LEFT CELLS CONTAINING A FULLY COMMUTATIVE ELEMENT

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Abstract. Let W be a finite or an affine Coxeter group and W_c the set of all the fully commutative elements in W. For any left cell L of W containing some fully commutative element, our main result of the paper is to prove that there exists a unique element (say w^L) in $L\cap W_c$ such that any $z\in L$ has the form $z=xw^L$ with $\ell(z)=\ell(x)+\ell(w^L)$ for some $x\in W$. This implies that L is left connected, verifying a conjecture of Lusztig in our case.

Introduction.

Let $W=(W,S)$ be a Coxeter group with S the distinguished generator set. The fully commutative elements of W were defined by Stembridge: $w\in W$ is fully commutative, if any two reduced expressions of w can be transformed from each other by only applying the relations $st=ts$ with $s,t\in S$ and $o(st)=2$ ($o(st)$ being the order of st), or equivalently, w has no reduced expression of the form $w=x(sts...)y$, where $sts...$ is a string of length $o(st)>2$ for some $s\neq t$ in S. The fully commutative elements were studied extensively by a number of people (see [3], [6], [7], [15], [16], [17], [18]). Let W_c be the set of all the fully commutative elements in W.

Let W be a finite or an affine Coxeter group. The aim of this paper is to prove a structural property for any left cell of W containing some element of W_c: if $z\in W$ satisfies

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$z \underset{L}{\sim} w$ for some $w \in F'_{\rm c}$ (see 1.8) then z is a left extension of w (see 1.1 and Theorem 2.1). So any $w \in F'_{\rm c}$ is the unique shortest element in the left cell of W containing it and that $F'_{\rm c}$ forms a representative set for all the left cells L of W with $L \cap W_{\rm c} \neq \emptyset$.

A subset K of W is *left connected*, if for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, ..., x_r = y$ in K with some $r \geqslant 0$ such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leqslant i \leqslant r$. Lusztig conjectured in [2] that if W is an affine Weyl group then any left cell L of W is left connected. The conjecture is supported by all the existing data. Then Theorem 2.1 verifies the conjecture in the case where L contains some element of $W_{\rm c}$.

Since the generalized Coxeter elements are fully commutative, this paper generalizes a result in my previous paper [14, Theorem 4.5]; the latter described any left cell of W containing some generalized Coxeter element.

Note that the conclusion of Theorem 2.1 was proved in [17] for the case where W is a Weyl or an affine Weyl group, using the knowledge of distinguished involutions of W in $W_{\rm c}$. The proof given in the present paper is independent of that in [17], without assuming the knowledge of distinguished involutions in $W_{\rm c}$, and is applicable to a more general case: W is a finite or an affine Coxeter group.

The contents of the paper are organized as follows. We collect some notations, terms and known results concerning cells and fully commutative elements of a Coxeter group W in Section 1. Then the main result of the paper is proved in Section 2.

§1. Some results on fully commutative elements.

Let (W, S) be a Coxeter system. In the Introduction we defined the set $W_{\rm c}$ of all the fully commutative elements of W. In this section, we collect some notations, terms and known results for later use.

1.1. Let $\leqslant$ be the Bruhat–Chevalley order and $\ell(w)$ the length function on W. Given $J \subseteq S$, let w_J be the longest element in the subgroup W_J of W generated by J. Call J *fully commutative* if the element w_J is so.

For $w, x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. In this case, we say that w is a *left* (resp., *right*) *extension* of y (resp., x), and say that y

(resp., x) is a *left* (resp., *right*) *retraction* of w. More generally, we say z is a *retraction* of w (or w is an *extension* of z), if $w = x \cdot z \cdot y$ for some $x, y \in W$.

We have the following results on the elements in W_{c}:

Lemma. (see [17, Lemma 1.1]) *For $w \in W_{\mathrm{c}}$, let $w = s_1 s_2 ... s_r$ be a reduced expression of w with $s_i \in S$.*

(1) The multi-set $\{s_1, s_2, ..., s_r\}$ only depends on w but not on the choice of a reduced expression.

(2) For any $s \in S$ with $sw \in W_{\mathrm{c}}$, the equation $sw = ws$ holds if and only if $ss_i = s_i s$ for any $1 \leqslant i \leqslant r$.

(3) If $s, t \in S$ satisfy $sw = wt \in W_{\mathrm{c}}$, then $s = t$.

(4) If $w \in W_{\mathrm{c}}$ then any retraction of w is also in W_{c}. In particular, if $w \in W_{\mathrm{c}}$ has an expression $w = x \cdot w_J \cdot y$ with $x, y \in W$ and $J \subseteq S$, then J is fully commutative.

1.2. Let $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$) be the preorder on W defined as in [8], and let $\underset{L}{\sim}$ (resp., $\underset{R}{\sim}$, $\underset{LR}{\sim}$) be the equivalence relation on W determined by $\underset{L}{\leqslant}$ (resp., $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$). The corresponding equivalence classes are called *left* (resp., *right*, *two-sided*) *cells* of W.

1.3. For any $w \in W$, let $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

Assume $m = o(st) > 2$ for some $s, t \in S$. A sequence of elements

$$\underbrace{ys,\ yst,\ ysts,\ \dots}_{m-1 \text{ terms}}$$

is called a *right $\{s, t\}$-string* (or just a *right string*) if $y \in W$ satisfies $\mathcal{R}(y) \cap \{s, t\} = \emptyset$.

We say that z is obtained from w by a *right $\{s, t\}$-star operation* (or a *right star operation* for brevity), if z, w are two neighboring terms in a right $\{s, t\}$-string. Clearly, a resulting element z of a right $\{s, t\}$-star operation on w, when it exists, need not be unique unless w is a terminal term of the right $\{s, t\}$-string containing it.

Similarly, we can define a *left $\{s, t\}$-string* and a *left $\{s, t\}$-star operation* on an element.

Lemma. *(1) If $x, y \in W$ can be obtained from each other by successively applying left (resp., right) star operations, then $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$).*

(2) The set $W_{\rm c}$ is invariant under star operations.

Proof. (1) follows easily from the definition of the relations $\underset{L}{\sim}$ and $\underset{R}{\sim}$ on W. (2) is just [17, Proposition 2.10]. □

From now on, we always assume that W is a finite or an affine Coxeter group unless otherwise specified.

1.4. In [10], [11], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ and proved the following results (we further assume that W is a Weyl or an affine Weyl group when the results involve the function a).

(a) $a(w_J) = \ell(w_J)$ for $J \subseteq S$ with W_J finite (see [10, Proposition 2.4] and [11, Proposition 1.2]). In particular, when J is fully commutative, we have $a(w_J) = |J|$, the cardinality of the set J.

(b) If $x \underset{LR}{\leqslant} y$ in W, then $a(x) \geqslant a(y)$. So $x \underset{LR}{\sim} y$ implies $a(x) = a(y)$ (see [10, Theorem 5.4]).

(c) If $w = x \cdot y$ then $w \underset{L}{\leqslant} y$ and $w \underset{R}{\leqslant} x$.

(d) If $x \underset{L}{\leqslant} y$ and if either $x \underset{LR}{\sim} y$ or $a(x) = a(y)$ then $x \underset{L}{\sim} y$ (see [11, Corollary 1.9], [12, Subsection 1.7 (i)] and [1, Corollary 3.3]).

(e) The relation $x \underset{L}{\leqslant} y$ (resp., $x \underset{R}{\leqslant} y$) implies $\mathcal{R}(x) \supseteq \mathcal{R}(y)$ (resp., $\mathcal{L}(x) \supseteq \mathcal{L}(y)$). In particular, the relation $x \underset{L}{\sim} y$ (resp., $x \underset{R}{\sim} y$) implies $\mathcal{R}(x) = \mathcal{R}(y)$ (resp., $\mathcal{L}(x) = \mathcal{L}(y)$) (see [8, Proposition 2.4]).

By the notation x—y in W, we mean that $\max\{\deg P_{x,y}, \deg P_{y,x}\} = \frac{1}{2}(|\ell(x)-\ell(y)|-1)$, where $P_{x,y}$ is the celebrated Kazhdan–Lusztig polynomial associated to the ordered pair (x, y) in W, and we stipulate that the degree of the zero polynomial is $-\infty$.

(f) If $x, y \in W$ with x—y are in some right $\{s, t\}$-strings (not necessarily in the same right string) for some $s, t \in S$ with $st \neq ts$, then there exist some $x', y' \in W$ which are obtained from x, y respectively by a right $\{s, t\}$-star operation and satisfy x'—y' (see [10 Subsection 10.4]).

1.5. By a *graph*, we mean a finite set of nodes together with a finite set of edges. Two nodes of a graph are *adjacent* if they are joined by an edge. A *directed graph* (or a *digraph* for brevity) is a graph with each edge oriented. A *directed edge* (i.e., an edge with orientation) with two incident nodes $\mathbf{v}, \mathbf{v}'$ is denoted by an ordered pair $(\mathbf{v}, \mathbf{v}')$, if the orientation is from $\mathbf{v}$ to $\mathbf{v}'$. A node $\mathbf{s}$ of a digraph $\mathbf{G}$ is a *source* (resp., a *sink*) if $(\mathbf{s}, \mathbf{s}')$ (resp., $(\mathbf{s}', \mathbf{s})$) is a directed edge of $\mathbf{G}$ for any node $\mathbf{s}'$ adjacent to $\mathbf{s}$. A source or a sink of $\mathbf{G}$ is also called an *extreme node*. A *directed path* ξ of $\mathbf{G}$ is a sequence of nodes $\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_r$ in $\mathbf{G}$ with $r \geqslant 0$ such that $(\mathbf{v}_{i-1}, \mathbf{v}_i)$ is a directed edge of $\mathbf{G}$ for $1 \leqslant i \leqslant r$. A *subdigraph* of a digraph $\mathbf{G}$ is a digraph which can be obtained from $\mathbf{G}$ by removing some nodes and all the directed edges incident to these removed nodes.

1.6. To an expression

(1.6.1) $$\chi: \quad w = s_1 s_2 ... s_r$$

(not necessarily reduced) of any $w \in W$ with $s_i \in S$, we associate a digraph $\mathbf{G}(\chi)$ as follows. The node set $\mathbf{V}(\chi)$ of $\mathbf{G}(\chi)$ is $\{\mathbf{s}_i \mid 1 \leqslant i \leqslant r\}$ (with the convention that $\mathbf{s}_i \neq \mathbf{s}_j$ for any $i \neq j$), and the directed edge set $\mathbf{E}(\chi)$ of $\mathbf{G}(\chi)$ consists of all the ordered pairs $(\mathbf{s}_i, \mathbf{s}_j)$ satisfying the conditions $i < j$, $s_i s_j \neq s_j s_i$ and that there does not exist any $i = h_0 < h_1 < ... < h_t = j$ with $t > 1$ such that $s_{h_{p-1}} s_{h_p} \neq s_{h_p} s_{h_{p-1}}$ for every $1 \leqslant p \leqslant t$. The digraph $\mathbf{G}(\chi)$ so obtained usually depends on the choice of an expression χ of w. However, if two expressions of w can be obtained from each other by only applying the relations of the form $st = ts$ for some $s, t \in S$ with $o(st) = 2$, then their corresponding digraphs are the same. In particular, when w is in W_{c} and an expression χ of w in (1.6.1) is reduced, the digraph $\mathbf{G}(\chi)$ only depends on the element w, but not on the particular choice of a reduced expression χ of w. In this case, it makes sense to denote $\mathbf{G}(\chi)$, $\mathbf{V}(\chi)$, $\mathbf{E}(\chi)$ by $\mathbf{G}(w)$, $\mathbf{V}(w)$, $\mathbf{E}(w)$, respectively. Call $\mathbf{G}(w)$ the *associated digraph* of w.

By the above construction of a digraph $\mathbf{G}(w)$ for $w \in W_{\mathrm{c}}$, there exists a natural map $\phi : \mathbf{s}_i \mapsto s_i$ from $\mathbf{V}(w)$ to S and hence $\mathbf{V}(w)$ can be regarded as a multi-set in S.

Note that the above definition of the digraph $\mathbf{G}(w)$ can be regarded as a reformulation of Viennot's notion of a heap (see [19]). The digraph $\mathbf{G}(w)$ is also a certain kind of dependence graph (see [5] for example).

1.7. By [18, Proposition 2.3], we see that an element w of W is in W_{c} if and only if there exists some (and hence any) reduced expression $\chi : w = s_1 s_2 ... s_r$ such that the following two conditions hold:

(1.7.1) for any pair $i < j$ with $s_i = s_j$, there exists a directed path in $\mathbf{G}(\chi)$ connecting the nodes $\mathbf{s}_i$ and $\mathbf{s}_j$.

(1.7.2) for any directed path $\mathbf{s}_{i_1}, \mathbf{s}_{i_2}, ..., \mathbf{s}_{i_m}$ in $\mathbf{G}(\chi)$ with $s_{i_h} = s_{i_{h+2}}$ for $1 \leqslant h \leqslant m-2$ and $m = o(s_{i_1} s_{i_2}) > 2$, there always exists another directed path with $\mathbf{s}_{i_1}, \mathbf{s}_{i_m}$ two extreme nodes.

For $w \in W_{\rm c}$, $\mathcal{L}(w)$ (resp., $\mathcal{R}(w)$) (see 1.3) is exactly the set of all $s \in S$ with $\phi^{-1}(s)$ containing a source (resp., a sink) of $\mathbf{G}(w)$. Let $s \in \mathcal{L}(w)$ (resp., $s \in \mathcal{R}(w)$). Then both $\mathcal{L}(w) \not\supseteq \mathcal{L}(sw)$ and $\mathcal{L}(w) \not\subseteq \mathcal{L}(sw)$ (resp., $\mathcal{R}(w) \not\supseteq \mathcal{R}(ws)$ and $\mathcal{R}(w) \not\subseteq \mathcal{R}(ws)$) hold if and only if the removal of the source (resp., sink) $\mathbf{s}$ from $\mathbf{G}(w)$ yields a new source (resp., sink) in the resulting digraph.

For $w \in W_{\rm c}$, there is an expression $w = x \cdot w_J \cdot y$ for some $J \subseteq S$ and $x, y \in W$ if and only if there is a node set $\mathbf{J}$ of $\mathbf{G}(w)$ with $\phi(\mathbf{J}) = J$ and $|\mathbf{J}| = |J|$ such that

(1.7.3) for any $\mathbf{s} \neq \mathbf{t}$ in $\mathbf{J}$, there is no directed path connecting $\mathbf{s}$ and $\mathbf{t}$ in $\mathbf{G}(w)$.

Denote by $n(w)$ the maximum possible cardinality of a node set $\mathbf{J}$ of $\mathbf{G}(w)$ satisfying condition (1.7.3). Then $n(w)$ is also the maximum possible value of $\ell(w_J)$ in an expression $w = x \cdot w_J \cdot y$, or equivalently, the maximum size of an antichain in the corresponding heap.

1.8. Let $F_{\rm c}$ be the set of all the elements w in $W_{\rm c}$ such that $\mathcal{L}(sw) \subset \mathcal{L}(w)$ (or equivalently, $\mathcal{L}(sw) = \mathcal{L}(w) \setminus \{s\}$) for any $s \in \mathcal{L}(w)$. Denote by $F'_{\rm c}$ the set of all the elements w in $F_{\rm c}$ such that $n(sw) < n(w) = |\mathcal{L}(w)|$ for any $s \in \mathcal{L}(w)$. Let $F''_c = F_{\rm c} \setminus F'_{\rm c}$.

Proposition. *Assume that W is a finite or an affine Coxeter group.*

(1) If $w \in F'_{\rm c}$ then any right retraction of w is also in $F'_{\rm c}$ (see [17, Lemma 3.14]*).*

(2) If W is irreducible and $w \in F''_{\rm c}$, then $s \leqslant w$ for any $s \in S$ (see [17, Lemma 3.11]*).*

Now assume that W is a Weyl or an affine Weyl group.

(3) $a(w) = n(w)$ for any $w \in W_{\rm c}$ (see [17, Theorem 3.1], [4, Theorem 4.1]*).*

(4) $a(w) = |\mathcal{L}(w)|$ for any $w \in F_{\rm c}$ (see [17, Corollary 3.18]*).*

The next result is concerned with some further properties of $w \in F'_{\mathrm{c}}$.

Lemma 1.9. (see [17, Lemma 3.15]) *Let W be a Weyl or an affine Weyl group.*

(1) For any $w \in F'_{\mathrm{c}}$, there exists a sequence of elements $x_0 = w, x_1, ..., x_r = w_K$ in F'_{c} with $K = \mathcal{L}(w)$ such that x_i can be obtained from x_{i-1} by a right star operation and $x_i < x_{i-1}$ for every $1 \leqslant i \leqslant r$.

(2) For any $w \in W_{\mathrm{c}}$, there exists some $y \in F'_{\mathrm{c}}$ such that y is a left retraction of w with $y \underset{L}{\sim} w$ and $n(y) = n(w)$.

§**2. Left cells of W containing some element of W_{c}.**

In this section, we consider all the left cells of W containing some $w \in W_{\mathrm{c}}$. Since any $w \in W_{\mathrm{c}}$ has the form $w = x \cdot y$ with $w \underset{L}{\sim} y$ for some $y \in F'_{\mathrm{c}}$ and $x \in W_{\mathrm{c}}$ (see Lemma 1.9 (2)), we may assume $w \in F'_{\mathrm{c}}$ without loss of generality. Say an element $x \in W$ satisfies *condition (A)*, if

(A) $x \underset{L}{<} sx$ (i.e., $x \underset{L}{\leqslant} sx$ but $x \underset{L}{\not\sim} y$) for any $s \in \mathcal{L}(x)$.

The main result of the paper is to prove

Theorem 2.1. *Let W be an irreducible finite or affine Coxeter group. Let $w \in F'_{\mathrm{c}}$ and $z \in W$ satisfy $z \underset{L}{\sim} w$.*

(1) If the element z satisfies condition (A), then $z = w$.

(2) If $z \in F'_{\mathrm{c}}$ then $z = w$.

(3) In general, we have $z = x \cdot w$ for some $x \in W_{\mathrm{c}}$.

We break the proof of Theorem 2.1 up into some lemmas.

Lemma 2.2. *Suppose that we are given $w \in F'_{\mathrm{c}}$ with $m = \ell(w)$. Let $z \in W$ satisfy $z \underset{L}{\sim} w$ and condition (A). Assume that we are in the following case:*

(1) W is an irreducible Weyl or affine Weyl group;

(2) $m > |\mathcal{L}(w)|$ and the assertion of Theorem 2.1 has been proved when $\ell(w) < m$;

(3) $w = w_J \cdot y$ satisfies $n(w) = J$ and $y \in W_{\mathrm{c}}$;

(4) w, z can be transformed to $w' = ws', z' = zs$ respectively by a right $\{s, s'\}$-star operation with $s' \in \mathcal{R}(y)$ and $w' \underset{L}{\sim} z'$, where $s, s' \in S$ satisfy $ss' \neq s's$.

Then we have

(a) $z = x \cdot w_{J'} \cdot y$ *with* $J' = J \setminus \{s\}$ *for some* $x \in W$*;*

(b) the element s *commutes with but is not equal to any* $v \in S$ *with* $v \leqslant w_{J'} \cdot ys'$*.*

Proof. Note that any $x \in F'_{\rm c}$ satisfies condition (A) by Proposition 1.8 (3) and 1.4 (d). We have $w' \in F'_{\rm c}$ by Proposition 1.8 (1).

We claim that z' does not satisfy condition (A). For otherwise, one would have $z' = w' = ws'$ by the assumption (2) with w', z' in the place of w, z respectively. By the condition $s' \in \mathcal{R}(y) \subseteq \mathcal{R}(w)$, we have $s \in \mathcal{R}(ws')$ since the transformation of sending w to ws' is a right $\{s, s'\}$-star operation. So $z = z's = ws's$ is a right retraction of w and hence is in $F'_{\rm c}$ by Proposition 1.8 (1). Since $\ell(z) < \ell(w) = m$, we have $z = w$ by the assumption (2) with z, w in the place of w, z respectively, which is impossible.

So there exists some $t \in \mathcal{L}(z')$ satisfying $tz' \underset{L}{\sim} z'$. We have $\mathcal{R}(z) = \mathcal{R}(w)$ by 1.4 (e) and the condition $z \underset{L}{\sim} w$. Then $s' \in \mathcal{R}(z)$ and $z' = z \cdot s$. There is a reduced expression of z':

$$z' = s_1 ... s_a s' s \qquad \text{with } s_i \in S. \tag{2.2.1}$$

We claim that we have a reduced expression of tz':

$$tz' = s_1 ... s_a s. \tag{2.2.2}$$

For otherwise, we would have a reduced expression either $tz' = s_1 ... s_a s'$ or $tz' = s_1 ... \widehat{s_i} ... s_a s' s$ for some $1 \leqslant i \leqslant a$ by the exchange condition on W, where $\widehat{s_i}$ means the deletion of the factor s_i. If $tz' = s_1 ... s_a s'$, then we would have $z' \underset{L}{<} tz'$ by [10 Corollary 5.5], a contradiction. Also, if $tz' = s_1 ... \widehat{s_i} ... s_a s' s$, then $tz = s_1 ... \widehat{s_i} ... s_a s'$, which can be obtained from tz' by a right $\{s, s'\}$-star operation, and hence $tz \underset{R}{\sim} tz' \underset{L}{\sim} z' \underset{R}{\sim} z$. This would imply $tz \underset{LR}{\sim} z$ and hence $tz \underset{L}{\sim} z$ by 1.4 (c)–(d), contradicting condition (A) on z since $t \in \mathcal{L}(z') = \mathcal{L}(z)$ by 1.4 (e).

By 1.4 (d), we can write $tz' = x \cdot z''$ for some $x, z'' \in W$, where z'' satisfies conditions $z'' \underset{L}{\sim} tz'$ and (A). Since $z'' \underset{L}{\sim} w'$, $w' \in F'_{\rm c}$ and $\ell(w') < \ell(w)$, we have $z'' = w'$ by the assumption (2) with w', z'' in the place of w, z respectively. So $tz' = x \cdot w'$. We can write

$y = y' \cdot s'$ for some $y' \in W$ by the fact that $s' \in \mathcal{R}(y)$. Then

$$z' = t \cdot x \cdot w_J \cdot y' \tag{2.2.3}$$

by the assumption (3). Hence

$$z = (t \cdot x \cdot w_J \cdot y')s. \tag{2.2.4}$$

By the fact $s \in \mathcal{R}(ws') = \mathcal{R}(w_J y')$ and by the exchange condition on W, we have either

$$z = t \cdot x \cdot w_{J'} \cdot y' \quad \text{with } J' \subset J \text{ and } |J'| = |J| - 1 \tag{2.2.5}$$

or

$$z = t \cdot x \cdot w_J \cdot y'' \quad \text{with } y'' < y' \text{ and } \ell(y'') = \ell(y') - 1. \tag{2.2.6}$$

We claim that the case (2.2.6) could not occur. For otherwise, since $|J| = a(w) = a(z)$ by 1.4 (b) and Proposition 1.8 (3), we would have $z \underset{L}{\sim} tz$ by 1.4 (d), contradicting the assumption that z satisfies condition (A) since $t \in \mathcal{L}(z)$. In particular, we have

$$w_{J'} y' s = w_J \cdot y' = w' \in F'_{\mathrm{c}} \subseteq W_{\mathrm{c}}. \tag{2.2.7}$$

This implies $J \setminus J' = \{s\}$ by Lemma 1.1 (3). So we get from (2.2.3) and (2.2.7) that

$$z' = t \cdot x \cdot w_{J'} \cdot y' \cdot s \tag{2.2.8}$$

and hence

$$tz' = x \cdot w_{J'} \cdot y' \cdot s. \tag{2.2.9}$$

On the other hand, we have

$$tz' = (t \cdot x \cdot w_{J'} \cdot y')s' \cdot s \tag{2.2.10}$$

by (2.2.8) and (2.2.1)–(2.2.2). Comparing (2.2.9) with (2.2.10), we get

$$t\cdot x\cdot w_{J'}\cdot y' = x\cdot w_{J'}\cdot y'\cdot s'. \tag{2.2.11}$$

This implies by (2.2.5) and (2.2.11) that

$$z = t\cdot x\cdot w_{J'}\cdot y' = x\cdot w_{J'}\cdot y'\cdot s' = x\cdot w_{J'}\cdot y \tag{2.2.12}$$

So (a) is proved.

For (b), the conclusion that s commutes with any $v\in S$ with $v\leqslant w_{J'}\cdot y'$ follows by (2.2.7) and Lemma 1.1 (2). If $s\leqslant w_{J'}\cdot y'$, then $s\leqslant y'$ and thus $\ell(w_J\cdot y')<\ell(w_J)+\ell(y')$ by $J=J'\cup\{s\}$, which is absurd. So $s\not\leqslant w_{J'}\cdot y' = w_{J'}\cdot ys'$. □

Lemma 2.3. *Keep all the assumptions of Lemma 2.2 on the elements $w\in F'_{\rm c}$ and $z\in W$ (in particular, $w=w_J\cdot y$ and $J=J'\cup\{s\}=\mathcal{L}(w)$). Then the element w can also be transformed to wu' by a right $\{u,u'\}$-star operation for some $u\in S$, $u'\in\mathcal{R}(y)$ with $uu'\neq u'u$ and $s\notin\{u,u'\}$.*

Proof. We have a reduced expression $y=y'\cdot s'$ of y for some $y'\in W$. Let $w_1=sw=w_{J'}\cdot y$. Then $w_1\in W_{\rm c}$. By (2.2.7), we have $w=s\cdot w_1=w_{J'}y'ss'$. We claim that w_1 can be transformed to w_1u' by a right $\{u,u'\}$-star operation for some $u'\in\mathcal{R}(y)$ and $u\in S$ with $uu'\neq u'u$. For otherwise, we would have $w_1^{-1}\in F_{\rm c}$. Then $|\mathcal{L}(w_1)|+1=|\mathcal{L}(w)|>|\mathcal{R}(w)|=|\mathcal{R}(w_1)|=|\mathcal{L}(w_1^{-1})|\geqslant|\mathcal{R}(w_1^{-1})|=|\mathcal{L}(w_1)|$ by the assumptions that $w\in F'_{\rm c}$ and $\ell(w)>|\mathcal{L}(w)|$, by the fact that $\mathbf{s}$ is not a sink of $\mathbf{G}(w)$ (since $\{s,s'\}\cap\mathcal{R}(w)=\{s'\}$ by the assumption in Lemma 2.2 (4)), and by Proposition 1.8 (4). This implies $|\mathcal{L}(w_1)|=|\mathcal{R}(w_1)|$ and hence $w_1^{-1}\in F''_{\rm c}$ (see 1.8) by the fact that $\ell(w_1^{-1})>|\mathcal{L}(w_1^{-1})|$. On the other hand, we have $s\not\leqslant w_1=w_{J'}\cdot y=w_{J'}\cdot y'\cdot s'$ by Lemma 2.2 (b). This is impossible by Proposition 1.8 (2). Our claim is proved. The claim implies that $u,u'\leqslant w_1$. Since $s\not\leqslant w_1$, we have $s\notin\{u,u'\}$. Hence $w=s\cdot w_1$ also can be transformed to wu' by a right $\{u,u'\}$-star operation. □

Lemma 2.4. *Theorem 2.1 is true when $W=H_3,H_4,I_2(m)$ ($m=5$ or $\geqslant 7$).*

Proof. By [9, Proposition 3.8], we see that the result is true in the case where $w \in F'_{\rm c}$ has a unique reduced expression. In particular, the result is true when $W = I_2(m)$. It remains to consider the case where $W \in \{H_3, H_4\}$, and $w \in F'_{\rm c}$ has more than one reduced expression. Now assume that we are in such a case. Let W_1 be the set of all the fully commutative elements of W each of which has more than one reduced expression. Then we get the following facts:

(i) First we claim that W_1 is a single two-sided cell of W.

By [15, 3.5], we know that W_1 is a union of two-sided cells of W. Then the claim follows by [1, Section 3] for $W = H_4$, where W_1 is the two-sided cell E in the notation of [1]; and by a direct calculation for $W = H_3$, where W_1 consists of 25 elements, which is a union of 5 left (resp., right) cells.

(ii) Next we claim that $W_1 = \{y \in W_{\rm c} \mid n(y) = 2\}$.

This is because $n(w) \leqslant 2$ for any $w \in W$, $n(x) = 1$ for any $x \in W_{\rm c} \setminus W_1$ and $n(y) > 1$ for any $y \in W_1$.

(iii) Let $w \in W_1$. When $W = H_3$, let $S = \{s_1, s_2, s_3\}$ satisfy $(s_1s_2)^5 = (s_2s_3)^3 = 1$. Define $z = s_1s_3 \cdot s_2 \cdot s_1 \cdot s_2 \cdot s_3$. Then we claim that w is in $F'_{\rm c}$ if and only if w is a right retraction of z with $\mathcal{L}(w) = \{s_1, s_3\}$. When $W = H_4$, let $S = \{s_1, s_2, s_3, s_4\}$ satisfy $(s_1s_2)^5 = (s_2s_3)^3 = (s_3s_4)^3 = 1$. Define $z_1 = s_1s_3 \cdot s_2 \cdot s_1 \cdot s_2 \cdot s_3 \cdot s_4$, $z_2 = s_1s_4$ and $z_3 = s_2s_4 \cdot s_3$. Then we claim that w is in $F'_{\rm c}$ if and only if w is a right retraction of z_i with $|\mathcal{L}(w)| = 2$ for some $1 \leqslant i \leqslant 3$. For, by (ii), we see that $w \in W_1$ is in $F'_{\rm c}$ if and only if $|\mathcal{L}(w)| = 2$ and $n(sw) = 1$ for any $s \in \mathcal{L}(w)$. Then the above two claims follow by a direct calculation.

By 1.4 (c)–(d), the above (i)–(ii) and Lemma 1.9 (2), we see that any $z \in W_1$ can be written in the form $z = x \cdot w$ for some $w \in F'_{\rm c} \cap W_1$ and some $x \in W$ with $z \underset{L}{\sim} w$. This, in particular, implies that $z \in W_1$ satisfies condition (A) only if $z \in F'_{\rm c}$. By (iii), we see that any $w \in F'_{\rm c} \cap W_1$ satisfies condition (A). By comparing their generalized τ-invariants (see [13, Section 4] for the definition), we see that two elements x, y of $F'_{\rm c} \cap W_1$ satisfy $x \underset{L}{\sim} y$ if and only if $x = y$. Hence our result on H_3 and H_4 follows. □

2.5. *Proof of Theorem 2.1.* By Lemma 2.4, we need only consider the case where W is an irreducible Weyl or affine Weyl group. Now assume that we are in such a case.

First we prove (2)–(3) under the assumption of (1). Since any element of $F'_{\rm c}$ satisfies condition (A) by 1.4 (d) and Proposition 1.8 (3)–(4), assertion (2) is an immediate consequence of (1). For (3), we can write $z = x \cdot z'$ for some $x, z' \in W$ with z' satisfying the conditions $z' \underset{L}{\sim} z$ and (A). Then we have $z' = w$ by (1) and hence $z = x \cdot w$.

So it remains to prove (1). We can write $w = w_J \cdot y$ with $J = \mathcal{L}(w)$ for some $y \in W_{\rm c}$. Hence $|J| = n(w) = a(w)$ by Proposition 1.8 (3)–(4). We apply induction on $\ell(y) \geqslant 0$. When $\ell(y) = 0$, we have $w = w_J$. The condition $z \underset{L}{\sim} w$ implies that $\mathcal{R}(z) = \mathcal{R}(w) = J$ and $|J| = a(z)$ by 1.4 (e), (b), (a). Hence $z = x \cdot w_J$ for some $x \in W$. Then condition (A) on z further implies $z = w_J$. Next assume $\ell(y) > 0$. By Lemma 1.9 (1), the element w can be transformed to $w' = ws'$ by a right $\{s, s'\}$-star operation for some $s \in S$, $s' \in \mathcal{R}(y)$ with $ss' \neq s's$. Then $w' \in F'_{\rm c}$ by Proposition 1.8 (1). At least one (say z') of zs' and zs is obtained from z by a right $\{s, s'\}$-star operation and satisfies $z' \underset{L}{\sim} w'$ by 1.4 (e)–(f). If $z' = zs'$, then z' is a right retraction of z with $z' \underset{R}{\sim} z$ by the fact that $s' \in \mathcal{R}(w) = \mathcal{R}(z)$. For any $r \in \mathcal{L}(z')$, we have $r \in \mathcal{L}(z)$ by 1.4 (e), hence $z' \underset{R}{\sim} z \underset{L}{<} rz \underset{R}{\leqslant} rzs' = rz'$ by the assumption that z satisfies condition (A). This implies $z' \underset{LR}{<} rz'$ and further $z' \underset{L}{<} rz$, i.e., z' also satisfies condition (A). By the inductive hypothesis, we have $z' = w'$ and hence $z = z's' = w's' = w$, as required.

Now assume $z' = zs$. Then all the assumptions (1)–(4) of Lemma 2.2 on w, z hold, where the assumptions (1)–(3) hold by our inductive hypothesis, while the assumption (4) holds by the above discussion and the assumption $z' = zs$. Hence by Lemmas 2.2 and 2.3, we have $z = x \cdot w_{J'} \cdot y$ with $J' = J \setminus \{s\}$ for some $x \in W$, and w can also be transformed to $w'' = wu'$ by a right $\{u, u'\}$-star operation for some $u \in S$ and $u' \in \mathcal{R}(y)$ with $uu' \neq u'u$ and $s \notin \{u, u'\}$. By the same argument as above, we can prove the following assertions:

(i) $w'' \in F'_{\rm c}$;

(ii) at least one (say z'') of zu' and zu is obtained from z by a right $\{u, u'\}$-star operation and satisfies $z'' \underset{L}{\sim} w''$;

(iii) if $z'' = zu'$ then $z'' = w''$ and hence $z = w$;

(iv) if $z'' = zu$ then $z = x' \cdot w_{J''} \cdot y$ with $J'' = J \setminus \{u\}$ for some $x' \in W$.

We claim that the cases of $z = x \cdot w_{J'} \cdot y$ and $z = x' \cdot w_{J''} \cdot y$ can't happen simultaneously. For otherwise, we would have $x \cdot u = x' \cdot s$. Since $s \neq u$, this implies $s \in \mathcal{R}(x)$, contradicting the fact that $xw_J = x \cdot w_J$ (see (2.2.3)). So we must have $z = w$ by the assertions (ii)–(iv). This completes our proof. □

Theorem 2.1 tells us that any $w \in F'_{\rm c}$ is the unique shortest element in the left cell L_w of W containing w and that any $z \in L_w$ has the form $z = x \cdot w$ for some $x \in W$.

Remark 2.6. (1) The proof of Lemma 2.2 follows the line of the corresponding part in the proof of [14, Theorem 4.7]. However, the remaining part in the proof of Theorem 2.1 is new.

(2) Corollaries 4.10 and 4.11 in [17] are the consequence of Theorem 2.1.

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