THE ENUMERATION OF COXETER ELEMENTS

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Let (W, S, Γ) be a Coxeter system: a Coxeter group W with S its distinguished generator set and Γ its Coxeter graph. In the present paper, we always assume that the cardinality $l = |S|$ of S is finite. A Coxeter element of W is by definition a product of all generators $s \in S$ in any fixed order. We use the notation $C(W)$ to denote the set of all the Coxeter elements in W. These elements play an important role in the theory of Coxeter groups, e.g. the determination of polynomial invariants, the Poincaré polynomial, the Coxeter number and the group order of W (see [1], [2], [3], [4], [5] for example). They are also important in representation theory (see [6]). In the present paper, we show that the set $C(W)$ is in one-to-one correspondence with the set $C(\Gamma)$ of all acyclic orientations of Γ. Then we use some graph-theoretic tricks to compute the cardinality $c(W)$ of the set $C(W)$ for any Coxeter group W. We deduce a recurrence formula for this number. Furthermore, we obtain some direct formulae of $c(W)$ for a large family of Coxeter groups, which include all the finite, affine and hyperbolic Coxeter groups.

The content of the paper is organized as below. In section 1, we discuss some properties of Coxeter elements for simplifying the computation of the value $c(W)$. In particular, we establish a bijection between the sets $C(W)$ and $C(\Gamma)$. Then among the other results, we give a recurrence formula of $c(W)$ in section 2. Subsequently we deduce some closed formulae of $c(W)$ for certain families of Coxeter groups in section 3.

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§1. Some properties of Coxeter elements.

Let (W, S, Γ) be a Coxeter system. We shall first make some reductions for the computation of the number $c(W)$. By abuse of notations, we shall identify an element of S with the corresponding vertex of Γ.

Lemma 1.1. *Let (W, S, Γ) and (W', S', Γ') be two Coxeter systems. Assume that there exists a bijection $\phi : S \longrightarrow S'$ such that $s, t \in S$ are adjacent (i.e. they are joined by an edge) in Γ if and only if $\phi(s)$, $\phi(t)$ are adjacent in Γ'. Then ϕ induces a bijection from the set $C(W)$ to the set $C(W')$ naturally.*

Proof. Notice that two Coxeter elements of a Coxeter group W are equal if and only if any reduced expressions of these Coxeter elements can be transformed from one to another by only using the commutative relations on the set S of W (i.e. the relations of the forms $st = ts$ for $s, t \in S$) successively [2, Ch. IV, §1, Proposition 5 and Exercise 13]. Thus we can define a map $\phi' : C(W) \longrightarrow C(W')$ as follows. Let $w = t_1 t_2 \cdots t_r$ be any reduced expression of an element $w \in C(W)$ with $t_i \in S$. We define $\phi'(w) = \phi(t_1)\phi(t_2)\cdots\phi(t_r)$. By the above remark, we see that the map ϕ' is well-defined and bijective. □

From this result, we see that the number $c(W)$ depends only on the underlying graph $|\Gamma|$, but not on the particular labellings of the edges of the graph Γ, where the graph $|\Gamma|$ is obtained from Γ by forgetting the labellings of all the edges. So from now on, we shall identify a graph Γ with its underlying graph $|\Gamma|$. Notice that a Coxeter graph Γ is always a simple graph, i.e. it contains no loop and no multi-edges. A graph mentioned in the present paper will be assumed simple.

The next result will reduce the problem to the case when the graph Γ is connected.

Lemma 1.2. *Let $\{\Gamma_i\}_{1\le i\le n}$ be the collection of all the connected components of the graph Γ. Let (W_i, S_i) be the standard parabolic subgroups of W corresponding to Γ_i. Then*

$$c(W) = \prod_{i=1}^{n} c(W_i).$$

Proof. Notice that for any $i \neq j$ in $\{1, 2, \cdots, n\}$, the elements of W_i commute with those of W_j. So each Coxeter element w of W can be expressed uniquely in the following form:

$$w = w_1 w_2 \cdots w_n, \qquad w_i \in C(W_i).$$

Conversely, any element of W of the above form is in the set $C(W)$. Hence our result follows easily from these facts. $\square$

1.3. In order to make further reduction, we need to investigate some properties for the reduced expressions of a Coxeter element of W. Given a reduced expression $\xi : s_1s_2\cdots s_l$ of a Coxeter element $w \in W$, we denote $s \underset{\xi}{\longrightarrow} t$ for $s, t \in S$, if the factor s occurs to the left of t in ξ.

Proposition. *Let $\xi : s_1s_2\cdots s_l$, $\zeta : t_1t_2\cdots t_l$ be reduced expressions of Coxeter elements w, y of W respectively with $s_i, t_j \in S$. Then $w = y$ if and only if for any adjacent pair $s, t \in S$ (see Lemma 1.1), the relations $s \underset{\xi}{\longrightarrow} t$ and $s \underset{\zeta}{\longrightarrow} t$ either both hold or both not.*

Proof. ($\Longrightarrow$) We see that applying the commutative relations of S to the expression ξ does not change the relations $s \underset{\xi}{\longrightarrow} t$ for any adjacent pair $s, t \in S$. So the implication in this direction follows from the remark at the beginning of the proof of Lemma 1.1.

($\Longleftarrow$) Apply induction on $l = |S| \geq 1$. It is nothing to prove for the case $l = 1$. Now assume $l > 1$. We have $t_i = s_1$ for some $i \geq 1$. By our condition, we have $t_jt_i = t_it_j$ for any $j < i$. So $\zeta' : t_it_1\cdots\hat{t_i}\cdots t_l$ is again a reduced expression of y, where the notation $\hat{t_i}$ means the deletion of the factor t_i. Now $\xi_0 : s_2\cdots s_l$ and $\zeta_0 : t_1\cdots\hat{t_i}\cdots t_l$ are reduced expressions of s_1w, s_1y respectively, the latter are Coxeter elements of the Coxeter group generated by $S' = S \setminus \{s_1\}$. For any adjacent pair $s, t \in S'$, the relations $s \underset{\xi_0}{\longrightarrow} t$ and $s \underset{\zeta_0}{\longrightarrow} t$ either both hold or both not by our condition. So we have $s_1w = s_1y$ by inductive hypothesis and hence $w = y$. $\square$

By the above proposition, it makes sense to write $s \underset{w}{\longrightarrow} t$ for any adjacent pair $s, t \in S$ if for some reduced expression ξ of w, the relation $s \underset{\xi}{\longrightarrow} t$ holds.

An orientation of a graph G is a directed graph (or digraph for brevity), which is obtained from G by assigning to each edge an orientation. Then we have actually defined an orientation of the Coxeter graph Γ from a Coxeter element w of W.

1.4 For our further discussion, we need some more terminologies and results in graph theory, which we introduce now. We also refer the reader to [8] for more detailed references.

Let G be a digraph. A vertex v of G is called a source (resp. a sink), if for any vertex u of G adjacent to v, we have $v \longrightarrow u$ (resp. $u \longrightarrow v$). A directed path ρ of G is a sequence of vertices $v_0, v_1, \cdots, v_r$ in G such that $v_{h-1} \longrightarrow v_h$ for all h, $1 \leq h \leq r$. The number r is the length of ρ. When $v_0 = v_r$, we also call ρ a directed cycle. A graph is acyclic, if it contains no directed cycle. The following are some simple properties of an acyclic digraph which we shall use later.

Lemma. *Let G be an acyclic digraph with $n \geq 2$ vertices. Then*

(1) G contains at least one source and one sink.

(2) Any vertex of G belongs to some directed path of G which is from a source to a sink.

(3) Any subgraph of G is again acyclic.

Proof. Any vertex of G belongs to some maximal directed path of G which can be shown to start with a source and end with a sink. So (1) and (2) follow. The result (3) is obvious. □

The following result establishes a relation between Coxeter elements and acyclic digraphs which is the key to the subsequent discussion.

Theorem 1.5. *Let (W, S, Γ) be a Coxeter system. Then there exists a bijection between the set $C(W)$ of Coxeter elements of W and the set $C(\Gamma)$ of acyclic orientations of Γ.*

Proof. It is easily seen that an orientation of Γ coming from a Coxeter element of W is always acyclic. For any given acyclic orientation α of Γ, we define a Coxeter element w of W as follows. Let I_1 be the set of all elements in S which are sources in α. Inductively, suppose that we have defined subsets $I_1, \cdots, I_i$ of S and that the set $F_i = S \setminus (\bigcup_{j=1}^{i} I_j)$ is non-empty. Then we define by I_{i+1} the set of all elements in F_i which are sources in the full subgraph of α with F_i its vertex set. By Lemma 1.4, we get a sequence of non-empty disjoint subsets I_j, $1 \leq j \leq p$, of S whose union is S. Note that s, t commute for any $s, t \in I_i$, $1 \leq i \leq p$. We set $w = \prod_{s \in I_1} s \prod_{t \in I_2} t \cdots \prod_{r \in I_p} r$. Then w is the Coxeter element of W whose corresponding acyclic orientation of Γ is just α. Hence our result follows by Proposition 1.3. □

Thus computing the number $c(W) = |C(W)|$ is equivalent to computing the number $c(\Gamma)$ of all the acyclic orientations of Γ.

By abuse of notations, we shall not distinct between a Coxeter element of W (resp. the set $C(W)$) and its corresponding acyclic orientation of Γ (resp. the set $C(\Gamma)$).

It is well known that there is a natural bijection between acyclic digraphs and posets. So Theorem 1.5 also establishes a relation between Coxeter elements and posets. We shall give some detailed discussion on this correspondence elsewhere.

The following corollary is concerned with the condition on a Coxeter element to have neighboring factors $s, t \in S$ in one of its reduced expressions. This result is crucial in the proof of Theorem 2.4.

Corollary 1.6. *Let (W, S, Γ) be a Coxeter system with $l = |S| \geq 2$. For $s \neq t$ in S, an element $w \in C(W)$ has a reduced expression of the form $\cdots st \cdots$ if and only if the orientation α of Γ determined by w satisfies the following condition.*

(A) *There is no directed path from t to s and also no directed path of length greater than 1 from s to t.*

Proof. ($\Longrightarrow$) Obvious.

($\Longleftarrow$) It is nothing to prove if $l = 2$. Now assume $l > 2$. We claim that there exists some source or sink r in α with $r \neq s, t$. For otherwise, the elements s, t would be the unique source and sink of α respectively by Lemma 1.4, (1), (2) and by the first half statement of condition (**A**). Again by Lemma 1.4, (2), any vertex r in α with $r \neq s, t$ (which does exist by the assumption $l > 2$) belongs to some directed path of α from s to t. But this is impossible by the last half statement of condition (**A**). Now assume that there exists a source r in α with $r \neq s, t$. Let $S' = S \setminus \{r\}$ and let α' be the digraph obtained from α by removing the vertex r and all the edges adjacent to r. Then rw is a Coxeter element of the Coxeter group generated by S' whose corresponding digraph is α'. By Lemma 1.4, (3), we see that α' is acyclic, which clearly satisfies condition (**A**). By inductive hypothesis, rw has a reduced expression of the form $\cdots st \cdots$ and so does the element w. The case that r is a sink in α with $r \neq s, t$ can be discussed in the same way but with rw replaced by wr. □

§2. A recurrence formula.

Theorem 1.5 and its corollary make it possible for us to use graph-theoretic methods in the study of Coxeter elements. We shall deduce some formulae to relate the numbers $c(W)$ as W varies over some different Coxeter groups. In particular, we get a recurrence formula of $c(W)$.

2.1. An edge E of a graph G is orient-free, if reversing the orientation of E in any acyclic orientation of G results in another acyclic orientation of G. For a given orient-free edge E of G, the process of reversing the orientation of E gives rise to a fixpoint-free involutive permutation on the set of all orientations of G. An edge of G is a bridge, if it does not belong to any cycle of G. Clearly, an edge of a graph is orient-free if and only if it is a bridge. So we get the following result.

Lemma. *In two Coxeter systems (W, S, Γ) and (W', S', Γ'), if Γ' is obtained from Γ by removing a bridge, then $c(W) = 2c(W')$.*

2.2. Assume that $s \neq t$ in S are not two termini of any path of Γ. Let Γ' be a graph obtained from Γ by fusing s and t into one new vertex z and let (W', S') be a Coxeter system whose Coxeter graph is isomorphic to Γ'. For example, we can define $S' = \{x' \mid x \in S, x \neq s, t\} \cup \{z\}$ such that for any $x, y \in S \setminus \{s, t\}$, $o(x'y') = o(xy)$, and $o(x'z) = \max\{o(xs), o(xt)\}$.

Lemma. *In the above setup, we have $c(W) = c(W')$.*

Proof. Notice that in the above definition of the set S', the orders $o(xs)$ and $o(xt)$ can't be both greater than 2 by our assumption. This implies that there is a natural bijection between the edge sets of Γ and Γ', which induces a bijection between the orientations of Γ and Γ'. Since the vertex z does not lie in any cycle of Γ' by the assumption on s, t, this bijection induces a bijective map from the acyclic orientations of Γ to those of Γ'. So our result follows immediately by Theorem 1.5. □

2.3. For any $s \neq t$ in S, we denote by $\Sigma(s, t)$ or $\Sigma_\Gamma(s, t)$ the set of all the elements $w \in C(W)$ having a reduced expression of the form $\cdots st \cdots$. Then it is easily seen that $\Sigma(s, t) = \Sigma(t, s)$ if and only if s, t are not adjacent. When s, t are adjacent, we have $\Sigma(s, t) \cap \Sigma(t, s) = \emptyset$ and $|\Sigma(s, t)| = |\Sigma(t, s)|$.

2.4. The following is our main result in this section, which can be regarded as a recurrence formula for the number $c(W)$.

Theorem. *Let (W, S, Γ), (W', S', Γ') and (W'', S'', Γ'') be three Coxeter systems. Assume that there are two elements $s, t \in S$ such that*

(1) s, t are adjacent in Γ.

(2) Γ' is isomorphic to the graph obtained from Γ by removing the edge joining s, t.

(3) Γ'' is isomorphic to the graph obtained from Γ by fusing two vertices s and t into a new vertex z.

Then we have the relation $c(W) = c(W') + c(W'')$.

Proof. We write $S' = \{r' \mid r \in S\}$ and $S'' = \{r'' \mid r \in S \setminus \{s, t\}\} \bigcup \{z\}$, where r' (resp. r'') is the vertex of Γ' (resp. Γ'') corresponding to the vertex r of Γ. Let $C_0(W)$ (resp.

$C_0(W'))$ be the complement of $\Sigma_\Gamma(s,t)\bigcup\Sigma_\Gamma(t,s)$ in $C(W)$ (resp. $\Sigma_{\Gamma'}(s',t')$ in $C(W')$).

By Corollary 1.6, we see that $\Sigma_\Gamma(s,t)$ can be regarded as the set of all acyclic orientations of Γ with $s\longrightarrow t$ which contains no directed path of length greater than 1 from s to t. Also, $\Sigma_{\Gamma'}(s',t')$ can be regarded as the set of all acyclic orientations of Γ' containing no directed path with s',t' its two end vertices. On the other hand, $C_0(W)$ (resp. $C_0(W')$) can be regarded as the set of all acyclic orientations of Γ (resp. Γ') containing a directed path of length greater than 1 with s,t (resp. s',t') its two end vertices. Note that in the case of Γ, this directed path determines the orientation of the edge joining s,t. Let η be the map from $C(W)$ to $C(W')$ defined by removing the edge joining s,t from any element of $C(W)$, where we regard $C(W)$ (resp. $C(W')$) as the set of acyclic orientations of Γ (resp. Γ') and regard Γ' as a subgraph of Γ. Then it is clear that η induces bijections from $\Sigma_\Gamma(s,t)$ (resp. $\Sigma_\Gamma(t,s)$) to $\Sigma_{\Gamma'}(s',t')$ and from $C_0(W)$ to $C_0(W')$. Thus by the observation of 2.3, to show our result, it is enough to establish a bijection between the sets $\Sigma_\Gamma(s,t)$ and $C(W'')$.

We observe that if $w\in S\setminus\{s,t\}$ is adjacent to both s and t, then for any acyclic orientation of Γ in $\Sigma_\Gamma(s,t)$, the relations $w\longrightarrow s$ and $w\longrightarrow t$ either both hold or both not.

Now we define a map $\psi:\Sigma_\Gamma(s,t)\longrightarrow C(W'')$. For $\alpha\in\Sigma_\Gamma(s,t)$, we define an orientation $\psi(\alpha)$ of Γ'' as follows. For any adjacent pair $u,v\in S\setminus\{s,t\}$, we set $u''\longrightarrow v''$ if $u\longrightarrow v$. For any $u\in S\setminus\{s,t\}$ adjacent to at least one of s,t (say it is adjacent to s for definity), we set $u''\longrightarrow z$ (resp. $z\longrightarrow u''$) if $u\longrightarrow s$ (resp. $s\longrightarrow u$). By the above remark, we see that $\psi(\alpha)$ is well-defined. We claim that $\psi(\alpha)$ is acyclic. For, if not, then there exists some directed cycle ρ in $\psi(\alpha)$. By the acyclicity of α, we see that ρ must contain z as its vertex. But this would imply that either α contains a directed cycle with at least one of s,t as its vertex, or α contains a directed path of length ≥ 2 from s to t.

Now that we get a map ψ from $\Sigma_\Gamma(s,t)$ to $C(W'')$. To show ψ is bijective, it is enough to find its inversing map. To any acyclic orientation α'' of Γ'', we define an orientation $\lambda(\alpha'')$ of Γ as below. For any adjacent pair $u'',v''\in S''\setminus\{z\}$, we set $u\longrightarrow v$, if $u''\longrightarrow v''$. For any $v''\in S''\setminus\{z\}$ adjacent to z, we set $v\longrightarrow r$ (resp. $r\longrightarrow v$) for any $r\in\{s,t\}$ adjacent to v, if $v''\longrightarrow z$ (resp. $z\longrightarrow v''$). Finally, we set $s\longrightarrow t$. By the acyclicity of α'', it is easily seen that the orientation $\lambda(\alpha'')$ so obtained is acyclic and is in $\Sigma_\Gamma(s,t)$. Therefore we get a map λ from $C(W'')$ to $\Sigma_\Gamma(s,t)$. We see that λ is the inversing map of ψ and hence ψ is bijective. $\square$

For a graph G, we denote by $v(G)$ (resp. $e(G)$) the number of vertices (resp. edges) of G. Then in Theorem 2.4, we have $v(\Gamma) = v(\Gamma') = v(\Gamma'')+1$ and $e(\Gamma) = e(\Gamma')+1 > e(\Gamma'')$. Thus Theorem 2.4 actually provides us a recurrence formula for the number $c(W)$, where the recurrent step is taken on the sum $v(\Gamma) + e(\Gamma) \geq 1$.

§3. Some closed formulae.

Keep all the notations given before. In particular, let (W, S, Γ) be a Coxeter system. In this section, we shall use the results of the previous sections to deduce some closed formulae for the number $c(W)$ in some special cases.

The following two extreme cases are the simplest ones.

Lemma 3.1. *(1) $c(W) = 1$ if $e(\Gamma) = 0$.*
(2) $c(W) = l!$ if Γ is a complete graph with $v(\Gamma) = l$.

In general, we have $1 \leq c(W) \leq l!$ if $v(\Gamma) = l$.

The next simplest cases are a tree and a cycle.

Lemma 3.2. *(1) If the graph Γ is a tree with $e(\Gamma) = l$, then $c(W) = 2^l$.*
(2) If the graph is a cycle with $e(\Gamma) = l$, then $c(W) = 2^l - 2$.

Proof. We know that the number of orientations of a graph Γ is equal to $2^{e(\Gamma)}$. When Γ is a tree, all the orientations of Γ are acyclic. When Γ is a cycle, all the orientations are acyclic except for two directed cycles. So our result follows by Theorem 1.5. $\square$

Now we want to deal with some slightly complicated cases. To do this, we need first establish two propositions.

Proposition 3.3. *Let Γ be a connected graph covered by two subgraphs Γ_1 and Γ_2. Assume that the intersection of Γ_1 and Γ_2 is a cut vertex x of Γ. Let (W_i, S_i) $(i = 1, 2)$ be the Coxeter system with Γ_i its Coxeter graph. Then $c(W) = c(W_1) \cdot c(W_2)$.*

Proof. Let Γ' be a graph consisting of two connected components Γ'_1 and Γ'_2, where Γ'_i $(i = 1, 2)$ is isomorphic to Γ_i. Let (W', S') be the Coxeter system with Γ' its Coxeter graph. Then by Lemma 1.2, we have $c(W') = c(W_1) \cdot c(W_2)$. Let us denote the vertex of Γ'_i corresponding to x (with respect to a fixed isomorphism $\phi_i : \Gamma_i \longrightarrow \Gamma'_i$) by x_i. Then up to isomorphism, Γ can be obtained from Γ' by fusing x_1 and x_2 into one vertex x. So we have $c(W) = c(W')$ by Lemma 2.2. This implies our assertion. $\square$

By the above result, we can split a graph Γ into some relatively simpler subgraphs Γ_i, $i = 1, 2, \cdots$ when Γ contains cut vertices, by which the computation of the number $c(W)$ can be reduced to the computation of these simpler numbers $c(\Gamma_i)$.

The next result provides some more simplification.

Proposition 3.4. *Let Γ be a connected graph covered by two subgraphs Γ_1 and Γ_2. Assume that the intersection of these two subgraphs is an edge (in other words, a tree with two vertices s, t). Let (W_i, S_i) $(i = 1, 2)$ be the Coxeter system with Γ_i its Coxeter graph. Then $c(W) = \frac{1}{2}c(W_1)c(W_2)$.*

Proof. Define a graph Γ' to be a disjoint union of the graphs Γ_1 and Γ_2, where the vertices s and t in Γ_i $(i = 1, 2)$ are denoted by s_i and t_i respectively. Thus the graph Γ can be obtained from Γ' by fusing the vertices s_1, s_2 into s, and t_1, t_2 into t. Let (W', S') be the Coxeter system with Γ' its Coxeter graph. We define the following four subsets of $C(W')$: for $\alpha \neq \alpha'$ in $\{s_1, t_1\}$, and $\beta \neq \beta'$ in $\{s_2, t_2\}$, let $\Sigma(\alpha, \alpha'; \beta, \beta')$ be the set of all the elements w of $C(W')$ such that there exists a reduced expression of w of the form $\cdots \alpha\beta \cdots \alpha'\beta' \cdots$. Then $C(W')$ is a disjoint union of these four subsets, each of which has the same cardinality $\frac{1}{4}c(W')$. There exists a well-defined map ϕ from the set $\Sigma(s_1, t_1; s_2, t_2)$ to $C(W)$ by replacing the factors $s_1 s_2$ and $t_1 t_2$ by s and t respectively in a reduced expression of an element of $C(W')$ of the form $\cdots s_1 s_2 \cdots t_1 t_2 \cdots$. Also, there exists a well-defined map ψ from the set $\Sigma(t_1, s_1; t_2, s_2)$ to $C(W)$ by the same replacement on the factors of a reduced expression of an element of the form $\cdots t_1 t_2 \cdots s_1 s_2 \cdots$. We see that both maps ϕ and ψ are injective and that $C(W)$ is a disjoint union of the sets $\operatorname{im}\phi$ and $\operatorname{im}\psi$. Hence our result follows. $\square$

Now we shall give some applications of Propositions 3.3 and 3.4.

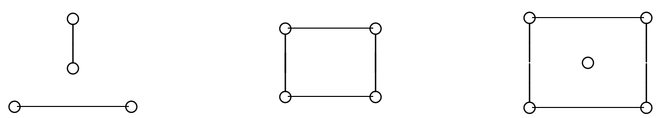
Proposition 3.5. *Let Γ be a graph containing exactly r cycles of m_1, m_2, $\cdots$, m_r edges respectively. Assume that no vertex of Γ belongs to more than two cycles and that no pair of cycles of Γ share more than one common vertex. Then*

$$c(W) = 2^n \prod_{i=1}^{r} (2^{m_i - 1} - 1),$$

where $n = l + r - \sum_{i=1}^{r} m_i$ and l is the number of edges of Γ.

Proof. Let Γ' be the graph obtained from Γ by removing all the edges not belonging to any cycle and by splitting each vertex belonging to two cycles into two vertices, one in each cycle. Then Γ' consists of r isolated cycles of $m_1, m_2, \cdots, m_r$ edges respectively, together with some isolated vertices. Let (W', S') be the Coxeter system with Γ' its Coxeter graph. Then $c(W') = \prod_{i=1}^{r}(2^{m_i} - 2)$ by Lemmas 1.2 and 3.2, (2). Thus our result follows by Lemma 2.1 and Proposition 3.3. □

Remark 3.6. The result of Lemma 3.2 covers all the irreducible finite and affine Coxeter groups. Then Proposition 3.5 further covers all the irreducible hyperbolic Coxeter groups with three exceptions whose Coxeter graphs are as below.



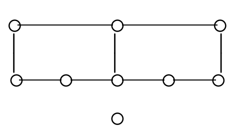
(a) (b) (c)

Figure 1.

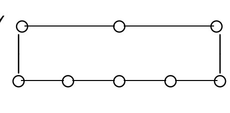
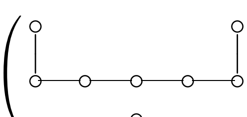
where (a) is a complete graph which has been included in Lemma 3.1; (b) and (c) will be included in Theorem 3.8. Theorem 2.4, together with Propositions 3.3, 3.4, can be used to simplify the calculation of the number $c(W)$ in many cases. In particular, this is the case when $s, t \in S$ in the theorem form a vertex cut set of the graph Γ (i.e. the removal of these vertices increases the number of connected components of Γ). Let us illustrate this point by some examples.

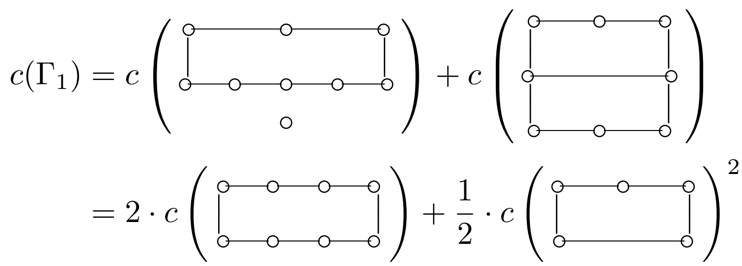
Examples 3.7. Let (W, S, Γ) be a Coxeter system. Recall that the notation $c(\Gamma)$ stands for the number $c(W)$.

(1) Let

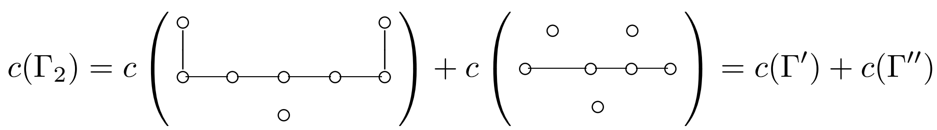
$\Gamma =$ 

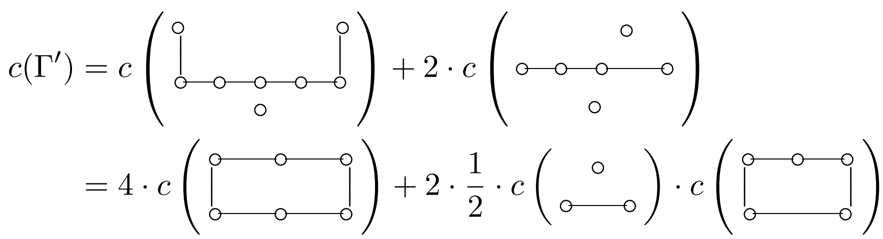
Then by Theorem 2.4 and Propositions 3.3, 3.4, we have

$c(\Gamma) = c\Bigg($  $\Bigg) + c\Bigg($  $\Bigg) = c(\Gamma_1) + c(\Gamma_2)$

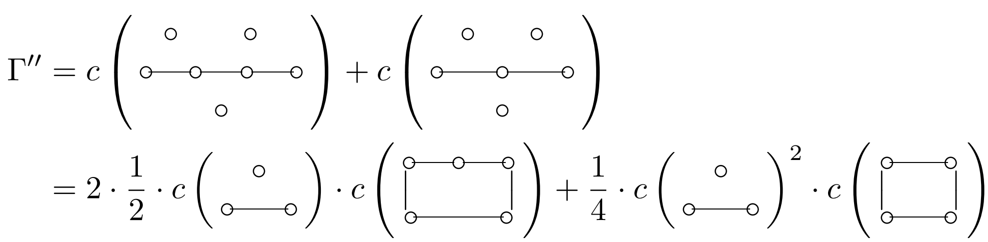


$$= 2 \cdot 254 + \frac{1}{2} \cdot 30^2 = 958.$$





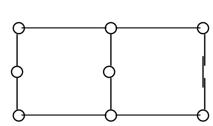
$$= 4 \cdot 62 + 6 \cdot 30 = 428.$$



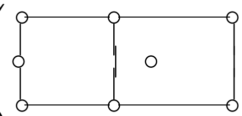
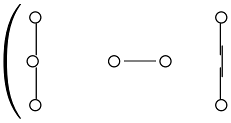
$$= 6 \cdot 30 + \frac{1}{4} \cdot 6^2 \cdot 14 = 306$$

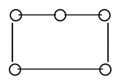
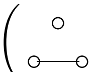
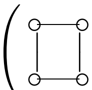
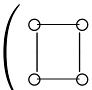
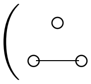
This implies $c(\Gamma) = 958 + 428 + 306 = 1692$.

(2) Suppose that the graph is as below.

$\Gamma =$ 

Then by Theorem 2.4 and Propositions 3.3, 3.4, we have

$c(\Gamma) = c\Big($  $\Big) - c\Big($  $\Big)$

$= \frac{1}{4} \cdot c\Big($  $\Big) \cdot c\Big($  $\Big) \cdot c\Big($  $\Big) - 2 \cdot c\Big($  $\Big) \cdot c\Big($  $\Big)$

$= \frac{1}{4} \cdot 30 \cdot 6 \cdot 14 - 2 \cdot 14 \cdot 6 = 462.$

We can deal with even more general cases than that in Examples 3.7, (2). Consider a Coxeter graph Γ satisfying the following conditions.

(i) There are $n+2$ vertices in Γ with $n = m(1) + m(2) + \cdots + m(r) - r$, where $r \geq 1$, $1 \leq m(1) \leq m(2) \leq \cdots \leq m(r)$ and $m(2) > 1$. These vertices are labelled by (i, j), $(1 \leq i \leq r,\ 1 \leq j < m(i))$, and x, y, respectively.

(ii) The edges of Γ are $\{(i,1), x\}$, $\{(i, m(i)-1), y\}$ $(1 \leq i \leq r)$, $\{(i,j),(i,j+1)\}$ $(1 \leq j < m(i)-1)$. In the case when $m(1) = 1$, $\{x, y\}$ is also an edge.

We use the notation $\lambda(m(1), m(2), \cdots, m(r))$ for the number $c(W)$ when the graph Γ satisfies conditions (i) and (ii). Then we have the following formula:

Theorem 3.8. *In the above setup, we have*

(3.8.1)
$$\begin{aligned} &\lambda(m(1), m(2), \cdots, m(r)) \\ =&2\prod_{i=1}^{r}(2^{m(i)}-1) - \prod_{i=1}^{r}(2^{m(i)}-2) \\ =&\sum_{0\leq t\leq r}(-1)^{r-1-t}(2^{r-t}-2)\sum_{1\leq i_1<\cdots<i_t\leq r} 2^{m(i_1)+\cdots+m(i_t)}. \end{aligned}$$

Proof. Let (W', S', Γ') be a Coxeter system such that Γ' is obtained from Γ by adding an edge $\{x, y\}$ if $m(1) > 1$ or that is isomorphic to Γ if $m(1) = 1$. Let (W'', S'', Γ'') be a Coxeter system such that Γ'' is obtained from Γ by fusing the vertices x and y into a new vertex. By repeatedly applying Propositions 3.3 and 3.4, we have

(3.8.2)
$$c(W') = \frac{1}{2^{r-1}}\prod_{i=1}^{r}(2^{m(i)+1}-2) = 2\prod_{i=1}^{r}(2^{m(i)}-1)$$

and

(3.8.3)
$$c(W'') = \prod_{i=1}^{r}(2^{m(i)}-2).$$

So by Theorem 2.4, the first equality of (3.8.1) follows from (3.8.2) and (3.8.3). Then the second equality of (3.8.1) can be obtained by directly calculation. □

Theorem 3.8 holds even without the restriction $m(i) \neq 1$ for $2 \leq i \leq r$. This can be seen directly from formula (3.8.1). On the other hand, we see that Examples 3.7, (2) is a special case of Theorem 3.8, where we have $r = 3$, $m(1) = 2$, $m(2) = 3$ and $m(3) = 4$. Thus by formula (3.8.1), we get

$$\lambda(2,3,4) = 2^{2+3+4} - 2(2^2 + 2^3 + 2^4) + 6 = 462,$$

just the same as that we got before. Figure 1, (b), (c) are also special cases of Theorem 3.8; the corresponding values of $c(W)$ are $\lambda(1,2,2) = 18$ and $\lambda(2,2,2) = 46$, respectively.

In particular, when $m(1) = m(2) = \cdots = m(r) = k \geq 1$, we denote the number $\lambda(\underbrace{k, k, \cdots, k}_{r \text{ factors}})$ simply by $\lambda(k^r)$. Then (3.8.1) becomes

Corollary 3.9.

$$\lambda(k^r) = 2(2^k - 1)^r - (2^k - 2)^r = \sum_{t=0}^{r} (-1)^{r-1-t} \binom{r}{t} (2^{r-t} - 2) 2^{tk}.$$

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