A COUNTER-EXAMPLE TO A CONJECTURE OF LUSZTIG

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Abstract. In [7], Lusztig defined two functions a and a' on a Coxeter group W and conjectured that they are the same. In the present paper, we give a counter-example to Lusztig's conjecture. Then we propose a new conjecture for the description of two-sided cells of W which can be regarded as an improvement of Lusztig's conjecture. Some examples are given to support our conjecture.

In his study of Hecke algebras with unequal parameters (see [7]), Lusztig defined two functions a and a' on a Coxeter group (W, S). The function a is defined in terms of structural coefficients of the Hecke algebra associated to W, while the function a' is determined by the values of a on the finite standard parabolic subgroups of W. Then Lusztig conjectured in [7, Subsection 13.12] that $a(w) = a'(w)$ for any $w \in W$ (see Conjecture 2.1).

In the present paper, we give a counter-example to the above conjecture of Lusztig in the case where W is the affine Weyl group of type $\widetilde{A}_{10}$ (see Theorem 4.1). Then we propose a conjecture for the description of two-sided cells in a Coxeter group (see Conjecture 4.3), which can be regarded as an improvement of the above conjecture of Lusztig. Some examples are given to support our conjecture (see Theorem 4.8).

The contents are organized as follows. In Section 1, we introduce the function a on a Coxeter group W in terms of the Hecke algebra associated to W. Then we state some

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conjectures of Lusztig involving the function a in Section 2. The affine Weyl group $\widetilde{A}_{n-1}$, $n \geqslant 2$, is introduced in Section 3. Finally, in Section 4, we provide a counter-example in $\widetilde{A}_{10}$ against Conjecture 2.1 and also propose a new conjecture to replace Conjecture 2.1.

§1. Function a on a Coxeter group.

The concepts introduced in this section follow Lusztig in [7].

1.1. Let W be a Coxeter group with S its Coxeter generator set. For $w \in W$, let $\ell(w)$ be the smallest integer $q \geqslant 0$ such that $w = s_1 s_2 \cdots s_q$ with $s_i \in S$. Call $\ell(w)$ the *length* of w and $w = s_1 s_2 \cdots s_q$ a *reduced expression* of w if $q = \ell(w)$. A map $L : W \to \mathbb{Z}$ is said to be a *weight function* for W if $L(ww') = L(w) + L(w')$ for any $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$. Any weight function for W satisfy the equation $L(s) = L(t)$ whenever $s, t \in S$ are conjugate in W. Hence L is a multiple of ℓ if the elements of S are pairwise conjugate.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v. For $s \in S$, set $v_s = v^{L(s)} \in \mathcal{A}$. Let $\mathcal{H}$ be the $\mathcal{A}$-algebra defined by the generators T_s $(s \in S)$ and the relations

$$(T_s - v_s)(T_s + v_s^{-1}) = 0 \qquad \text{for } s \in S$$

$$T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots \qquad \text{(both sides have } o(ss') \text{ factors)}$$

where $o(ss')$ is the order of the product ss' for any $s \neq s'$ in S such that $o(ss') < \infty$; $\mathcal{H}$ is called the *Hecke algebra* associated to W, L.

For $w \in W$, we define $T_w \in \mathcal{H}$ by $T_w = T_{s_1} T_{s_2} \cdots T_{s_q}$, where $w = s_1 s_2 \cdots s_q$ is a reduced expression in W. Then T_w is independent of the choice of a reduced expression and hence is well-defined. The set $\{T_w \mid w \in W\}$ forms an $\mathcal{A}$-basis of $\mathcal{H}$.

1.2. Let $^- : \mathcal{A} \to \mathcal{A}$ be the ring involution which sends v^n to v^{-n} for any $n \in \mathbb{Z}$. Then there is a unique ring homomorphism $^- : \mathcal{H} \to \mathcal{H}$ which is $\mathcal{A}$-semilinear with respect to $^- : \mathcal{A} \to \mathcal{A}$ and satisfies $\overline{T}_w = T_{w^{-1}}^{-1}$ for any $w \in W$.

1.3. For any $n \in \mathbb{Z}$, let $\mathcal{A}_{\leqslant n} = \bigoplus_{m \leqslant n} \mathbb{Z}v^m$, $\mathcal{A}_{<n} = \bigoplus_{m<n} \mathbb{Z}v^m$, $\mathcal{H}_{\leqslant 0} = \bigoplus_{w \in W} \mathcal{A}_{\leqslant 0} T_w$ and $\mathcal{H}_{<0} = \bigoplus_{w \in W} \mathcal{A}_{<0} T_w$.

It is known that for any $w \in W$, there exists a unique element $c_w \in \mathcal{H}_{\leqslant 0}$ such that $\overline{c_w} = c_w$ and $c_w \equiv T_w \mod \mathcal{H}_{<0}$. The set $\{c_w \mid w \in W\}$ forms another $\mathcal{A}$-basis of $\mathcal{H}$.

1.4. For $z \in W$, define $D_z \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, \mathcal{A})$ by $D_z(c_w) = 0$ if $w \in W \setminus \{z\}$ and $D_z(c_z) = 1$.

For $w, w' \in W$, we write $w \longleftarrow_{\mathcal{L}} w'$ (respectively, $w \longleftarrow_{\mathcal{R}} w'$) if $D_w(c_s c_{w'}) \neq 0$ (respectively, $D_w(c_{w'} c_s) \neq 0$) for some $s \in S$. We write $w \leqslant_{\mathcal{L}} w'$ (respectively, $w \leqslant_{\mathcal{R}} w'$) if there exist $w = w_0, w_1, ..., w_r = w'$ in W such that $w_{i-1} \longleftarrow_{\mathcal{L}} w_i$ (respectively, $w_{i-1} \longleftarrow_{\mathcal{R}} w_i$) for any $1 \leqslant i \leqslant r$. We write $w \leqslant_{\mathcal{LR}} w'$ if there exist $w = w_0, w_1, ..., w_r = w'$ in W such that either $w_{i-1} \longleftarrow_{\mathcal{L}} w_i$ or $w_{i-1} \longleftarrow_{\mathcal{R}} w_i$ for any $1 \leqslant i \leqslant r$.

The relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ are preorders on W. Let $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$, $\sim_{\mathcal{LR}}$ be the associated equivalence relations. The equivalence classes on W for $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$, $\sim_{\mathcal{LR}}$ are called *left cells, right cells, two-sided cells* of W, L, respectively. The preorder $\leqslant_{\mathcal{L}}$ (respectively, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$) on W induces a partial ordering on the set of left cells (respectively, right cells, two-sided cells) of W.

1.5. From now on, assume $L(s) > 0$ for $s \in S$. For $x, y, z \in W$, define $h_{x,y,z} \in \mathcal{A}$ by

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

We say that W, L is *bounded* if there exists some $N \in \mathbb{N}$ such that $v^{-N} h_{x,y,z} \in \mathcal{A}_{\leqslant 0}$ for all $x, y, z \in W$. It is known that W, L is bounded when W is either a finite or an affine Coxeter group.

From now on, assume that W, L is *bounded.* Then for any $z \in W$, there exists a unique integer $a(z)$, $0 \leqslant a(z) \leqslant N$ such that

$$h_{x,y,z} \in v^{a(z)} \mathbb{Z}[v^{-1}] \qquad \text{for all } x, y \in W,$$
$$h_{x,y,z} \notin v^{a(z)-1} \mathbb{Z}[v^{-1}] \qquad \text{for some } x, y \in W.$$

The function a plays an important role in the theory of the $\mathcal{A}$-algebra $\mathcal{H}$, in particular, in the study of cells of W. When W is either a finite or an affine Coxeter group and when L is constant on S, the function a is constant on any two-sided cell Ω of W (see [4, Theorem 5.4]).

§2. Some conjectures of Lusztig involving the a-function.

For $x, y, z \in W$, the notation $z = x \cdot y$ means that $z = xy$ and $\ell(z) = \ell(x) + \ell(y)$. In this case, we have $z \leqslant_{\mathcal{L}} y$ and $z \leqslant_{\mathcal{R}} x$.

For $w \in W$, let $Z(w)$ be the set of all $y \in W$ such that $w = u \cdot y \cdot u'$ (i.e., $\ell(w) = \ell(u) + \ell(y) + \ell(u')$) for some $u, u' \in W$ and some $y \in W_I$ with W_I finite, where W_I is the subgroup of W generated by $I \subset S$. Define $a'(w) = \max_{y \in Z(w)} a(y)$. Lusztig proposed the following

Conjecture 2.1. (see [7, Subsection 13.12]) *We have $a(w) = a'(w)$ for any $w \in W$.*

2.2. In [7, Chapter 14], Lusztig proposed a number of conjectures involving the function a, the followings are two of them:

P4. If $z' \leqslant_{\mathcal{LR}} z$ in W then $a(z') \geqslant a(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$ then $a(z') = a(z)$.

P11. If $z' \leqslant_{\mathcal{LR}} z$ and $a(z') = a(z)$ in W then $z' \sim_{\mathcal{LR}} z$.

Conjectures P4 and P11 hold when W is either a finite or an affine Weyl group and the weight function L is constant on S (see [4, Theorem 5.4] and [6, Corollary 1.9]).

We have $w \leqslant_{\mathcal{LR}} y$ for any $y \in Z(w)$. So if P4 is true, then the inequality $a(w) \geqslant a'(w)$ holds in general.

2.3. Denote $x <_{\mathcal{LR}} y$ in W, if $x \leqslant_{\mathcal{LR}} y$ and $x \nsim_{\mathcal{LR}} y$. Denote $y \leqslant w$ in W, if there is a reduced expression $w = s_1 s_2 \cdots s_r$ with $s_i \in S$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_t}$ for some $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant r$. Call $\leqslant$ the *Bruhat-Chevalley ordering* on W. For any $w \in W$, set

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Consider the following set:

(2.3.1) $$Y(W)=\{w\in W\mid w<_{\mathcal{LR}} sw, wt\ \text{ for any } s\in\mathcal{L}(w), t\in\mathcal{R}(w)\}.$$

Lemma 2.4. *Suppose that P4 and P11 hold for W. Then Conjecture 2.1 holds if and only if any $z\in Y(W)$ lies in W_I for some $I\subset S$ with W_I finite.*

Proof. Under our assumption, the set $Y(w)$ can be described as follows.

(2.4.1) $$Y(W)=\{w\in W\mid a(sw), a(wt)<a(w)\ \text{ for any } s\in\mathcal{L}(w), t\in\mathcal{R}(w)\}.$$

First assume that Conjecture 2.1 holds. Let $z\in Y(W)$. Then there exists an expression $z=u\cdot y\cdot u'$ for some $u,u'\in W$ and some $y\in W_I$, $I\subset S$ with W_I finite and $a(y)=a(z)$. By the assumption of $z\in Y(W)$, we must have $u=u'=1$. So $z\in W_I$, as required.

Next assume that any $z\in Y(W)$ lies in W_I for some $I\subset S$ with W_I finite. It is easily seen that any $w\in W$ has an expression $w=u\cdot y\cdot u'$ for some $u,u',y\in W$ with $y\in Y(w)$ and $a(y)=a(w)$. Hence $y\in W_I$ for some $I\subset S$ with W_I finite by our assumption. So $a(w)\geqslant a'(w)\geqslant a(y)=a(w)$, Conjecture 2.1 holds. □

§3. The affine Weyl group of type $\widetilde{A}_{n-1}$, $n\geqslant 2$.

The most part of the results stated in 3.1-3.4 can be found in [8].

3.1. Let W be the affine Weyl group of type $\widetilde{A}_{n-1}$, $n\geqslant 2$ (by abuse of notation, we denote W by $\widetilde{A}_{n-1}$). Lusztig described $\widetilde{A}_{n-1}$ as a permutation group on the set $\mathbb{Z}$ as follows (see [3, Subsection 3.6] and [8, Chapter 4]):

$$\widetilde{A}_{n-1}=\left\{w:\mathbb{Z}\to\mathbb{Z}\ \middle|\ (i+n)w=(i)w+n\ \forall\, i\in\mathbb{Z}; \sum_{k=1}^{n}(k)w=\sum_{k=1}^{n}k\right\}.$$

Any $w\in\widetilde{A}_{n-1}$ can be identified with an $\infty\times\infty$ permutation matrix $(a_{ij})_{i,j\in\mathbb{Z}}$, where

$$a_{ij}=\begin{cases}1, & \text{if } j=(i)w,\\ 0, & \text{otherwise.}\end{cases}\qquad \text{(see [8, Subsection 4.1])}$$

The Coxeter generator set $S=\{s_0,s_1,...,s_{n-1}\}$ of $\widetilde{A}_{n-1}$ are given by setting

$$(t)s_i=\begin{cases} t, & \text{if } t\not\equiv i,i+1 \ (\bmod\ n),\\ t+1, & \text{if } t\equiv i \ (\bmod\ n),\\ t-1, & \text{if } t\equiv i+1 \ (\bmod\ n).\end{cases}\qquad \text{for } 0\leqslant i<n \ \text{ and } \ t\in\mathbb{Z}.$$

For any $w\in\widetilde{A}_{n-1}$ and $0\leqslant i<n$, we have

$$\text{(3.1.1)}\qquad \begin{aligned} s_i\in\mathcal{L}(w) &\iff (i)w>(i+1)w,\\ s_i\in\mathcal{R}(w) &\iff (i)w^{-1}>(i+1)w^{-1}.\end{aligned}\qquad \text{(see [8, Corollary 4.2.3])}$$

For any $i,j\in\mathbb{Z}$, the condition $a_{ij}=1$ implies that $a_{i+qn,j+qn}=1$ for all $q\in\mathbb{Z}$. In particular, for any (maximal) proper subset $J\subset S$, the matrix $(a_{ij})_{i,j\in\mathbb{Z}}$ of an element z of W_J contains an $n\times n$ diagonal matrix block $M(z)$ which is a permutation matrix and determines the matrix z by periodically extension. More precisely, there exists some $1\leqslant c\leqslant n$ such that $M(z)=(a_{ij})_{c<i,j\leqslant c+n}$ is a permutation matrix.

3.2. Fix a positive integer n. A *partition* of n is by definition a weakly decreasing sequence of non-negative integers $\lambda=(\lambda_1,\lambda_2,...,\lambda_r)$ with $\sum_{i=1}^r\lambda_i=n$. Two partitions are regarded as the same if one can be obtained from the other by adding some zero parts at the end. Let Λ_n be the set of all partitions of n and let $\Lambda=\bigcup_{n\geqslant 1}\Lambda_n$. Let $\lambda=(\lambda_1,\lambda_2,...,\lambda_r)$ and $\mu=(\mu_1,\mu_2,...,\mu_t)$ be in Λ. Write $\lambda\leqslant\mu$, if $\lambda_1+\cdots+\lambda_k\leqslant\mu_1+\cdots+\mu_k$ for any $k\geqslant 1$, where we stipulate $\lambda_p=\mu_q=0$ for $p>r$ and $q>t$. This defines a partial ordering on the set Λ.

3.3. For any $w\in\widetilde{A}_{n-1}$, call $\{i_1,i_2,...,i_t\}\subset\mathbb{Z}$ to be a w-*chain*, if $i_1<i_2<\cdots<i_t$ and $(i_1)w>(i_2)w>\cdots>(i_t)w$. In terms of matrix entries, by a w-chain $\{i_1,i_2,...,i_t\}$, it means that the non-zero entries of the matrix w at the rows $i_1,i_2,...,i_t$ are going down to the left monotonously. Let $k\in\mathbb{N}$. By a (w,k)-chain-family, we mean a union of k w-chains $\bigcup_{j=1}^k\{i_{j1},i_{j2},...,i_{jm_j}\}$ such that $i_{jh}\not\equiv i_{j'h'}\ (\bmod\ n)$ for any $(j,h)\neq(j',h')$. Let $d_k(w)$ be the maximal possible cardinal of a (w,k)-chain-family. Then $d_k(w)\leqslant n$ for any $k\geqslant 1$ and there is some $r\in\mathbb{N}$ such that $d_1(w)<d_2(w)<\cdots<d_r(w)=n$. Let $\lambda_1(w)=d_1(w)$ and $\lambda_i(w)=d_i(w)-d_{i-1}(w)$ for $1<i\leqslant r$. By a result of C. Greene (see [2]), we see that $\lambda(w)=(\lambda_1(w),\lambda_2(w),...,\lambda_r(w))$ is a partition of n. So Lusztig defined

a map $\psi : \widetilde{A}_{n-1} \to \Lambda_n$ by sending w to $\lambda(w)$ (see [3, Subsection 3.6] and [8, Definition 5.3]).

The above definitions of a w-chain, a (w,k)-chain-family and a partition $\psi(w)$ is also applicable to any permutation in the symmetric group S_m over the set $\{1,2,...,m\}$ (or equivalently, to any $m \times m$ permutation matrix).

A submatrix $w' = (a_{ij})_{i \in I, j \in J}$ of $w = (a_{ij})_{i,j \in \mathbb{Z}}$ with $I, J \subset \mathbb{Z}$ is called *n-distinguished* if $h \not\equiv k \pmod n$ for any $h \neq k$ either both in I or both in J (hence $|I|, |J| \leqslant n$). The following fact can be seen easily: suppose that w' is an n-distinguished submatrix $(a_{ij})_{i \in I, j \in J}$ of $w = (a_{ij})_{i,j \in \mathbb{Z}} \in \widetilde{A}_{n-1}$, then any w'-chain is also a w-chain. This implies that $\psi(w') \leqslant \psi(w)$.

3.4. For any $\lambda \in \Lambda_n$, it is known (see [5, Theorem 6] and [8, Theorem 17.4]) that the inverse image $\psi^{-1}(\lambda)$ forms a two-sided cell (denoted by Ω_λ) of $\widetilde{A}_{n-1}$. Moreover, we see by [11, Theorem B] that the map ψ induces an order-reversing bijection from the set $\mathrm{Cell}(\widetilde{A}_{n-1})$ of two-sided cells of $\widetilde{A}_{n-1}$ to the set Λ_n, that is,

$$(3.4.1) \qquad \lambda \geqslant \mu \text{ in } \Lambda_n \iff \Omega_\lambda \leqslant_{\mathcal{LR}} \Omega_\mu \text{ in } \mathrm{Cell}(\widetilde{A}_{n-1}).$$

Note that for any proper subset I of S, the subgroup W_I of $\widetilde{A}_{n-1}$ is always finite.

Lemma 3.5. *If $w \in \widetilde{A}_{n-1}$ has an expression $w = u \cdot y \cdot u'$ for some $u, u' \in \widetilde{A}_{n-1}$ and some $y \in W_I$, $I \subset S$, with $\psi(w) = \psi(y)$, then the matrix w has an n-distinguished $n \times n$ permutation submatrix w' with $\psi(w') = \psi(w)$.*

Proof. The $\infty \times \infty$ matrix y satisfies $\psi(y) = \psi(w)$ by our assumption. There is a sequence $w_0 = y, w_1, ..., w_r = w$ in $\widetilde{A}_{n-1}$ with some $r \geqslant 0$ such that for every $1 \leqslant i \leqslant r$, either $w_i = t_i \cdot w_{i-1}$ or $w_i = w_{i-1} \cdot t_i$ for some $t_i \in S$. In general, we have $\psi(w') \leqslant \psi(w)$ for any n-distinguished $n \times n$ permutation submatrix w' of w. So by induction on $\ell(w) - \ell(y) \geqslant 0$, we need only to show that if $w, y \in \widetilde{A}_{n-1}$ satisfy that $y \in \{sw, wt \mid s \in \mathcal{L}(w), t \in \mathcal{R}(w)\}$ and that y' is an n-distinguished $n \times n$ permutation submatrix of y, then w has an n-distinguished $n \times n$ permutation submatrix w' with $\psi(w') \geqslant \psi(y')$. Say $y = (b_{ij})_{i,j \in \mathbb{Z}}$

and $y' = (b_{ij})_{i\in I,j\in J}$ with $I, J \subset \mathbb{Z}$ and $|I| = |J| = n$. By symmetry, we need only to consider the case where $w = s_c \cdot y$ for some $s_c \in \mathcal{L}(w)$ with $0 \leqslant c < n$. Then w is obtained from y by transposing the $(c+qn)$-th row and the $(c+1+qn)$-th row for all $q \in \mathbb{Z}$. Take the n-distinguished $n \times n$ submatrix $w' = (a_{ij})_{i\in I',j\in J}$ of the matrix $w = (a_{ij})_{i,j\in\mathbb{Z}}$ with $I' = (I)s_c$. Then w' is an n-distinguished $n \times n$ permutation submatrix of w and satisfies $w' = y'$ except that $(\overline{c} \cup \overline{c+1}) \cap I$ consists of two consecutive integers, where $\overline{a} = \{a + qn \mid q \in \mathbb{Z}\}$ for any $a \in \mathbb{Z}$. In the latter case, we have $w' = s \cdot y'$ in S_n for some $s \in \mathcal{L}(w')$. So $\psi(w') \geqslant \psi(y')$ in either case by (3.4.1). □

Remark 3.6. I wonder if the converse for the result in Lemma 3.5 holds. That is, when the matrix $w \in \widetilde{A}_{n-1}$ has an n-distinguished $n \times n$ permutation submatrix w' with $\psi(w') = \psi(w)$, is there always an expression $w = u \cdot y \cdot u'$ for some $u, u' \in \widetilde{A}_{n-1}$ and $y \in W_I$, $I \subset S$ such that $\psi(y) = \psi(w')$?

§4. A counter-example to Conjecture 2.1.

For any $I \subseteq S$ with W_I finite, denote by w_I the longest element in W_I. When $I = \{s_{i_1}, s_{i_2}, ..., s_{i_r}\}$, we simply write w_I by $w_{i_1 i_2 \cdots i_r}$.

Theorem 4.1. *Conjecture 2.1 does not hold in general.*

Proof. We need only to provide a counter-example to Conjecture 2.1. Take $w = w_{1456789} \cdot s_3s_4s_{10}s_9 \cdot s_8s_7s_6s_5s_6s_7s_8 \cdot s_2s_1s_3s_2 \cdot s_0s_1s_{10}s_0 \cdot s_2s_1s_3s_2$ in $\widetilde{A}_{10}$ (i.e., $n = 11$ in $\widetilde{A}_{n-1}$). As a permutation on $\mathbb{Z}$, we have

$$((1)w, (2)w, ..., (11)w) = (2, 1, 9, 15, 14, 8, 7, 6, 0, -1, 5).$$

The matrix form of w is as in Fig. 1. We shall provide two proofs for the failure of Conjecture 2.1 for the element w.

First Proof: We have

$$\mathcal{L}(w) = \{s_1, s_4, s_5, s_6, s_7, s_8, s_9\} \quad \text{and} \quad \mathcal{R}(w) = \{s_{10}, s_0, s_1, s_2, s_3, s_5, s_6, s_7, s_8\}.$$

Hence $\mathcal{L}(w) \cup \mathcal{R}(w) = S$. This implies $w \notin W_I$ for any $I \subset S$. It is easily seen from Figure 1 that $\psi(sw), \psi(wt) < \psi(w)$ for any $s \in \mathcal{L}(w)$ and $t \in \mathcal{R}(w)$. Hence $w \in Y(W)$

by (3.4.1). So Conjecture 2.1 fails to hold for w by Lemma 2.4 (note that the weight function L of $\widetilde{A}_{n-1}$ on S is always constant and hence $\widetilde{A}_{n-1}$ satisfies P4 and P11 by [4, Theorem 5.4] and [6, Corollary 1.9]).

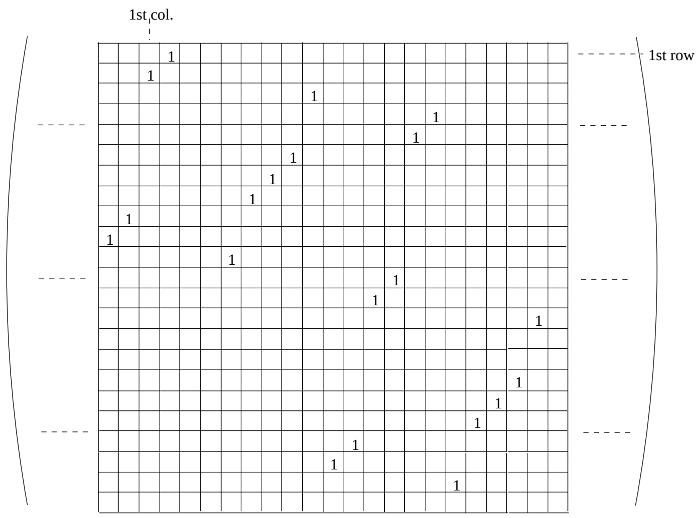

Figure 1.

Second Proof: $\{4,5,6,7,8,9,10\}$ and $\{3,6,7,8,11\} \cup \{4,5,12,13,20,21\}$ are respectively $(w,1)$- and $(w,2)$-chain-families of maximal possible cardinal. So $d_1(w) = 7$ and $d_2(w) = 11$. Hence $\psi(w) = (7,4)$. We see from Figure 1 that there is no 11-distinguished 11×11 permutation submatrix w' with $\psi(w') = (7,4)$. This implies the failure of Conjecture 2.1 for w by Lemma 3.5. $\square$

Now that Conjecture 2.1 fails to hold in general, it is desirable to propose a new conjecture to replace it. We shall do it in the remaining part of the section.

4.2. Let Cell(W) be the set of all two-sided cells in W. Then Cell(W) is a poset with respect to the partial ordering $\leqslant_{\mathcal{LR}}$.

For $w \in W$, denote by $\Omega(w)$ the two-sided cell of W containing w. Define

$$T(w) = \{\Omega \in \text{Cell}(W) \mid \Omega \cap Z(w) \neq \emptyset\},$$

$$B(w) = \{\Omega \in \text{Cell}(W) \mid \Omega \leqslant_{\mathcal{LR}} \Omega',\ \forall\, \Omega' \in T(w)\}.$$

Then $\Omega(w) \in B(w)$.

Conjecture 4.3. *(1) For any $w \in W$, there is a unique maximal element (denoted by $\Omega'(w)$) in the set $B(w)$ with respect to the partial ordering $\leqslant_{\mathcal{LR}}$.*

(2) The equation $\Omega(w) = \Omega'(w)$ holds for any $w \in W$.

4.4. Let $\overline{S}$ be the set of all $I \subseteq S$ with W_I finite. For any $w \in W$, define

$$E(w) = \{\Omega(w_I) \in \text{Cell}(W) \mid w = u \cdot w_I \cdot u' \text{ for some } u, u' \in W, I \in \overline{S}\},$$

$$F(w) = \{\Omega \in \text{Cell}(W) \mid \Omega \leqslant_{\mathcal{LR}} \Omega', \ \forall \Omega' \in E(w)\}.$$

We consider the following auxiliary statement.

Statement 4.5. *(1) For any $w \in W$, there is a unique maximal element (say $\Omega''(w)$) in the set $F(w)$ with respect to the partial ordering $\leqslant_{\mathcal{LR}}$.*

(2) The equation $\Omega(w) = \Omega''(w)$ holds for any $w \in W$.

Statement 4.5 is stronger than Conjecture 4.3.

Lemma 4.6. *Statement 4.5 implies Conjecture 4.3.*

Proof. We have the relations $E(w) \subseteq T(w)$ and $B(w) \subseteq F(w)$ for any $w \in W$ in general. Now assume that Statement 4.5 is true. Then $\Omega(w)$ is the unique maximal element in $F(w)$ under the partial ordering $\leqslant_{\mathcal{LR}}$. Since $\Omega(w) \in B(w)$, $\Omega(w)$ is also the unique maximal element in $B(w)$. So Conjecture 4.3 is also true. □

We have some examples to support both Conjecture 4.3 and Statement 4.5.

Theorem 4.7. (see [10, Theorem 3.1]) *Let $w \in \widetilde{A}_{n-1}$ be with $d_1(w) < d_2(w) < \cdots < d_t(w) = n$. Then for any k, $1 \leqslant k \leqslant t$, there exists an expression*

$$w = x \cdot w_J \cdot y$$

for some $x, y \in \widetilde{A}_{n-1}$ and $J \in \overline{S}$ with $d_k(w) = d_k(w_J)$. On the other hand, $w \neq x' \cdot w_I \cdot y'$ for any $x', y' \in \widetilde{A}_{n-1}$ and $I \in \overline{S}$ with $d_h(w_I) > d_h(w)$ for some h, $1 \leqslant h \leqslant t$.

Theorem 4.8. *Both Conjecture 4.3 and Statement 4.5 are true for* $W \in \{\widetilde{A}_{n-1}, \widetilde{D}_4\}$.

Proof. By Lemma 4.6, we need only to verify Statement 4.5 for $W \in \{\widetilde{A}_{n-1}, \widetilde{D}_4\}$. Statement 4.5 for $W = \widetilde{A}_{n-1}$ is an immediate consequence of Theorem 4.7 and (3.4.1). Now assume $W = \widetilde{D}_4$. Let s_0, s_1, s_2, s_3, s_4 be the Coxeter generator set of W with s_2 corresponding to the branching node of its Coxeter graph. Then any $w \in W$ with $a(w) \neq 7$ has an expression $w = x \cdot w_J \cdot y$ for some $x, y \in W$ and $J \in \overline{S}$ with $a(w) = a(w_J)$ by [9, Theorem 1.1] and [12, Theorem B]. Let $W_{(i)} = \{z \in W \mid a(z) = i\}$ for any $i \in \mathbb{N}$. Then by a result of J. Du in [1, Theorem 4.6], $W_{(7)}$ is a two-sided cell of W such that any $w \in W_{(7)}$ has an expression $w = x \cdot s_i s_2 s_k s_i s_2 s_i s_j s_2 s_i \cdot y$ for some $x, y \in W$ and some distinct $i, j, k \in \{0, 1, 3, 4\}$, where $z := s_i s_2 s_k s_i s_2 s_i s_j s_2 s_i = w_{ik2} \cdot s_j s_2 s_i = s_i s_2 s_k \cdot w_{ij2}$. We have $\Omega(w_{ik2}) \neq \Omega(w_{ij2})$ in $E(w)$ by [1, Theorem 3.9]. Hence $F(w) = \{W_{(7)}, W_{(12)}\}$. So $\Omega(w) = W_{(7)}$ is the unique maximal element in $F(w)$ with respect to $\leqslant_{\mathcal{LR}}$. This verifies Statement 4.5 for $W = \widetilde{D}_4$. $\square$

One may find more examples to support Statements 4.5 and hence Conjecture 4.3. However, Statement 4.5 is not true in general. The following is a counter-example.

Example 4.9. In the Weyl group W of type B_3, let s, r, t be its Coxeter generators satisfying that $(sr)^3 = (rt)^4 = (st)^2 = 1$. Take the weight function L on W to be the length function ℓ. Consider the element $w = srtrsr$. We have

$$w \sim_{\mathcal{L}} tsrtrsr \sim_{\mathcal{L}} trtrsr \sim_{\mathcal{R}} trtrs \sim_{\mathcal{R}} trtr := w_{rt},$$

So $\Omega(w) = \Omega(w_{rt})$. It is easily seen that $E(w) = \{\Omega(w_I) \mid I \in \{\{s, r\}, \{s, t\}, \{s\}, \{r\}, \{t\}, \{\emptyset\}\}$. Clearly, $F(w)$ contains the element $\Omega(w_{sr})$ with $\Omega(w_{rt}) <_{\mathcal{LR}} \Omega(w_{sr})$. This violates Statement 4.5.

Remark 4.10. Under the assumption of P4 and P11, Conjecture 2.1 amounts to assert that $T(w) \cap B(w) = \{\Omega(w)\}$ for any $w \in W$. Hence Conjecture 2.1 implies Conjecture 4.3 in this case. Now Theorem 4.1 shows that it is possible that $\Omega(w) \notin T(w)$ for some $w \in W$, negating Conjecture 2.1. So one can only expect the validity of Conjecture 4.3.

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