LEFT-CONNECTEDNESS OF SOME LEFT CELLS IN CERTAIN COXETER GROUPS OF SIMPLY-LACED TYPE

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Abstract. Let W be an irreducible finite or affine Weyl group of simply-laced type. We show that any $w \in W$ with $a(w) \leqslant 6$ satisfies Condition (C): $w = x \cdot w_J \cdot y$ for some $x, y \in W$ and some $J \subseteq S$ with W_J finite and $\ell(w_J) = a(w)$ (see 0.1-0.2 for the notation w_J, W_J, $\ell(w)$ and $a(w)$). We also show that if L is a left cell of W all of whose elements satisfy Condition (C), then the distinguished involution d_L of W in L satisfies $d_L = \lambda(z^{-1}, z) = {z'}^{-1} \cdot w_J \cdot z'$ for any $z = w_J \cdot z' \in E_{\min}(L)$ with $J = \mathcal{L}(z)$ (see 1.6. for the notation $\lambda(z^{-1}, z)$, and 0.3. for $\mathcal{L}(z)$, $E_{\min}(L)$ and $E(L)$), verifying a conjecture of mine in [10, Conjecture 8.10] in our case. If $E(L) = E_{\min}(L)$ then we show that the left cell L is left-connected, verifying a conjecture of Lusztig in our case.

§0. Introduction.

0.1. Let $\mathbb{N}$ (respectively, $\mathbb{Z}$) be the set of all non-negative integers (respectively, integers). Let W be a Coxeter group with S a distinguished generator set. In order to construct representations of W and the associated Hecke algebra $\mathcal{H}$, Kazhdan and Lusztig defined the concept of left, right and two-sided cells of W (see [4]). Later Lusztig defined a function $a : W \to \mathbb{N}$ and a finite set of distinguished involutions of W, both of which play an important role in the representation theory of W and $\mathcal{H}$ (see [5] [6]). Lusztig proved in [6] that each left cell of W contains a unique distinguished involution.

0.2. Let $\leqslant$ (respectively, ℓ) be the Bruhat-Chevalley order (respectively, the length function) on W. For any $J \subseteq S$, denote by w_J the longest element in the subgroup W_J of W generated by J whenever W_J is finite. For $x, y \in W$, we use the notation $w = x \cdot y$ to mean $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. Consider the following condition on $w \in W$:

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(C) $w = x \cdot w_J \cdot y$ for some $x, y \in W$ and some $J \subseteq S$ with $\ell(w_J) = a(w)$.

0.3. In the present paper, we focus our attention on the case where W is an irreducible finite or affine Weyl group of simply-laced type, i.e., $W \in \{\widetilde{A}_l, A_l, \widetilde{D}_m, D_m, \widetilde{E}_i, E_i \mid l \geqslant 1, m \geqslant 4, i = 6, 7, 8\}$.

A subset K of W is *left-connected*, if for any $x, y \in K$, there exist $x_0 = x, x_1, ..., x_r = y$ in K such that $x_{i-1}x_i^{-1} \in S$ for every $1 \leqslant i \leqslant r$.

For any $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$. For any left cell L of W, define $E(L) = \{w \in L \mid sw \notin L \ \forall\ s \in \mathcal{L}(w)\}$ and $E_{\min}(L) = \{w \in L \mid \ell(w) \leqslant \ell(x) \ \forall\ x \in L\}$.

By the exchanging condition on a Coxeter system (see [3, Subsection 1.7]), one can show, for any $w \in W$, that the group W_J with $J := \mathcal{L}(w)$ is always finite and that $w = w_J \cdot (w_J w)$ (see Lemma 1.11).

The main results of the paper can be stated as follows.

Theorem A. *Suppose that Condition (C) holds on all elements in a left cell L of W.*

(1) If $w = w_J \cdot x \in E_{\min}(L)$ with $J = \mathcal{L}(w)$, then the distinguished involution (see (1.4.2) for the definition) in L has the expression $x^{-1} \cdot w_J \cdot x$.

(2) If $E_{\min}(L) = E(L)$ then L is left-connected.

Theorem B. *Let W be an irreducible finite or affine Weyl group of simply-laced type. Let L be a left cell of W with $m := a(L) \leqslant 6$.*

(1) Any $w \in L$ satisfies Condition (C).

(2) Any $w \in E(L)$ has the form $w = w_J \cdot y$ for some $y \in W$ and some $J \subseteq S$ with $\ell(w_J) = m$.

0.4. Let W be a group as in Theorem B. By Theorems A, B, we see that any distinguished involution d of W with $a(d) \leqslant 6$ has an expression described as in Theorem A (1).

The assertions of Theorems A, B have been verified in the following cases:

(a) W is a finite or affine Weyl group and L is a left cell of W containing a fully-commutative element (see [13]).

(b) The group W is as in (a) and L is a left cell of W contained in the lowest two-sided cell of W under a certain partial order (see [8] and 1.3).

The equation $E_{\min}(L) = E(L)$ always holds for any left cell L of W in cases (a)–(b) above (see [8], [13]). Hence these left cells of W are left-connected by Theorem A (2).

0.5. Under the assumption of Theorem B on the left cell L, we conjecture that the equation $E_{\min}(L) = E(L)$ in Theorem A (2) always holds for any left cell L of W with $a(L) \leqslant 6$. This conjecture has been verified in the case where $W = \widetilde{E}_i$, $i = 6, 7, 8$, and $a(L) = 4$ (see [14, Lemma 6.4]). Hence by Theorem A (2), we conclude that these left cells L are left-connected.

The above facts support a conjecture of Lusztig in [1] that any left cell of an affine Weyl group is left-connected.

In general, the assertions (1)–(2) of Theorem B are not valid without the assumptions that $a(w) \leqslant 6$ and W is of simply-laced type. There exist counter-examples in the following two cases: one is when $W = \widetilde{A}_5$ and $a(w) = 7$, and the other is when $W = B_3$ and $a(w) = 4$ (see Remark 2.5). So the assertions (1)–(2) of Theorem B are the best possible in this sense.

0.6. The assertions (1) and (2) in Theorem B are equivalent (see Lemma 2.1). The major part of the paper is devoted to prove Theorem B (2), or equivalently, Proposition 2.4. Let Γ be the Coxeter graph of W. Under the assumption in Proposition 2.4, let $w = w_J \cdot w_I \cdot x \in W$ be with $J = \mathcal{L}(w)$ and $I = \mathcal{L}(w_J w)$. By considering all the possible subgraphs Γ' of Γ with the vertex sets $I \cup J$, the proof of Proposition 2.4 is based on a case-by-case analysis. By Lemma 1.10, we need only to show that for any $w = w_J \cdot y \in W$ with $J = \mathcal{L}(w)$ and $a(w) > \ell(w_J) \leqslant 5$, there exists some $z \in J$ such that either $\{zw, w\}$ is a primitive pair, or zw can be obtained from w by a star operation.

0.7. The contents are organized as follows. Section 1 is the preliminaries, some notation, concept and known results are collected there for later use. We prove Theorem A, and also prove Theorem B by assuming Proposition 2.4 in Section 2. After a short preparation in Section 3, we prove Proposition 2.4 in Sections 4–6 in the cases where

$w \in W$ satisfies $U(k)$, $k = 2, 3, 4, 5$, respectively (see 3.3 for the notation $U(k)$).

§1. Preliminaries.

1.1. Let (W, S, Γ) be the *Coxeter system* of an irreducible finite or affine Weyl group W with S a distinguished generator set and Γ the corresponding Coxeter graph. We further assume that W is of *simply-laced type*, i.e., one of types A, D, E, $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E}$.

To any $w \in W$, we associate two subsets of S as follows.

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

1.2. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of all Laurent polynomials in an indeterminate u with integer coefficients. The *Hecke algebra* $\mathcal{H}$ of W over $\mathcal{A}$ is an associative $\mathcal{A}$-algebra with two sets of $\mathcal{A}$-bases $\{T_x \mid x \in W\}$ and $\{C_w \mid w \in W\}$, satisfying the relations

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S, \end{cases} \tag{1.2.1}$$

and

$$C_w = \sum_{y \leqslant w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y, \tag{1.2.2}$$

where $P_{y,w} \in \mathbb{Z}[u]$ satisfies that $P_{w,w} = 1$, $\quad P_{y,w} = 0$ if $y \not\leqslant w$ and $\deg P_{y,w} \leqslant \frac{1}{2}(\ell(w) - \ell(y) - 1)$ if $y < w$. The $P_{y,w}$'s are called *Kazhdan-Lusztig polynomials* (see [4]). Denote y—w or w—y, if $y < w$ and $\deg P_{y,w} = \frac{1}{2}(\ell(w) - \ell(y) - 1)$. The relation w—y is not easy to be checked in general since it involves the complicated computation of Kazhdan-Lusztig polynomials. However, we have

(1.2.3) $\quad w$—y if $y < w$ and $\ell(y) = \ell(w) - 1$.

1.3. The preorders $\underset{\mathrm{L}}{\leqslant}$, $\underset{\mathrm{R}}{\leqslant}$, $\underset{\mathrm{LR}}{\leqslant}$ and the associated equivalence relations $\underset{\mathrm{L}}{\sim}$, $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$ on W are defined as in [4]. The equivalence classes of W with respect to $\underset{\mathrm{L}}{\sim}$ (respectively, $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$) are called *left cells* (respectively, *right cells, two-sided cells*). The preorder $\underset{\mathrm{L}}{\leqslant}$ (respectively, $\underset{\mathrm{R}}{\leqslant}$, $\underset{\mathrm{LR}}{\leqslant}$) on W induces a partial order on the set of all left (respectively, right, two-sided) cells of W.

1.4. For any $x, y, z \in W$, define $h_{x,y,z} \in \mathcal{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z.$$

In [5], Lusztig defined a function $a : W \to \mathbb{N}$ by

$$a(z) = \min\{k \in \mathbb{N} \mid u^k h_{x,y,z} \in \mathbb{Z}[u],\ \forall\, x, y \in W\} \qquad \text{for } z \in W. \tag{1.4.1}$$

The following are some known properties of the function a:

(1) If $x \underset{\mathrm{LR}}{\leqslant} y$ then $a(x) \geqslant a(y)$. In particular, $x \underset{\mathrm{LR}}{\sim} y$ implies $a(x) = a(y)$. So we may define the a-value $a(X)$ on a left (respectively, right, two-sided) cell X of W to be $a(x)$ for any $x \in X$ (see [5]).

(2) $a(w_J) = \ell(w_J)$ for any $J \subseteq S$ with W_J finite (see [5]).

(3) If $x, y, w \in W$ satisfy $w = x \cdot y$ (see Introduction) then call w a *left* (respectively, *right*) *extension* of y (respectively, x), and call y (respectively, x) a *left* (respectively, *right*) *retraction* of w. In this case, we have the relations $x \underset{\mathrm{R}}{\geqslant} w \underset{\mathrm{L}}{\leqslant} y$ and $a(w) \geqslant a(x), a(y)$ (see [5]).

(4) If $a(x) = a(y)$ and $x \underset{\mathrm{L}}{\leqslant} y$ (respectively, $x \underset{\mathrm{R}}{\leqslant} y$) then $x \underset{\mathrm{L}}{\sim} y$ (respectively, $x \underset{\mathrm{R}}{\sim} y$) (see [6]).

Let $\delta(z) = \deg P_{1,z}$ for $z \in W$, where 1 is the identity of the group W. Define

$$\mathcal{D} = \{w \in W \mid \ell(w) = 2\delta(w) + a(w)\}. \tag{1.4.2}$$

Then Lusztig proved in [6] that

(5) $\mathcal{D}$ is a finite set of involutions (called *distinguished involutions* by Lusztig) and that each left (respectively, right) cell of W contains a unique element of $\mathcal{D}$.

For any $x \in W$, we have

(6) $h_{x^{-1},x,d} \neq 0$ for $d \in \mathcal{D}$ with $d \underset{\mathrm{L}}{\sim} x$ (see [6]).

1.5. For any left cell L of W, define

$$E(L) = \{w \in L \mid sw \notin L, \ \forall\ s \in \mathcal{L}(w)\},$$

$$E_{\min}(L) = \{w \in L \mid \ell(w) \leqslant \ell(x), \ \forall\ x \in L\}.$$

By 1.4 (3)–(4), we can equivalently define $E(L) = \{w \in L \mid a(sw) < a(w), \ \forall\ s \in \mathcal{L}(w)\}$. Clearly, the relation $E_{\min}(L) \subseteq E(L)$ holds in general. The equality $E_{\min}(L) = E(L)$ holds if and only if all the elements in $E(L)$ have the same length.

1.6. We proved in [8] that for any $x, y \in W$, there exists a unique element $w \in W$ satisfying that $h_{x,y,w} \neq 0$ and that any $z \in W$ with $h_{x,y,z} \neq 0$ satisfies $z \leqslant w$. Denote such an element w by $\lambda(x, y)$. Given a reduced expression $x = s_r s_{r-1} \cdots s_1$ of x with $s_i \in S$, define a sequence $y_0 = y, y_1, ..., y_r$ in W such that for every $1 \leqslant i \leqslant r$, we set $y_i = y_{i-1}$ if $y_{i-1} > s_i y_{i-1}$ and $y_i = s_i y_{i-1}$ if $y_{i-1} < s_i y_{i-1}$. Then we showed in [8] that $\lambda(x, y)$ is equal to the last term y_r of the sequence and that $\lambda(x, y)$ is a left (respectively, right) extension of y (respectively, x).

In particular, we have

(1.6.1) $\quad d \leqslant \lambda(x^{-1}, x)$ for any $d \in \mathcal{D}$ and $x \in W$ with $d \underset{\mathrm{L}}{\sim} x$

by 1.4 (6). We conjectured in [10, Conjecture 8.10] that $d = \lambda(x^{-1}, x)$ for any $x \in E_{\min}(L_d)$, where L_d is the left cell of W containing d.

1.7. Given $s \neq t$ in S with $o(st) = 3$, a set of the form $\{sy, tsy\}$ or $\{ty, sty\}$ is called a *left* $\{s,t\}$*-string* (or a *left string* in short), if $y \in W$ satisfies $\mathcal{L}(y) \cap \{s, t\} = \emptyset$.

An element x is obtained from w by a *left* $\{s,t\}$*-star operation* (or a *left star operation* in short), if $\{x, w\}$ is a left $\{s,t\}$-string. Note that the resulting element x of a left $\{s,t\}$-star operation on w is unique whenever it exists.

Let us reformulate a result by Kazhdan-Lusztig as follows.

Lemma 1.8. (see [4, Theorem 4.2]) *Let* $w_1, w_2, y_1, y_2 \in W$ *and* $s, t \in S$ *be with* $o(st) = 3$*. If both* $\{w_1, w_2\}$ *and* $\{y_1, y_2\}$ *are left* $\{s,t\}$*-strings, then* $w_1 — y_1$ *if and only if* $w_2 — y_2$*.*

1.9. Two elements $x, y \in W$ form a *left primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, ..., x_r$ and $y_0 = y, y_1, ..., y_r$ in W such that the following conditions are satisfied.

(a) Both $\{x_{i-1}, x_i\}$ and $\{y_{i-1}, y_i\}$ are left $\{s_i, t_i\}$-strings for every $1 \leqslant i \leqslant r$ and some $s_i, t_i \in S$ with $o(s_i t_i) = 3$.

(b) x_i—y_i (see 1.2) for some (and then for all by Lemma 1.8) $0 \leqslant i \leqslant r$.

(c) Either $\mathcal{L}(x) \nsubseteq \mathcal{L}(y)$ and $\mathcal{L}(y_r) \nsubseteq \mathcal{L}(x_r)$, or $\mathcal{L}(y) \nsubseteq \mathcal{L}(x)$ and $\mathcal{L}(x_r) \nsubseteq \mathcal{L}(y_r)$ hold.

Note that elements in a left string forms a left primitive pair, where we take $r = 0$ in the above definition.

The following result is well known.

Lemma 1.10. (see [10, Section 3]) *If $\{x, y\}$ is a left primitive pair, then $x \underset{L}{\sim} y$.*

Similarly, we can define a right $\{s, t\}$-string, a right $\{s, t\}$-star operation and a right primitive pair. We also have the ‘ right-handed ’ versions of the results in Lemmas 1.8 and 1.10.

Since only the “ left-handed ” versions of these concepts will be applied in this paper, we shall omit the adjective “ left ”by simply call them a *string*, a *star operation* and a *primitive pair* respectively from now on.

Next two results will be useful in subsequent sections.

Lemma 1.11. *Any $w \in W$ can be expressed uniquely in the form $w = w_J \cdot w_1$ with $J = \mathcal{L}(w)$ for some $w_1 \in W$.*

Proof. We prove our result via the Tits representation of the Coxeter system (W, S) (see [3, Subsection 5.3]). The root system of W has the form $\Phi = \{(\alpha_s)x \mid s \in S, x \in W\}$ with $\Delta_S = \{\alpha_s \mid s \in S\}$ a simple system. Let Φ^+ and Φ^- be the corresponding positive and negative systems of Φ respectively. It is known by [3, Proposition 5.6] that for any $w \in W$, $\ell(w)$ is equal to the cardinality of the set $\{\alpha \in \Phi^+ \mid (\alpha)w \in \Phi^-\}$. For any $I \subseteq S$, let $\Delta_I := \{\alpha_s \mid s \in I\}$ and let Φ_I^+ be the set of all elements in Φ^+ which are in the space spanned by Δ_I. Then $(\Delta_J)w \subseteq \Phi^-$ and hence $(\Phi_J^+) \subseteq \Phi^-$. This implies that the subgroup W_J of W generated by J is finite by the fact that $\ell(w_J) = |\Phi_J^+| \leqslant \ell(w)$. Hence $w = w_J \cdot w_1$ for some $w_1 \in W$ by the exchanging condition on Coxeter groups (see [3, Subsection 1.7]). □

Lemma 1.12. *Let $w \in W$ and $s \in S$.*

(1) If $s \notin \mathcal{L}(w)$ then $(\mathcal{L}(sw) \setminus \{s\}) \subseteq \mathcal{L}(w)$;

(2) Let $w = w_J \cdot w_1$ be with $J = \mathcal{L}(w)$. Then for any $s \in \mathcal{L}(w_1)$, there is some $t \in J$ with $st \neq ts$.

Proof. It is well known that in a Coxeter system (W, S), if $x, y \in W$ and $r \in S$ satisfy $xr > x$ and $ry > y$ then $xry > xy$ (see [9, Theorem 1]).

For any $t \in (\mathcal{L}(sw) \setminus \{s\})$, we have $sw > tsw$, and also $tsw > tw$ by the above result, the assumption $sw > w$ and the fact $ts > t$. This implies $\ell(tw) \leqslant \ell(sw) - 2 = \ell(w) - 1$ and hence $tw < w$, proving (1). Then (2) follows by the facts that $J \cap \mathcal{L}(w_1) = \emptyset$ and that $\{r \in \mathcal{L}(w_1) \mid rs = rs \ \forall\ s \in J\} \subset \mathcal{L}(w) = J$. □

§2. Proof of Theorems A, B.

In this section, we prove Theorem A, and also prove Theorem B by assuming Proposition 2.4. Assume that W is an irreducible finite or affine Weyl group of simply-laced type throughout the section.

Lemma 2.1. *In Theorem B, the assertions (1) and (2) are equivalent.*

Proof. Let L be a left cell of W with $a(L) = m$. Let us first prove (2) by assuming (1). If $w \in E(L)$, then there exists an expression $w = x \cdot w_J \cdot y$ with $\ell(w_J) = m$ for some $x, y \in W$ and some $J \subseteq S$ by (1). We claim $x = 1$. For otherwise, take any $s \in \mathcal{L}(x)$. Then $s \in \mathcal{L}(w)$ and $m = a(w) \geqslant a(sw) = a(sx \cdot w_J \cdot y) \geqslant a(w_J) = m$ by 1.4 (2)-(3), which implies $a(w) = a(sw)$ and hence $w \underset{L}{\sim} sw$ by 1.4 (3)–(4), contradicting the assumption of $w \in E(L)$. The claim is proved and hence (2) follows. Next let us prove (1) by assuming (2). Given $w \in L$, we can find some $x, y \in W$ with $w = x \cdot y$ and $y \in E(L)$. Then $y = w_J \cdot y'$ for some $y' \in W$ and some $J \subseteq S$ with $\ell(w_J) = m$ by (2). Hence $w = x \cdot w_J \cdot y'$, as required. □

2.2. *Proof of Theorem A.* We first show assertion (1). Let d be the distinguished involution of W in L. By our assumption, we can write $d = x \cdot w_J \cdot y$ for some $x, y \in W$ and $J \subseteq S$ with $\ell(w_J) = a(w) := m$. Choose an expression of such kind with $\ell(y)$ smallest possible. Then $w_J \cdot y \underset{\mathrm{L}}{\sim} d$ by 1.4 (2)–(4). We also have $\ell(x) \geqslant \ell(y)$ by the fact that both d and w_J are involutions. Hence

$$\ell(d) \geqslant 2\ell(y) + m. \tag{2.2.1}$$

We claim that $w_J \cdot y$ is in $E_{\min}(L)$. For, take any $z \in E_{\min}(L)$ with $z \underset{\mathrm{L}}{\sim} d$. By Lemma 2.1, we can write $z = w_I \cdot z'$ for some $z' \in W$ and $I \subseteq S$ with $\ell(w_I) = m$. By 1.6 (in particular, by (1.6.1)), we have $d \leqslant \lambda(z^{-1}, z) = \lambda(z'^{-1}, w_I \cdot z')$ and hence

$$\ell(d) \leqslant \ell(\lambda(z^{-1}, z)) = \ell(\lambda(z'^{-1}, w_I \cdot z')) \leqslant 2\ell(z') + \ell(w_I) = 2\ell(z') + m \tag{2.2.2}$$

Since $w_J \cdot y \underset{\mathrm{L}}{\sim} d \underset{\mathrm{L}}{\sim} z$, this implies that

$$2\ell(y) + m \geqslant 2\ell(z') + m \geqslant \ell(d) \geqslant 2\ell(y) + m \tag{2.2.3}$$

by (2.2.1)–(2.2.2). So all the equalities in (2.2.3) should hold. Hence $w_J \cdot y$ is in the set $E_{\min}(L)$, as claimed. This further implies that $d = \lambda(z^{-1}, z) = z'^{-1} \cdot w_I \cdot z'$ for any $z = w_I \cdot z' \in E_{\min}(L)$, proving assertion (1).

Next we prove assertion (2). Let d_L be the distinguished involution of W in L. For any $w \in L$, there exists a left retraction w' of w in $E(L)$. Hence by 1.4 (3), there exist a sequence of elements $x_0 = w, x_1, ..., x_r = w'$ in L such that $x_{i-1}x_i^{-1} \in S$ and $\ell(x_i) = \ell(x_{i-1}) - 1$ for $1 \leqslant i \leqslant r$. By our assumption, we have $w' \in E_{\min}(L)$. By assertion (1), we have $d_L = \lambda(w'^{-1}, w')$, which is a left extension of w' (see 1.4 (3) and 1.6). Hence there exist $y_0 = w', y_1, ..., y_t = d_L$ in L such that $y_{i-1}y_i^{-1} \in S$ and $\ell(y_i) = \ell(y_{i-1}) + 1$ for $1 \leqslant i \leqslant t$. So L is left-connected. □

Next we want to prove Theorem B. First consider the case of $W \in \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$.

Proposition 2.3. *Assertion (1) in Theorem B holds when* $W \in \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$.

Proof. (a) First assume $W = \widetilde{A}_{n-1}$, $n \geqslant 2$. By a partition λ of n, we mean a sequence of integers $\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ with $\sum_{i=1}^r \lambda_i = n$. Write $\lambda = (\lambda_1, ..., \lambda_r)$ and call λ_i a part of λ. Let Λ_n be the set of all partitions of n. A partition λ can also be denoted by $(a_1^{e_1} a_2^{e_2} \cdots a_k^{e_k})$ with $a_1 > a_2 > \cdots > a_k$ if the a_i's are all distinct parts of λ with multiplicities $e_i > 0$, $1 \leqslant i \leqslant k$. By [7, Theorem 17.4], there exists a surjection $\phi : \widetilde{A}_{n-1} \to \Lambda_n$ which induces a bijection from the set of two-sided cells of $\widetilde{A}_{n-1}$ to Λ_n, where the two-sided cell consisting of the identity element of $\widetilde{A}_{n-1}$ corresponds to the partition (1^n), and the lowest two-sided cell of $\widetilde{A}_{n-1}$ under the partial order $\underset{\mathrm{LR}}{\leqslant}$ corresponds to the partition (n) (see 1.3). The Coxeter graph Γ of $\widetilde{A}_{n-1}$ is a circle with the nodes $0, 1, 2, ..., n-1$ arranging on the circle clockwise. Any $J \subset S$ (identifying S with $\{0, 1, ..., n-1\}$) can be decomposed into a disjoint union $J = \cup_{i=1}^r J_i$ of non-empty maximal subsets J_i consisting of consecutive nodes along the circle Γ. We may assume $|J_1| \geqslant \cdots \geqslant |J_r|$ by relabelling J_i's if necessary. Then $\phi(w_J) = (|J_1|+1, |J_2|+1, ..., |J_r|+1, 1, ..., 1) \in \Lambda_n$ by adding a proper number of parts 1 at the end. We have $a(w_J) = \ell(w_J) = \sum_{i=1}^r \binom{|J_i|+1}{2}$ by 1.4 (2), where $\binom{k}{h} := \frac{k!}{h!(k-h)!}$. In general, for any $w \in \widetilde{A}_{n-1}$ with $\phi(w) = (\lambda_1, ..., \lambda_r) \in \Lambda_n$, we have

(2.3.1) $$a(w) = \sum_{i=1}^r \binom{\lambda_i}{2} \qquad \text{(see [10, (6.27)]).}$$

So the inequality $a(w) \leqslant 6$ holds if and only if the partition $\phi(w)$ is in the following list:

(2.3.2) $$\begin{array}{ccccccc} (41^{n-4}), & (3^2 1^{n-6}), & (32^3 1^{n-9}), & (2^6 1^{n-12}), & (32^2 1^{n-7}), & (2^5 1^{n-10}), & \\ (321^{n-5}), & (2^4 1^{n-8}), & (31^{n-3}), & (2^3 1^{n-6}), & (2^2 1^{n-4}), & (21^{n-2}), & (1^n), \end{array}$$

Note that any partition $\lambda = (\lambda_1, ..., \lambda_r, 1, ..., 1)$ in (2.3.2) with $\lambda_r > 1$ satisfies that

(2.3.3) for any $\mu_1 \geqslant \cdots \geqslant \mu_k > 0$ in $\mathbb{N}$ with $\sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i$ and $k \leqslant r$, the equation $\mu_j = \lambda_j$ holds for every $1 \leqslant j \leqslant k$.

The result in [11, Theorem 3.1] asserts that if $w \in \widetilde{A}_{n-1}$ is such that $\phi(w) = (\lambda_1, ..., \lambda_r)$, then for any $1 \leqslant k \leqslant r$, there exists an expression $w = x \cdot w_{J_k} \cdot y$ with some $x, y \in \widetilde{A}_{n-1}$, $J_k \subset S$ such that $\phi(w_{J_k}) = (\mu_1, ..., \mu_k, 1, ..., 1) \in \Lambda_n$ satisfies $\sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i$. By this result, we see that if $\lambda \in \Lambda_n$ satisfies (2.3.3) then any $w \in \phi^{-1}(\lambda)$ satisfies Condition (C). In particular, Theorem B (1) holds for any $w \in \widetilde{A}_{n-1}$ with $a(w) \leqslant 6$.

(b) Next assume $W = \widetilde{D}_4$ with its Coxeter graph as in Fig. 1.

First assume $\mathcal{L}(w) = \{s, r\}$ with $a(w) > 2$. Then $w = srt \cdot w_1$ for some $1 \neq w_1 \in \widetilde{D}_4$ with $\mathcal{L}(w_1) \subseteq \{u, v, s, r\}$. If $s \in \mathcal{L}(w_1)$ then rw can be obtained from w by an $\{r, t\}$-star operation and hence $rw \underset{\text{L}}{\sim} w$. Similarly we can show that $sw \underset{\text{L}}{\sim} w$ in the case of $r \in \mathcal{L}(w_1)$. Now assume $\mathcal{L}(w_1) \subseteq \{u, v\}$. We claim that $\mathcal{L}(w_1) = \{u, v\}$. For otherwise, $|\mathcal{L}(w_1)| = 1$, say $\mathcal{L}(w_1) = \{u\}$ without loss of generality. Then $w = srtut \cdot w_2$ for some $w_2 \in \widetilde{D}_4$, which would imply $\mathcal{L}(w) \supseteq \{s, r, u\}$, contradicting the assumption of $\mathcal{L}(w) = \{s, r\}$. The claim is proved. Next we claim that $\{sw, w\}$ is a primitive pair. For, let $z_0 = w$, $z_1 = t \cdot z_0$, $z_2 = u \cdot z_1$, $z_0' = sw$, $z_1' = rz_0'$, $z_2' = tz_1'$. Then $\{z_0, z_1\}$, $\{z_0', z_1'\}$ are $\{t, r\}$-strings, and $\{z_1, z_2\}$, $\{z_1', z_2'\}$ are $\{t, u\}$-strings. We see that z_0—z_0' by (1.2.3) and hence z_i—z_i' for $i = 1, 2$ by Lemma 1.8. We have $\mathcal{L}(z_0) \not\subseteq \mathcal{L}(z_0')$ and $\mathcal{L}(z_2') \not\subseteq \mathcal{L}(z_2)$ since $s \in \mathcal{L}(z_0) \setminus \mathcal{L}(z_0')$ and $v \in \mathcal{L}(z_2') \setminus \mathcal{L}(z_2)$ by Lemma 1.12. This proves the claim (In the subsequent discussion, we shall indicate many pairs of elements to be primitive. Their proofs are more or less similar to that in the above, which will be left to the readers in most cases). So $sw \underset{\text{L}}{\sim} w$.

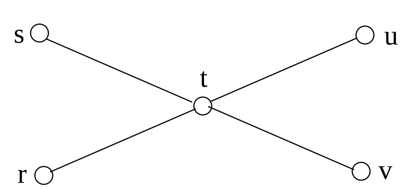


Fig. 1.

Next assume $\mathcal{L}(w) = \{s, r, u\}$ with $a(w) > 3$. Then $w = srut \cdot w_1$ for some $1 \neq w_1 \in \widetilde{D}_4$. We claim that $\mathcal{L}(w_1) \cap \{s, r, u\} \neq \emptyset$. For otherwise, $\mathcal{L}(w_1) \cap \{s, r, u\} = \emptyset$. Then $w = srutvt \cdot w_2$ for some $w_2 \in \widetilde{D}_4$, which would imply $\mathcal{L}(w) \supseteq \{s, r, u, v\}$, contradicting the assumption of $\mathcal{L}(w)=\{s,r,u\}$. The claim is proved. We may assume $s\in\mathcal{L}(w_1)$ without loss of generality. Then $w=sruts\cdot w_2$ for some $w_2\in\widetilde{D}_4$. If w_2 is a right extension of ut then rw can be obtained from w by an $\{r,t\}$-star operation; otherwise, $\{rw,w\}$ is a primitive pair. We have $rw\underset{\mathrm{L}}{\sim}w$ in either case.

Next assume $\mathcal{L}(w)=\{s,t\}$ with $a(w)>3$. Then $w=sts\cdot w_1$ for some $w_1\in\widetilde{D}_4$. Set $I=\mathcal{L}(w_1)$. If $|I|=1$, then $w=stsrt\cdot w_2$ for some $w_2\in\widetilde{D}_4$ by relabelling S if necessary. Hence tw can be obtained from w by a $\{t,r\}$-star operation. If $|I|\geqslant 2$, then either $w=stsrut\cdot w_2$ or $w=stsruv\cdot w_2$ for some $w_2\in\widetilde{D}_4$ by relabelling S if necessary. Thus $\{sw,w\}$ is a primitive pair and hence $sw\underset{\mathrm{L}}{\sim}w$.

Finally, assume $a(w)>4$. Then $a(w)\geqslant 6$ since there is no element w in $\widetilde{D}_4$ with $a(w)=5$ by [12]. By the results of [12], we see that any $w\in\widetilde{D}_4$ with $a(w)\geqslant 6$ has an expression of the form $w=x\cdot w_J\cdot y$ for some $x,y\in\widetilde{D}_4$ and $J\subset S$ with $\ell(w_J)=6$.

Hence Theorem B (1) holds for $W=\widetilde{D}_4$. □

Proposition 2.4. *Let W be an irreducible finite or affine Weyl group of simply-laced type. For $w\in W$, write $w=w_J\cdot w_1$ with $J=\mathcal{L}(w)$ for some $w_1\in W$. If w satisfies $a(w)>\ell(w_J)\leqslant 5$ then there exists some $s\in J$ such that $sw\underset{\mathrm{L}}{\sim}w$.*

The proof of Proposition 2.4 is long and shall be given by a case-by-case argument in Sections 4–6.

Remark 2.5. (1) In Proposition 2.4, it cannot be removed for the assumption of W having simply-laced type. For example, in the Weyl group $W=B_3$, let $S=\{s,r,t\}$ satisfy $o(sr)=3$ and $o(rt)=4$. Take $w=srtrsr$. Then $a(w)=4$ by observing

$$w\underset{\mathrm{L}}{\sim}tsrtrsr\underset{\mathrm{L}}{\sim}trtrsr\underset{\mathrm{R}}{\sim}trtrs\underset{\mathrm{R}}{\sim}trtr:=w_{rt},$$

while $a(w_{rt})=\ell(w_{rt})=4$ by 1.4 (2). However, w has no expression of the form $w=x\cdot w_{rt}\cdot y$ for any $x,y\in W$.

(2) A more general statement than Proposition 2.4 is as follows:

(2.5.1) For any $w\in W$ with $J=\mathcal{L}(w)\subseteq S$ and $\ell(w_J)<a(w)$, there exists some $s\in J$ with $sw\underset{\mathrm{L}}{\sim}w$.

However, the statement (2.5.1) is not always true, even when W is of simply-laced type. A counter-example occurs when $W = \widetilde{A}_5$. Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5\}$ be a Coxeter generator set satisfying $o(s_0s_5) = o(s_is_{i+1}) = 3$ for $0 \leqslant i < 5$. Take $w = w_{234} \cdot s_1s_2s_5s_4 = s_3s_2s_4s_3 \cdot w_{12}w_{45}$, where $w_{234} := s_2s_3s_2s_4s_3s_2$, $w_{12} = s_1s_2s_1$ and $w_{45} = s_4s_5s_4$. Then $\phi(w) = (42)$, the partition having one part 4 and one part 2. Hence $a(w) = 7$ by (2.3.1). The element w satisfies $a(sw) < a(w)$ for any $s \in \mathcal{L}(w) = \{s_2, s_3, s_4\}$. But $\ell(w_{234}) = 6 < a(w)$. Note that the partition (42) does not satisfy Condition (2.3.3).

(3) In the group $\widetilde{A}_{n-1}$, we can show that a partition $\lambda \in \Lambda_n$ satisfy Condition (2.3.3) if and only if all elements in $\phi^{-1}(\lambda)$ satisfy Condition (C). This generalizes Theorem B.

We can prove Theorem B by assuming Proposition 2.4.

2.6. *Proof of Theorem B.* Assertions (1)–(2) follow by Proposition 2.3 and Lemma 2.1 when $W \in \{\widetilde{A}_{n-1}, \widetilde{D}_4 \mid n \geqslant 2\}$. Now assume $W \notin \{\widetilde{A}_{n-1}, \widetilde{D}_4 \mid n \geqslant 2\}$. By Lemma 2.1, it is enough to show assertion (1) for $w \in W$ with $a(w) \leqslant 6$. Apply induction on $\ell(w) \geqslant 0$. Assertion (1) clearly holds when $\ell(w) = 0$, (i.e., $w = 1$). Now assume $\ell(w) > 0$. We may write $w = w_J \cdot w_1$ with $J = \mathcal{L}(w)$ for some $w_1 \in W$ by Lemma 1.11. If $a(w) = \ell(w_J)$ then we are done. If $a(w) > \ell(w_J)$ then $\ell(w_J) \leqslant 5$ and hence $sw \underset{L}{\sim} w$ for some $s \in J$ by Proposition 2.4. Since $a(sw) = a(w) \leqslant 6$ by 1.4 (1), assertion (1) holds for sw by inductive hypothesis and by the fact $\ell(sw) < \ell(w)$, So $sw = x \cdot w_I \cdot y$ for some $I \subset S$ and some $x, y \in W$ with $a(sw) = \ell(w_I)$. Hence $w = sx \cdot w_I \cdot y$, assertion (1) holds also for w. $\square$

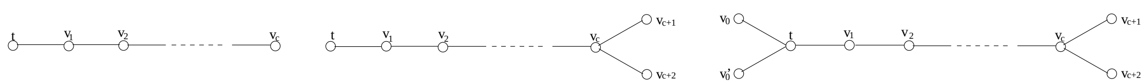
§3. Some more results on Coxeter groups.

The remaining part of the paper shall be devoted to proving Proposition 2.4. Since the assertion of Proposition 2.4 in the case of $W \in \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$ is a direct consequence of Proposition 2.3, we shall always assume $W \notin \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$ from now on unless otherwise specified, hence the Coxeter graph Γ of W contains no circle and the number of edges incident to any given node in Γ is at most 3.

In this section, we collect some more results on Coxeter groups for later use.

Lemma 3.1. *Suppose that Γ is the Coxeter graph of W and that $w \in W$, $s, t \in S$ satisfy $\mathcal{L}(w) = \{s\}$, $st \neq ts$ and $t \notin \mathcal{L}(sw)$. Then one of the following cases must occur:*
(1) $w = v_1 v_2 \cdots v_c$ for some $c \geqslant 1$ with $v_1 = s$ and a subgraph of Γ in Fig. 2 (a).
(2) w is a right extension of $v_1 v_2 \cdots v_c v_{c+1} v_{c+2}$ for some $c \geqslant 1$ with $v_1 = s$ and a subgraph of Γ in Fig. 2 (b). In this case, if t is a branching node then $W = \widetilde{D}_n$ for some $n > 4$ and w is a retraction of $(v_1 v_2 \cdots v_c v_{c+1} v_{c+2} v_c v_{c-1} \cdots v_1 t v_0 v_0' t)^k$ for some $k \geqslant 1$, where Γ is as in Fig. 2 (c).

Proof. This follows directly from the classification of an irreducible finite and affine Weyl group of simply-laced type and by the assumption that $W \notin \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$.
□



(a) (b) (c)

Fig. 2.

Lemma 3.2. *Assume that $w \in W$ satisfies $w \neq w_J$ for any $J \subseteq S$. Set $J := \mathcal{L}(w)$. Then $I := \mathcal{L}(w_J w) \neq \emptyset$ and $w = w_J \cdot w_I \cdot w_1$ for some $w_1 \in W$. We have $zw \underset{\mathrm{L}}{\sim} w$ for some $z \in J$ if one of the following cases occurs:*

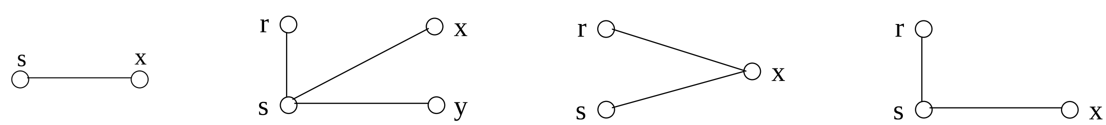
(1) There exist some $s \in J$ and $x \in I$ with $sx \neq xs$ such that s, x commute with any $z \in J \setminus \{s\}$ (see Fig. 3 (a)).

(2) There are some $x \neq y$ in I and some $r \neq s$ in J such that x, y, s, r commute with any $z \in J \setminus \{s, r\}$ and that $xs \neq sx$, $ys \neq sy$, $sr \neq rs$ (hence $rx = xr$, $ry = yr$, $xy = yx$ since the Coxeter graph of W contains no circle, see Fig. 3 (b)).

(3) There exist some $x \in I$ and some $r \neq s$ in J with $sr = rs$, $rx \neq xr$, $sx \neq xs$ and $r \in \mathcal{L}(w_1)$ such that s, r, x commute with any $z \in (J \cup I) \setminus \{s, r, x\}$ (see Fig. 3 (c)).

(4) There exist some $x \in I$ and $r, s \in J$ with $sr \neq rs$, $sx \neq xs$, $rx = xr$, $s \in \mathcal{L}(w_1)$ such that s, r, x commute with any $z \in (J \cup I) \setminus \{s, r, x\}$ (see Fig. 3 (d)).

Proof. The assertions $I \neq \emptyset$ and $w = w_J \cdot w_I \cdot w_1$ follow by Lemma 1.11 and the assumption $w \neq w_J$. In the case (2) with $s, r \in J$ and $sr \neq rs$, if w_1 is a right extension of sx (respectively, sy) then sw can be obtained from w by an $\{s, y\}$- (respectively, $\{s, x\}$-) star operation; otherwise, we claim that $\{rw, w\}$ is a primitive pair. For, we have $w = rsr \cdot xy \cdot w_1$. Let $z_0 = w$, $z_1 = x \cdot z_0$, $z_0' = rz_0$, $z_1' = sz_0'$. Then both $\{z_0, z_1\}$ and $\{z_0', z_1'\}$ are $\{s, x\}$-strings. We have $z_0 — z_0'$ by (1.2.3) and hence $z_1 — z_1'$ by Lemma 1.8. So $\mathcal{L}(z_0) \not\subseteq \mathcal{L}(z_0')$ and $\mathcal{L}(z_1') \not\subseteq \mathcal{L}(z_1)$ since $r \in \mathcal{L}(z_0) \setminus \mathcal{L}(z_0')$ and $y \in \mathcal{L}(z_1') \setminus \mathcal{L}(z_1)$ by Lemma 1.12. This proves the claim. So $rw \underset{L}{\sim} w$ by Lemma 1.10. In all the other cases, sw can be obtained from w by an $\{s, x\}$-star operation. So the result follows by 1.7 and Lemma 1.10. □



(a) (b) (c) (d)

Fig. 3.

3.3. For any $w = w_J \cdot w_I \cdot w_1 \in W$ with $J = \mathcal{L}(w)$ and $I = \mathcal{L}(w_J w)$, denote by $U(k)$ (respectively, $U(k; i)$) the collection of the assumptions (i)–(ii) (respectively, (i)–(iii)) on W and/or w below:

(i) $W \notin \{\widetilde{D}_4, \widetilde{A}_{n-1} \mid n \geqslant 2\}$;

(ii) $a(w) > k = \ell(w_J)$;

(iii) $i = \ell(w_I)$;

In the subsequent sections, we shall frequently apply the results (iv)–(v) below:

(iv) The classification of irreducible finite and affine Weyl groups;

(v) Lemmas 3.1 and 3.2.

§4. On the elements w satisfying $U(2)$ or $U(3)$.

Sections 4–6 are devoted to the proof of Proposition 2.4. The proof is divided into a series of lemmas. In these sections, we always assume W being an irreducible finite or affine Weyl group of simply-laced type with $W \notin \{\widetilde{A}_{n-1}, \widetilde{D}_4 \mid n \geqslant 2\}$.

In the present section, we consider the case where $w \in W$ satisfies $U(2)$ or $U(3)$ (see 3.3 for the notation). The results of the section are contained in Lemmas 4.1–4.3.

Lemma 4.1. *Let $w \in W$ satisfy $U(2)$ with $\mathcal{L}(w) = \{s, r\}$ (hence $sr = rs$). Then either $sw \underset{\mathrm{L}}{\sim} w$ or $rw \underset{\mathrm{L}}{\sim} w$ holds.*

Proof. By Lemma 3.2 (1), we need only to consider the case where w satisfies $U(2; 1)$, i.e., $w = sr \cdot t \cdot w_1$ with $\mathcal{L}(srw) = \{t\}$, $w_1 \in W$, $tz \neq zt$ for any $z \in \{s, r\}$. By $U(2)$ on w, we need only to consider the case where w_1 is a right extension of $v_1 v_2 \cdots v_c v_{c+1} v_{c+2}$ for some $c \geqslant 1$, where v_i, $1 \leqslant i \leqslant c+2$, are as in Fig. 2 (b). Then we claim that $\{sw, w\}$ is a primitive pair. For, let $z_0 = w$, $z_1 = t \cdot w$ and $z_i = v_{i-1} \cdot z_{i-1}$ for $1 < i \leqslant c+2$. Also, let $z_0' = sw$, $z_1' = rz_0'$, $z_2' = tz_1'$, $z_i' = v_{i-2} z_{i-1}'$ for $2 < i \leqslant c+2$. We see that z_{c+2}' is obtained from z_0' by the same sequence of star operations as z_{c+2} from z_0. We have z_0—z_0' by (1.2.3) and hence z_i—z_i' for $1 \leqslant i \leqslant c+2$ by Lemma 1.8. We also have $\mathcal{L}(z_0) \not\subseteq \mathcal{L}(z_0')$ and $\mathcal{L}(z_{c+2}') \not\subseteq \mathcal{L}(z_{c+2})$ since $s \in \mathcal{L}(z_0) \setminus \mathcal{L}(z_0')$ and $v_{c+2} \in \mathcal{L}(z_{c+2}') \setminus \mathcal{L}(z_{c+2})$ by Lemma 1.12. This proves our claim. Hence $sw \underset{\mathrm{L}}{\sim} w$ by Lemma 1.10. □

Lemma 4.2. *Assume that $w \in W$ satisfies $U(3)$ with $\mathcal{L}(w) = J := \{s, r, u\}$ and $|J| = 3$. Then there exists some $z \in J$ with $zw \underset{\mathrm{L}}{\sim} w$.*

Proof. By Lemmas 1.11, 3.2 and $U(3)$ on w, we need only to consider the cases where any $t \in \mathcal{L}(w_J w)$ commutes with at most one element in $\{s, r, u\}$, i.e., the cases (1)–(2) below by 3.3 (iv):

(1) $w = sru \cdot t \cdot w_1$ for some $w_1 \in W$, where $\mathcal{L}(w_J w) = \{t\}$, and Γ has a subgraph in Fig. 4 (a) or (b).

(2) $w = sru \cdot tv \cdot w_1$ for some $w_1 \in W$, where $\mathcal{L}(w_J w) = \{t, v\}$, and Γ has a subgraph in Fig. 4 (c).

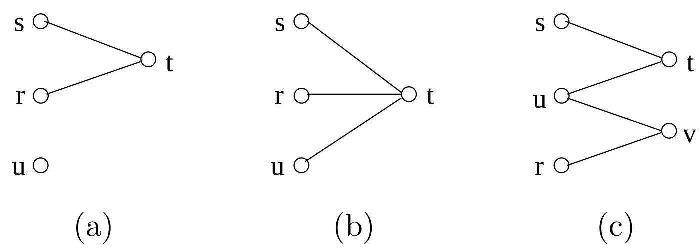


Fig. 4.

First assume that we are in the case of Fig. 4 (a). We have $\mathcal{L}(w_1) \cap J \subseteq \{s, r\}$ by $U(3;1)$ on w. If $\mathcal{L}(w_1) \cap J \neq \emptyset$, then either sw or rw can be obtained from w by a star operation. Now assume $\mathcal{L}(w_1) \cap J = \emptyset$. Then by 3.3 (iv)–(v) and $U(3;1)$ on w, we have $w_1 = v_1 v_2 \cdots v_c \cdot w_2$ for some $w_2 \in W$ and $c \geqslant 2$ with $u = v_c$, and Γ has a subgraph in Fig. 2 (a).

We claim that $\{sw, w\}$ is a primitive pair. For, let $z_0 = w$, $z_1 = t \cdot z_0$, $z_i = v_{i-1} \cdot z_{i-1}$ for $1 < i \leqslant c$. Also, let $z'_0 = sw$, $z'_1 = rz'_0$, $z'_2 = tz'_1$, $z'_i = v_{i-2} z'_{i-1}$ for $2 < i \leqslant c$. We see that z'_c is obtained from z'_0 by the same sequence of star operations as z_c from z_0. We have z'_0—z_0 by (1.2.3) and hence z_i—z'_i for $1 \leqslant i \leqslant c$ by Lemma 1.8. We also have $\mathcal{L}(z_0) \not\subseteq \mathcal{L}(z'_0)$ and $\mathcal{L}(z'_c) \not\subseteq \mathcal{L}(z_c)$ since $s \in \mathcal{L}(z_0) \setminus \mathcal{L}(z'_0)$ and $v_c \in \mathcal{L}(z'_c) \setminus \mathcal{L}(z_c)$ by Lemma 1.12 and the facts $u = v_c$ and $v_{c-1} v_c \neq v_c v_{c-1}$. So our claim is proved.

Next assume that we are in the case of Fig. 4 (b). Then by $U(3;1)$ on w, we must have $\mathcal{L}(w_1) \cap J \neq \emptyset$, say $s \in \mathcal{L}(w_1)$ without loss of generality. We can show that $\{rw, w\}$ is a primitive pair by considering the elements $z_0 = w$, $z_1 = t \cdot z_0$, $z'_0 = rw$, $z'_1 = uz'_0$ and the facts z_0—z'_0, z'_1—z_1, $r \in \mathcal{L}(z_0) \setminus \mathcal{L}(z'_0)$ and $s \in \mathcal{L}(z'_1) \setminus \mathcal{L}(z_1)$.

Finally assume that we are in the case of Fig. 4 (c). If $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$, then uw can be obtained from w by a star operation. If $\mathcal{L}(w_1) \cap \{s, r\} = \emptyset$, then by 3.3 (iv)–(v) and $U(3;1)$ on w, we need only to consider the case of $u \in \mathcal{L}(w_1)$ with one of the following cases occurring:

(i) w_1 is a right extension of either ut or uv.

(ii) $w = sur \cdot tv \cdot uy \cdot tx \cdot szy \cdot w_2$ for some $w_2 \in W$ with $W = \widetilde{E}_7$ as in Fig. 5 (a).

(iii) $w = sur \cdot tv \cdot uyz \cdot w_2$ for some $w_2 \in W$ with $W = \widetilde{D}_6$ as in Fig. 5 (b).

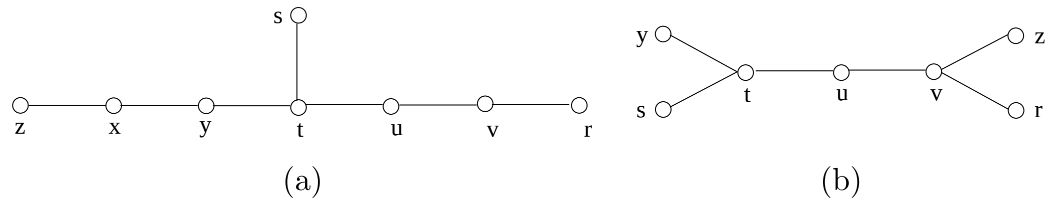


Fig. 5.

In the case (i), either rw or sw can be obtained from w by a star operation; while in any of the cases (ii)–(iii), $\{uw, w\}$ is a primitive pair. As an example, let us explain why $\{uw, w\}$ is a primitive pair in the case (ii). Let $z_0 = w$, $z_1 = t \cdot z_0$, $z_2 = v \cdot z_1$, $z_3 = y \cdot z_2$, $z_4 = u \cdot z_3$, $z_5 = x \cdot z_4$, $z_6 = t \cdot z_5$, $z_7 = s \cdot z_6$, $z_8 = z \cdot z_7$, $z'_0 = uw$, $z'_1 = sz'_0$, $z'_2 = rz'_1$, $z'_3 = tz'_2$, $z'_4 = vz'_3$, $z'_5 = yz'_4$, $z'_6 = uz'_5$, $z'_7 = tz'_6$, $z'_8 = xz'_7$. Then z'_8 is obtained from z'_0 by the same sequence of star operations as z_8 from z_0. We have z'_0—z_0 by (1.2.3) and hence z'_i—z_i for $1 \leqslant i \leqslant 8$ by Lemma 1.8. We also have $\mathcal{L}(z_0) \nsubseteq \mathcal{L}(z'_0)$ and $\mathcal{L}(z'_8) \nsubseteq \mathcal{L}(z_8)$ since $u \in \mathcal{L}(z_0) \setminus \mathcal{L}(z'_0)$ and $y \in \mathcal{L}(z'_8) \setminus \mathcal{L}(z_8)$ by Lemma 1.12. This implies that $\{uw, w\}$ is a primitive pair.

So our result follows by Lemma 1.10. □

Lemma 4.3. *Assume that $w \in W$ satisfies $U(3)$ with $\mathcal{L}(w) = J := \{s, t\}$ (hence $st \neq ts$). Then there exists some $z \in J$ with $zw \underset{L}{\sim} w$.*

Proof. By 3.3 (iv)–(v) and $U(3)$ on w, we need only to consider the following cases:

(1) $w = sts \cdot r \cdot w_1$ for some $w_1 \in W$ with $\mathcal{L}(w_1) \cap J = \emptyset$, where Γ has a subgraph in Fig. 6 (a);

(2) $w = sts \cdot rv \cdot w_1$ for some $w_1 \in W$ with $\mathcal{L}(w_1) \cap J = \emptyset$, where Γ has a subgraph in Fig. 6 (b).

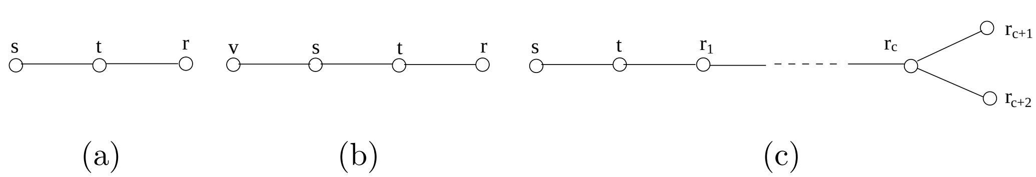


Fig. 6.

By Lemma 1.11, 3.3 (iv)–(v) and $U(3)$ on w, we have $w_1 = r_2 r_3 \cdots r_c r_{c+1} r_{c+2} \cdot w_2$ for some $w_2 \in W$ and $c \geqslant 0$, where Γ has a subgraph in Fig. 6 (c) with $r_1 = r$ (by relabelling s, t if necessary in the case (2)). We claim that $\{sw, w\}$ is a primitive pair. For, let $z_0 = w$, $z_i = r_i \cdot z_{i-1}$ for $1 \leqslant i \leqslant c+1$, and let $z'_0 = sw$, $z'_1 = tz'_0$, $z'_i = r_{i-1} z'_{i-1}$ for $1 < i \leqslant c+1$. Then z'_{c+1} is obtained from z'_0 by the same sequence of star operations as z_{c+1} from z_0. So z'_0—z_0 by (1.2.3) and hence z'_i—z_i for $1 \leqslant i \leqslant c+1$ by Lemma 1.8. We have $\mathcal{L}(z_0) \not\subseteq \mathcal{L}(z'_0)$ and $\mathcal{L}(z'_{c+1}) \not\subseteq \mathcal{L}(z_{c+1})$ since $s \in \mathcal{L}(z_0) \setminus \mathcal{L}(z'_0)$ and $r_{c+2} \in \mathcal{L}(z'_{c+1}) \setminus \mathcal{L}(z_{c+1})$ by Lemma 1.12. This proves the claim.

So our result follows by Lemma 1.10. □

§5. On the elements w satisfying $U(4)$.

Again assume that (W, S) is an irreducible finite or affine Weyl group of simply-laced type with $W \notin \{\widetilde{A}_{n-1}, \widetilde{D}_4 \mid n \geqslant 2\}$ throughout the section.

As before, any $w \in W$ not of the form w_K, $K \subseteq S$, can be written as $w = w_J \cdot w_I \cdot w_1$ with $J = \mathcal{L}(w)$, $I = \mathcal{L}(w_J w) \neq \emptyset$ and some $w_1 \in W$ by Lemma 1.11.

Lemma 5.1. *Assume that $w \in W$ satisfies $U(4;1)$. Then $zw \underset{L}{\sim} w$ for some $z \in J$.*

Proof. Write $w = w_J \cdot w_I \cdot w_1$ with $J = \mathcal{L}(w)$, $I = \mathcal{L}(w_J w)$ and some $w_1 \in W$. By 3.3 (iv)–(v) and $U(4;1)$ on w, we need only to consider the case of Γ having a subgraph Γ' as one in Fig. 7 (1)–(4), where J (respectively, I) is the vertex set of Γ' at the left (respectively, right) column:

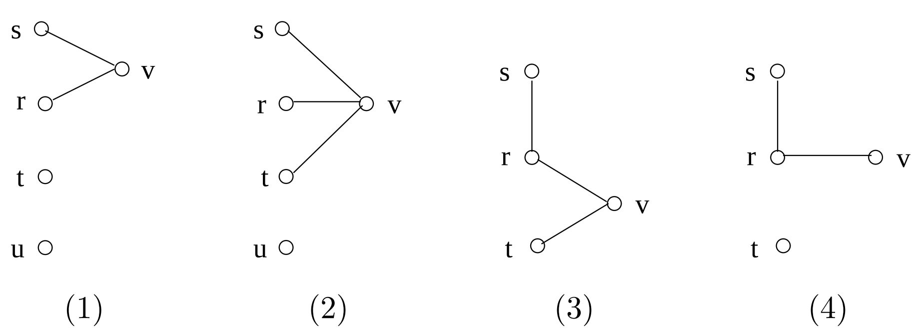


Fig. 7.

By Lemma 1.11 and $U(4;1)$ on w, we see that $\mathcal{L}(w_1) \cap J \neq \emptyset$ in any of the cases (2)–(3). In the case (2), we have $\mathcal{L}(w_1) \cap J \subseteq \{s, r, t\}$. We may assume $s \in \mathcal{L}(w_1)$

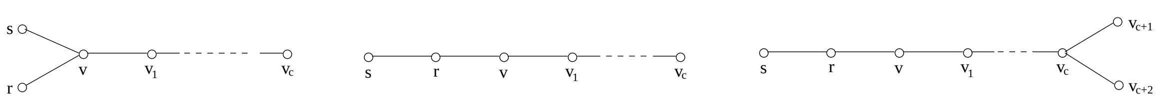
without loss of generality. Then $\{rw, w\}$ is a primitive pair. In the case (3), we have $\mathcal{L}(w_1) \cap J \subseteq \{t, r\}$. If $t \in \mathcal{L}(w_1)$, then $\{sw, w\}$ is a primitive pair; if $\mathcal{L}(w_1) \cap J = \{r\}$, then we must have $w = srst \cdot v \cdot rs \cdot w_2$ for some $w_2 \in W$ by Lemma 1.11, 3.3 (iv)–(v) and $U(4;1)$ on w. In this case, tw can be obtained from w by a $\{t, v\}$-star operation.

If $\mathcal{L}(w_1) \cap J \neq \emptyset$ in the case (1), then either rw or sw can be obtained from w by a star operation; if $r \in \mathcal{L}(w_1)$ in the case (4), then rw can be obtained from w by an $\{r, v\}$-star operation. If $\mathcal{L}(w_1) \cap J = \emptyset$ in any of the cases (1), (4), then by 3.3 (iv)–(v) and $U(4;1)$ on w, one of the following cases must occur:

(i) $w_1 = v_1 v_2 \cdots v_c \cdot w_2$ in the case (1) for some $w_2 \in W$ and $c \geqslant 2$ with $v_c \in \{t, u\}$, where Γ has a subgraph in Fig. 8 (a);

(ii) $w_1 = v_1 v_2 \cdots v_c \cdot w_2$ in the case (4) for some $w_2 \in W$ and $c \geqslant 2$ with $v_c = t$, where Γ has a subgraph in Fig. 8 (b);

(iii) $w_1 = v_1 v_2 \cdots v_c v_{c+1} v_{c+2} \cdot w_2$ in the case (4) for some $w_2 \in W$ and $c \geqslant 0$ with $t \neq v_i$, $1 \leqslant i \leqslant c+2$, where Γ has a subgraph in Fig. 8 (c).



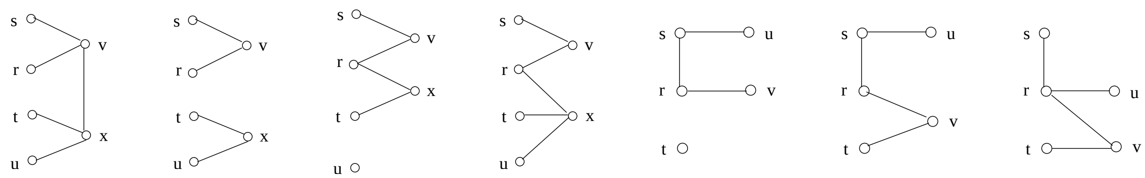
(a) (b) (c)

Fig. 8.

In either of the cases (i)-(iii), $\{sw, w\}$ is a primitive pair. So our result follows by Lemma 1.10. □

Lemma 5.2. *If* $w \in W$ *satisfies* $U(4;2)$, *then* $zw \underset{L}{\sim} w$ *for some* $z \in J$.

Proof. We may write $w = w_J \cdot w_I \cdot w_1$ for some $w_1 \in W$ with $J = \mathcal{L}(w)$ and $I = \mathcal{L}(w_J w)$ by Lemma 1.11. By 3.3 (iv)–(v) and $U(4;2)$ on w, we need only to consider the case of Γ having a subgraph Γ' as one in Fig. 9 (1)–(7), where the vertices at the left (respectively, right) column of Γ' belong to J (respectively, I).



(1) (2) (3) (4) (5) (6) (7)

Fig. 9.

First assume the case (1). Hence $W = \widetilde{D}_5$. Since $a(w) > 4$, w can be a right retraction of neither $\alpha_{k,h} := (srtu\cdot vxv)^k\cdot(srvxtuxv)^h\cdot tu$ nor $\beta_{k,h} := (srtu\cdot vxv)^k\cdot(tuxvsrvx)^h\cdot sr$ for any $k > 0$, $h \geqslant 0$ by 1.4 (3) and by the fact of $a(\alpha_{k,h}) = a(\beta_{k,h}) = 4$. Thus w must be a right extension of $z = (srtu\cdot vxv)^k\cdot z'$ for some $z' \in \{sv, rv, tx, ux\}$ and $k > 0$. Say $z' = sv$ without loss of generality. Hence

$$w = \begin{cases} sv\cdot rtu\cdot vxv\cdot(srtu\cdot vxv)^{k-1}\cdot s\cdot w_2, & \text{if } k \text{ is even,}\\ rv\cdot stu\cdot vxv\cdot(srtu\cdot vxv)^{k-1}\cdot s\cdot w_2, & \text{if } k \text{ is odd,}\end{cases}$$

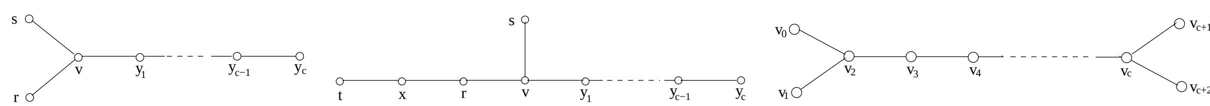
for some $w_2 \in W$. Hence sw (respectively, rw) can be obtained from w by an $\{s, v\}$-(respectively, $\{r, v\}$-) star operation if k is even (respectively, odd).

Next assume the case (2). If $\mathcal{L}(w_1)\cap J \neq \emptyset$, we may assume $s \in \mathcal{L}(w_1)$ without loss of generality, then rw can be obtained from w by an $\{r, v\}$-star operation. If $\mathcal{L}(w_1)\cap J = \emptyset$, then either v or x is a branching node of Γ. If both v and x are branching nodes of Γ, then $W = \widetilde{D}_{c+2}$ for some $c \geqslant 4$. By 3.3 (iv)–(v) and $U(4; 2)$ on w, we see that

$$\begin{aligned} w &= v_0v_1v_{c+1}v_{c+2}\cdot v_2v_c\cdot v_3v_4\cdots v_cv_{c+1}v_{c+2}v_cv_{c-1}\cdot w_2\\ &= v_0v_1v_2v_3\cdots v_{c-1}\cdot v_{c+1}v_{c+2}v_cv_{c+1}v_{c+2}v_c\cdot v_{c-1}v_c\cdot w_2\end{aligned}$$

for some $w_2 \in W$ with $(s, r, t, u, v, x) = (v_0, v_1, v_{c+1}, v_{c+2}, v_2, v_c)$ up to a graph automorphism on Γ, where Γ is in Fig. 10 (c). We claim that $\{v_0w, w\}$ is a primitive pair. For, let $z_0 = w$, $z'_0 = v_0w$, $z_i = v_{i+1}\cdot z_{i-1}$ and $z'_i = v_iz'_{i-1}$ for $1 \leqslant i \leqslant c-1$. Then z'_{c-1} is obtained from z'_0 by the same sequence of star operations as z_{c-1} from z_0. So z'_0—z_0 by

(1.2.3) and hence z_i'—z_i for any $1 \leqslant i < c$ by Lemma 1.8. Also, we have $\mathcal{L}(z_0) \nsubseteq \mathcal{L}(z_0')$ and $\mathcal{L}(z_{c-1}') \nsubseteq \mathcal{L}(z_{c-1})$ since $v_0 \in \mathcal{L}(z_0) \setminus \mathcal{L}(z_0')$ and $v_{c+1} \in \mathcal{L}(z_{c-1}') \setminus \mathcal{L}(z_{c-1})$. This proves our claim. Now assume that exactly one (say v without loss of generality) of v, x is a branching node of Γ, then by Lemma 1.11, we have $w = srtu \cdot vx \cdot y_1y_2 \cdots y_c$ for some $c \geqslant 2$ with Γ having a subgraph in Fig. 10 (a) and $y_c \in \{t, u\}$, Then $\{w, sw\}$ is a primitive pair.



(a) (b) (c)

Fig. 10.

Next assume the case (3). By 3.3 (iv)–(v) and $U(4;2)$ on w, either $\mathcal{L}(w_1) \cap \{s, t\} \neq \emptyset$, or w_1 is a right extension of rv or rx. In the former case, rw can be obtained from w by a star operation; in the latter case, tw or sw could be obtained from w by a star operation.

Next assume the case (4). By $U(4;2)$ on w, we must have $\mathcal{L}(w_1) \cap J \neq \emptyset$. If $s \in \mathcal{L}(w_1)$, then rw can be obtained from w by an $\{r, v\}$-star operation; if $\mathcal{L}(w_1) \cap \{t, u\} \neq \emptyset$, then either $\{tw, w\}$ or $\{uw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap J = \{r\}$, then w_1 is a right extension of either rx or rv. Hence sw can be obtained from w by an $\{s, v\}$-star operation in the former case, and $\{tw, w\}$ is a primitive pair in the latter case.

Assume the case (5). If $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$, then either sw or rw can be obtained from w by a star operation. Now assume $\mathcal{L}(w_1) \cap \{s, r\} = \emptyset$. By 3.3 (iv)–(v) and $U(4;2)$ on w, one of the following cases must occur by relabelling elements of S if necessary:

(i) $w = srst \cdot uv \cdot u_1u_2 \cdots u_c \cdot w_2$ for some $w_2 \in W$ and some $c \geqslant 2$ with $t = u_c$, where Γ has a subgraph in Fig. 11 (a);

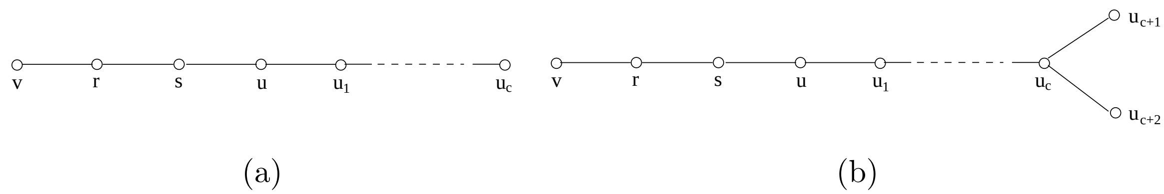


Fig. 11.

(ii) $w = srst \cdot uv \cdot u_1u_2 \cdots u_cu_{c+1}u_{c+2} \cdot w_2$ for some $w_2 \in W$ and some $c \geqslant 0$, where Γ has a subgraph in Fig. 11 (b), and $t \neq u_i$ for any $1 \leqslant i \leqslant c+2$.

In either of the cases (i), (ii), $\{rw, w\}$ is a primitive pair.

Now assume the case (6). If $s \in \mathcal{L}(w_1)$, then sw can be obtained from w by an $\{s, u\}$-star operation; if $\emptyset \neq \mathcal{L}(w_1) \cap J \subseteq \{r, t\}$, then either $\{sw, w\}$ or $\{rw, w\}$ is a primitive pair. Now assume $\mathcal{L}(w_1) \cap \{s, r, t\} = \emptyset$. By 3.3 (iv)–(v) and $U(4; 2)$ on w, we have $w = srst \cdot uv \cdot u_1u_2 \cdots u_cu_{c+1}u_{c+2} \cdot w_2$ for some $w_2 \in W$ and some $c \geqslant 0$, where Γ has a subgraph in Fig. 12. Hence $\{rw, w\}$ is a primitive pair.

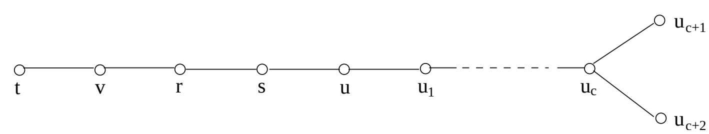


Fig. 12.

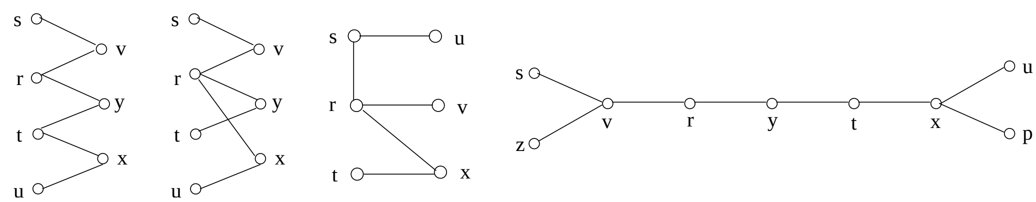
Finally assume the case (7). By $U(4; 2)$ on w, we have $\mathcal{L}(w_1) \cap \{t, r\} \neq \emptyset$. If $t \in \mathcal{L}(w_1)$, then $\{sw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap \{r, t\} = \{r\}$, then w_1 must be a right extension of rv, $rusr$ or rsu_1, where $u_1 \in S \setminus \{r, t\}$ satisfies either $u_1u \neq uu_1$ or $u_1v \neq vu_1$. When w_1 is a right extension of rv (respectively, $rusr$), the element rw (respectively, tw) can be obtained from w by an $\{r, u\}$- (respectively, $\{t, v\}$-) star operation; when w_1 is a right extension of rsu_1, $\{sw, w\}$ is a primitive pair.

Hence our result follows by Lemma 1.10. $\square$

Lemma 5.3. *If $w \in W$ satisfies $U(4; 3)$, then $zw \underset{L}{\sim} w$ for some $z \in J$.*

Proof. Write $w = w_J \cdot w_I \cdot w_1$ with $J = \mathcal{L}(w)$ and $I = \mathcal{L}(w_Jw)$ for some $w_1 \in W$. By 3.3 (iv)–(v) and $U(4; 3)$ on w, we need only to consider the case of Γ having a subgraph

Γ' as one in Fig. 13 (a)–(c), where J (respectively, I) consists of all vertices of Γ' at the left (respectively, right) column:



(a) (b) (c) (d)

Fig. 13.

In the case (a), we have $\mathcal{L}(w_1)\cap J \neq \emptyset$ by $U(4;3)$ on w. If $\mathcal{L}(w_1)\cap\{s,u\} \neq \emptyset$ then either rw or tw can be obtained from w by a star operation. Now assume $\mathcal{L}(w_1)\cap J \subseteq \{r,t\}$. By 3.3 (iv)–(v) and $U(4;3)$ on w, we see that w_1 is a right extension of rv, ry, ty, tx or $rtzp$, where the last case occurs only when $W = \widetilde{D}_8$ (see Fig. 13 (d)). If w_1 is a right extension of rv, ry, ty or tx then at least one of the elements tw, sw, uw, rw can be obtained from w by a star operation. If w_1 is a right extension of $rtzp$, then $\{sw, w\}$ is a primitive pair.

In the case (b), we have $W = \widetilde{E}_6$. If $\mathcal{L}(w_1)\cap\{s,t,u\} \neq \emptyset$, then rw can be obtained from w by a star operation. If $\mathcal{L}(w_1)\cap\{s,t,u\} = \emptyset$, then $\mathcal{L}(w_1) = \{r\}$. By 3.3 (iv)–(v) and $U(4;3)$ on w, we see that w_1 must be a right extension of $rvyr$, $rvxr$ or $rxyr$. Say $rvyr$ without loss of generality. Then uw can be obtained from w by a $\{u,x\}$-star operation.

In the case (c), $W \in \{E_i, \widetilde{E}_i \mid i = 6,7,8\}$. By $U(4;3)$ on w, we have $\mathcal{L}(w_1)\cap\{s,r,t\} \neq \emptyset$. If $s \in \mathcal{L}(w_1)$, then sw can be obtained from w by an $\{s,u\}$-star operation. If $t \in \mathcal{L}(w_1)$, then $\{sw, w\}$ is a primitive pair. Now assume $\mathcal{L}(w_1)\cap\{s,r,t\} = \{r\}$. Then w_1 must be a right extension of either rv or rx by 3.3 (iv)–(v) and $U(4;3)$ on w. We have a primitive pair $\{rw, w\}$ if w_1 is a right extension of rv. The element rw can be obtained from w by an $\{r,v\}$-star operation if w_1 is a right extension of rx. $\square$

§6. On the elements w satisfying $U(5)$.

Let $w = w_J \cdot w_I \cdot w_1 \in W$ be with $J = \mathcal{L}(w)$ and $I = \mathcal{L}(w_J w)$ for some $w_1 \in W$. Then w satisfies $U(5)$ only if the Coxeter graph of W has a subgraph Γ' as one in Fig. 14 with $I \cup J$ its vertex set, where all vertices of Γ' at left (respectively, right) column belong to J (respectively, I).

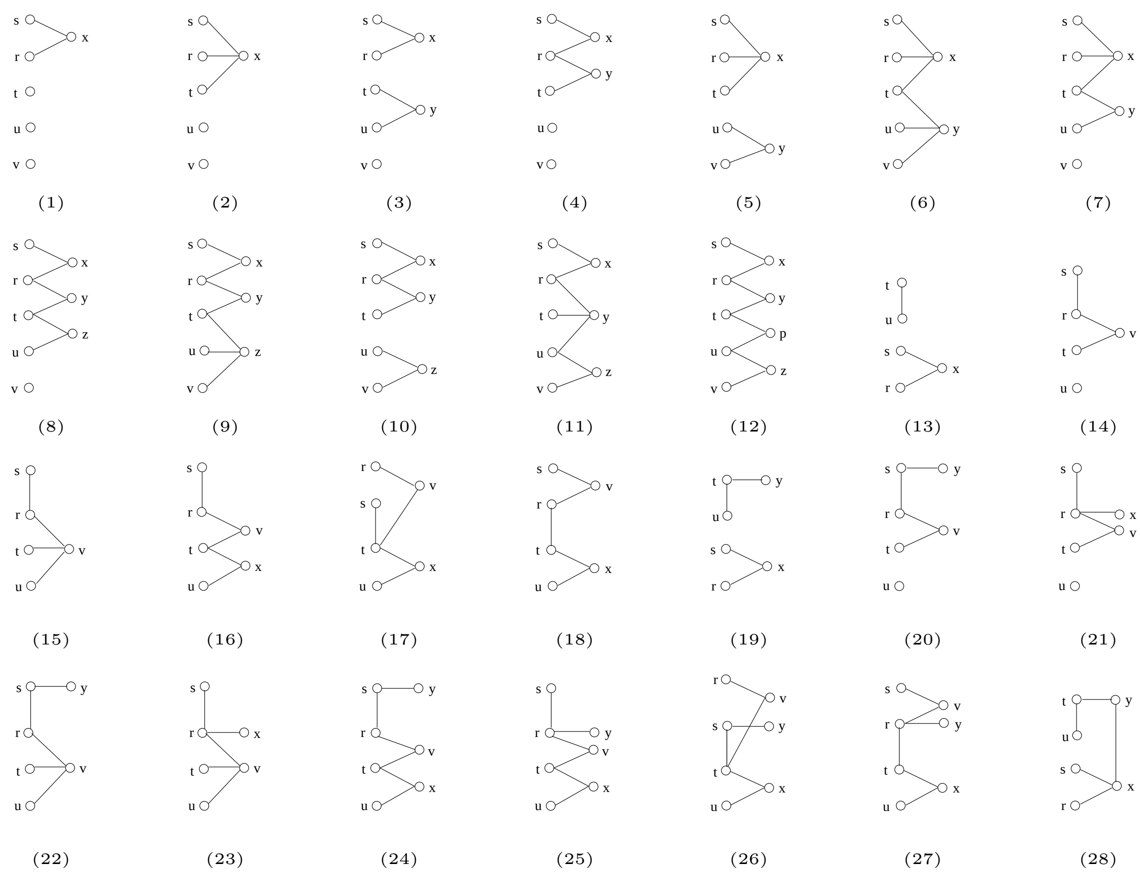


Fig. 14

Lemma 6.1. *If $w \in W$ satisfies $U(5)$, then $zw \underset{L}{\sim} w$ for some $z \in J$.*

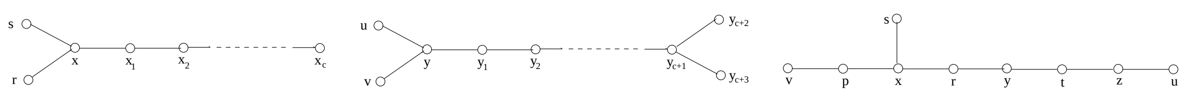
Proof. We prove our result by a case-by-case argument. By 1.7, Lemma 1.10, 3.3 (iv)–(v) and $U(5)$ on w, we need only to show that in any of the cases (1)-(28), there exists some $z \in J$ such that either $\{zw, w\}$ is a primitive pair, or zw is obtained from w by a star operation.

(1) If $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$, then sw or rw can be obtained from w by a star operation. If $\mathcal{L}(w_1) \cap \{s, r\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5; 1)$ on w, the graph Γ contains x

as a branching node and any $y \in \mathcal{L}(w_1)$ satisfies $yx \neq xy$, hence there is an expression $w = w_J w_I \cdot x_1 x_2 \cdots x_c \cdot w_2$ for some $c \geqslant 2$ and $w_2 \in W$ such that $x_c \in \{t, u, v\}$ and Γ has a subgraph as in Fig. 15 (a). Then $\{sw, w\}$ is a primitive pair.

(2) By 3.3 (iv)–(v) and $U(5;1)$ on w, we have $\emptyset \neq \mathcal{L}(w_1) \subseteq \{s, r, t\}$, say $s \in \mathcal{L}(w_1)$ without loss of generality. Then $\{rw, w\}$ is a primitive pair.

(3) If $\mathcal{L}(w_1) \cap \{s, r, t, u\} \neq \emptyset$, we may assume $s \in \mathcal{L}(w_1)$ without loss of generality, then rw can be obtained from w by an $\{r, x\}$-star operation. If $\mathcal{L}(w_1) \cap \{s, r, t, u\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that at least one (say x without loss of generality) of x, y is a branching node of Γ such that $w_1 = x_1 x_2 \cdots x_c \cdot w_2$ for some $c \geqslant 2$ and $w_2 \in W$ satisfying $x_c \in \{t, u, v\}$, where Γ has a subgraph as in Fig. 15 (a). Hence $\{sw, w\}$ is a primitive pair.



(a) (b) (c)

Fig. 15.

(4) By 3.3 (iv)–(v) and $U(5;2)$ on w, we must have $\emptyset \neq \mathcal{L}(w_1) \cap J \subseteq \{s, r, t\}$. If $\mathcal{L}(w_1) \cap \{s, t\} \neq \emptyset$, then rw can be obtained from w by a star operation. If $\mathcal{L}(w_1) \cap J = \{r\}$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, the element w_1 must be a right extension of either rx or ry. Hence either tw or sw can be obtained from w by a star operation.

(5) First assume $\mathcal{L}(w_1) \cap \{s, r, t\} \neq \emptyset$, say $s \in \mathcal{L}(w_1)$ without loss of generality. If w_1 is a right extension of $srtxs$ then sw can be obtained from w by an $\{s, x\}$-star operation; otherwise, $\{rw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap \{u, v\} \neq \emptyset$, say $u \in \mathcal{L}(w_1)$, then vw can be obtained from w by a $\{v, y\}$-star operation. If $\mathcal{L}(w_1) \cap \{s, r, t, u, v\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that $W = \widetilde{D}_n$, $n \geqslant 7$, y is a branching node of Γ and $w_1 = y_1 y_2 \cdots y_c y_{c+1} y_{c+2} y_{c+3} y_{c+1} \cdot w_2$ for some $c > 1$ such that Γ is in Fig. 15 (b), $\{y_c, y_{c+2}, y_{c+3}\} = \{s, r, t\}$ and $y_{c+1} = x$. We claim that $\{uw, w\}$ is a primitive pair. For, let $z_0 = w$, $z_1 = y \cdot z_0$, $z_i = y_{i-1} \cdot z_{i-1}$ for $1 < i \leqslant c$, and $z'_0 = uw$, $z'_1 = vz'_0$, $z'_2 = yz'_1$, $z'_i = y_{i-2} z'_{i-1}$ for $2 < i \leqslant c$. Then z'_c is obtained from z'_0 by the same sequence of star

operations as z_c from z_0. We have z_0—z_0' by (1.2.3) and hence z_i'—z_i for any $1 \leqslant i \leqslant c$ by Lemma 1.8. We also have $\mathcal{L}(z_0) \nsubseteq \mathcal{L}(z_0')$ and $\mathcal{L}(z_c') \nsubseteq \mathcal{L}(z_c)$ since $u \in \mathcal{L}(z_0) \setminus \mathcal{L}(z_0')$ and $y_c \in \mathcal{L}(z_c') \setminus \mathcal{L}(z_c)$ by Lemma 1.12. This proves our claim.

(6) $W = \widetilde{D}_6$. If $\mathcal{L}(w_1) \cap \{s, r, u, v\} \neq \emptyset$, then $\{tw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap J = \{t\}$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, the element w_1 must be a right extension of either tx or ty, say tx without loss of generality. Then $\{uw, w\}$ is a primitive pair.

(7) By 3.3 (iv)–(v) and $U(5;2)$ on w, we have $\mathcal{L}(w_1) \subseteq \{s, r, t, u\}$. If $u \in \mathcal{L}(w_1)$ then tw can be obtained from w by a $\{t, y\}$-star operation. If $\mathcal{L}(w_1) \cap \{s, r\} \neq \emptyset$, say $s \in \mathcal{L}(w_1)$ without loss of generality, then $\{rw, w\}$ is a primitive pair. If we are not in any of the above cases, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that w_1 must be a right extension of either tx or ty. In the former case, uw can be obtained from w by a $\{u, y\}$-star operation; while in the latter case, $\{sw, w\}$ is a primitive pair.

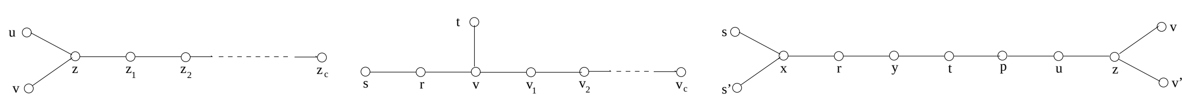
(8) If $\mathcal{L}(w_1) \cap \{s, u\} \neq \emptyset$, say $s \in \mathcal{L}(w_1)$ without loss of generality, then rw can be obtained from w by an $\{r, x\}$-star operation. If $\mathcal{L}(w_1) \cap \{s, u\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5;3)$ on w, we see that w_1 must be a right extension of rx, ry, ty, tz or pv, where the last case occurs only if $W = \widetilde{E}_8$ with Γ in Fig. 15 (c). If w_1 is a right extension of rx (respectively, ry), then tw (respectively, sw) can be obtained from w by a $\{t, y\}$- (respectively, $\{s, x\}$-) star operation. Similarly for the cases where w_1 is a right extension of ty or tz. If w_1 is a right extension of pv then $\{sw, w\}$ is a primitive pair.

(9) We have $W \in \{D_n, \widetilde{D}_n, \widetilde{E}_8 \mid n \geqslant 8\}$. If $s \in \mathcal{L}(w_1)$ then rw can be obtained from w by an $\{r, x\}$-star operation. If $\mathcal{L}(w_1) \cap \{u, v\} \neq \emptyset$, say $u \in \mathcal{L}(w_1)$ without loss of generality, then $\{vw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap \{s, u, v\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5;3)$ on w, we see that w_1 must be a right extension of rx, ry, ty or tz. If w_1 is a right extension of rx (respectively, ry, tz), then tw (respectively, sw, rw) can be obtained from w by a $\{t, y\}$- (respectively, $\{s, x\}$-, $\{r, y\}$-) star operation. If w_1 is a right extension of ty, then $\{uw, w\}$ is a primitive pair.

(10) If $\mathcal{L}(w_1) \cap \{s, t, u, v\} \neq \emptyset$, then at least one of rw, vw and uw can be obtained

from w by a star operation. If $\mathcal{L}(w_1)=\{r\}$, then by 3.3 (iv)–(v) and $U(5;3)$ on w, the element w_1 must be a right extension of either rx or ry. When w_1 is a right extension of rx (respectively, ry), the element tw (respectively, sw) can be obtained from w by a $\{t,y\}$- (respectively, $\{s,x\}$-) star operation. If $\mathcal{L}(w_1)\cap\{s,r,t,u,v\}=\emptyset$, then by 3.3 (iv)–(v) and $U(5;3)$ on w, we see that z is a branching node of Γ, $W\in\{D_n,\widetilde{D}_n\mid n\geqslant 9\}$, and $w_1=z_1z_2\cdots z_c\cdot w_2$ for some $w_2\in W$ and $c\geqslant 3$, where Γ has a subgraph as in Fig. 16 (a) with $(z_{c-1},z_c)\in\{(s,x),(t,y)\}$. In this case, $\{uw,w\}$ is a primitive pair.

(11) We have $W=\widetilde{E}_7$. If $\mathcal{L}(w_1)\cap\{s,v\}\neq\emptyset$, then either rw or uw can be obtained from w by a star operation. If $t\in\mathcal{L}(w_1)$, then $\{rw,w\}$ is a primitive pair. If we are not in any of the above cases, then w_1 must be a right extension of rx, ry, uy or uz by 3.3 (iv)–(v) and $U(5;3)$ on w. If w_1 is a right extension of ry (respectively, uy), then sw (respectively, vw) can be obtained from w by an $\{s,x\}$- (respectively, $\{v,z\}$-) star operation. If w_1 is a right extension of rx (respectively, uz), then $\{uw,w\}$ (respectively, $\{rw,w\}$) is a primitive pair.



(a) (b) (c)

Fig. 16.

(12) By $U(5;4)$ on w, we have $\mathcal{L}(w_1)\cap J\neq\emptyset$. If $\mathcal{L}(w_1)\cap\{s,v\}\neq\emptyset$, then either rw or uw can be obtained from w by a star operation. If $\mathcal{L}(w_1)\cap J\subseteq\{r,t,u\}$, then by 3.3 (iv)–(v) and $U(5;4)$ on w, the element w_1 must be a right extension of rx, ry, ty, tp, up, uz or $rtus'v'$, where the last case occurs only if $W=\widetilde{D}_{10}$ and Γ is as in Fig. 16 (c). Then tw (respectively, sw, uw, rw, vw, tw) can be obtained from w by a star operation if w_1 is a right extension of rx (respectively, ry, ty, tp, up, uz). If w_1 is a right extension of $rtus'v'$ then $\{sw,w\}$ is a primitive pair.

(13) If $\mathcal{L}(w_1)\cap\{s,r\}\neq\emptyset$, then sw or rw can be obtained from w by a star operation. If $\mathcal{L}(w_1)\cap\{s,r\}=\emptyset$, then by 3.3 (iv)–(v) and $U(5;1)$ on w, we see that x is a branching

node of Γ and any $y \in \mathcal{L}(w_1)$ satisfies $yx \neq xy$, hence $w_1 = x_1x_2\cdots x_c \cdot w_2$ for some $c \geqslant 3$ and $w_2 \in W$ such that $\{x_{c-1}, x_c\} = \{t, u\}$ and Γ has a subgraph as in Fig. 15 (a). Then $\{sw, w\}$ is a primitive pair (comparing with case (1)).

(14) We claim that $\mathcal{L}(w_1) \cap \{r, t\} \neq \emptyset$. For otherwise, by Lemma 1.11, 3.3 (iv)–(v) and $U(5;1)$ on w, Γ would have a subgraph as in Fig. 16 (b) with v a branching node, $w_1 = v_1v_2\cdots v_c$ for some $c \geqslant 2$, and $v_i = u$ for at most one i, $1 < i \leqslant c$. Then $a(w) = 5$, a contradiction. The claim is proved. Now that $\mathcal{L}(w_1) \cap \{r, t\} \neq \emptyset$. If $t \in \mathcal{L}(w_1)$ then $\{sw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap \{r, t\} = \{r\}$, then $w_1 = rs \cdot w_2$ for some $w_2 \in W$, so tw can be obtained from w by a $\{t, v\}$-star operation.

(15) By 3.3 (iv)–(v) and $U(5;1)$ on w, we see that $\mathcal{L}(w_1) \subseteq \{r, t, u\}$ and that w_1 is a right extension of rtv, ruv, tuv or rs. Then at least one of $\{rw, w\}$, $\{sw, w\}$ and $\{uw, w\}$ is a primitive pair.

(16) We have $\mathcal{L}(w_1) \cap \{r, t, u\} \neq \emptyset$ by 3.3 (iv)–(v) and $U(5;2)$ on w. If $u \in \mathcal{L}(w_1)$, then tw can be obtained from w by a $\{t, x\}$-star operation. If $\mathcal{L}(w_1) \cap \{r, t, u\} \subseteq \{t, r\}$, then w_1 must be a right extension of rs, tv or tx by 3.3 (iv)–(v) and $U(5;2)$ on w. When w_1 is a right extension of rs (respectively, tv), the element tw (respectively, uw) can be obtained from w by a $\{t, v\}$- (respectively, $\{u, x\}$-) star operation. When w_1 is a right extension of tx, we have a primitive pair $\{sw, w\}$.

(17) By 3.3 (iv)–(v) and $U(5;2)$ on w, we have $W \in \{E_i, \widetilde{E}_i \mid i = 6, 7, 8\}$ and $\emptyset \neq \mathcal{L}(w_1) \subseteq \{t, r, u\}$ with t the unique branching node of Γ. If $\mathcal{L}(w_1) \cap \{r, u\} \neq \emptyset$, then $\{sw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) = \{t\}$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that w_1 is a right extension of tvx, $tvst$ or $txst$. Then either $\{rw, w\}$ or $\{sw, w\}$ is a primitive pair.

(18) We have $\mathcal{L}(w_1) \cap J \neq \emptyset$ by $U(5;2)$ on w. If $\mathcal{L}(w_1) \cap \{s, u\} \neq \emptyset$, then either $\{rw, w\}$ or $\{tw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap \{s, u\} = \emptyset$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that w_1 must be a right extension of trv, $trytyr$, rtx or $rtzrzt$, where w_1 is a right extension of $trytyr$ (respectively, $rtzrzt$) only if Γ has a subgraph as in Fig. 17 (a) (respectively, Fig. 17 (b)). By symmetry, we need only to consider

the case where w_1 is a right extension of trv or $trytyr$. If w_1 is a right extension of trv then uw can be obtained from w by a $\{u,x\}$-star operation. If w_1 is a right extension of $trytyr$ then $\{rw,w\}$ is a primitive pair.

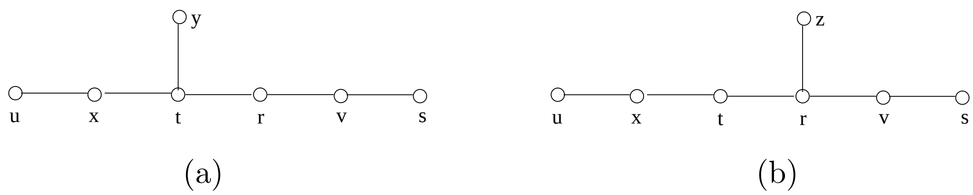


Fig. 17.

(19) We have $\mathcal{L}(w_1)\cap J\subseteq\{s,r,t\}$. If $\mathcal{L}(w_1)\cap\{s,r,t\}\neq\emptyset$, then sw, rw or uw can be obtained from w by a star operation. If $\mathcal{L}(w_1)\cap\{s,r,t\}=\emptyset$, then by 3.3 (iv)–(v) and $U(5;2)$ on w, we see that $w_1=x_1x_2\cdots x_c\cdot w_2$ for some $c\geqslant 4$ and $w_2\in W$ with $(x_{c-2},x_{c-1},x_c)\in\{(y,t,u),(u,t,y)\}$ and Γ having a subgraph as in Fig. 15 (a). Then $\{sw,w\}$ is a primitive pair (comparing with case (13)).

(20) We have $\mathcal{L}(w_1)\cap J\subseteq\{s,r,t\}$. If $t\in\mathcal{L}(w_1)$ (respectively, $r\in\mathcal{L}(w_1)$) then $\{sw,w\}$ (respectively, $\{rw,w\}$) is a primitive pair. If $s\in\mathcal{L}(w_1)$ then sw can be obtained from w by an $\{s,y\}$-star operation. Now assume $\mathcal{L}(w_1)\cap J=\emptyset$. By 3.3 (iv)–(v) and $U(5;2)$ on w, we see that w_1 is a right extension of either (i) $y_1y_2\cdots y_c$ with $y_c=u$ for some $c\geqslant 2$ and Γ has a subgraph as in Fig. 18 (a), or (ii) $y_1y_2\cdots y_cy_{c+1}y_{c+2}$ with $y_i\neq u$ for any $1\leqslant i\leqslant c+2$, some $c\geqslant 0$ and Γ has a subgraph as in Fig. 18 (b). In either case, $\{rw,w\}$ is a primitive pair (comparing with case (14)).

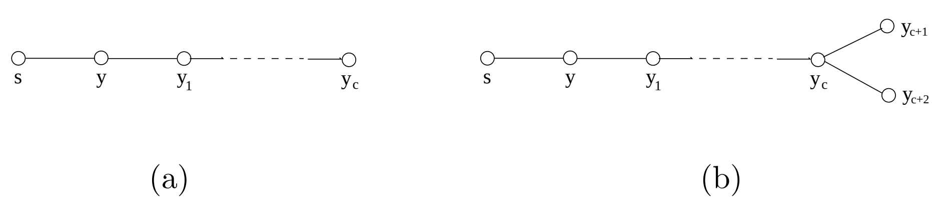


Fig. 18.

(21) We have $\mathcal{L}(w_1)\cap J\subseteq\{r,t\}$ by 3.3 (iv)–(v) and $U(5;2)$ on w. If $t\in\mathcal{L}(w_1)$, then $\{sw,w\}$ is a primitive pair. If $\mathcal{L}(w_1)\cap J=\{r\}$, then w_1 must be a right extension of either rv or $rxsr$. In the former case, rw can be obtained from w by an $\{r,x\}$-star

operation; in the latter case, tw can be obtained from w by a $\{t, v\}$-star operation (comparing with case (14)).

(22) If $s \in \mathcal{L}(w_1)$ then sw can be obtained from w by an $\{s, y\}$-star operation. If $s \notin \mathcal{L}(w_1)$ then by 3.3 (iv)–(v) and $U(5; 2)$ on w, we see that w_1 is a right extension of tuv, trv, ruv, rsy or $y_1 y_2 \cdots y_c y_{c+1} y_{c+2}$ for some $c \geqslant 1$, where in the last case, Γ has a subgraph as in Fig. 18 (b). Hence at least one of $\{sw, w\}$, $\{rw, w\}$, $\{uw, w\}$ is a primitive pair (comparing with case 15).

(23) We have $W = \widetilde{D}_5$ and $\mathcal{L}(w_1) \subseteq \{r, t, u\}$. If $\mathcal{L}(w_1) \cap \{t, u\} \neq \emptyset$ then $\{sw, w\}$ is a primitive pair. Now assume $\mathcal{L}(w_1) = \{r\}$. By 3.3 (iv)–(v) and $U(5; 2)$ on w, we see that w_1 is a right extension of either rxv or $rxsr$. In the former case, rw can be obtained from w by an $\{r, x\}$-star operation; while in the latter case, $\{tw, w\}$ is a primitive pair (comparing with case 15).

(24) If $u \in \mathcal{L}(w_1)$ (respectively, $s \in \mathcal{L}(w_1)$) then tw (respectively, sw) can be obtained from w by a $\{t, x\}$- (respectively, $\{s, y\}$-) star operation. If $r \in \mathcal{L}(w_1)$ then $\{rw, w\}$ is a primitive pair. If $\mathcal{L}(w_1) \cap J = \{t\}$ then w_1 must be a right extension of either tv or tx by 3.3 (iv)–(v) and $U(5; 3)$ on w, hence either uw can be obtained from w by a $\{u, x\}$-star operation, or $\{rw, w\}$ is a primitive pair. Now assume $\mathcal{L}(w_1) \cap J = \emptyset$. By 3.3 (iv)–(v) and $U(5; 3)$ on w, we see that w_1 must be a right extension of $y_1 y_2 \cdots y_c y_{c+1} y_{c+2}$ for some $c \geqslant 0$ with Γ having a subgraph as in Fig. 18 (b). So $\{rw, w\}$ is a primitive pair (comparing with case 16).

(25) We have $W \in \{D_n, \widetilde{D}_n, E_8, \widetilde{E}_8 \mid n \geqslant 7\}$. By 3.3 (iv)–(v) and $U(5; 3)$ on w, we have $\emptyset \neq \mathcal{L}(w_1) \cap J \subseteq \{r, t, u\}$. If $u \in \mathcal{L}(w_1)$, then tw can be obtained from w by a $\{t, x\}$-star operation. If $\mathcal{L}(w_1) \cap J \subseteq \{r, t\}$, then w_1 must be a right extension of tv, tx, rv or $rysr$. When w_1 is a right extension of tv (respectively, rv), the element uw (respectively, rw) can be obtained from w by a $\{u, x\}$- (respectively, $\{r, y\}$-) star operation. When w_1 is a right extension of tx or $rysr$, $\{sw, w\}$ is a primitive pair (comparing with case 16).

(26) We have $W = \widetilde{E}_6$ and $\emptyset \neq \mathcal{L}(w_1) \subseteq J$. If $s \in \mathcal{L}(w_1)$ then sw can be obtained

from w by an $\{s,y\}$-star operation. If $\mathcal{L}(w_1)\cap\{r,u\}\neq\emptyset$ then $\{sw,w\}$ is a primitive pair. Now assume $\mathcal{L}(w_1)=\{t\}$. Then by 3.3 (iv)–(v) and $U(5;3)$ on w, we see that w_1 must be a right extension of tv or tx. Hence $\{tw,w\}$ is a primitive pair (comparing with case 17).

(27) We have $W\in\{E_i,\widetilde{E}_i\mid i=7,8\}$ and $\emptyset\neq\mathcal{L}(w_1)\subseteq J$ by 3.3 (iv)–(v) and $U(5;3)$ on w. If $r\in\mathcal{L}(w_1)$ then rw can be obtained from w by an $\{r,y\}$-star operation. If $u\in\mathcal{L}(w_1)$ (respectively, $\mathcal{L}(w_1)\cap\{s,t\}\neq\emptyset$) then $\{rw,w\}$ (respectively, $\{tw,w\}$) is a primitive pair. (comparing with case 18).

(28) By 3.3 (iv)–(v) and $U(5;3)$ on w, we see that w_1 must be a right extension of sx, rx, ty, tu or $srxy$. If w_1 is a right extension of sx (respectively, rx, ty) then rw (respectively, sw, tw) can be obtained from w by an $\{r,x\}$- (respectively, $\{s,x\}$-, $\{t,y\}$-) star operation. If w_1 is a right extension of tu (respectively, $srxy$) then $\{sw,w\}$ (respectively, $\{uw,w\}$) is a primitive pair (comparing with case 19). □

Now we are ready to prove Proposition 2.4.

6.2. *Proof of Proposition 2.4.* Let $w=w_J\cdot w_1\in W$ be as in Proposition 2.4 with $n:=a(w)>m:=\ell(w_J)\leqslant 5$. By Lemma 2.1 and Proposition 2.3, we may assume $W\notin\{\widetilde{A}_{n-1},\widetilde{D}_4\mid n\geqslant 2\}$. Then w satisfies $U(m)$. If $m=1$, say $J=\{s\}$, then sw can be obtained from w by a star operation. If $m>1$, then our result follows by Lemmas 4.1-4.3, 5.1-5.3 and 6.1. □

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