. Hence we may assume $0 < (j)w^{(1)} - (j+2)w^{(1)} < 2n+2$ by replacing $w^{(1)}$ by some element in the same left-connected component of $E_{(k,2n+2-k)}$ if necessary.

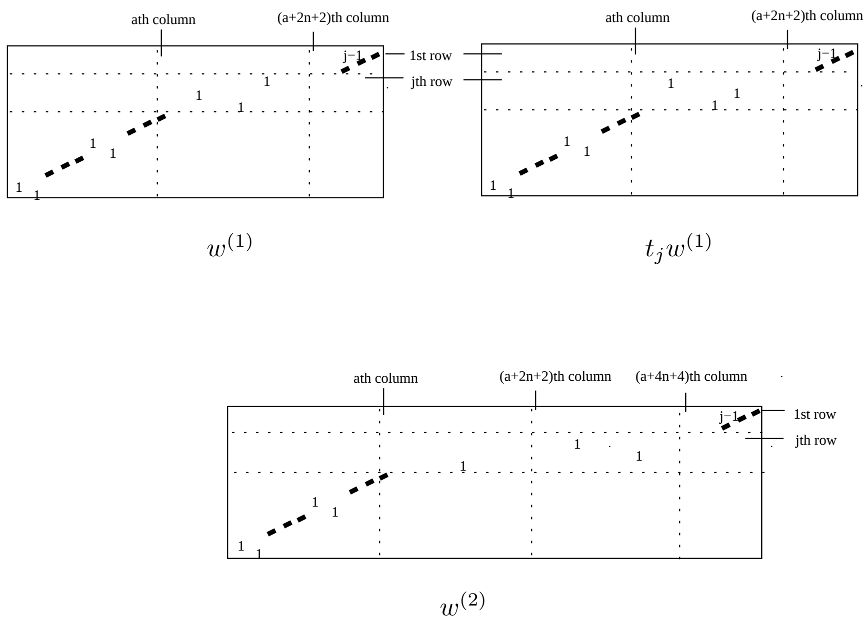
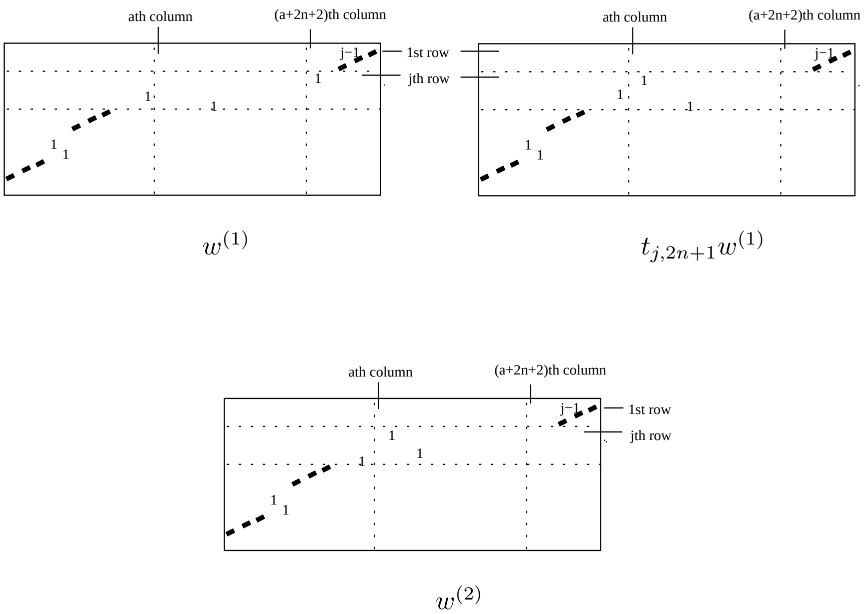


Figure 2



$w^{(1)}$ $t_{j,2n+1}w^{(1)}$

$w^{(2)}$

Figure 3

Define $w^{(2)}$ to be $x_{j+1}t_jw^{(1)}$ if $(j)w^{(1)} - (j+1)w^{(1)} < 2n+2$ and to be $t_{j+1}t_{j,2n+1}w^{(1)}$ if $(j)w^{(1)} - (j+1)w^{(1)} > 2n+2$.

In Figure 2 (resp., Figure 3), we display the corresponding parts for the matrix forms of $w^{(1)}$, $t_jw^{(1)}$, $w^{(2)}$ in the case of $a < (j+1)w^{(1)} < (j)w^{(1)} < a+2n+2$ (resp., $w^{(1)}$, $t_{j,2n+1}w^{(1)}$, $w^{(2)}$ in the case of $(j+1)w^{(1)} < a < a+2n+2 \leqslant (j)w^{(1)}$) for some $a \in \mathbb{Z}$, where the symbol q (or in short) stands for a rectangular submatrix with q rows for some $q \in [n]$ each row contains a unique non-zero entry which is 1, the entries 1 are going down to the left.

We see from the above graphs that $w^{(2)}$ satisfies the conditions 5.1 (i), (ii), (iii′), hence $w^{(1)}, w^{(2)}$ are in the same left-connected component of $E_{(k,2n+2-k)}$ by Lemma 5.2. We have that $j+2 \prec_{w^{(2)}} j, j+1$ if $j = 1$ and that $j+2 \prec_{w^{(2)}} j, j+1 \prec_{w^{(2)}} j-1$ if $j > 1$ and that $j, j+1$ are $w^{(2)}$-uncomparable. We also have $p \prec_{w^{(1)}} q$ if and only if $p \prec_{w^{(2)}} q$ for any other pairs of integers p, q in $[n]$.

By repeatedly applying the above process, we can eventually find an element $w^{(r)}$ in the left-connected component of $E_{(k,2n+2-k)}$ containing w with some $r \in \mathbb{N}$ such that $w^{(r)}$ satisfies the condition (5.4.1) below.

(5.4.1) There exist n distinct w-wild heads $i'_1, i'_2, \cdots, i'_l, j'_1, j'_2, ..., j'_m$ in $[2n+1]_{n+1}$ such that $(j'_m, i'_l, j'_{m-1}, i'_{l-1}, ..., j'_{m-l+1}, i'_1, j'_{m-l}, ..., j'_2, j'_1) = (1, 2, ..., n)$, where $j'_p \prec_{w^{(r)}} j'_{p+1}$ and

$i'_q \prec_{w^{(r)}} i'_{q+1}$ and $j'_{m-l+q-1} \prec_{w^{(r)}} i'_q \prec_{w^{(r)}} j'_{m-l+q+1}$ and $j'_{m-1} \prec_{w^{(r)}} i_l$ for any $p \in [m-1]$ and $q \in [l-1]$ and that $i'_{q'}, j'_{m-l+q'}$ are $w^{(r)}$-uncomparable for any $q' \in [l]$.

Hence we see by 2.3 that there exists $w' \in F'_{(k,2n+2-k)}$ such that $w', w^{(r)}$ (and further w', w) are in the same left-connected component of $E_{(k,2n+2-k)}$.

5.5. Lemma. *The set $F'_{(k,2n+2-k)}$ is contained in a right-connected component of $E_{(k,2n+2-k)}$.*

Proof. Let $J = \{t_1, t_2, ..., t_n\}$. Then $w_J = [2n+1, 2n, ..., n+2]$. Define the set $F' = \{z'zw \mid w \in F'_{(k,2n+2-k)}\}$ with $z = t_1t_3 \cdots t_{2l-3}t_{2l-1}$ and $z' = (t_0t_1 \cdots t_{n-1}t_n)^n$. For any $w \in F'_{(k,2n+2-k)}$, we have $\mathcal{L}(z'zw) = J$ and $\ell(zw) = \ell(z) + \ell(w)$ and $\ell(z'zw) = \ell(zw) - \ell(z')$ and $(i)z'zw = (i)zw - 2n - 2$ for any $i \in [n]$ by Corollary 1.9. A direct check shows that $w_0 := z^{-1}(z')^{-1}w_J$ satisfies the condition (5.3.1) with w_0 in the place of w and hence is in $F'_{(k,2n+2-k)}$. This implies that $w \mapsto z'zw$ is an order-preserving bijection from the set $F'_{(k,2n+2-k)}$ to F' with w_0, w_J the unique shortest elements in $F'_{(k,2n+2-k)}$, F', respectively. Any $w \in F'_{(k,2n+2-k)}$ satisfies $z'zw = w_Jx$ with $\ell(z'zw) = \ell(w_J) + \ell(x)$ for some $x \in \widetilde{C}_n$, hence $w = z^{-1}(z')^{-1}w_Jx = w_0x$ satisfies $\ell(w) = \ell(w_0) + \ell(x)$. The element w is in the right-connected component of $E_{(k,2n+2-k)}$ containing w_0 by Corollary 1.20. □

Since $F'_{(k,2n+2-k)} \subseteq E_{(k,2n+2-k)} \cap \Omega$, it makes sense to define the set $\mathbf{T}_{(k,2n+2-k)} := \{T(w) \mid w \in F'_{(k,2n+2-k)}\}$ by 2.5. Let

$$\mathbf{a} := (\underbrace{2, ..., 2}_{l \text{ times}}, \underbrace{1, ..., 1}_{m-l \text{ times}}, 2, \underbrace{1, ..., 1}_{m-l \text{ times}}, \underbrace{2, ..., 2}_{l \text{ times}}) \in \widetilde{\Lambda}_{2n+2}.$$

Then $\xi(\mathbf{T}) = \mathbf{a}$ for any $\mathbf{T} \in \mathbf{T}_{(k,2n+2-k)}$.

5.6. Lemma. *There are exactly $n!2^m$ left cells of $\widetilde{C}_n$ in the set $E_{(k,2n+2-k)}$.*

Proof. By Lemmas 1.16, 1.19, 2.6 and 5.4, we need only to enumerate the set $\mathbf{T}_{(k,2n+2-k)}$. We see that $\mathbf{T} = (T_1, T_2, ..., T_{2m+1}) \in \xi^{-1}(\mathbf{a})$ is in $\mathbf{T}_{(k,2n+2-k)}$ if and only if $T_i = \overline{T_{2m+2-i}}$ for any $i \in [m]$ and $T_{m+1} = \{n+1, 2n+2\}$. When the equivalent conditions hold, the generalized tabloid $\mathbf{T}$ is determined uniquely by the m-tuple $(T_1, T_2, ..., T_m)$. Now the number of the choices for the set T_1 is $2n(n-1)$. Recurrently, suppose that $a \in [m-1]$ and that all the T_b, $b \in [a]$, have been chosen. Then the number of the choices for T_{a+1} is $2(n-2a)(n-2a-1)$ if $a \in [l-1]$ and is $2(n-l-a)$ if $a \in [l, m-1]$. This implies that the cardinal of the set $\mathbf{T}_{(k,2n+2-k)}$ is $n!2^m$, our result follows. □

5.7. Theorem. *(1) If $k = 2m \in [n+1, 2n]$ is even, then $E_{(k,2n+2-k)} = \emptyset$.*

Now assume $k = 2m+1 \in [n+1, 2n]$ odd.

(2) The set $E_{(k,2n+2-k)}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$.

(3) The set $E_{(k,2n+2-k)}$ is infinite.

(4) The set $E_{(k,2n+2-k)}$ contains $n!2^m$ left cells, each of which is left-connected.

Proof. The assertion (1) follow by Proposition 3.2. Then (2) and (4) are the consequences of Lemmas 5.2 and 5.4-5.5. Finally, (3) follows since the number of the choices for the integer $(j_m)w$ in the condition 5.1 (i) on $w \in E_{(k,2n+2-k)}$ is infinite. $\square$

6. The left and two-sided cells of the affine Weyl group $\widetilde{C}_3$.

We shall study the cells of the weighted Coxeter group $\widetilde{C}_3 = (\widetilde{C}_3, \widetilde{\ell}_7)$ in this section.

Recall the notation E_λ defined in 1.18 for any $\lambda \in \Lambda_{2n+2}$ and the group automorphism η of $\widetilde{C}_n$ defined in 1.10. Denote by $n(\lambda)$ the number of left cells of $\widetilde{C}_n$ in E_λ. Fix $\lambda \in \Lambda_8$. We shall prove that the set E_λ contains at most two two-sided cells of $\widetilde{C}_3$. When E_λ is a union of two two-sided cells (say E'_λ , E''_λ) of $\widetilde{C}_3$, denote by $n'(\lambda)$, $n''(\lambda)$ the numbers of left cells of $\widetilde{C}_3$ in E'_λ , E''_λ, respectively. The results on the cells of $\widetilde{C}_3$ can be summarized as follows.

6.1. Theorem. *Let $\widetilde{C}_3 = (\widetilde{C}_3, \widetilde{\ell}_7)$ be the weighted Coxeter group with η its automorphism defined in 1.10. Let $\lambda \in \Lambda_8$.*

(1) The set E_λ forms a single two-sided cell of $\widetilde{C}_3$ if $\lambda \in \{\mathbf{71}, \mathbf{53}, \mathbf{51^3}, \mathbf{421^2}, \mathbf{3^22}, \mathbf{3^21^2}, \mathbf{31^5}, \mathbf{2^21^4}, \mathbf{1^8}\}$.

(2) The set E_λ is a union of two two-sided cells of $\widetilde{C}_3$ if $\lambda \in \{\mathbf{521}, \mathbf{32^21}, \mathbf{321^3}\}$.

(3) $E_\lambda = \emptyset$ if $\lambda \in \{\mathbf{8}, \mathbf{62}, \mathbf{61^2}, \mathbf{4^2}, \mathbf{431}, \mathbf{42^2}, \mathbf{41^4}, \mathbf{2^4}, \mathbf{2^31^2}, \mathbf{21^6}\}$; $E_\lambda \neq \emptyset$ is finite if $\lambda \in \{\mathbf{32^21}, \mathbf{321^3}, \mathbf{2^21^4}, \mathbf{1^8}\}$, and is infinite if $\lambda \in \{\mathbf{71}, \mathbf{53}, \mathbf{521}, \mathbf{51^3}, \mathbf{421^2}, \mathbf{3^22}, \mathbf{3^21^2}, \mathbf{31^5}\}$.

(4) η interchanges the two-sided cells E'_λ, E''_λ for any $\lambda \in \{\mathbf{521}, \mathbf{32^21}, \mathbf{321^3}\}$.

(5) The numbers $n(\lambda)$ for all $\lambda \in \Lambda_8$ with $E_\lambda \neq \emptyset$ are listed in the following table.

λ	$\mathbf{71}$	$\mathbf{53}$	$\mathbf{521}$	$\mathbf{51^3}$	$\mathbf{421^2}$	$\mathbf{3^22}$	$\mathbf{3^21^2}$	$\mathbf{32^21}$	$\mathbf{321^3}$	$\mathbf{31^5}$	$\mathbf{2^21^4}$	$\mathbf{1^8}$
$n(\lambda)$	48	24	12	24	6	8	12	6	2	6	2	1

where $n'(\mathbf{521}) = n''(\mathbf{521}) = 6$ and $n'(\mathbf{32^21}) = n''(\mathbf{32^21}) = 3$ and $n'(\mathbf{321^3}) = n''(\mathbf{321^3}) = 1$.

(6) Each left cell of $\widetilde{C}_3$ is left-connected.

6.2. We have $E_\lambda = \emptyset$ for $\lambda \in \{\mathbf{8}, \mathbf{62}, \mathbf{61^2}, \mathbf{4^2}, \mathbf{431}, \mathbf{42^2}, \mathbf{41^4}, \mathbf{2^4}, \mathbf{2^31^2}, \mathbf{21^6}\}$ by Propositions 3.2-3.3. To prove Theorem 6.1, we need only to consider the sets E_λ with $\lambda \in \Delta := \{\mathbf{521}, \mathbf{421^2}, \mathbf{3^22}, \mathbf{3^21^2}, \mathbf{32^21}, \mathbf{321^3}, \mathbf{2^21^4}\}$ by Theorems 4.6 and 5.7. We shall do this by a case-by-case argument, which can be sketched as follows. Given $\lambda \in \Delta$. We usually define a subset (say F_λ) of E_λ and then prove that F_λ has a non-empty intersection with each left-connected component of E_λ by 2.3 and that there are no two elements of F_λ belong to the same left cell of $\widetilde{C}_3$ by either Lemma 1.17 or Lemma 2.6. This implies by Lemmas 1.16, 1.17

and 1.19 that each left cell of E_λ is left-connected and that F_λ is a representative set for the left cells in E_λ. We get the number $n(\lambda)$ simply by enumerating the set F_λ. Finally, when E_λ forms a single two-sided cell of $\widetilde{C}_3$, we prove such a conclusion usually by proving that the set F_λ is contained in some two-sided-connected component of E_λ; when E_λ is a union of two two-sided cells of $\widetilde{C}_3$, the argument becomes more subtle.

6.3. Consider the partial order $\preceq_w$ on $[8]$ with respect to a fixed $w \in \widetilde{C}_3$. The following equivalent conditions hold by Lemma 2.2:

(1) $\psi(w) = \mathbf{521}$ if and only if there are distinct w-wild heads $i, j, k \in [7]_4$ such that $\{i, j, k\}$ is neither a w-chain nor a w-antichain and that either $\{i, j, 4\}$ or $\{i, j, 8\}$ is a w-antichain.

(2) $\psi(w) = \mathbf{421^2}$ if and only if there are one w-wild head i and two w-tame heads j, k in $[7]_4$ such that $\bar{i} \prec_w j \prec_w k \prec_w i$.

(3) $\psi(w) = \mathbf{3^22}$ if and only if there are pairwise w-uncomparable w-wild heads i, j, k in $[7]_4$ such that $4 \prec_w i$ and $8 \prec_w k$.

(4) $\psi(w) = \mathbf{3^21^2}$ if and only if one of the conditions (4a)-(4c) holds for some distinct $i, j, k \in [7]_4$:

(4a) Two w-uncomparable w-wild heads i, j and one w-tame head k satisfy either that $4 \prec_w i$ and $8 \prec_w j$ or that both k and $\overline{k}$ are w-comparable with at least one of i, j;

(4b) One w-wild head i and two w-tame heads j, k satisfy either $\bar{i} \not\prec_w j \prec_w k \prec_w i$ or $\bar{i} \prec_w j \prec_w k \not\prec_w i$;

(4c) i, j, k are all w-tame heads and compose a w-chain.

(5) $\psi(w) = \mathbf{32^21}$ if and only if one of the conditions (5a)-(5c) holds for some distinct $i, j, k \in [7]_4$:

(5a) i, j, k are all w-wild heads such that either $\{4, i, j, k\}$ or $\{8, i, j, k\}$ is a w-antichain;

(5b) Two w-wild heads i, j and one w-tame head k satisfy that exactly one of $\{k, i, j\}$ and $\{\overline{k}, i, j\}$ is a w-antichain and that either $\{4, i, j\}$ or $\{8, i, j\}$ is a w-antichain;

(5c) One w-wild head i and two w-tame heads j, k satisfy $\bar{i} \not\prec_w j \prec_w k \not\prec_w i$.

(6) $\psi(w) = \mathbf{321^3}$ if and only if there are two w-wild heads i, j and one w-tame head k in $[7]_4$ such that $\{i, j, k, \overline{k}\}$ and one of $\{4, i, j\}$, $\{8, i, j\}$ are w-antichains.

(7) $\psi(w) = \mathbf{2^21^4}$ if and only if there are distinct w-tame heads $i, j, k \in [7]_4$ such that the set $\{i, j, k\}$ forms neither a w-chain nor a w-antichain.

By making use of the above description of E_λ, $\lambda \in \Delta$, we shall prove Theorem 6.1 in 6.4-6.9.

6.4. Denote by $E'_{\mathbf{521}}$ (resp., $E''_{\mathbf{521}}$) the set of all such $w \in \widetilde{C}_3$ that there are distinct w-wild heads $i, j, k \in [7]_4$ satisfying either that $\{4, i, k\}$ is a w-antichain and $i \prec_w j$ or that $\{4, j, k\}$ is a w-antichain and $i \prec_w j, k$ (resp., either that $\{8, i, k\}$ is a w-antichain and $i \prec_w j$ or that $\{8, j, k\}$ is a w-antichain and $i \prec_w j, k$). We see that $E^{-1} = E$ for any $E \in \{E'_{\mathbf{521}}, E''_{\mathbf{521}}\}$ and that $E_{\mathbf{521}} = E'_{\mathbf{521}} \cup E''_{\mathbf{521}}$ by 6.3 (1). Let $F_{\mathbf{521}} = F'_{\mathbf{521}} \cup F''_{\mathbf{521}}$ with

$$F'_{\mathbf{521}} = \{[-2,-3,-1], [-2,-5,-1], [-3,-6,-1], [-3,-7,-2], [-3,-9,-2], [-1,-3,-2]\},$$

$$F''_{\mathbf{521}} = \{[5,7,6],\ [5,9,6],\ [5,10,7],\ [6,11,7],\ [6,13,7],\ [6,7,5]\}.$$

The group automorphism η of $\widetilde{C}_3$ stabilizes the sets $E_{\mathbf{521}}$, $F_{\mathbf{521}}$ and interchanges $E'_{\mathbf{521}}$, $E''_{\mathbf{521}}$ (resp., $F'_{\mathbf{521}}$, $F''_{\mathbf{521}}$). By 6.3 (1) and 2.3, we see that for any w in $E_{\mathbf{521}}$ (resp., $E'_{\mathbf{521}}$, $E''_{\mathbf{521}}$) there is some w' in $F_{\mathbf{521}}$ (resp., $F'_{\mathbf{521}}$, $F''_{\mathbf{521}}$) such that w', w are in the same left-connected component of $E_{\mathbf{521}}$.

In $F'_{\mathbf{521}}$, let $x := [-2,-3,-1]$. Then $[-1,-3,-2] = xt_1$, $[-2,-5,-1] = xt_3$, $[-3,-6,-1] = xt_3t_2$, $[-3,-7,-2] = xt_3t_2t_1$, $[-3,-9,-2] = xt_3t_2t_1t_0$, $\mathcal{R}(x) = \{t_0, t_2\}$, $\mathcal{R}(xt_1) = \{t_0, t_1\}$, $\mathcal{R}(xt_3) = \{t_0, t_3\}$, $\mathcal{R}(xt_3t_2) = \{t_0, t_2\}$, $\mathcal{R}(xt_3t_2t_1) = \{t_1\}$, $\mathcal{R}(xt_3t_2t_1t_0) = \{t_0\}$.

Also, in $F''_{\mathbf{521}}$, let $y := [5,7,6]$. Then $[6,7,5] = yt_2$, $[5,9,6] = yt_0$, $[5,10,7] = yt_0t_1$, $[6,11,7] = yt_0t_1t_2$, $[6,13,7] = yt_0t_1t_2t_3$, $\mathcal{R}(y) = \{t_1, t_3\}$, $\mathcal{R}(yt_2) = \{t_2, t_3\}$, $\mathcal{R}(yt_0) = \{t_0, t_3\}$, $\mathcal{R}(yt_0t_1) = \{t_1, t_3\}$, $\mathcal{R}(xt_0t_1t_2) = \{t_2\}$, $\mathcal{R}(yt_0t_1t_2t_3) = \{t_3\}$.

The above data can be displayed by two graphs in Figure 4 below (see 1.13):

$$\underset{[-1,-3,-2]}{\boxed{\mathbf{01}}} \text{———} \underset{[-2,-3,-1]}{\boxed{\mathbf{02}}} \text{- - - - - -} \underset{[-2,-5,-1]}{\boxed{\mathbf{03}}} \text{———} \underset{[-3,-6,-1]}{\boxed{\mathbf{02}}} \text{———} \underset{[-3,-7,-2]}{\boxed{\mathbf{1}}} \text{- - - - - -} \underset{[-3,-9,-2]}{\boxed{\mathbf{0}}}\ .$$

$$\underset{[6,7,5]}{\boxed{\mathbf{23}}} \text{———} \underset{[5,7,6]}{\boxed{\mathbf{13}}} \text{- - - - - -} \underset{[5,9,6]}{\boxed{\mathbf{03}}} \text{———} \underset{[5,10,7]}{\boxed{\mathbf{13}}} \text{———} \underset{[6,11,7]}{\boxed{\mathbf{2}}} \text{- - - - - -} \underset{[6,13,7]}{\boxed{\mathbf{3}}}$$

Figure 4

Hence we see from Figure 4 that the elements of $F'_{\mathbf{521}}$ (resp., $F''_{\mathbf{521}}$) are in the same right-connected component of $E_{\mathbf{521}}$ and that the elements of $F_{\mathbf{521}}$ have pairwise different generalized τ-invariants. This implies by Lemmas 1.16, 1.17 and 1.19 that each left-connected component of $E_{\mathbf{521}}$ lies in either $E'_{\mathbf{521}}$ or $E''_{\mathbf{521}}$ and forms a left cell of $\widetilde{C}_3$. This further implies by Lemma 1.3 that each of $E'_{\mathbf{521}}$ and $E''_{\mathbf{521}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_n$.

6.5. Let $F_{\mathbf{421^2}} = F'_{\mathbf{421^2}} \cup F''_{\mathbf{421^2}}$ be with

$$F'_{\mathbf{421^2}} = \{[7,3,2], [9,3,2], [10,3,1]\} \quad \text{and} \quad F''_{\mathbf{421^2}} = \{[3,1,-6], [2,1,-5], [2,1,-3]\}.$$

Then by 6.3 (2) and 2.3, we see that for any $w \in E_{\mathbf{421^2}}$, there is some $w' \in F_{\mathbf{421^2}}$ such that w', w are in the same left-connected component of $E_{\mathbf{421^2}}$. From Figure 5, we see that each of the sets $F'_{\mathbf{421^2}}$, $F''_{\mathbf{421^2}}$ is right-connected. Take $x = [7,3,2] \in F'_{\mathbf{421^2}}, y = [2,1,-3] \in F''_{\mathbf{421^2}}$ and $z = [11,2,1] \in E_{\mathbf{421^2}}$. Then $z = xt_0t_1t_2 = t_1t_2t_3y$ with $\ell(z) = \ell(x) + \ell(t_0t_1t_2) = \ell(t_1t_2t_3) + \ell(y)$. Hence x, y, z are in the same two-sided-connected component of $E_{\mathbf{421^2}}$ by Corollary 1.20. This implies that the set $E_{\mathbf{421^2}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_3$ by Lemmas 1.16, 1.17 and 1.19.

Figure 5

Let $x = [7,3,2] \in F'_{\mathbf{421^2}}$ and $y = [2,1,-3] \in F''_{\mathbf{421^2}}$. By regarding x, y as elements in $\widetilde{A}_7$ and by Corollary 1.9, we have $\widetilde{\mathcal{R}}(x) = \widetilde{\mathcal{R}}(y) = \{s_1, s_2, s_5, s_6\}$, $\widetilde{\mathcal{R}}(xs_3) = \widetilde{\mathcal{R}}(ys_3) = \{s_1, s_3, s_5, s_6\}$, $\widetilde{\mathcal{R}}(xs_3s_4) = \widetilde{\mathcal{R}}(ys_3s_4) = \{s_1, s_4, s_6\}$, $\widetilde{\mathcal{R}}(xs_3s_4s_6) = \mathcal{R}(ys_3s_4s_5) = \{s_1, s_4, s_5\}$, $\widetilde{\mathcal{R}}(xs_3s_6) = \widetilde{\mathcal{R}}(ys_3s_4s_5s_3) = \{s_1, s_3, s_5\}$ and $\widetilde{\mathcal{R}}(xs_6) = \{s_1, s_2, s_5\} \neq \{s_2, s_5\} = \widetilde{\mathcal{R}}(ys_3s_4s_5s_3s_2)$. This implies that x, y have different generalized τ-invariants. Hence we see from Figure 5 that the elements of $F_{\mathbf{421^2}}$ have pairwise different generalized τ-invariants. So $F_{\mathbf{421^2}}$ forms a representative set for the left cells of $\widetilde{C}_3$ in $E_{\mathbf{421^2}}$ by Lemma 1.17.

6.6. Let $F_{\mathbf{3^22}} = F'_{\mathbf{3^22}} \cup F''_{\mathbf{3^22}}$ be with

$$F'_{\mathbf{3^22}} = \{[-1,5,6], [-2,5,7], [-3,6,7], [-5,6,7]\},$$

$$F''_{\mathbf{3^22}} = \{[-2,-1,5], [-3,-1,6], [-3,-2,7], [-3,-2,9]\}.$$

Then by 6.3 (2) and 2.3, we see that for any $w \in E_{\mathbf{3^22}}$, there is some $w' \in F_{\mathbf{3^22}}$ such that w', w are in the same left-connected component of $E_{\mathbf{3^22}}$. From Figure 6, we see that each of the sets $F'_{\mathbf{3^22}}$, $F''_{\mathbf{3^22}}$ is right-connected. Take $x = [-1,5,6] \in F'_{\mathbf{3^22}}$, $y = [-2,-1,5] \in F''_{\mathbf{3^22}}$ and $z = [-5,-1,6] \in E_{\mathbf{3^22}}$. Then $z = xt_0t_1t_2 = t_1t_2t_3y$ with $\ell(z) = \ell(x) + \ell(t_0t_1t_2) = \ell(t_1t_2t_3) + \ell(y)$. Hence x, y, z are in the same two-sided-connected component of $E_{\mathbf{3^22}}$ by Corollary 1.20. This implies that the set $E_{\mathbf{3^22}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_3$ by Lemmas 1.16, 1.17 and 1.19.

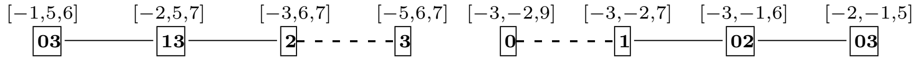


Figure 6

From Figure 6, we see that the elements of $F_{\mathbf{32_2}}$ have pairwise different generalized τ-invariants. So $F_{\mathbf{32_2}}$ forms a representative set for the left cells of $\widetilde{C}_3$ in $E_{\mathbf{32_2}}$ by Lemma 1.17.

6.7. Let $F_{\mathbf{32_12}} = F'_{\mathbf{32_12}} \cup F''_{\mathbf{32_12}}$ be with

$$F'_{\mathbf{32_12}} = \{[-1,2,5],[-2,1,5],[-1,3,6],[-3,1,6],[-5,1,6],[-2,3,7],[-2,3,9]\},$$

$$F''_{\mathbf{32_12}} = \{[3,2,-1],[3,1,-2],[5,2,1],[6,3,1],[3,2,1]\}.$$

By 6.3 (4) and 2.3, we see that for any $w \in E_{\mathbf{32_12}}$, there is some $w' \in F_{\mathbf{32_12}}$ such that w', w are in the same left-connected component of $E_{\mathbf{32_12}}$. From Figure 7, we see that each of the sets $F'_{\mathbf{32_12}}$, $F''_{\mathbf{32_12}}$ is right-connected and that the elements of $F_{\mathbf{32_12}}$ have pairwise different generalized τ-invariants.

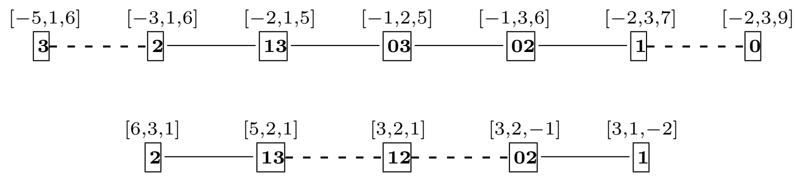


Figure 7

Take $x = [-1,2,5] \in F'_{\mathbf{32_12}}$, $y = [3,2,1] \in F''_{\mathbf{32_12}}$ and $z = [-3,2,7] \in E_{\mathbf{32_12}}$. Then $z = xy$ with $\ell(z) = \ell(x) + \ell(y)$. Hence x, y, z are in the same two-sided-connected component of $E_{\mathbf{32_12}}$ by Corollary 1.20. This implies that the set $E_{\mathbf{32_12}}$ is two-sided-connected and forms a single two-sided cell of $\widetilde{C}_3$ by Lemmas 1.16, 1.17 and 1.19.

We see from Figure 7 that the elements of $F_{\mathbf{32_12}}$ have pairwise different generalized τ-invariants. This implies by Lemma 1.17 that $F_{\mathbf{32_12}}$ forms a representative set for the left cells of $\widetilde{C}_3$ in $E_{\mathbf{32_12}}$.

6.8. Let $E'_{\mathbf{322_1}} = \bigcup_{k=1}^{3} E_k$ and $E''_{\mathbf{322_1}} = \bigcup_{k=4}^{6} E_k$ be with

$E_1 = \{[2,1,5],[2,5,1],[2,5,7]\}$, $E_2 = \{[3,1,6],[3,6,1],[3,6,7]\}$,

$E_3 = \{[5,1,6],[5,6,1],[5,6,7]\}$, $E_4 = \{[-1,3,2],[3,-1,2],[-3,-1,2]\}$,

$E_5 = \{[-2,3,1],[3,-2,1],[-3,-2,1]\}$, $E_6 = \{[-2,3,-1],[3,-2,-1],[-3,-2,-1]\}$.

Figure 8

We see from 6.3 (5) that $E_{\mathbf{32^21}} = E'_{\mathbf{32^21}}\dot{\cup}E''_{\mathbf{32^21}}$ and that the set E_k is a left-connected component of $E_{\mathbf{32^21}}$ for any $k \in [6]$. From Figure 8, we see that the set $E'_{\mathbf{32^21}}$ (resp., $E''_{\mathbf{32^21}}$) is two-sided-connected and that the elements occurring as vertices of the graphs in Figure 8 have pairwise different generalized τ-invariants. So the set E_k forms a left cell of $E_{\mathbf{32^21}}$ for any $k \in [6]$ by Lemmas 1.16, 1.17 and 1.19.

We have $E^{-1} = E$ for any $E \in \{E'_{\mathbf{32^21}}, E''_{\mathbf{32^21}}\}$. Hence each of the sets $E'_{\mathbf{32^21}}$ $E''_{\mathbf{32^21}}$ forms a single two-sided cell of $\widetilde{C}_3$ by Lemma 1.3.

6.9. By 6.3 (6)-(7), we have

$$E_{\mathbf{321^3}} = \{[1,5,6],[-2,-1,3]\} \quad \text{and} \quad E_{\mathbf{2^21^4}} = \{[2,1,3],[2,3,1],[1,3,2],[3,1,2]\}.$$

Since both $x := [1,5,6]$ and $y := [-2,-1,3]$ are involutions with $\mathcal{R}(x) = \{t_3\} \neq \{t_0\} = \mathcal{R}(y)$, each of the sets $E'_{\mathbf{321^3}} = \{[1,5,6]\}$ and $E''_{\mathbf{321^3}} = \{[-2,-1,3]\}$ forms both a two-sided cell and a left cell in $\widetilde{C}_3$ by Lemma 1.3. The group automorphism η of $\widetilde{C}_3$ interchanges $E'_{\mathbf{321^3}}$ and $E''_{\mathbf{321^3}}$. On the other hand, we see that the set $E_{\mathbf{2^21^4}}$ is two-sided-connected and hence forms a single two-sided cell of $\widetilde{C}_3$ by Lemma 1.16, 1.17 and 1.19. The set $E_{\mathbf{2^21^4}}$ consists of two left-connected components $E_1 := \{[2,1,3],[2,3,1]\}$ and $E_2 := \{[1,3,2],[3,1,2]\}$ with $\mathcal{R}(E_1) = \{t_1\} \neq \{t_2\} = \mathcal{R}(E_2)$. This in turn implies that both E_1 and E_2 are left cells of $\widetilde{C}_3$ by 1.2.

References

[1] C. Greene, *Some partitions associated with a partially ordered set*, J. Comb. Theory (A) **20** (1976), 69-79.
[2] G. Lusztig, *Some examples in square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc., **277** (1983), 623–653.
[3] G. Lusztig, *The two-sided cells of the affine Weyl group of type $\widetilde{A}_n$*, in book “ Infinite Dimensional Groups with Applications ”, pp. 275–283, (V. kac, ed.), MSRI. Publictions 4, Spring-Verelag, 1985.
[4] G. Lusztig, *Hecke algebras with unequal parameters*, CRM Monograph Series, vol. 18, AMS, USA, 2003.
[5] J. Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Math., **1179**, Springer-Verlag, 1986.
[6] J. Y. Shi, *The partial order on two-sided cells of certain affine Weyl groups*, J. Algebra. **179** **(2)** (1996), 607-621.
[7] J. Y. Shi, *Cells of the affine Weyl group $\widetilde{C}_n$ in a quasi-split case*, preprint.
[8] J. Y. Shi, *Cells of the affine Weyl group $\widetilde{C}_n$ in a quasi-split case, II*, preprint.

Department of Mathematics, East China Normal University, Shanghai, 200241, P. R. China