COXETER ELEMENTS AND KAZHDAN-LUSZTIG CELLS

Jian-yi Shi

Fakultät für Mathematik Universität
Bielefeld, D-33501, Bielefeld, Germany
and
Department of Mathematics, East China
Normal University, Shanghai, 200062, P.R.China

Abstract. By the correspondence between Coxeter elements of a Coxeter system (W, S, Γ) and the acyclic orientations of the Coxeter graph Γ, we study some properties of elements in the set $C_0(W)$. We show that when W is of finite, affine or hyperbolic type, any $w \in C_0(W)$ satisfies $w \underset{\mathrm{LR}}{\sim} w_J$ with $\ell(w_J) = |J| = m(w)$ for some $J \subset S$. Now assume that W is of finite or affine type. We give an explicit description for all the distinguished involutions d of W with $d \underset{\mathrm{L}}{\sim} w$ for some $w \in E(W)$, which verifies a conjecture proposed in [12, Conjecture 8.10] in our case. We show that any left cell of W containing some element of $C_0(W)$ is left-connected, which verifies a conjecture of Lusztig in [3] in our case.

Introduction.

Let (W, S, Γ) be a Coxeter system: a Coxeter group W with S the distinguished generator set and Γ the Coxeter graph. We always assume S finite in the present paper. A Coxeter element of W is by definition a product of all generators $s \in S$ in any fixed order. Let $C(W)$ be the set of all the Coxeter elements in W and let $C_0(W) = \bigcup_{J \subseteq S} C(W_J)$, where W_J is the standard parabolic subgroup of W generated by J. It is known that the elements of $C(W)$ are in 1-1 correspondence with the acyclic orientations of Γ (here and later when considering orientations, Γ is only regarded as a graph consisting of a set of nodes and a set of edges, the

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labelings of whose edges are usually neglected, see [15, Theorem 1.5]). In the present paper, we shall use this result to make some further investigation on the properties of the elements in $C_0(W)$. We mainly consider the cases where W is of finite, affine or hyperbolic type. We show that any $w \in C_0(W)$ is in the two-sided cell of W containing some element of the form w_J with $\ell(w_J) = |J| = m(w)$ (see 1.6 and Theorem 3.5), where w_J is the longest element in the group W_J. Now assume that W is of finite or affine type. Then we further show that $m(w)$ is precisely $a(w)$, the a-value of w defined by Lusztig (see 1.3 and Theorem 3.5). We define a subset $E(W)$ of $C_0(W)$, which consists of minimal elements in certain left cells of W (see the definition preceding Lemma 2.5). Then we give an explicit expression for any distinguished involution d of W with $d \underset{\mathrm{L}}{\sim} w$ for some $w \in E(W)$ (see 1.3 (f) and Theorem 4.3). More precisely, if $w = w_I \cdot y$ with $m(w) = |I|$, then $d = y^{-1} \cdot w_I \cdot y$. The result verifies a conjecture proposed in [12, Conjecture 8.10] in our case (see Conjecture 1.5).

A subset K of W is *left-connected*, if, for any $x, y \in K$, there exists a sequence of elements $x_0 = x, x_1, ..., x_r = y$ in K such that $x_{i-1}x_i^{-1} \in S$ for $1 \leqslant i \leqslant r$. Lusztig conjectured that any left cell L of an affine Weyl group (and hence also of a Weyl group) is left-connected (see [3]). This conjecture has been verified in the following cases: (1) W has type $\widetilde{A}_l$, $l \geqslant 1$ [10, Theorem 18.2.1]; (2) L is in the lowest two-sided cell of W [11, Corollary 1.2]; (3) L is of a-value $\leqslant 1$ (easy). In the present paper, we show that any $w \in E(W)$ is the unique minimal element in the left cell L_w of W containing w and that L_w is a left-connected subset of W (see Theorems 4.7 and 4.8). This verifies the conjecture of Lusztig in the case where L contains an element of $C_0(W)$.

When (W, S, Γ) is of finite or affine type, we give a necessary and sufficient condition on the relation $w_I \underset{\mathrm{LR}}{\sim} w_J$ for $I, J \subseteq S$ with $w_I, w_J \in C_0(W)$.

The organization of the paper is as follows. Section 1 is served as preliminaries, where we collect some notations, definitions, and results, that will be needed in the rest of the paper. We deduce some graph-theoretic results in Section 2 by making use of the correspondence between Coxeter elements and the acyclic orientations of the related graph. Assuming W to be of finite, affine or hyperbolic type, we study the properties of $w \in C_0(W)$ in terms of $m(w)$ and $a(w)$ in Section 3. In Sections 4-5, we assume that W is of finite or affine type. In Section 4, we describe all the distinguished involutions d with $d \underset{\mathrm{L}}{\sim} w$ for some $w \in E(W)$. Then we show that $w \in E(W)$ is the unique minimal element in the left cell L_w of W containing w and that L_w is left-connected. We give a brief discussion in Section 5 for the relation $\underset{\mathrm{LR}}{\sim}$ on the elements of $C_0(W)$ of the form w_I, $I \subset S$. Finally, in Appendix, we list all the Coxeter graphs of finite, affine and hyperbolic types.

§1. Preliminaries.

Let (W, S, Γ) be a Coxeter system. In this section, we collect some concepts, terminologies and known results for the subsequent usage.

1.1. Let $\leqslant$ be the Bruhat order on W and let ℓ be the length function of W with respect to S. Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. The *Hecke algebra* $\mathcal{H}(W)$ associated to W is an associative algebra over A with two A-bases $\{T_w \mid w \in W\}$ and $\{C_w \mid w \in W\}$. The multiplication is determined by

$$T_w T_{w'} = T_{ww'}, \qquad \text{if } \ell(ww') = \ell(w) + \ell(w');$$
$$(T_s - u^{-1})(T_s + u) = 0 \qquad \text{for } s \in S.$$

The relation between two bases are as below.

$$C_w = \sum_{y \leqslant w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y, \qquad \text{for } \ w \in W,$$

where the $P_{y,w}(u)$'s are known as *Kazhdan-Lusztig polynomials*, satisfying that for any $w, y \in W$, $P_{w,w}(u) = 1$, $P_{y,w}(u) = 0$ for $y \nleqslant w$ and $\deg P_{y,w}(u) \leqslant 1/2(\ell(w) - \ell(y) - 1)$ for $y < w$.

For $w \in W$ and $s \in S$, we have

$$C_s C_w = \begin{cases} (u^{-1} + u) C_w, & \text{if } \ sw < w, \\ C_{sw} + \sum\limits_{\substack{y \prec w \\ sy < y}} \mu(y, w) C_y, & \text{if } \ sw > w, \end{cases} \tag{1.1.1}$$

where $\mu(y, w)$ is the coefficient of $u^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}(u)$, and the notation $y \prec w$ means $\mu(y, w) \neq 0$.

1.2. Let $\underset{\mathrm{L}}{\leqslant}$ (resp. $\underset{\mathrm{R}}{\leqslant}$, resp. $\underset{\mathrm{LR}}{\leqslant}$) be the preorder on W defined as in [4], and let $\underset{\mathrm{L}}{\sim}$ (resp. $\underset{\mathrm{R}}{\sim}$, resp. $\underset{\mathrm{LR}}{\sim}$) be the equivalence relation on W determined by $\underset{\mathrm{L}}{\leqslant}$ (resp. $\underset{\mathrm{R}}{\leqslant}$, resp. $\underset{\mathrm{LR}}{\leqslant}$). The corresponding equivalence classes are called *left* (resp. *right*, resp. *two-sided*) *cells* of W. $\underset{\mathrm{L}}{\leqslant}$ (resp. $\underset{\mathrm{R}}{\leqslant}$, resp. $\underset{\mathrm{LR}}{\leqslant}$) induces a partial order on the set of left (resp. right, resp. two-sided) cells of W.

(a) If $w = x \cdot y$ then $w \underset{\mathrm{L}}{\leqslant} y$ and $w \underset{\mathrm{R}}{\leqslant} x$, where the notation $w = x \cdot y$ for $x, y, w \in W$ means that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$.

To each $w \in W$, we associate two sets $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

(b) Let $x, y \in W$ satisfy $x^{-1}y \in S$ (resp. $xy^{-1} \in S$). If $\mathcal{R}(x) \overset{\not\supseteq}{\not\subseteq} \mathcal{R}(y)$ (resp. $\mathcal{L}(x) \overset{\not\supseteq}{\not\subseteq} \mathcal{L}(y)$), then $x \underset{\mathrm{R}}{\sim} y$ (resp. $x \underset{\mathrm{L}}{\sim} y$).

1.3. Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ for a Coxeter group W in [6]. When W is of finite or affine type, Lusztig showed in [6, 7] the following results.

(a) There exists some $N \in \mathbb{N}$ such that $a(z) \leqslant N$ for $z \in W$.

Write $C_x C_y = \sum_z h_{x,y,z} C_z$ with $h_{x,y,z} \in A$ for $x, y \in W$.

(b) For any $x, y \in W$, the coefficients of $h_{x,y,z}$ are all nonnegative.

(c) $a(w_J) = \ell(w_J)$ for $J \subseteq S$. In particular, when J consists of mutually commuting elements, we have $a(w_J) = |J|$, the cardinality of the set J.

(d) If $x \underset{\mathrm{LR}}{\leqslant} y$ in W, then $a(x) \geqslant a(y)$. So $x \underset{\mathrm{LR}}{\sim} y$ implies $a(x) = a(y)$, i.e., the function a is constant on a two-sided cell of W.

(e) If $x \underset{\mathrm{LR}}{\sim} y$ and $x \underset{\mathrm{L}}{\leqslant} y$, then $x \underset{\mathrm{L}}{\sim} y$.

Let $\delta(z) = \deg P_{1,z}(u)$ for $z \in W$, where 1 is the identity element of W.

(f) $\mathcal{D}_0(W) = \{z \in W \mid \ell(z) - 2\delta(z) - a(z) = 0\}$ is a finite set of involutions (called *distinguished involutions* of W).

(g) Each left cell of W contains a unique distinguished involution.

(h) Let d be a distinguished involution of W. Then for any $x \in W$ with $x \underset{\mathrm{L}}{\sim} d$, we have $h_{x^{-1},x,d} \neq 0$.

Distinguished involutions play an important role in the representations of the group W

and the associated Hecke algebra $\mathcal{H}(W)$ (see, e.g., [7, 8, 9]).

1.4. Write $T_xT_y = \sum_z f_{x,y,z}T_z$ for $x, y \in W$ with $f_{x,y,z} \in A$. There exists a unique element $w := \lambda(x,y) \in W$ with $f_{x,y,w} \neq 0$ such that $f_{x,y,z} \neq 0$ only if $z \leqslant w$. The element $\lambda(x,y)$ can be described as follows. Given a reduced expression $x = s_1s_2\cdots s_r$ with $s_i \in S$. Let $1 \leqslant j_1 < ... < j_t \leqslant r$ be the subsequence of $1, ..., r$ satisfying:

$$s_{j_k}s_{j_k+1}\cdots\widehat{s_{j_{k+1}}}\cdots\widehat{s_{j_t}}\cdots s_ry < s_{j_k+1}\cdots\widehat{s_{j_{k+1}}}\cdots\widehat{s_{j_t}}\cdots s_ry$$
$$s_ms_{m+1}\cdots\widehat{s_{j_p}}\cdots\widehat{s_{j_t}}\cdots s_ry > s_{m+1}\cdots\widehat{s_{j_p}}\cdots\widehat{s_{j_t}}\cdots s_ry$$

for $1 \leqslant k \leqslant t$ and $j_{p-1} < m < j_p$, $1 \leqslant p \leqslant t+1$, where we stipulate $j_0 = 0$, $j_{t+1} = r+1$, and $\widehat{s_h}$ means the deletion of the factor s_h. Then $\lambda(x,y)$ is just the element $s_1\cdots\widehat{s_{j_1}}\cdots\widehat{s_{j_2}}\cdots\widehat{s_{j_t}}\cdots s_ry$ (see [11, Proposition 2.3]).

The following conjecture on distinguished involutions was proposed in [12, Conjecture 8.10].

Conjecture 1.5. *Let W be of finite or affine type. Let $w \in W$ be such that $w \underset{\mathrm{L}}{<} sw$* (i.e., $w \underset{\mathrm{L}}{\leqslant} sw$ and $w \underset{\mathrm{L}}{\not\sim} sw$, or equivalently, $a(sw) < a(w)$) *for any $s \in \mathcal{L}(w)$. Then $d = \lambda(w^{-1}, w)$ is the distinguished involution with $d \underset{\mathrm{L}}{\sim} w$.*

The conjecture has been verified in the cases where w is in the lowest two-sided cell of W with respect to the partial order $\underset{\mathrm{LR}}{\leqslant}$ (see [11, Theorem 6.1]). In Section 4, we shall verify the conjecture when w is in $C_0(W)$.

1.6. For $w \in C_0(W)$, denote by $m(w)$ the maximal possible value of $\ell(w_J)$ in an expression $w = x \cdot w_J \cdot y$.

1.7. Assume that the order m of the product of $s, t \in S$ is greater than 2. A sequence of elements

$$\underbrace{ys,\ yst,\ ysts,\ \ldots}_{m-1 \text{ terms}}$$

is called a *right* $\{s,t\}$*-string* (or just a *right string*) if $y \in W$ satisfies $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

We say that z is obtained from w by a *right* $\{s,t\}$*-star operation* (or a *right star operation* for brevity), if z, w are two neighboring terms in a right $\{s,t\}$-string. Clearly, a resulting element z of a right $\{s,t\}$-star operation on w, when it exists, need not be unique unless w is a terminus of the right $\{s,t\}$-string containing it.

Similarly, we can define a *left* $\{s,t\}$*-string* and a *left* $\{s,t\}$*-star operation* on an element.

We state a result which can reduce many problems on $w \in C_0(W)$ to the case where w is $C(W)$ and Γ is connected.

Let $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_r$ be the decomposition of Γ into a disjoint union of connected components Γ_i, $1 \leqslant i \leqslant r$. Then we have the corresponding decomposition of (W,S,Γ) into a direct product of Coxeter subsystems (W_i,S_i,Γ_i), $1 \leqslant i \leqslant r$. Any $w \in W$ has a unique expression of the form $w = w_1w_2\cdots w_r$ with $w_i \in W_i$. It is easy to show the following

Lemma 1.8. *In the above setup, we have*

(1) $a(w) = \sum_i a(w_i)$.

(2) $m(w) = \sum_i m(w_i)$.

(3) Let $y = y_1y_2\cdots y_r \in W$ *be with* $y_i \in W_i$. *Then* y *can be obtained from* w *by star operations if and only if so do* y_i *from* w_i *for* $1 \leqslant i \leqslant r$.

(4) Assume that W *is of finite or affine type. Then* w *is a distinguished involution of* W *if and only if* w_i *is a distinguished involution of* W_i *for* $1 \leqslant i \leqslant r$.

1.9. By a *graph*, we mean a finite set of nodes together with a finite set of edges. A graph is always assumed to be *simple* (i.e., no loop and no multi-edges). Two nodes of a graph are *adjacent* if they are joined by an edge. In a graph G, the *degree* $d_G(v)$ of a node v is the number of edges incident on v; v is a *branch node* if $d_G(v) > 2$, and a *terminus* if $d_G(v) \leqslant 1$. A *directed graph* (or a *digraph* for brevity) is a graph with each edge orientated. An *orientation* α of a graph G is a digraph obtained from G by orientating all its edges. A *directed edge* (i.e., an edge with orientation) with two incident nodes v, v' is denoted by an ordered pair (v, v'), if the orientation is from v to v'.

1.10. Given a digraph α. A node v of α is a *source* (resp. a *sink*) if (v, v') (resp. (v', v)) is a directed edge of α for any node v' adjacent to v. An isolated node is the only node which is both a source and a sink. A source or a sink of α is also called an *extreme node*. A *directed path* ξ of α is a sequence of nodes $v_0, v_1, \ldots, v_r$ in α such that (v_{i-1}, v_i) is a directed edge of

α for $1 \leqslant i \leqslant r$. ξ is *maximal* if ξ is not properly contained in any other directed path of α. ξ is a *directed cycle*, if $v_0 = v_r$. A digraph is *acyclic* if it contains no directed cycle.

Let $O(G)$ be the set of all acyclic orientations of a graph G. The following result was proved in [15, Theorem 1.5].

Theorem 1.11. *For a Coxeter system (W, S, Γ), there is a bijective map $\phi : C(W) \longrightarrow O(\Gamma)$ as follows. Let $w = s_1 s_2 \cdots s_l$ be a reduced expression of $w \in C(W)$ with $s_h \in S$. Then for any adjacent nodes s_i, s_j in Γ, (s_i, s_j) is a directed edge in $\phi(w)$ if and only if $i < j$* (here and later, we shall not distinguish an element of S and the corresponding node in Γ).

Call $\alpha = \phi(w)$ the *associated digraph* of $w \in C(W)$.

For $w \in W$, we have that $w \in C(W)$ if and only if $w^{-1} \in C(W)$. A node v is a sink (resp. source) in $\phi(w)$ if and only if v is a source (resp. sink) in $\phi(w^{-1})$. Therefore one need only deal with either of source and sink but not both in many cases.

§2. The values $m(w)$, $n(\alpha)$ and the set $E(W)$.

In this section, we deduce from Theorem 1.11 some graph-theoretic results on $w \in C_0(W)$ concerning the values $m(w)$, $n(\alpha)$ and the set $E(W)$.

Lemma 2.1. *Let α be an acyclic orientation of a graph G. Then*

(i) Each terminus of G is an extreme node of α.

(ii) Each node of G is contained in some maximal directed path of α, which starts with a source and ends with a sink.

(iii) Assume $G = \Gamma$ for a Coxeter system (W, S, Γ). Let $w \in C(W)$ be with α the associated digraph. Then $\mathcal{L}(w)$ (resp. $\mathcal{R}(w)$) is exactly the set of all sources (resp. sinks) of α.

(iv) Keep the assumption of (iii). Let $s \in \mathcal{L}(w)$ (resp. $s \in \mathcal{R}(w)$). Then $\mathcal{L}(w) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{L}(sw)$ (resp. $\mathcal{R}(w) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{R}(ws)$) if and only if the removal of s from α yields a new source (resp. sink) in the resulting digraph.

Proof. The proof of (i) and (ii) is straightforward. (iii) follows from Theorem 1.11. Then

(iv) is an easy consequence of (iii). □

Lemma 2.2. *Given $w \in C(W)$ with α the associated digraph. We have an expression $w = x \cdot w_J \cdot y$ for some $J \subseteq S$ and $x, y \in W$ if and only if the following condition holds.*
(2.2.1) For any $s \neq t$ in J, there is no directed path from s to t in α.

Proof. ($\Longrightarrow$) Note that here the set J must consist of mutually commuting elements. Then our result follows immediately from Theorem 1.11.

($\Longleftarrow$) Apply induction on $m = |S| - |J| \geqslant 0$. If $m = 0$, then α contains no edge and hence $w = w_J$. Now assume $m > 0$. We claim that there exists some extreme node of α in $S \setminus J$ (the complement of J in S). For otherwise, all the extreme nodes of α belong to J. Take any $s \in S \setminus J$. Then by Lemma 2.1 (ii), s belongs to some maximal directed path of α starting with a source (say r) and ending with a sink (say t). Then $r \neq t$ are in J, contradicting (2.2.1). Now assume that there is some source r of α not in J. Then $w' = rw$ is in $C(W_I)$ with $I = S \setminus \{r\}$, whose associated digraph α' is obtained from α by removing the node r and all the edges incident on r. J is a subset of I satisfying condition (2.2.1) with α' instead of α. Since $|I| - |J| = m - 1 < m$, we can write $w' = x' \cdot w_J \cdot y'$ for some $x', y' \in W_I$ by inductive hypothesis. So we get $w = r \cdot w' = rx' \cdot w_J \cdot y'$. The case where there is some sink of α in $S \setminus J$ can be argued similarly. This proves our result. □

Let $w \in C_0(W)$ be with α the associated digraph. Denote by $n(\alpha)$ or $n(w)$ the maximal possible cardinality of a node set J of Γ satisfying condition (2.2.1). Then an immediate consequence of Lemma 2.2 is as below.

Corollary 2.3. *$m(w) = n(\alpha)$ for any $w \in C(W)$ with α the associated digraph.*

Next result is concerned with the number $n(\alpha)$ when α varies.

Lemma 2.4. *Given $w \in C(W)$ with α the associated digraph. Let $v \in S$ be a source (resp. a sink) of α whose removal from α yields a new source (resp. sink) v' in the resulting digraph α'. Then $n(\alpha) = n(\alpha')$.*

Proof. We have $n(\alpha) \geqslant n(\alpha')$ in general. Now let $J \subseteq S$ be a set of nodes of α satisfying condition (2.2.1) and equation $|J| = n(\alpha)$. By Lemma 2.2, we need only find a set $J' \subseteq S \setminus \{v\}$

with $|J'| = |J|$ such that for any $s \neq t$ in J', there is no directed path of α' from s to t. When $v \notin J$, we can take $J' = J$. Now assume $v \in J$. We shall show our result when v is a source of α. The case where v is a sink of α can be argued similarly. Then (v, v') is a directed edge in α with $v' \notin J$. We claim that there is no directed path in α' joining v' with any element v'' in $J \setminus \{v\}$. There is no directed path in α' from v' to v'' since there is no directed path in α from v to v''. If there is some directed path in α' from v'' to v', then this contradicts the assumption that v' is a source of α'. Thus we can take $J' = (J \setminus \{v\}) \cup \{v'\}$. Our result follows. □

By Lemma 2.1 (iv), we see that the above lemma is amount to asserting that the number $n(w)$ (i.e., $n(\alpha)$) is invariant under the star operations on w.

Denote by $E(W)$ the set of all elements $w \in C_0(W)$ such that $m(sw) < m(w)$ for any $s \in \mathcal{L}(w)$. The following lemma describe the elements of $E(W)$.

Lemma 2.5. *Let $w \in C_0(W)$.*

(1) $w \in E(W)$ if and only if there is no expression $w = x \cdot w_J \cdot y$ with $|J| = m(w)$, $x, y \in W$ and $\ell(x) > 0$.

(2) $|\mathcal{L}(w)| = m(w)$ for $w \in E(W)$, that is, any $w \in E(W)$ has the form $w = w_I \cdot y$ with $|I| = m(w)$ and $y \in W$.

Proof. We have an expression $w = x \cdot w_J \cdot y$ with $|J| = m(w)$ and $x, y \in W$ by the definition of $m(w)$. If $w \in E(W)$, then we must have $\ell(x) = 0$. On the other hand, if $w \notin E(W)$, then there exists some $s \in \mathcal{L}(w)$ with $m(sw) = m(w)$. Hence we have an expression $sw = x' \cdot w_I \cdot y'$ for some $I \subset S$ and $x', y' \in W$ with $|I| = m(sw) = m(w)$. So $w = s \cdot sw = sx' \cdot w_I \cdot y'$ and $\ell(sx') = \ell(s) + \ell(x') > 0$. This proves (1). Then (2) is an immediate consequence of (1). □

§3. A relation between $C_0(W)$ and $C_1(W)$.

In this section, assume that the Coxeter system (W, S, Γ) is of finite, affine or hyperbolic type and that Γ is connected. Let $C_1(W)$ be the set consisting of all the elements of $C_0(W)$ of the form w_J, $J \subseteq S$. We shall show a relation between the sets $C_0(W)$ and $C_1(W)$. The main result of the section is Theorem 3.5.

Lemma 3.1. *Let α be an acyclic digraph containing at most one branch node which has degree three whenever it occurs. Assume that there is a node s of α which is not extreme. Then there is some node v of α satisfying one of the following conditions.*

(1) v is a source of α whose removal from α yields a new source.

(2) v is a sink of α whose removal from α yields a new sink.

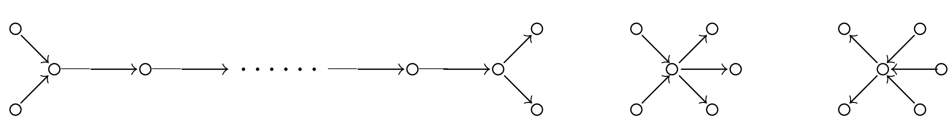
Proof. By Lemma 2.1 (ii), there exists a maximal directed path $\xi : v_0, v_1, ..., v_r$ of α passing through s. Clearly, v_0 (resp. v_r) is a source (resp. a sink) of α and $r \geqslant 2$. If either $r > 2$ or that ξ contains no branch node of α, then at least one of v_1, v_{r-1} is not a branch node of α. If v_1 (or v_{r-1}) is not a branch node of α then v_1 (or v_{r-1}) will be a new source (or sink) after the removal of v_0 (or v_r) from α. Next assume $r = 2$ and that $v_1 = s$ is a branch node of α. Let t be the node other than v_0, v_2 which is adjacent to s. Then there are two possibilities for the directed edge incident to t, s:

(a) (t, s); (b) (s, t).

In case (a), s becomes a new sink after the removal of v_2 from α. In case (b), s becomes a new source after the removal of v_0 from α. So our result follows. □

Lemma 3.2. *Assume that there are either more than one branch nodes, or one branch node with degree greater than 3 in Γ for a Coxeter system (W, S, Γ). Let $w \in C_0(W)$ be with α the associated digraph. Then by applying star operations, we can transform w either to an element of $C_0(W)$ the nodes of whose associated digraph are all extreme, or to an element associated with one of the following digraphs.*

(3.2.1)



(a) (b) (c)

where the number of nodes in (a) is greater than 4.

Proof. We may assume that there is a nonextreme node in α since otherwise we have nothing to do. Take all the possible maximal directed paths $\xi : v_0, v_1, ..., v_r$ in α with $r \geqslant 1$. We can remove v_0 (resp. v_r) to yield a new source (resp. sink) of the resulting digraph if v_0 (resp. v_r) is the unique node t of α with the directed edge (t, v_1) (resp. (v_{r-1}, t)). By applying induction on the number of nodes of α and by a case-by-case checking (see the list of graphs in Appendix), we see that after a certain sequence of such removals of nodes from α (or equivalently by a certain sequence of star operations on w), the resulting digraph will either have the node set consisting of extreme nodes or be one of those in (3.2.1). This proves our result. $\square$

Lemmas 3.1 and 3.2 cover all the cases where (W, S, Γ) is of finite, affine or hyperbolic type. Thus we can always apply star operations to transform $w \in C_0(W)$ either to an element all the nodes of whose associated digraph are extreme, or to an element whose associated digraph is one of those in (3.2.1).

Next we consider the case where the nodes of the digraph α associated to $w \in C_0(W)$ are all extreme.

Lemma 3.3. *Given $w \in C(W)$ with α the associated digraph. Assume that all the nodes of α are extreme.*

(i) $w = w_I w_J$ for some disjoint $I, J \subset S$ with $S = I \cup J$.

(ii) Assume $|I| \geqslant |J|$ (resp. $|I| \leqslant |J|$) in (i). Then by applying right (resp. left) star operations, we can transform w to the element w_I (resp. w_J) unless Γ is one of the following graphs: (A.5), (A.6) (with the number of nodes even) *and (A.9)(b).*

Proof. (i) Take I to be the set of all sources in α and let $J = S \setminus I$. Then I and J satisfy the required condition.

(ii) Note that we assume Γ connected. By (1), we can write $w = w_I w_J$ for some disjoint $I, J \subset S$ with $I \cup J = S$. Note that elements in I and in J occur alternately as nodes in the graph Γ. We need only show our result in the case of $|I| \geqslant |J| = m$. Then the other case can be argued similarly. By a case-by-case checking (see the list of graphs in Appendix), we see that when Γ is not one of the three excepted graphs, the elements of J can always be arranged into a sequence $\xi : v_1, v_2, ..., v_m$ satisfying:

(3.3.1) For any i, $1 \leqslant i \leqslant m$, there is some $r_i \in I$ such that r_i is adjacent to v_i but not to any v_k, $k > i$.

Now we define a sequence of elements $x_0 = w$, x_1, ..., x_m such that $x_i = x_{i-1}v_i$ for $1 \leqslant i \leqslant m$. Then we have $v_i \in \mathcal{R}(x_{i-1}) \setminus \mathcal{R}(x_i)$, $r_i \in \mathcal{R}(x_i) \setminus \mathcal{R}(x_{i-1})$ and so $\mathcal{R}(x_i) \underset{\not\subseteq}{\overset{\not\supseteq}{}} \mathcal{R}(x_{i-1})$ for $1 \leqslant i \leqslant m$. We also have $x_m = w_I$. This implies that w can be transformed to w_I by right star operations. $\square$

The following result is concerned with the exceptional cases of Lemmas 3.2 and 3.3. We shall use the concept of a primitive pair in the proof. A *left* (resp. *right*) *primitive pair* consists of two elements in a Coxeter group W which are in the same left (resp. right) cell of W. A left or right primitive pair is also called a *primitive pair*. For the precise definition of a primitive pair, we refer the reader to [16, 3.5].

Lemma 3.4. *(i) Let Γ be as in (A.5) with $l = 2m \geqslant 4$. The subscripts of elements of $S = \{s_i \mid 1 \leqslant i \leqslant l\}$ are compatible with the labelings of nodes of Γ* (the same holds for (ii)-(iv)). *Let $w = w_I w_J$ be with $I = \{s_1, s_3, ..., s_{2m-1}\}$ and $J = \{s_2, s_4, ..., s_{2m}\}$. Then $w_I \underset{\mathrm{R}}{\sim} w \underset{\mathrm{L}}{\sim} w_J$. So $a(w) = m$ when W is of type $\widetilde{A}_{l-1}$.*

(ii) Let Γ be as in (A.6). Let $w = s_0 s_1 s_2 \cdots s_{l-1} s_l$. In the case of $l = 2m-1 \geqslant 5$, let $y = w_I w_J$, where $I = \{s_0, s_1, s_3, s_5, ..., s_{2m-3}\}$ and $J = \{s_2, s_4, s_6, ..., s_{2m-2}, s_{2m-1}\}$. Then $w \underset{\mathrm{L}}{\sim} s_{l-1}s_l$ and $y \underset{\mathrm{LR}}{\sim} w_K$ with $K = \{s_0, s_1, s_4, s_6, ..., s_{2m-2}, s_{2m-1}\}$. In particular, this implies $a(w) = 2$ and $a(y) = m+1$.

(iii) Let Γ be as in (A.9) (c). Let $w = s_1 s_5 s_0 s_2 s_3 s_4$ (resp. $w = s_2 s_3 s_4 s_0 s_1 s_5$). Then $w \underset{\mathrm{LR}}{\sim} w_{234}$, where $w_{234} := w_J$ with $J = \{s_2, s_3, s_4\}$.

(iv) Let Γ be as in (A.9) (b). Let $w = s_2 s_4 s_1 s_3 s_5$ (resp. $w = s_1 s_3 s_5 s_2 s_4$). Then $w \underset{\mathrm{LR}}{\sim} w_{135}$.

Proof. (i) $\{s_1 w, w\}$ is a left primitive pair in W. $s_1 w$ can be transformed to w_J by left star operations. So $w \underset{\mathrm{L}}{\sim} w_J$ and hence $a(w) = a(w_J) = |J| = m$ by 1.2 (b) and 1.3 (c),(d). Similarly we can show $w \underset{\mathrm{R}}{\sim} w_I$.

(ii) $\{s_0 w, w\}$ is a left primitive pair in W. $s_0 w$ can be transformed to $s_{l-1}s_l$ by left star operations. So $w \underset{\mathrm{L}}{\sim} s_{l-1}s_l$ and $a(w) = 2$. On the other hand, $s_0 s_1 w_J$ can be obtained from

y by left star operations. Let $K = (J \setminus \{s_2\}) \bigcup \{s_0, s_1\}$, which consists of $m+1$ mutually commuting elements in S. Then $w_K = s_0 s_1 w_J s_2$ can be obtained from $s_0 s_1 w_J$ by a right $\{s_0, s_2\}$- star operation. This implies $y \underset{\mathrm{LR}}{\sim} w_K$ and hence $a(y) = m+1$ by 1.2 (b) and 1.3 (c),(d).

(iii) Let $w = s_1 s_5 s_0 s_2 s_3 s_4$, $x = s_1 w$ and $y = w_{234}$. Then $\{w, x\}$ is a left primitive pair in W, $x = s_5 s_0 y$ and $\mathcal{L}(x) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{L}(s_0 y) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{L}(y)$. This implies $w \underset{\mathrm{L}}{\sim} x \underset{\mathrm{L}}{\sim} w_{234}$ and hence $w \underset{\mathrm{LR}}{\sim} w_{234}$ by 1.2 (b). The case of $w = s_2 s_3 s_4 s_0 s_1 s_5$ can be argued similarly.

(iv) Let $y = w_{135}$. Let $x = s_2 w$ (resp. $x = w s_2$) for $w = s_2 s_4 s_1 s_3 s_5$ (resp. $w = s_1 s_3 s_5 s_2 s_4$). Then the argument is similar to that in (iii). $\square$

Theorem 3.5. *Let $w \in C_0(W)$ be with α the associated digraph. Then $w \underset{\mathrm{LR}}{\sim} w_I$ for some $I \subseteq S$ with I consisting of $m(w)$ mutually commuting elements. In particular, when W is of finite or affine type, we have $a(w) = m(w)$.*

Proof. By Lemmas 3.1, 3.2 and 3.3, we see that any element $w \in C_0(W)$ can be transformed by star operations to an element of the form w_I ($I \subseteq S$) unless the graph Γ contains one of those in (A.5), (A.6), (A.9)(b),(c) as its subgraph. In these exceptional cases, we can again transform w to an element of the form w_I ($I \subseteq S$) via star operations, together with passing from one element to another in a certain primitive pair. We know that if $y, w \in C_0(W)$ are obtained from one to another by a star operation then $w \underset{\mathrm{LR}}{\sim} y$ and $m(w) = m(y)$ by Corollary 2.3 and Lemma 2.4. We also see that if $y, w \in C_0(W)$ form a primitive pair occurring as in the proof of Lemma 3.4 then we again have $w \underset{\mathrm{LR}}{\sim} y$ and $m(w) = m(y)$. Finally, it is obvious that $m(w_I) = |I|$ for any $I \subseteq S$ consisting of mutually commuting elements. This proves the first assertion of the theorem. Then the last assertion follows by the first one and 1.3 (c). $\square$

Remark 3.6. The last conclusion of Theorem 3.5 also holds for any Coxeter system of hyperbolic type which satisfies the conditions 1.3 (a),(b).

§4. Left cells containing some $w \in E(W)$.

In this section, assume that (W, S, Γ) is of finite or affine type and that Γ is connected. Let $w \in C_0(W)$ be with α the associated digraph.

Recall the definition of $\mathcal{D}_0(W)$, $E(W)$ given in 1.3 (f) and in Section 2 (preceding Lemma 2.5). The main results of the section are Theorems 4.3, 4.7 and 4.8.

We know from Lemma 2.5 that any $w \in E(W)$ has the form $w = w_I \cdot y$ for some $I \subseteq S$ with $|I| = m(w)$. The first two results of the section, i.e., Lemmas 4.1 and 4.2, are concerned with the relations among the sets I, $\mathcal{L}(y)$ and $\mathcal{R}(y)$. Lemma 4.1 is also valid for any Coxeter group of finite, affine or hyperbolic type.

Lemma 4.1. *Let $w = w_I \cdot y \in E(W)$ be with $m(w) = |I|$. Then for any $t \in \mathcal{L}(y)$, there are at least two nodes $s \neq r$ in I adjacent to t in Γ.*

Proof. It is clear that for any element in $\mathcal{L}(y)$, there must be some element in I adjacent to it by the condition $m(w) = |I|$. Suppose that the lemma fails to hold for some $t \in \mathcal{L}(y)$. Then there is only one element (say s) in I adjacent to t. Let $K = (I \setminus \{s\}) \cup \{t\}$. Then elements in K are pairwise commutative and so $sw = w_K \cdot ty$. But this implies $m(sw) \geqslant |K| = |I|$, contradicting the assumption of $w \in E(W)$. $\square$

Lemma 4.2. *Let $w = w_I \cdot y$ be in $E(W)$ with $|I| = m(w)$ and $\ell(y) > 0$.*

(1) There exists at least one element s in I which is adjacent to exactly one element of $\mathcal{L}(y)$.

(2) There exists some $s \in \mathcal{R}(y)$ whose removal from α yields a new sink.

Proof. Let I' be the subset of I consisting of all the elements s adjacent to some element of $\mathcal{L}(y)$. Then $I' \neq \emptyset$ by Lemma 4.1 and by the assumption of $\ell(y) > 0$. If there are at least two distinct elements of $\mathcal{L}(y)$ adjacent to s for any $s \in I'$, then by Lemma 4.1, there exists a circle in $I' \cup \mathcal{L}(y)$. So W must have type $\widetilde{A}_n$ for some $n > 1$ and we must have $w \in C(W)$. But then we have $S \subseteq I' \cup \mathcal{L}(y)$. Since $I \cap \mathcal{L}(y) = \emptyset$ and since any of I and $\mathcal{L}(y)$ consists of mutually commuting elements, this implies $I = I'$ and $|I| = |\mathcal{L}(y)|$, which contradicts the assumption of $w \in E(W)$. So (1) is proved.

By (1), we can take some $s \in I$ adjacent to exactly one element, say t, of $\mathcal{L}(y)$. Consider a longest directed path of α passing through s, t: $r_0 = s, r_1 = t, r_2, ..., r_k$ (note that here we use a stronger adjective " longest " rather than " maximal "). Then $r_k \in \mathcal{R}(y)$. If either $k = 1$, or that r_{k-1} is not a branch node of α, then the removal of r_k yields a new sink

r_{k-1}. Now assume $k > 1$ and that r_{k-1} is a branch node of α. We claim that there is no node r'_k, $r'_k \neq r_k$, of α with a directed edge (r_{k-1}, r'_k). Suppose that such a node r'_k does exist. We see that $r_1 \in \mathcal{L}(y)$ must be a branch node by Lemma 4.1. If $k = 2$, i.e., $r_1 = r_{k-1}$, then W has type $\widetilde{D}_4$ and $w = s_0s_1s_2s_3s_4$ with s_2 the branch node. But this contradicts the assumption of $w \in E(W)$. If $k > 2$, then W has type $\widetilde{D}_l$ for some $l > 4$, and we have $w = s_0s_1s_2\cdots s_{l-2}s_{l-1}s_l$ (see (A.6) for the factors s_i). But this again contradicts the assumption of $w \in E(W)$. Now according to the claim, we see that the removal of r_k yields a new sink r_{k-1}. So (2) is proved. $\square$

Next result describes all the distinguished involutions d of W with $d \underset{\mathrm{L}}{\sim} w$ for some $w \in E(W)$.

Theorem 4.3. *Assume that $w = w_I \cdot y$ is in $E(W)$ with $m(w) = |I|$. Then $d = y^{-1} \cdot w_I \cdot y$ is the distinguished involution of W with $d \underset{\mathrm{L}}{\sim} w$.*

Proof. It is enough to show our result in the case of $w \in C(W) \cap E(W)$ by Lemma 1.8.

By Lemma 4.2, we see that there is a reduced expression $y = t_1t_2\cdots t_k$ with $t_i \in S$ such that, let $y_j = t_1\cdots t_j$ for $0 \leqslant j \leqslant k$ (with the convention that $y_0 = 1$), we have

$$\mathcal{R}(w_Iy_j) \overset{\not\supseteq}{\underset{\not\subseteq}{}} \mathcal{R}(w_Iy_{j+1}) \quad \text{for } 0 \leqslant j < k. \tag{4.3.1}$$

We have $w_Iy_j \in E(W)$ with $m(w_Iy_j) = |I|$ for $0 \leqslant j \leqslant k$. Since $w_Iy_j \in C_0(W)$, the condition (4.3.1) is amount to that for $1 \leqslant j \leqslant k$, t_j is adjacent to some $s_j \in \mathcal{R}(w_Iy_{j-1})$ in Γ.

Now we show our result by applying induction on $k \geqslant 0$. It is obvious in the case of $k = 0$. Now assume $k \geqslant 1$. By (4.3.1), there is some $s_k \in \mathcal{R}(w_Iy_{k-1})$ adjacent to t_k in Γ. By inductive hypothesis, the element $d_{k-1} = y_{k-1}^{-1} \cdot w_I \cdot y_{k-1}$ is the distinguished involution of W with $d_{k-1} \underset{\mathrm{L}}{\sim} w_Iy_{k-1}$. To show that $d_k = y_k^{-1}w_Iy_k = t_kd_{k-1}t_k$ is the distinguished involution of W with $d_k \underset{\mathrm{L}}{\sim} w = w_Iy_k$, it is enough to show that $d_{k-1}t_k$ is the first term of the left $\{s_k, t_k\}$-string containing it by [13, Proposition 5.12]. By Lemma 4.1, we see that there are at least two factors in S (not necessarily distinct), one being s_k, in the reduced expression $d_{k-1} = t_{k-1}\cdots t_1 \cdot w_I \cdot t_1\cdots t_{k-1}$ which are adjacent to t_k in Γ.

Since $t_k \not\leqslant d_{k-1}$, $s_k \in \mathcal{R}(d_{k-1}) = \mathcal{L}(d_{k-1})$ and $s_k t_k \neq t_k s_k$, we have $s_k \in \mathcal{L}(d_{k-1}t_k)$ and $t_k \notin \mathcal{L}(s_k d_{k-1} t_k) \cup \mathcal{L}(d_{k-1}t_k)$. That is, $d_{k-1}t_k$ is the first term of the left $\{s_k, t_k\}$-string containing it. Clearly, $d_k = t_k \cdot d_{k-1} \cdot t_k = t_k \cdot y_{k-1}^{-1} \cdot w_I \cdot y_{k-1} \cdot t_k = y_k^{-1} \cdot w_I \cdot y_k$. This proves our result. $\square$

Let us record a result for later use, which is a consequence of a result of Lusztig (see [6, Corollary 5.5]).

Lemma 4.4. *Suppose that $w \in W$ and $s \in \mathcal{L}(w)$ satisfy $\mathcal{R}(sw) \subsetneq \mathcal{R}(w)$. Then $w \underset{L}{<} sw$.*

By a subexpression ζ of an expression $s_1 s_2 \cdots s_r$ with $s_i \in S$, we mean that ζ has the form $s_{i_1} s_{i_2} \cdots s_{i_t}$ for some subsequence $i_1, i_2, ..., i_t$ of $1, 2, ..., r$.

In the expression $C_x C_y = \sum_z h_{x,y,z} C_z$ with $h_{x,y,z} \in A$, we see from (1.1.1) that if $h_{x,y,z} \neq 0$ then z has a reduced expression which is a subexpression of $t_1 t_2 \cdots t_a t'_1 t'_2 \cdots t'_b$, where $t_j, t'_k \in S$ and, $x = t_1 t_2 \cdots t_a$, $y = t'_1 t'_2 \cdots t'_b$ are reduced expressions of x, y, respectively.

Lemma 4.5. *Let $w = w_I \cdot y \in E(W)$ be with $m(w) = |I|$ and let $z \in W$ satisfy $z \underset{L}{\sim} w$. Then $w \leqslant z$.*

Proof. Let $z = s_1 s_2 \cdots s_r$ be a reduced expression with $s_i \in S$. Then by (1.1.1) and 1.3 (b), we have

$$C_{s_r} C_{s_{r-1}} \cdots C_{s_1} C_{s_1} \cdots C_{s_{r-1}} C_{s_r} = C_{z^{-1}} C_z + \sum_{v \in W} a_v C_v.$$

for some $a_v \in A_+$, where A_+ is the set of the Laurent polynomials in u with nonnegative integer coefficients. Since $z \underset{L}{\sim} w$, we have

$$h_{z^{-1}, z, y^{-1} \cdot w_I \cdot y} \neq 0$$

in the expression $C_{z^{-1}} C_z = \sum_x h_{z^{-1},z,x} C_x$ by 1.3 (h) and Theorem 4.3. This implies that $y^{-1} \cdot w_I \cdot y$ has a reduced expression which is a subexpression of the expression $s_r s_{r-1} \cdots s_2 s_1 s_1 s_2 \cdots s_{r-1} s_r$. Let α be the associated digraph of w. By Lemma 4.1, there are at least two distinct elements of I adjacent to s in α for any $s \in \mathcal{L}(y)$. So $y^{-1} \cdot w_I \cdot y$ has no reduced expression of the form $z_1 \cdot sts \cdot z_2$ with $s, t \in S$, $st \neq ts$, and $z_1, z_2 \in W$. This implies that any two reduced expressions of $y^{-1} \cdot w_I \cdot y$ can be transformed from one to another by applying the relations of the form $st = ts$ for various commuting pairs s, t in S. So for any maximal directed path $t_1, ..., t_v$ in α, there exists a subexpression $t_v t_{v-1} \cdots t_2 t_1 t_2 \cdots t_{v-1} t_v$ in any reduced expression of $y^{-1} \cdot w_I \cdot y$ (note that $t_i \in I$ if and only if $i = 1$). This implies that $t_v t_{v-1} \cdots t_2 t_1 t_2 \cdots t_{v-1} t_v$ is also a subexpression of $s_r s_{r-1} \cdots s_2 s_1 s_1 s_2 \cdots s_{r-1} s_r$. Hence $t_1 t_2 \cdots t_v$ is a subexpression of $s_1 s_2 \cdots s_r$. So $w \leqslant z$. $\square$

The assertion of Lemma 4.5 will be further strenghtened in Theorem 4.7. Let us first show the following

Lemma 4.6. *Let $w \in E(W)$. Then the following statements are equivalent:*

(1) Any $z \in W$ with $z \underset{L}{\sim} w$ has an expression of the form $z = x \cdot w$ for some $x \in W$.

(2) If $z \in W$ satisfies $z \underset{L}{\sim} w$ and

$$z \underset{L}{<} sz \qquad \text{for any } s \in \mathcal{L}(z), \tag{4.6.1}$$

then $z = w$.

Proof. (1) $\Longrightarrow$ (2): Assume that $z \in W$ satisfies $z \underset{L}{\sim} w$ and (4.6.1). By (1), we can write $z = x \cdot w$ for some $x \in W$. If $\ell(x) > 0$, take $s \in \mathcal{L}(x)$, then $s \in \mathcal{L}(z)$ and $z \underset{L}{\leqslant} sz = sx \cdot w \underset{L}{\leqslant} w \underset{L}{\sim} z$ by 1.2 (a). This implies $sz \underset{L}{\sim} z$, contradicting the condition (4.6.1). So we must have $\ell(x) = 0$, i.e., $z = w$.

(2) $\Longrightarrow$ (1): Let $z \in W$ be with $z \underset{L}{\sim} w$. If z satisfies (4.6.1), then $z = w = 1 \cdot w$, the result is true. Now assume that z does not satisfy (4.6.1). Then there exists some $s_1 \in \mathcal{L}(z)$ with $z_1 := s_1 z \underset{L}{\sim} z$. If z_1 still does not satisfy (4.6.1), then we can find some $s_2 \in \mathcal{L}(z_1)$ with $z_2 := s_2 z_1 \underset{L}{\sim} z_1$. In this way, we find a sequence of elements $z_1 > z_2 > ...$, with $z_i \underset{L}{\sim} w$ for all $i \geqslant 1$. Clearly, such a sequence must be finite and hence we eventually find an element z_r, $z_r \underset{L}{\sim} w$, satisfying (4.6.1). This implies $z_r = w$ by (2). Let $x = s_1 s_2 \cdots s_r$. Then $z = x \cdot w$, as required. $\square$

Theorem 4.7. *Let $w \in E(W)$. If $z \in W$ satisfies the conditions $z \underset{L}{\sim} w$ and (4.6.1), then $z = w$.*

Proof. We can write $w = w_I \cdot y$ with $m(w) = |I|$ by Lemma 4.2. Applying induction on $\ell(y) \geqslant 0$. When $\ell(y) = 0$, we have $w = w_I$. The condition $z \underset{\mathrm{L}}{\sim} w$ implies that $\mathcal{R}(z) = \mathcal{R}(w) = I$ and hence $z = x \cdot w_I$ for some $x \in W$. Then the condition (4.6.1) on z further implies $z = w_I$ by Lemma 4.6. Next assume $\ell(y) > 0$. By Lemma 4.2, there exists a reduced expression $y = t_1 t_2 \cdots t_r$ with $t_i \in S$ such that $w' := wt_r$ can be obtained from w by a right $\{s, t_r\}$-star operation for some $s \in \mathcal{R}(wt_r)$ with $st_r \neq t_r s$. Clearly, $w' \in E(W)$. At least one (say z') of zt_r and zs is obtained from z by a right $\{s, t_r\}$-star operation and satisfies $z' \underset{\mathrm{L}}{\sim} w'$. If $z' = zt_r$, then z' also satisfies (4.6.1) with z' in the place of z. By inductive hypothesis, we have $z' = w'$ and hence $z = z' t_r = w' t_r = w$, the result is proved.

Now assume $z' = zs$. We claim that z' does not satisfy (4.6.1) with z' in the place of z. For otherwise, one would have $z' = w' = w_I \cdot t_1 \cdots t_{r-1}$ by inductive hypothesis and hence $z = z's = w_I \cdot t_1 \cdots t_{r-1} s$, which contradicts the fact $w \leqslant z$ by Lemma 4.5 since $t_r \leqslant w$ and $t_r \not\leqslant z$. So there exists some $t \in \mathcal{L}(z')$ satisfying $tz' \underset{\mathrm{L}}{\sim} z'$. Since $t_r \in \mathcal{R}(z)$ and $z' = z \cdot s$, there is a reduced expression

$$z' = s_1' \cdots s_a' t_r s \qquad \text{with } s_i' \in S. \tag{4.7.1}$$

We claim

$$tz' = s_1' \cdots s_a' s. \tag{4.7.2}$$

For otherwise, we would have either $tz' = s_1' \cdots s_a' t_r$ or $tz' = s_1' \cdots \widehat{s_i'} \cdots s_a' t_r s$ by the exchanging condition. When $tz' = s_1' \cdots s_a' t_r$, we have $z' \underset{\mathrm{L}}{<} tz'$ by Lemma 4.4, a contradiction. Also, when $tz' = s_1' \cdots \widehat{s_i'} \cdots s_a' t_r s$, the element tz can be obtained from tz' by a right $\{s, t_r\}$-star operation, and hence $tz \underset{\mathrm{R}}{\sim} tz' \underset{\mathrm{L}}{\sim} z' \underset{\mathrm{R}}{\sim} z$. This implies $tz \underset{\mathrm{LR}}{\sim} z$ and hence $tz \underset{\mathrm{L}}{\sim} z$ by 1.3 (e), contradicting the condition (4.6.1) on z since $t \in \mathcal{L}(z)$.

By the facts $tz' \underset{\mathrm{L}}{\sim} w'$ and $w' \in E(W)$, we have an expression $tz' = x \cdot w_I \cdot t_1 \cdots t_{r-1}$ for some $x \in W$ by inductive hypothesis and by Lemma 4.6. Then $z' = t \cdot x \cdot w_I \cdot t_1 \cdots t_{r-1}$ and hence $z = (t \cdot x \cdot w_I \cdot t_1 \cdots t_{r-1})s$. Since z satisfies (4.6.1), we must have $z = t \cdot x \cdot w_{I'} \cdot t_1 \cdots t_{r-1}$ for some $I' \subset I$ with $|I'| = |I| - 1$ by the exchanging condition and by the fact $s \in$

$\mathcal{R}(w_I t_1 \cdots t_{r-1})$. So by (4.7.1) and (4.7.2), we get $z = x \cdot w_{I'} \cdot y$. Since $w_{I'} t_1 \cdots t_{r-1} s$ and $w_I t_1 \cdots t_{r-1}$ are two reduced expressions of some element in $C_0(W)$, this implies $I \setminus I' = \{s\}$.

So far we have shown that if w can be transformed to $w' = wt_r$ by a right $\{s, t_r\}$-star operation with $t_r \in \mathcal{R}(y)$, then either $z = w$ (if $z' = zt_r$) or $z = x \cdot w_{I'} \cdot y$ with $I' = I \setminus \{s\}$ (if $z' = zs$) for some $x \in W$. When $z = x \cdot w_{I'} \cdot y$ (i.e., when $z' = zs$), the element s is not adjacent to any $v \in S$ with $v \leqslant w$ and $v \neq t_r$ by the fact $w_{I'} t_1 \cdots t_{r-1} s = w_I t_1 \cdots t_{r-1}$.

Suppose that we are in the case of $z = x \cdot w_{I'} \cdot y$ (i.e., when $z' = zs$) and that the element w can also be transformed to wt'_r by a right $\{s', t'_r\}$-star operation with $t'_r \in \mathcal{R}(y)$ and $\{s', t'_r\} \neq \{s, t_r\}$. Then $s \neq s'$. By the same argument as above, we can show that either $z = w$ or $z = x' \cdot w_{I''} \cdot y$ with $I'' = I \setminus \{s'\}$ for some $x' \in W$. We claim that the case of $z = x \cdot w_{I'} \cdot y = x' \cdot w_{I''} \cdot y$ never happens. For otherwise, we would have $x \cdot s' = x' \cdot s$ and hence $s \in \mathcal{R}(x)$, contradicting the fact $xw_I = x \cdot w_I$ (see an expression of z'). So we must have $z = w$, which shows the result.

Thus to show our result, it suffices to show that in the case of $z = x \cdot w_{I'} \cdot y$ (i.e., when $z' = zs$), the element w can also be transformed to wt'_r by a right $\{s', t'_r\}$-star operation for some $s' \in S$, $t'_r \in \mathcal{R}(y)$ with $\{s', t'_r\} \neq \{s, t_r\}$.

If there is a nonextreme node in the associated digraph α of w, then we can find a maximal directed path $s_0, s_1, ..., s_b$ in α with $b \geqslant 2$. Hence by Lemma 4.1, s_1 must be a branch node of α such that (s'_0, s_1) is a directed edge of α for some $s'_0 \in I$ with $s'_0 \neq s_0$. Since W is of finite or affine type with Γ connected, and since s_1 is not a node of the directed edge (s, t_r), we have $d_\Gamma(s_1) = 3$ and that w can be transformed to ws_b by a right $\{s_b, s_{b-1}\}$-star operation. Clearly, $\{s_b, s_{b-1}\} \neq \{s, t_r\}$ and $s_b \in \mathcal{R}(y)$.

Now suppose that all the nodes of α are extreme. Since $w \in E(W)$ and since s is a terminus of the underlying graph Γ' of α, there must exist a terminus s', $s' \neq s, t_r$, of Γ' which is not isolated by the facts that Γ' is a subgraph of a connected Coxeter graph of affine type and that Γ' is not a cycle. Let s'' be the node of α adjacent to s'. We claim that s'' is a sink of α. For otherwise, $s''w$ could be obtained from w by a left $\{s', s''\}$-star operation and hence $s''w \underset{\mathrm{L}}{\sim} w$ by 1.2 (b), contradicting the assumption of $w \in E(W)$. Then w can be transformed to ws'' by a right $\{s', s''\}$-star operation.

Therefore our proof is completed. □

Note that in the definition of left-connectedness for a subset K of W (see Introduction), it is equally well if we fix an element $x \in K$ and let another element y run through K. Also, note that if a left cell L of W contains an element of $C_0(W)$ then L contains an element of $E(W)$, too. The next result asserts that any left cell of W containing an element of $C_0(W)$ is left-connected, which verifies a conjecture of Lusztig in our case (see [3]).

Theorem 4.8. *Let L be a left cell of W. Then $|L \cap E(W)| \leqslant 1$. L is left-connected in the case of $|L \cap E(W)| = 1$.*

Proof. The first assertion follows by Theorem 4.7. Now assume $\{w\} = L \cap E(W)$. Then any $z \in L$ has the form $z = x \cdot w$ for some $x \in W$ by Theorem 4.7. Take a reduced expression $x = s_1 s_2 \cdots s_r$ with $s_i \in S$. Define a sequence $\xi : x_0, x_1, ..., x_r$ such that $x_0 = z$ and $x_i = s_i x_{i-1}$ for $1 \leqslant i \leqslant r$. Then we have $z = x_0 \underset{\mathrm{L}}{\leqslant} x_1 \underset{\mathrm{L}}{\leqslant} \cdots \underset{\mathrm{L}}{\leqslant} x_r = w \underset{\mathrm{L}}{\sim} z$ by 1.2 (a), i.e., ξ is in L. This implies that L is left-connected. □

Remark 4.9. (1) Theorem 4.7 shows that any $w \in E(W)$ is the unique minimal element in the left cell of W containing w (in the sense of Lemma 4.6 (1)). Let $w = w_I \cdot y \in E(W)$ be with $m(w) = |I|$. Since $\lambda(w^{-1}, w) = y^{-1} \cdot w_I \cdot y$, Theorem 4.3 verifies Conjecture 1.5 in the case where $w \in C_0(W)$.

(2) In [6, 7], Lusztig deduced his results in 1.3 under the assumption of a certain positivity condition on the Hecke algebra $\mathcal{H}(W)$ and of an up-bounded condition on the a-values of W (see [6, Sect. 3.1 and 7, Sect. 1.1 (d)] and also 1.3 (a), (b)). The results in this section are mainly based on these results of Lusztig. Thus one may expect to extend our results to the other Coxeter systems which satisfy these two conditions.

§5. The relation $\underset{\mathrm{LR}}{\sim}$ on the elements in $C_1(W)$.

Let (W, S, Γ) be of finite or affine type. Recall the notation $C_1(W)$ defined preceding Lemma 3.1. For $w_I, w_J \in C_1(W)$, we ask when the relation $w_I \underset{\mathrm{LR}}{\sim} w_J$ holds. Clearly, a necessary condition for $w_I \underset{\mathrm{LR}}{\sim} w_J$ is $|I| = |J|$. Thus it is natural to ask in which circumstance it is also a sufficient condition ? We give a brief answer to the problem.

5.1. Call any of the following cases an exceptional case, where the types are for Γ, and I, J are subsets of S with $w_I, w_J \in C_1(W)$.

(1) $\widetilde{D}_l$, $l \geqslant 4$; (2) D_l, $l \geqslant 4$;

(3) $\widetilde{B}_{l-1}$, $l \geqslant 4$; (4) $\widetilde{E}_7$ and $|I| = |J| \in \{3, 4\}$;

(5) E_7 and $|I| = |J| = 3$; (6) $\widetilde{E}_8$ and $|I| = |J| = 4$.

Then we have

Proposition 5.2. *Let W be of finite or affine type. Then besides the above exceptional cases, for any $w_I, w_J \in C_1(W)$, we have $w_I \underset{\text{LR}}{\sim} w_J$ if and only if $|I| = |J|$.*

Proof. Since W is of finite or affine type, it is enough to show one implication: if $|I| = |J|$ with $w_I, w_j \in C_1(W)$ then $w_I \underset{\text{LR}}{\sim} w_J$. Note that we are assumed not in any of the exceptional cases (1)-(6) of 5.1. So by a case-by-case checking, it is easily seen that the star operations alone are enough to transform w_I to w_J if either Γ is a tree, or Γ is a circle with $|I| < (1/2)|S|$. Hence $w_I \underset{\text{LR}}{\sim} w_J$ in these cases. When Γ is a circle (i.e., of type $\widetilde{A}_l$, $l > 1$) with $|I| = (1/2)|S|$, the assertion $w_I \underset{\text{LR}}{\sim} w_J$ is a consequence of [10, Theorem 17.4]. $\square$

5.3. Now we consider the exceptional cases. Let (W, S, Γ) be of type $\widetilde{D}_l$ $(l \geqslant 4)$ with the nodes of Γ indexed as in (A.6). $I \subset S$ with $w_I \in C_1(W)$ has type $A_1 \times ... \times A_1$ ($|I|$ factors) if $\{s_0, s_1\}, \{s_{l-1}, s_l\} \nsubseteq I$, type $D_2 \times A_1 \times ... \times A_1$ ($|I| - 1$ factors) if exactly one of $\{s_0, s_1\}$, $\{s_{l-1}, s_l\}$ is contained in I, and type $D_2 \times D_2 \times A_1 \times ... \times A_1$ ($|I| - 2$ factors) if $\{s_0, s_1, s_{l-1}, s_l\} \subseteq I$.

Proposition 5.4. *(1) Let (W, S, Γ) be of type $\widetilde{D}_l$ $(l \geqslant 4)$ with Γ as in (A.6). For $w_I, w_J \in C_1(W)$, we have $w_I \underset{\text{LR}}{\sim} w_J$ if and only if I and J have the same type, except for the case where l is even* (say $l = 2k$), *$|I| = |J| = k$, and that both I and J have the type $A_1 \times ... \times A_1$* ($k$ factors). *In this excepted case, we have $w_I \underset{\text{LR}}{\sim} w_J$ if and only if $\{I, J\}$ is either $\{\{s_0, s_3, s_5, ..., s_{l-3}, s_{l-1}\}, \{s_1, s_3, s_5, ..., s_{l-3}, s_l\}\}$ or $\{\{s_0, s_3, s_5, ..., s_{l-3}, s_l\}, \{s_1, s_3, s_5, ..., s_{l-3}, s_{l-1}\}\}$.*

(2) Let (W', S', Γ') be of type D_l $(l \geqslant 4)$ with Γ' obtained from the graph Γ in (1) by removing the node 0 (see (A.2)). *Then for any $w_I, w_J \in C_1(W')$, we have $w_I \underset{\text{LR}}{\sim} w_J$ in W' if and only if they are so in W.*

(3) Let (W'', S'', Γ'') be of type $\widetilde{B}_{l-1}$ $(l \geqslant 4)$ with Γ'' as in (A.2). There is a natural bijection ϕ from S'' to the set S' in (2) which preserves the labelings of the nodes in the respective graphs. Then for any $w_I, w_J \in C_1(W'')$, we have $w_I \underset{\mathrm{LR}}{\sim} w_J$ in W'' if and only if $w_{\phi(I)} \underset{\mathrm{LR}}{\sim} w_{\phi(J)}$ in W'.

Proposition 5.5. *(1) Let (W, S, Γ) be of type $\widetilde{E}_7$ with Γ as in (A.7)(b).*

(1a) For $w_I, w_J \in C_1(W)$ with $|I| = |J| = 3$, we have $w_I \underset{\mathrm{LR}}{\sim} w_J$ if and only if I and J are either both in or both not in the set $\{\{s_2, s_5, s_7\}, \{s_0, s_2, s_3\}\}$.

(1b) For $w_I, w_J \in C_1(W)$ with $|I| = |J| = 4$, we have $w_I \underset{\mathrm{LR}}{\sim} w_J$ if and only if I and J either both contain or both do not contain s_2.

(2) Let (W', S', Γ') be of type E_7 with Γ' obtained from the graph Γ in (1) by removing the node 0 (see (A.3)(b)). *Then for $w_I, w_J \in C_1(W')$ with $|I| = |J| = 3$, we have $w_I \underset{\mathrm{LR}}{\sim} w_J$ in W' if and only if they are so in W.*

Proposition 5.6. *Let (W, S, Γ) be of type $\widetilde{E}_8$ with Γ as in (A.8). Then for $w_I, w_J \in C_1(W)$ with $|I| = |J| = 4$, we have $w_I \underset{\mathrm{LR}}{\sim} w_J$ if and only if I and J either both are or both are not $\{s_0, s_2, s_5, s_7\}$.*

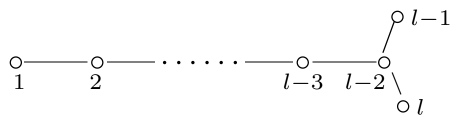
The results in Propositions 5.4-5.6 can be shown by [16, Algorithm 2.7], [9, Theorem 4.8] and some related results in [1, 5, 14]. The details are omited.

Remark 5.7. Let (W, S, Γ) be of hyperbolic type. I don't know if one can always obtain $\ell(w_I) = \ell(w_J)$ from the relation $w_I \underset{\mathrm{LR}}{\sim} w_J$ for $I, J \subset S$? I conjecture that the answer should be affirmative. On the other hand, if $w_I, w_J \in C_1(W)$ satisfy $|I| = |J|$, then we can show $w_I \underset{\mathrm{LR}}{\sim} w_J$ in all the cases except that Γ is the one in (A.2), (A.3) (b), (A.8) or (A.9) (c). It is interesting to give a necessary and sufficient condition for the relation $\underset{\mathrm{LR}}{\sim}$ on $C_1(W)$, in particular, in the excepted cases ?

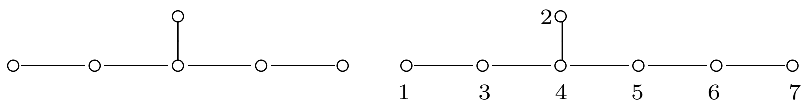
Appendix.

Here we list all the graphs (up to isomorphism) occurring in the finite, affine and hyperbolic cases (see [2, 2.4, 2.5 and 6.9], remember that we neglect the labelings of edges in the original Coxeter graphs). We label the nodes only for those graphs which are cited in the context. Let $n(\Gamma)$ be the number of nodes in Γ.

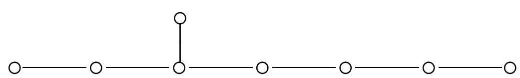
(A.1) $(n(\Gamma) \geqslant 1)$

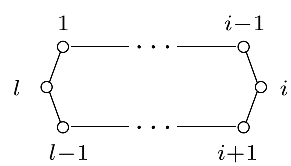
(A.2) 

$(l \geqslant 4)$

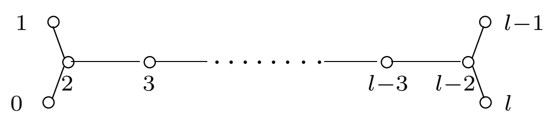
(A.3) 

(a) (b)

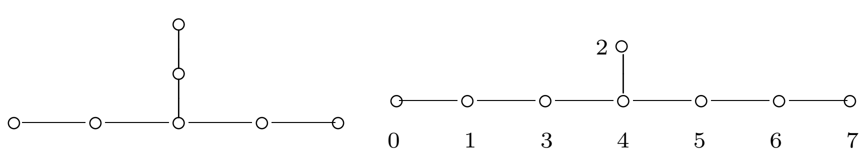
(A.4) 

(A.5) 

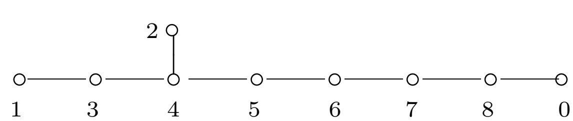
$(l \geqslant 3)$

(A.6) 

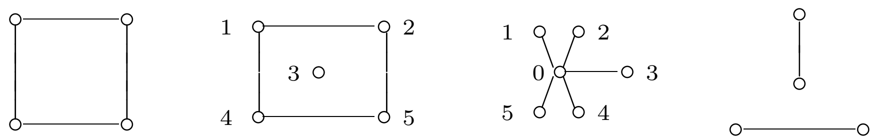
$(l \geqslant 4)$

(A.7) 

(a) (b)

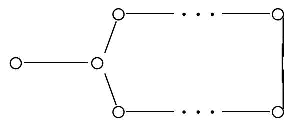
(A.8) 

(A9)



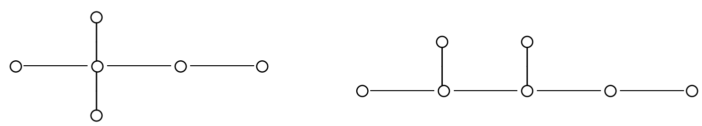
(a) (b) (c) (d)

(A.10)



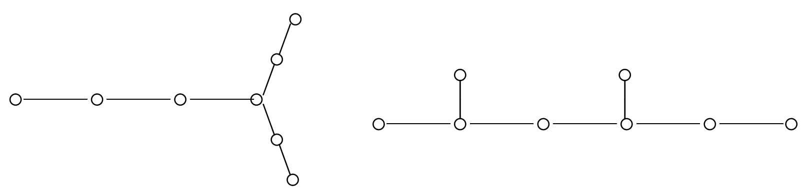
$(4 \leqslant n(\Gamma) \leqslant 9)$

(A.11)



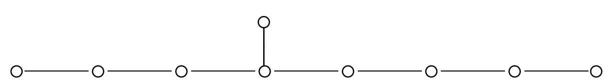
(a) (b)

(A.12)

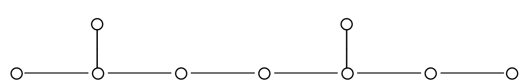


(a) (b)

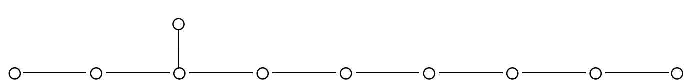
(A.13)



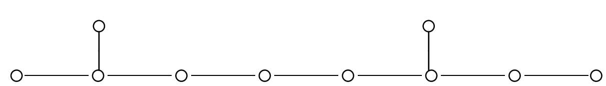
(A.14)



(A.15)



(A.16)



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