THE VERIFICATION OF A CONJECTURE ON LEFT CELLS OF CERTAIN COXETER GROUPS

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Abstract. Let W be a crystallographic group with its a-function upper bounded. In [15], the author showed that if $x \underset{L}{\sim} y$ in W then $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$, and conjectured that the reverse conclusion should also be true. In the present paper, we show this conjecture in the cases when W is an irreducible Weyl group W' and when W is an irreducible affine Weyl group W_a with the following cases excepted: W_a has type $\widetilde{F}_4$, the element x in that conjecture satisfies $\mathcal{R}(x) \in \{\{s_0, s_1, s_2\},\ \{s_3, s_4\}\}$ and $a(x) \in \{6, 7, 9, 10, 13, 16\}$.

§1. Introduction.

1.1 Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\leq$ be the Bruhat order on W. For $w \in W$, we denote by $\ell(w)$ the length of w. Let $A = \mathbb{Z}[u]$ be the ring of polynomials in an indeterminate u with integer coefficients. For each ordered pair $y, w \in W$, Kazhdan and Lusztig associated a polynomial $P_{y,w} \in A$, (known as a Kazhdan-Lusztig polynomial), which satisfies the conditions: $P_{y,w} = 0$ if $y \not\leq w$, $P_{w,w} = 1$, and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$ (see [3]). For $y, w \in W$, we write y—w if either $\deg P_{y,w}$ or $\deg P_{w,y}$ reaches $(1/2)(|\ell(w) - \ell(y)| - 1)$. The following fact is simple and useful: if $x, y \in W$ satisfy $y < x$ and $\ell(y) = \ell(x) - 1$, then we have y—x.

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To each element $x \in W$, we associate two subsets of S as below.

(1.1.1) $$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

1.2 The preorders $\underset{L}{\leq}, \underset{R}{\leq}, \underset{LR}{\leq}$ and the associated equivalence relations $\underset{L}{\sim}, \underset{R}{\sim}, \underset{LR}{\sim}$ on W are defined in [3]. The equivalence classes of W with respect to $\underset{L}{\sim}$ (resp. $\underset{R}{\sim}, \underset{LR}{\sim}$) are called left (resp. right , two-sided) cells. The set of all left (resp. right, two-sided) cells of W is partially ordered under the relation $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}, \underset{LR}{\leq}$).

1.3 Lusztig defined a function a: $W \longrightarrow \mathbb{N}\bigcup\{\infty\}$ which satisfies the following properties in the case when W is a crystallographic group (see [4]):

(1) $x \underset{LR}{\leq} y \Longrightarrow a(x) \geq a(y)$. In particular, $x \underset{LR}{\sim} y \Longrightarrow a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ on a (left, right or two-sided) cell Γ of W by $a(x)$ for any $x \in \Gamma$.

(2) Let $x = yz$ with $\ell(x) = \ell(y) + \ell(z)$ for $x, y, z \in W$. Then $x \underset{L}{\leq} z$, $x \underset{R}{\leq} y$ and hence $a(x) \geq a(y), a(z)$.

Certain special families of crystallographic groups satisfy more properties as below.

(3) If W is either a Weyl group or an affine Weyl group, then $a(z) \leq |\Phi|/2$, for any $z \in W$, where Φ is the root system determined by W;

(4) Let G be the connected reductive algebraic group over $\mathbb{C}$ associated with an irreducible affine Weyl group W_a. Then there exists a bijection $\mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ from the set of unipotent conjugacy classes in G to the set of two-sided cells in W_a. This bijection satisfies the equation $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$, where u is any element in $\mathbf{u}$, and $\dim \mathfrak{B}_u$ is the dimension of the variety of Borel subgroups of G containing u.

The above properties of function a were shown by Lusztig in his papers [4][5][6].

1.4 To each element $x \in W$, we associate a set $\Sigma(x)$ of all left cells Γ of W satisfying the condition that there exists some element $y \in \Gamma$ with y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$.

It is obvious that if W is a crystallographic group then any $\Gamma \in \Sigma(x)$ is in the two-sided cell of W containing x.

The following result was shown in the auther's previous paper [15].

Theorem. *Let W be a crystallographic group with a-function upper bounded. If $x \underset{L}{\sim} y$ in W, then $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$.*

From now on, we always assume that W is a crystallographic group with a-function upper bounded. Note that Weyl groups and affine Weyl groups are such groups. By this theorem, we can use the notations $\Sigma(\Gamma)$ and $\mathcal{R}(\Gamma)$ for any left cell Γ of W, which are by definition $\Sigma(x)$ and $\mathcal{R}(x)$ respectively for any $x \in \Gamma$.

In [15], the author proposed the following conjecture by which the converse of the above theorem should also be true.

Conjecture 1.5. *Let W be a crystallographic group with a-function upper bounded. For $x, y \in W$, we have the equivalence:*

$$x \underset{L}{\sim} y \iff \mathcal{R}(x) = \mathcal{R}(y) \text{ and } \Sigma(x) = \Sigma(y).$$

1.6 The main result of the present paper is to verify Conjecture 1.5 in the cases when W is an irreducible Weyl group W' and when W is an irreducible affine Weyl group W_a with the following cases excepted: W_a has type $\widetilde{F}_4$, the element x in that conjecture satisfies $\mathcal{R}(x) \in \{\{s_0, s_1, s_2\}, \{s_3, s_4\}\}$ and $a(x) \in \{6, 7, 9, 10, 13, 16\}$. Conjecture 1.5 may be proved in these excepted cases by explicitly finding a representative set of left cells of W_a of type $\widetilde{F}_4$. But this should involve considerable amount of computation. I don't know any simpler way to prove it.

Remark 1.7 (1) In the above conjecture, the condition $\mathcal{R}(x) = \mathcal{R}(y)$ on the right hand side is necessary. For example, let (W_a, S) be the affine Weyl group of type $\widetilde{B}_2$ with

$S = \{s_0, s_1, s_2\}$ such that the order $o(s_0 s_2)$ of the product $s_0 s_2$ is 2. Then $s_0 \underset{L}{\not\sim} s_2$ but $\Sigma(s_0) = \Sigma(s_2) = \{\Gamma_{s_1}\}$. Also, let $x = s_0 s_2 s_1 s_2$ and $y = s_0 s_2 s_1 s_0$. We have $x \underset{L}{\not\sim} y$ but $\Sigma(x) = \Sigma(y) = \{\Gamma_{s_0 s_2 s_1}, \Gamma_{s_0 s_2}\}$. Here the notation Γ_w ($w \in W_a$) stands for the left cell of W_a containing w.

(2) The above conjecture could be restated in a graphic way. Given a two-sided cell Ω of W, we define a directed graph $\mathfrak{T}(\Omega)$ as follows. Its vertex set consists of all left cells of W in Ω. Two of its vertices, say Γ and Γ', are jointed by an arrow: $\Gamma \longrightarrow \Gamma'$, if for some $x' \in \Gamma'$, there exists an element $x \in \Gamma$ with x—x' and $\mathcal{R}(x) \not\subseteq \mathcal{R}(x')$. In this case, the required element $x \in \Gamma$ exists for any $x' \in \Gamma'$ by Theorem 1.4. Such a graph is called the left cell graph of Ω. Clearly, for any left cell Γ of W, the set $\Sigma(\Gamma)$ consists of all left cells Γ'' of W with $\Gamma'' \longrightarrow \Gamma$.

The above conjecture is equivalent to the following

Conjecture 1.5$'$. *For any two-sided cell Ω of W, there exists no graphic automorphism of the left cell graph $\mathfrak{T}(\Omega)$ of Ω which only transposes two vertices and leaves all the other vertices fixed, where a graphic automorphism σ of $\mathfrak{T}(\Omega)$ consists of the following data:*
(1) a permutation σ_1 of vertices i of $\mathfrak{T}(\Omega)$ with $\mathcal{R}(i) = \mathcal{R}(\sigma_1(i))$,
(2) a permutation of arrows of $\mathfrak{T}(\Omega)$ which sends an arrow $i \longrightarrow j$ to the arrow $\sigma_1(i) \longrightarrow \sigma_1(j)$.

The content of the paper is organized as follows. §§2–3 are served as preliminaries. We introduce some results on alcove forms and sign types of elements of affine Weyl groups and Weyl groups in §2, and on strings in §3. Then we prove Conjecture 1.5 in the above listed cases in §§4–6. More precisely, we do the unsaturated cases of $\mathcal{R}(x)$ in §4, do the saturated cases of $\mathcal{R}(x)$ for Weyl groups in §5 and for affine Weyl groups in §6.

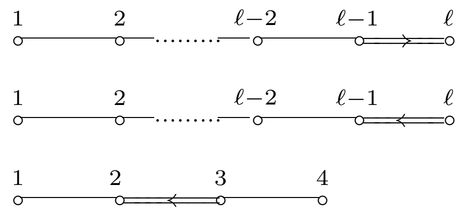
§2. Some results on alcove forms and sign types of elements.

2.1 Let E be the euclidean space spanned by an irreducible root system Φ of type $X \in \{B_\ell,\ C_\ell,\ F_4 \mid \ell \geq 2\}$. Let $\Delta = \{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_\ell\}$ be a choice of simple root system of Φ whose indices are compatible with the corresponding Dynkin diagrams:

type B_ℓ:

type C_ℓ:

type F_4:



Let Φ^+ be the positive root system of Φ determined by Δ. Let W_a be the affine Weyl group of type $\widetilde{X^\vee}$ acting on E in the usual way, where $X^\vee$ is the type dual to X ($X^\vee = X$ unless $X = B_\ell,\ C_\ell$. In the latter cases, we have $B_\ell^\vee = C_\ell$ and $C_\ell^\vee = B_\ell$). Let $s_i = s_{\alpha_i}$ be the simple reflections on E with respect to α_i, $1 \leq i \leq \ell$, and $s_0 = s_{\alpha_0} \cdot T_{-\alpha_0}$, where $-\alpha_0$ is the highest short root of Φ and $T_{-\alpha_0}$ is the translation on E by $-\alpha_0$. Then $S = \{s_0, s_1, \cdots, s_\ell\}$ forms a set of Coxeter generators of W_a.

2.2 Let $H^1_{\alpha;k} = H^1_{-\alpha;-k} = \{v \in E \mid k < \langle v, \alpha^\vee \rangle < k+1\}$ for any $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, where $\langle\ ,\ \rangle$ is an inner product on E with $\langle \alpha, \alpha \rangle = 1$ for any short root $\alpha \in \Phi$ and $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$ is the dual root of α. By an alcove, we mean a non-empty set of E of the form $\bigcap_{\alpha \in \Phi} H^1_{\alpha;k_\alpha}$ with all $k_\alpha \in \mathbb{Z}$. The simply transitive action of W_a on the set of alcoves of E enables us to identify any element $w \in W_a$ with an alcove:

$$A_w = \bigcap_{\alpha \in \Phi} H^1_{\alpha;k(w,\alpha)}, \qquad \text{for some set of integers } k(w,\alpha)$$

The latter is called the alcove form of w. This correspondence is determined uniquely by the following properties.

(1) $k(e, \alpha) = 0,\ \forall\ \alpha \in \Phi$, where e is the identity of W_a;

(2) If $w' = ws_i$ ($0 \le i \le \ell$), then

$$(2.2.1) \qquad k(w', \alpha) = k(w, (\alpha)\bar{s}_i) + k(s_i, \alpha)$$

with
$$k(s_i, \alpha) = \begin{cases} 0, & \text{if } \alpha \neq \pm\alpha_i; \\ -1, & \text{if } \alpha = \alpha_i; \\ 1, & \text{if } \alpha = -\alpha_i. \end{cases}$$

where $\bar{s}_i = s_i$ for $1 \le i \le \ell$; and $\bar{s}_0 = s_{\alpha_0}$.

Since any alcove $\bigcap_{\alpha \in \Phi} H^1_{\alpha; k_\alpha}$ of E is determined completely by a Φ-tuple $(k_\alpha)_{\alpha \in \Phi}$ or a Φ^+-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ over $\mathbb{Z}$, we can simply write $(k_\alpha)_{\alpha \in \Phi}$ or $(k_\alpha)_{\alpha \in \Phi^+}$ for an alcove $\bigcap_{\alpha \in \Phi} H^1_{\alpha; k_\alpha}$. Not any of the Φ-tuples $(k_\alpha)_{\alpha \in \Phi}$ over $\mathbb{Z}$ gives rise to an alcove of E. It is so if and only if the following inequality

$$(2.2.2) \qquad \begin{aligned} |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 &\le |\alpha + \beta|^2 (k_{\alpha+\beta} + 1) \\ &\le |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha + \beta|^2 - 1 \end{aligned}$$

holds for any $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$ (see [9]).

2.3 The Weyl group W' of the root system Φ could be regarded as a standard parabolic subgroup of the affine Weyl group W_a, which is generated by the set $S' = \{s_1, s_2, \cdots, s_\ell\}$. Thus an element $w \in W_a$ is in W' if and only if the alcove form $(k(w, \alpha))_{\alpha \in \Phi^+}$ of w satisfies the condition: $k(w, \alpha) \in \{0, -1\}$ for any $\alpha \in \Phi^+$.

The following are some simple properties for the alcove form of an element in W_a.

Proposition 2.4. *Let $(k_\alpha)_{\alpha \in \Phi^+}$ be the alcove form of an element of W_a.*

(1) If $\{\alpha, \beta, \alpha + \beta\}$ forms a subsystem of Φ^+ of type A_2, then

$$k_\beta < 0 \Longrightarrow k_{\alpha+\beta} \le k_\alpha;$$

$$k_\beta \ge 0 \Longrightarrow k_{\alpha+\beta} \ge k_\alpha.$$

(2) If $\{\alpha,\ \beta,\ \alpha+\beta,\ 2\alpha+\beta\}$ forms a subsystem of Φ^+ of type B_2, then

$$\begin{cases} k_\beta \geq 0 \Longrightarrow k_\alpha \leq k_{\alpha+\beta}; \\ k_\beta < 0 \Longrightarrow k_\alpha \geq k_{\alpha+\beta}; \end{cases} \qquad \begin{cases} k_\alpha \geq 0 \Longrightarrow k_\beta \leq k_{2\alpha+\beta}; \\ k_\alpha < 0 \Longrightarrow k_\beta \geq k_{2\alpha+\beta}. \end{cases}$$

Proof. This could be deduced straightforward from inequality (2.2.2). □

Analogous results also hold for a subsystem of Φ^+ of type G_2.

2.5 For $w,\ w' \in W_a$, we say that w' is a left extension of w if $\ell(w') = \ell(w) + \ell(w'w^{-1})$. Then the following results on the alcove form $(k(w,\alpha))_{\alpha\in\Phi}$ of an element $w \in W_a$ are known.

Proposition [9][10]**.** *(1) $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|$, where the notation $|x|$ stands for the absolute value of x;*

(2) $\mathcal{R}(w) = \{s_i \mid k(w,\alpha_i) < 0\}$;

(3) Let $w' = (k(w',\alpha))_{\alpha\in\Phi} \in W_a$. Then w' is a left extension of w if and only if the inequalities $k(w',\alpha)k(w,\alpha) \geq 0$ and $|k(w',\alpha)| \geq |k(w,\alpha)|$ hold for any $\alpha \in \Phi$.

We shall give a certain arrangement of entries k_α for the alcove form $(k_\alpha)_{\alpha\in\Phi^+}$ of an element in W_a of type $\widetilde{X}$, $X \in \{B_\ell,\ C_\ell,\ F_4\}$, for the convenience of the later use. To do this, we need only arrange the roots of Φ^+ of the corresponding types and then the entries of the alcove form of an element in W_a could be arranged accordingly.

2.6 $X = B_\ell$ ($\ell \geq 3$).

Let us do the case of $\ell = 5$ and then the general cases could be dealt with similarly.

Denote a root $\alpha = \sum_{i=1}^{\ell} a_i\alpha_i$ by its coordinate form $(a_1, a_2, \cdots, a_\ell)$. We arrange the roots of Φ^+, which has type C_5, in the following way:

$$
(2.6.1) \qquad (\alpha)_{\alpha\in\Phi^+} = \begin{array}{c}
(1,1,1,1,0) \\
(1,1,1,0,0) \;\; (0,1,1,1,0) \\
(1,1,0,0,0) \;\; (0,1,1,0,0) \;\; (0,0,1,1,0) \\
(1,0,0,0,0) \;\; (0,1,0,0,0) \;\; (0,0,1,0,0) \;\; (0,0,0,1,0) \\
(2,2,2,2,1) \;\; (0,2,2,2,1) \;\; (0,0,2,2,1) \;\; (0,0,0,2,1) \;\; (0,0,0,0,1) \\
(1,2,2,2,1) \;\; (0,1,2,2,1) \;\; (0,0,1,2,1) \;\; (0,0,0,1,1) \\
(1,1,2,2,1) \;\; (0,1,1,2,1) \;\; (0,0,1,1,1) \\
(1,1,1,2,1) \;\; (0,1,1,1,1) \\
(1,1,1,1,1)
\end{array} \cdots\cdots \text{the middle row}
$$

Let us divide the above diamond into three parts.

(i) The middle row.

(ii) The top triangle, which consists of all the entries above the middle row.

(iii) The lower triangle, which consists of all the entries below the middle row.

We list some properties for this arrangement of roots in Φ^+.

(a) The ones lying in the middle row are all the long roots of Φ^+, and the heights of these roots are monotonously decreasing from left to right. Thus the root at the rightend (resp. leftend) is the simple long (resp. the highest) root of Φ^+.

(b) The roots in the top triangle form a subsystem of Φ^+ of type $A_{\ell-1}$, and the roots in the i-th row from the bottom of this triangle have the same height i. In particular, the bottom row of this triangle consists of all the simple short roots of Φ^+.

(c) The heights of roots in the lower triangle are monoteneously getting larger along the alignment from northeast to southwest and also from southeast to northwest. In particular, the highest short root of Φ^+ is at the leftend of the row just below the middle one.

(d) A subset of Φ^+ forms a subsystem of type B_2 if and only if it is the vertex set of a subdiamond with two of its vertices lying in the middle row.

Properties (a), (b) and (d) determine the positions of roots of Φ^+ in diamond (2.6.1) completely.

2.7 $X = C_\ell$ ($\ell \geq 2$).

Again,we do the case of $\ell = 5$ as an illustration. We arrange the roots of Φ^+ of type B_5 in the following way:

$$
\begin{array}{c}
(1,1,1,1,0)\\
(1,1,1,0,0)\ \ (0,1,1,1,0)\\
(1,1,0,0,0)\ \ (0,1,1,0,0)\ \ (0,0,1,1,0)\\
(1,0,0,0,0)\ \ (0,1,0,0,0)\ \ (0,0,1,0,0)\ \ (0,0,0,1,0)\\
(1,1,1,1,1)\ \ (0,1,1,1,1)\ \ (0,0,1,1,1)\ \ (0,0,0,1,1)\ \ (0,0,0,0,1)\\
(1,2,2,2,2)\ \ (0,1,2,2,2)\ \ (0,0,1,2,2)\ \ (0,0,0,1,2)\\
(1,1,2,2,2)\ \ (0,1,1,2,2)\ \ (0,0,1,1,2)\\
(1,1,1,2,2)\ \ (0,1,1,1,2)\\
(1,1,1,1,2)
\end{array}
\tag{2.7.1}
$$

Note that this arrangement of Φ^+ could be obtained from (2.6.1) by replacing each entry by the corresponding dual root up to an overall scale factor. In particular, the highest short root of Φ^+ is placed at the leftend of the middle row. Moreover, some properties for this arrangement of Φ^+, similar to 2.6 (a)–(d), could be deduced by using duality.

2.8 Let Φ^+ be any positive root system. For each $\alpha \in \Phi^+$, we define $\delta(\alpha)$ to be the length t of a sequence $\xi:\ \gamma_1,\ \gamma_2,\ \cdots,\ \gamma_t = \alpha$ in Φ^+ with $\gamma_1 \in \Delta$ such that for every $i,\ 1 < i \leq t$, there exists some $\beta \in \Delta$ with $s_\beta(\gamma_{i-1}) = \gamma_i$ and $\gamma_{i-1} < \gamma_i$. By [14], we know that the number t is only dependent on α but independent of the choice of a sequence ξ. So $\delta(\alpha)$ is well defined. Note that when α is a short root in Φ^+, the number $\delta(\alpha)$ coincides with the height of α in the usual sense.

Now let Φ^+ be of type F_4. We arrange the roots of Φ^+ in the following way.

$$
\begin{array}{c}
(2,4,3,2)\\
(2,4,3,1)\\
(2,4,2,1)\\
(2,2,2,1)\\
(2,2,1,1)\ (0,2,2,1)\\
(0,2,1,1)\ (2,2,1,0)\\
(0,0,1,1)\ (0,2,1,0)\\
(0,0,0,1)\ (0,0,1,0)
\end{array}
\qquad\qquad
\begin{array}{c}
(2,3,2,1)\\
(1,3,2,1)\\
(1,2,2,1)\\
(1,2,1,1)\\
(1,2,1,0)\ (1,1,1,1)\\
(0,1,1,1)\ (1,1,1,0)\\
(0,1,1,0)\ (1,1,0,0)\\
(0,1,0,0)\ (1,0,0,0)
\end{array}
\tag{2.8.1}
$$

The left part of (2.8.1) consists of all the long roots of Φ^+ and that the roots in the i-th row from bottom have the same δ-value i.

2.9 A Φ^+-sign type (or a sign type in short when it is clear in the context) is a Φ^+-tuple $(X_\alpha)_{\alpha\in\Phi^+}$ with $X_\alpha \in \{\bigcirc,+,-\}$ for any $\alpha \in \Phi^+$. To each $w \in W_a$ with its alcove form

$(k_\alpha)_{\alpha\in\Phi^+}$, we associate a Φ^+-sign type $\zeta(w)=(X_\alpha)_{\alpha\in\Phi^+}$ by setting

$$X_\alpha=\begin{cases}+, & \text{if } k_\alpha>0;\\ -, & \text{if } k_\alpha<0;\\ \bigcirc, & \text{if } k_\alpha=0,\end{cases}$$

for any $\alpha\in\Phi^+$. Thus $w\longrightarrow\zeta(w)$ is a map from W_a to the set of all Φ^+-sign types. By abuse of terminology, we identify a Φ^+-sign type X with the set $\zeta^{-1}(X)$ of W_a and call $\zeta^{-1}(X)$ a Φ^+-sign type (or a sign type in short) if $\zeta^{-1}(X)\neq\emptyset$. It is known that there exists a unique shortest element in each sign type of W_a (see [10]). It is also known that each left cell in the lowest two-sided cell of an affine Weyl group W_a is exactly a single sign type (see [11][12]). For more results on sign types of elements of W_a, we refer the reader to [8][10][13].

§3. Strings.

3.1 A sequence of elements in W of the form

(3.1.1) $$\underbrace{ys,\ yst,\ ysts,\ \ldots}_{m-1\text{ terms}}$$

is called an $\{s,t\}$-string (or just call it a string) if $s,t\in S$ and $y\in W$ satisfy the conditions that the order $o(st)$ of the product st is m and $\mathcal{R}(y)\cap\{s,t\}=\emptyset$. The number $m-1$ is called the length of this string. Clearly, when (3.1.1) is an $\{s,t\}$-string, the sequence

(3.1.2) $$\underbrace{yt,\ yts,\ ytst,\ \ldots}_{m-1\text{ terms}}$$

is also an $\{s,t\}$-string.

Suppose that we are given two $\{s,t\}$-strings $x_1,\ x_2,\ \ldots,\ x_{m-1}$ and $y_1,\ y_2,\ \ldots,\ y_{m-1}$ with $o(st)=m$. Then the following result is known.

Proposition 3.2 [16]**.** *In the setup of 3.1, we have*

(1) If $m = 3$, then

$$x_1 \underset{L}{\sim} y_1 \iff x_2 \underset{L}{\sim} y_2;$$

$$x_1 \underset{L}{\sim} y_2 \iff x_2 \underset{L}{\sim} y_1;$$

(2) If $m = 4$, then

(a) $x_1 \underset{L}{\sim} y_2 \iff x_2 \underset{L}{\sim} y_1 \iff x_2 \underset{L}{\sim} y_3 \iff x_3 \underset{L}{\sim} y_2;$

(b) $x_1 \underset{L}{\sim} y_1 \iff x_3 \underset{L}{\sim} y_3;$

(c) $x_1 \underset{L}{\sim} y_3 \iff x_3 \underset{L}{\sim} y_1;$

(d) $x_2 \underset{L}{\sim} y_2 \iff$ *either* $x_1 \underset{L}{\sim} y_1$ *or* $x_1 \underset{L}{\sim} y_3 \iff$ *either* $x_1 \underset{L}{\sim} y_1$ *or* $x_3 \underset{L}{\sim} y_1$

$\iff$ *either* $x_3 \underset{L}{\sim} y_1$ *or* $x_3 \underset{L}{\sim} y_3 \iff$ *either* $x_1 \underset{L}{\sim} y_3$ *or* $x_3 \underset{L}{\sim} y_3$.

Remark 3.3 When $m \geq 6$, the analogue of Proposition 3.2 could be deduced.

3.4 Fix a left cell Γ of W. Suppose that there exist some $s, t \in S$ with $t \in \mathcal{R}(\Gamma)$, $s \notin \mathcal{R}(\Gamma)$ and $o(st) = m \geq 3$. Then each element $x \in \Gamma$ belongs to a unique $\{s, t\}$-string ξ_x. We denote by $i(x, s, t)$ the ordinal of x in string ξ_x, i.e. x is the $i(x, s, t)$-th term of ξ_x. Define Γ^* to be the set of all elements y of W satisfying the following condition: y is in some $\{s, t\}$-string of W containing a term x in Γ such that $i(y, s, t) = m - i(x, s, t)$. Then it is known that Γ^* is also a left cell of W (see [4]). Call the map $\Gamma \longrightarrow \Gamma^*$ a (right) star operation with respect to $\{s, t\}$ (or just call it a star operation if no danger of confusion). This map is involutive. Note that the left cells Γ and Γ^* need not be distinct in general. But they are distinct in the case of $m = 3$. Moreover, when $m = 3$, we have $\Gamma^* \in \Sigma(\Gamma)$ and $\Gamma \in \Sigma(\Gamma^*)$.

3.5 Given an element $x \in W$, We consider the set $M(x)$ of all elements y such that there exist a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in W with some $r \geq 0$, where for every i, $1 \leq i \leq r$, the conditions $x_{i-1}^{-1} x_i \in S$ and $\mathcal{R}(x_{i-1}) \underset{\not\subseteq}{\not\supseteq} \mathcal{R}(x_i)$ are satisfied. Clearly, if

$x' \in M(x)$ then $x \underset{R}{\sim} x'$.

§4. Proof of Conjecture 1.5 (unsaturated cases).

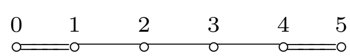
4.1 By Theorem 1.4, to show Conjecture 1.5, it suffices to show the following

Assertion. *If* $\Gamma,\ \Gamma'$ *are left cells of* W *with* $K = \mathcal{R}(\Gamma) = \mathcal{R}(\Gamma')$ *and* $\Sigma(\Gamma) = \Sigma(\Gamma')$*, then* $\Gamma = \Gamma'$.

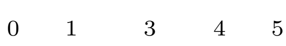
When $K = \emptyset$, the left cells Γ and Γ' of W in the above assertion are both equal to $\{e\}$. So in the subsequent discussion, we always assume $K \neq \emptyset$.

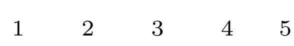
In the present section, we shall show this assertion mainly in the case when K is an unsaturated subset of S (see the definition in 4.2).

4.2 Let $D(S)$ be the Coxeter graph corresponding to the Coxeter generator set S. For any subset $I \subseteq S$, we denote by $D(I)$ the full subgraph of $D(S)$ with I its vertex set and by $D^\circ(I)$ the graph obtained from $D(I)$ by removing all the non-simply-laced edges. For example, let (W, S) be the affine Weyl group of type $\widetilde{C}_5$. Then $D(S)$ is



Let $I = \{s_0,\ s_1,\ s_3,\ s_4,\ s_5\}$ and $I' = \{s_1,\ s_2,\ s_3,\ s_4,\ s_5\}$. Then $D(I)$ and $D(I')$ (resp. $D^\circ(I)$ and $D^\circ(I')$) are



and


(resp.


and


),

respectively. Call a non-empty subset $J \subseteq S$ to be saturated if the subgroup of W generated by J is finite and if $D^\circ(J)$ consists of some connected components of $D^\circ(S)$; call J to be unsaturated if otherwise. Thus in the above example, I' is saturated but I is not.

Proposition 4.3. *Suppose that* Γ *and* Γ' *are left cells of* W *with* $\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')=K$*. If* K *is unsaturated, then* $\Sigma(\Gamma)=\Sigma(\Gamma')\Longleftrightarrow\Gamma=\Gamma'$

Proof. ($\Leftarrow$) By Theorem 1.4.

($\Rightarrow$) By the condition, there exist some $s\in S-K$ and $t\in K$ with $o(st)=3$. Let Γ^* (resp. Γ'^*) be the image of Γ (resp. Γ') under the (right) star operation with respect to $\{s,t\}$ (see 3.4). Then by 3.4, Γ^* is a left cell of W belonging to $\Sigma(\Gamma)$ and hence also to $\Sigma(\Gamma')$. Since $s\in\mathcal{R}(\Gamma^*)-\mathcal{R}(\Gamma')$ and $t\in\mathcal{R}(\Gamma')-\mathcal{R}(\Gamma^*)$, this implies from 3.4 that $\Gamma^*=\Gamma'^*$ and so $\Gamma=\Gamma'$. $\square$

Corollary 4.4. *Let* W_a *be an affine Weyl group of type* $\widetilde{A}_\ell$ *(* $\ell\geq 1$ *),* $\widetilde{D}_m$ *(* $m\geq 4$ *) or* $\widetilde{E}_n$ *(* $n=6,7,8$ *). Then for any left cells* Γ *and* Γ' *of* W_a *with* $\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')$*, the equivalence "* $\Sigma(\Gamma)=\Sigma(\Gamma')\Longleftrightarrow\Gamma=\Gamma'$ *" holds.*

Proof. Since $K=\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')$ is never saturated under our assumption, the result follows from Proposition 4.3 immediately. $\square$

Corollary 4.5. *Let* (W',S') *be a Weyl group of type* A_ℓ *(* $\ell\geq 1$ *),* D_m *(* $m\geq 4$ *) or* E_n *(* $n=6,7,8$ *). Then for any left cells* Γ *and* Γ' *of* W' *with* $\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')$*, we have* $\Sigma(\Gamma)=\Sigma(\Gamma')\Longleftrightarrow\Gamma=\Gamma'$*.*

Proof. Let $K=\mathcal{R}(\Gamma)$. Then K is saturated only if $K=S'$. But when $K=S'$, we have $\Gamma=\Gamma'$ which consists of a single element (i.e. the longest element of W') and $\Sigma(\Gamma)=\Sigma(\Gamma')=\emptyset$. So our result follows by this observation and by Proposition 4.3. $\square$

4.6 The remaining two sections are devoted to show Conjecture 1.5 in the cases when W is an irreducible Weyl group W' and when W is an irreducible affine Weyl group W_a with the following cases excepted: W_a has type $\widetilde{F}_4$, the element x in conjecture 1.5 satisfies the conditions $\mathcal{R}(x)\in\{\{s_0,s_1,s_2\},\ \{s_3,s_4\}\}$ and $a(x)\in\{6,7,9,10,13,16\}$. By Theorem 1.4

and Proposition 4.3, to show Conjecture 1.5, we need only to show Assertion 4.1 in the case when K is saturated. By Corollaries 4.4 and 4.5, we need only to consider the types C_ℓ ($\ell \geq 2$), F_4 and G_2 in the Weyl group cases and the types $\widetilde{B}_\ell$ ($\ell \geq 3$), $\widetilde{C}_m$ ($m \geq 2$), $\widetilde{F}_4$ and $\widetilde{G}_2$ in the affine Weyl group cases. Our proof will proceed by using case-by-case argument.

We record a simple fact for later use

Lemma. *If x—y and $\mathcal{R}(x)\overset{\not\supseteq}{\not\subseteq}\mathcal{R}(y)$ for $x, y \in W$, then $x^{-1}y \in S$. Moreover, if $\Gamma_0 \in \Sigma(\Gamma)$ satisfies $\mathcal{R}(\Gamma)\overset{\not\supseteq}{\not\subseteq}\mathcal{R}(\Gamma_0)$ for a left cell Γ of W, then $\Gamma \in \Sigma(\Gamma_0)$.*

§5. The case when W is a Weyl group (W', S').

5.1 Recall that the alcove form $(k(w,\alpha))_{\alpha\in\Phi^+}$ of an element $w \in W'$ satisfies the condition: $k(w,\alpha) \in \{0, -1\}$ for $\alpha \in \Phi^+$.

By Corollary 4.5, to show Assertion 4.1, we need only to consider the cases when W' has type C_ℓ ($\ell \geq 2$), F_4 or G_2 and when $\mathcal{R}(\Gamma)$ is saturated. The case $\mathcal{R}(\Gamma) = S'$ could be treated as that in the proof of Corollary 4.5. So in the subsequent discussion, we always assume $\emptyset \neq \mathcal{R}(\Gamma) \subsetneqq S'$ and so we have $\Sigma(\Gamma) \neq \emptyset$. We shall argue case-by-case and by contrary.

Let Γ and Γ' be left cells of W' with $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = K$ and $\Sigma(\Gamma) = \Sigma(\Gamma')$. Suppose $\Gamma \neq \Gamma'$.

5.2 First assume that W' has type C_ℓ, $\ell \geq 2$. The alcove forms of elements of W' are defined over the positive root system Φ^+ of type B_ℓ. In particular, when $\ell \geq 4$, the set of all long roots of Φ^+ forms a subsystem of type D_ℓ with $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{\ell-1},\ \alpha_{\ell-1} + 2\alpha_\ell$ its simple root system. The possible saturated proper subsets of S' are $\{s_1,\ s_2,\ \cdots,\ s_{\ell-1}\}$ and $\{s_\ell\}$. First assume $\mathcal{R}(\Gamma) = \{s_1,\ s_2,\ \cdots,\ s_{\ell-1}\}$. Then there exists some $\Gamma'' \in \Sigma(\Gamma) = \Sigma(\Gamma')$ with $s_{\ell-1} \notin \mathcal{R}(\Gamma'')$ and $s_\ell \in \mathcal{R}(\Gamma'')$. By Lemma 4.6 and the assumption $\Gamma \neq \Gamma'$, there exists some $\{s_{\ell-1}, s_\ell\}$-string x, y, z with $y \in \Gamma''$ such that either $x \in \Gamma$, $z \in \Gamma'$ or $x \in \Gamma'$, $z \in \Gamma$. By symmetry, we may assume $x \in \Gamma$, $z \in \Gamma'$ without loss of generality. Thus by (2.2.1) and Proposition 2.5, we have $k(z, \alpha_i) = -1$ for $1 \leq i < \ell$, $k(z, \alpha_\ell) = 0$ and $k(z, \alpha_{\ell-1} + \alpha_\ell) = k(z, \alpha_{\ell-1} + 2\alpha_\ell) = -1$ in the alcove form $(k(z, \alpha))_{\alpha \in \Phi^+}$ of z. By Proposition 2.4, this implies that for any $\alpha \in \Phi^+$,

$$k(z, \alpha) = \begin{cases} 0, & \text{if } \alpha = \alpha_\ell; \\ -1, & \text{otherwise.} \end{cases}$$

i.e. $z = w_0 s_\ell$, where w_0 is the longest element in W'. Hence $x = w_0 s_\ell s_{\ell-1} s_\ell$. Since w_0 is in the center of W', this implies that both x and z are involutions. So we get $x \underset{L}{\sim} z$ from the fact $x \underset{R}{\sim} z$. But then we get $\Gamma = \Gamma'$, contradicting our hypothesis. Next assume $\mathcal{R}(\Gamma) = \{s_\ell\}$. As mentioned in 2.3, $W' = \langle s_1, s_2, \cdots, s_\ell \rangle$ could be regarded as a subgroup of the affine Weyl group $W_a = \langle s_0, s_1, \cdots, s_\ell \rangle$ of type $\widetilde{C}_\ell$. There exists a unique left cell Γ_0 (resp. Γ'_0) of W_a containing Γ (resp. Γ'). Clearly, Γ_0 and Γ'_0 are in the same two-sided cell of W_a with $\mathcal{R}(\Gamma_0) = \mathcal{R}(\Gamma'_0) = \{s_\ell\}$. By [7], there exists at most one left cell F of W_a satisfying $\mathcal{R}(F) = \{s_\ell\}$ in a given two-sided cell. This implies $\Gamma_0 = \Gamma'_0$ and hence $\Gamma = \Gamma'$, contradicting our hypothesis, too.

5.3 Next assume that W' has type G_2.

The possible saturated proper subsets of S' are $\{s_1\}$ and $\{s_2\}$. But if $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = \{s_i\}$, then $\Gamma = \Gamma'$. This gives rise to a contradiction.

5.4 Finally assume that W' has type F_4.

The possible saturated proper subsets of S' are $\{s_1,\ s_2\}$ and $\{s_3,\ s_4\}$. The permutation $\psi:\ s_i \mapsto s_{5-i}$ ($1 \leq i \leq 4$) on S' could be extended uniquely to an automorphism of W' which induces a permutation Ψ on the set Ξ of left cells of W' with $\Psi(\Sigma(\Gamma)) = \Sigma(\Psi(\Gamma))$ for any $\Gamma \in \Xi$. So we need only to consider the case when $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = \{s_1,\ s_2\}$. Thus by Lemma 4.6 and the assumption $\Gamma \neq \Gamma'$, there must exist some $\{s_2,\ s_3\}$-string

x, y, z with $y \in \Gamma''$ such that either $x \in \Gamma$, $z \in \Gamma'$ or $x \in \Gamma'$, $z \in \Gamma$. Without loss of generality, we may assume $x \in \Gamma$ and $z \in \Gamma'$. Let $w = s_1s_2s_3s_2s_1s_2s_3s_2$ and $w' = s_1s_2s_3s_4s_2s_1s_3s_2s_3s_4s_1s_2s_3s_2s_1s_2s_3s_2$. Then the alcove forms of these two elements are

$$(5.4.1)\qquad \underset{w}{\begin{array}{cccc} & 0 & & 0 \\ & 0 & & 0 \\ & 0 & & 0 \\ & 0 & & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}} \quad \text{and} \quad \underset{w'}{\begin{array}{cccc} & -1 & & -1 \\ & -1 & & -1 \\ & -1 & & -1 \\ & -1 & & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}}$$

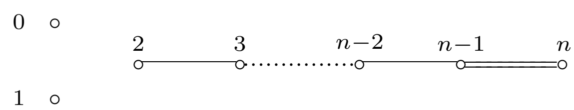
respectively. Then by applying Propositions 2.4 and 2.5 to the alcove forms of elements, we can see that z (resp. w') is a left extension of w (resp. z) (see 2.5). We can also check that w' is in $M(w)$ (see 3.5) and that both w and w' are involutions. This implies $w \underset{L}{\sim} w' \underset{L}{\leq} z \underset{L}{\leq} w$ by 3.5 and 1.3(2), i.e. $z \underset{L}{\sim} w$. Thus by Proposition 3.2(2), we have $x \underset{L}{\sim} ws_2s_3$. But it is easily seen that $ws_2s_3 \in M(w)$ and that both w and ws_2s_3 are involutions. So by 3.5, we have $w \underset{L}{\sim} ws_2s_3$ and hence $z \underset{L}{\sim} x$. This implies $\Gamma = \Gamma'$, again giving rise to a contradiction.

§6.The affine Weyl group (W_a, S) case.

6.1 To show Assertion 4.1, we need only to consider the case when W_a has the type $\widetilde{B}_n$ ($n \geq 3$), $\widetilde{C}_m$ ($m \geq 2$), $\widetilde{F}_4$ or $\widetilde{G}_2$. Let Γ and Γ' be left cells of W_a with $\Sigma(\Gamma) = \Sigma(\Gamma')$ and with $K = \mathcal{R}(\Gamma) = \mathcal{R}(\Gamma')$ saturated. Then $K \subsetneqq S$ and there exist some $t \in S - K$ and $s \in K$ with $o(ts) > 3$. Thus there must exist some $\Gamma'' \in \Sigma(\Gamma) = \Sigma(\Gamma')$ satisfying $t \in \mathcal{R}(\Gamma'')$ and $s \notin \mathcal{R}(\Gamma'')$. So by Lemma 4.6, we have $\Gamma' \rightleftarrows \Gamma'' \rightleftarrows \Gamma$ (see Remark 1.7(2)). Thus we see that each $\{s, t\}$-string ξ intersecting one of Γ, Γ' and Γ'' must intersects all the others, and that the middle term of ξ should be in Γ'' in the case when $\Gamma \neq \Gamma'$ and $m = 4$.

We shall show Assertion 4.1. We argue by contrary in the case when W_a has type $\widetilde{B}_n$ or $\widetilde{C}_n$. Thus in 6.2–6.3, we assume $\Gamma \neq \Gamma'$.

6.2 First assume that W_a has type $\widetilde{B}_n$ ($n \geq 3$). The corresponding extended Dynkin diagram is as below.



The possible saturated subsets K of S are $\{s_0, s_1, \cdots, s_{n-1}\}$ and $\{s_n\}$.

Figure 1.

(a) First consider the case $K = \{s_0, s_1, \cdots, s_{n-1}\}$.

We have $t = s_n$ and $s = s_{n-1}$. By Propositions 2.4 and 2.5, we see from 6.1 and the assumption $\Gamma \neq \Gamma'$ that the left cell Γ'' satisfies $\mathcal{R}(\Gamma'') = \{s_0, s_1, \cdots s_{n-2}, s_n\}$ and that for any given element $w'' \in \Gamma''$, there exist some elements $w \in \Gamma$ and $w' \in \Gamma'$ such that either w, w'', w' or w', w'', w form an $\{s_{n-1}, s_n\}$-string. By symmetry, we may assume the former without loss of generality. Thus by (2.2.1) and Proposition 2.5, in the alcove form $(k(w, \alpha))_{\alpha \in \Phi}$ of w, we have $k(w, \alpha_i) < 0$, $\forall\ 0 \leq i \leq n-1$, $k(w, \alpha_n) \geq 0$ and $k(w, \alpha_{n-1} + \alpha_n)$, $k(w, 2\alpha_{n-1} + \alpha_n) \geq 0$. So by Proposition 2.4, the element w has the sign type as in Figure 1 (a), where the notation $\oplus$ stands for the sign $+$ or $\bigcirc$ and $*$ for an undetermined sign. We see that all the entries of w in its top (resp. lower) triangle are of negative (resp. positive) values. The entries in its middle row weakly increase from left to right, and all the entries in this row are non-negative with possible one exception. But this implies from (2.2.1) and (2.6.1) that the element $w' = ws_ns_{n-1}$ has the sign type as in Figure 1 (b). This means $\mathcal{R}(w') = \{s_0, s_1, \cdots, s_{n-3}, s_{n-1}\}$ if $n > 3$ or $\mathcal{R}(w') = \{s_2\}$ if $n = 3$ by Proposition 2.5, contradicting the assumption $\mathcal{R}(w') = \mathcal{R}(\Gamma') = \{s_0, s_1, \cdots, s_{n-1}\}$.

FIGURE 2.

(b) $K = \{s_n\}$.

We have $t = s_{n-1}$ and $s = s_n$. By Propositions 2.4 and 2.5, we see from 6.1 and the assumption $\Gamma \neq \Gamma'$ that the left cell Γ'' satisfies $\mathcal{R}(\Gamma'') = \{s_{n-1}\}$ and that for any given element $w'' \in \Gamma''$, there exist some elements $w \in \Gamma$ and $w' \in \Gamma'$ such that either w, w'', w' or w', w'', w form an $\{s_{n-1}, s_n\}$-string. Again, we may assume the former without loss of generality. So in the alcove form $(k(w, \alpha))_{\alpha \in \Phi}$ of w, we have $k(w, \alpha_i) \geq 0$, $\forall\, 0 \leq i \leq n-1$, $k(w, \alpha_n) < 0$ and $k(w, \alpha_{n-1} + \alpha_n)$, $k(w, 2\alpha_{n-1} + \alpha_n) \geq 0$ by Proposition 2.5 and (2.2.1). So by Proposition 2.4, the element w has the sign type as in Figure 2. We see that all the entries of w in its top (resp. lower) triangle are of non-negative (resp. zero) values. The entries in its middle row weakly decrease from left to right and only the rightmost entry is negative. Again by Propositions 2.4 and 2.5, we see that this holds only when $w = s_n$. In this case, $w' = ws_{n-1}s_n = s_ns_{n-1}s_n \underset{L}{\sim} w$ and hence $\Gamma = \Gamma'$ which also gives rise to a contradiction.

6.3 Next assume that W_a has type $\widetilde{C}_n$ ($n \geq 2$).

The possible saturated subsets K of S are $\{s_1, s_2, \cdots, s_{n-1}\}$, $\{s_0, s_1, s_2, \cdots, s_{n-1}\}$, $\{s_1, s_2, \cdots, s_{n-1}, s_n\}$, $\{s_0\}$, $\{s_n\}$ and $\{s_0, s_n\}$. It is known that there exists a unique left cell Γ of W_a satisfying the condition $\mathcal{R}(\Gamma) = \{s_0, s_1, s_2, \cdots, s_{n-1}\}$ (resp. $\mathcal{R}(\Gamma) = \{s_1, s_2, \cdots, s_{n-1}, s_n\}$) (see [11][12]). On the other hand, the cases $K = \{s_0\}$ and $K = \{s_n\}$ are symmetric. It is known that there exists at most one left cell Γ with $\mathcal{R}(\Gamma) = \{s_0\}$ in a given two-sided cell of W_a (see [7]). Thus we need only to consider the cases when $K = \{s_1, s_2, \cdots, s_{n-1}\}$, and $\{s_0, s_n\}$. The cases $n = 2, 3$ can be checked directly (refer to [4][1]). So in the subsequent discussion, we always assume $n > 3$.

(a) First assume $K = \{s_1, s_2, \cdots, s_{n-1}\}$.

By Propositions 2.4 and 2.5, we see from 6.1 and the assumption $\Gamma \neq \Gamma'$ that the left cell Γ'' satisfies $\mathcal{R}(\Gamma'') = \{s_1, s_2, \cdots, s_{n-2}, s_n\}$ and that for any $w'' \in \Gamma''$, there exist some elements $w \in \Gamma$ and $w' \in \Gamma'$ such that either w, w'', w' or w', w'', w form an $\{s_{n-1}, s_n\}$-string. As before, we may assume w, w'', w' to be an $\{s_{n-1}, s_n\}$-string. Let $(k(w', \alpha))_{\alpha \in \Phi}$ be the alcove form of w'. Then by 2.7, (2.2.1) and Proposition 2.5, we have $k(w', \alpha_i) < 0$, $\forall,\ 1 \leq i \leq n-1$, $k(w', \alpha_0)$, $k(w', \alpha_n) \geq 0$, and $k(w', \alpha_{n-1} + \alpha_n)$, $k(w', \alpha_{n-1} + 2\alpha_n) < 0$. Hence by Proposition 2.4, the sign type of w' should be as in Figure 3 (a). We see that all but one entries of w' indexed by roots in Φ^+ are negative, the only non-negative entry of w' is at the rightend of its middle row. Then the sign type of the element $w = w's_{n-1}s_n$

should be as in Figure 3 (b), where all the entries of w indexed by roots in Φ^+ are negative with three exceptions, these three non-negative entries are $k(w, \alpha_n)$, $k(w, \alpha_{n-1} + \alpha_n)$ and $k(w, \alpha_{n-1} + 2\alpha_n)$.

FIGURE 3.

Let $y' = w's_0$ and $y = ws_0$. Then y', y have the sign types as in Figure 4 (a), (b), respectively. Let Γ_x be the left cell of W_a containing x for $x \in W_a$. Then by (2.2.1), 2.7 and Proposition 2.5, we have $\mathcal{R}(y') = \{s_0, s_2, s_3, \cdots, s_{n-1}\}$ and hence $\Gamma_{y'} \in \Sigma(\Gamma')$. Also, we have $\Gamma_y \in \Sigma(\Gamma) = \Sigma(\Gamma')$ with $\mathcal{R}(\Gamma_y) = \mathcal{R}(\Gamma_{y'})$. Since w' is the first term in the $\{s_0, s_1\}$-string containing y', we see from Lemma 4.6 that $\Sigma(\Gamma')$ contains a unique left cell F of W_a satisfying $s_0 \in \mathcal{R}(F)$ and $s_1 \notin \mathcal{R}(F)$. This implies $\Gamma_{y'} = \Gamma_y$, i.e. $y \underset{L}{\sim} y'$. Now by Proposition 3.2 and the assumption $w' \underset{L}{\not\sim} w$, this implies $w \underset{L}{\sim} y's_1$. But it is easily seen from (2.2.1), 2.7 and Proposition 2.5 that $\mathcal{R}(y's_1) = \{s_1, s_3, \cdots, s_{n-1}\} \neq \mathcal{R}(w)$ which is not allowed by Theorem 1.4.

(b) Finally assume $K = \{s_0, s_n\}$.

FIGURE 4.

The left cell Γ'' satisfies $\mathcal{R}(\Gamma'') = \{s_0, s_{n-1}\}$ by the assumption $n > 3$. Thus $\Gamma, \Gamma' \in \Sigma(\Gamma'')$ by Lemma 4.6. Given an element $w'' \in \Gamma''$, there exist some elements $w \in \Gamma$ and $w' \in \Gamma'$ such that either w, w'', w' or w', w'', w form an $\{s_{n-1}, s_n\}$-string by the assumption $\Gamma \neq \Gamma'$. Again, we need only to consider the case when w is the first term in this $\{s_{n-1}, s_n\}$-string. Thus in the alcove form $(k(w, \alpha))_{\alpha \in \Phi}$ of w, we have $k(w, \alpha_i) \geq 0$, $\forall\ 1 \leq i \leq n-1$, $k(w, \alpha_0)$, $k(w, \alpha_n) < 0$ and $k(w, \alpha_{n-1} + \alpha_n)$, $k(w, \alpha_{n-1} + 2\alpha_n) \geq 0$ by the assumption $n > 3$. So by Proposition 2.4, the element w has the sign type as in Figure 5. We see that all but one entries of w indexed by positive roots are non-negative, and that the entry at the leftend (resp. rightend) of its middle row is positive (resp. negative).

Let

$$\lambda \quad = \quad \begin{array}{ccc} & k(w, \alpha_1) & \\ k(w, -\alpha_0) & & k(w, -\alpha_0 - 2\alpha_1) \\ & k(w, -\alpha_0 - \alpha_1) & \end{array}.$$

Then the possible sign types $\boldsymbol{\lambda}$ of λ are $\begin{smallmatrix} & \oplus & \\ + & & \bigcirc \\ & + & \end{smallmatrix}$, $\begin{smallmatrix} & \bigcirc & \\ + & & \bigcirc \\ & \bigcirc & \end{smallmatrix}$ and $\begin{smallmatrix} & \oplus & \\ + & & + \\ & + & \end{smallmatrix}$, where the notation $\oplus$

Figure 5.

stands for the sign $+$ or $\bigcirc$. When $\boldsymbol{\lambda} = +\overset{\oplus}{\underset{+}{\,}}\bigcirc$, the element w is the second term in the $\{s_0, s_1\}$-string containing it. Let $y = ws_0$ and $y' = w's = ws_{n-1}s_ns$, where $s \in \{s_0, s_1\}$ is chosen such that $y' \underset{L}{\sim} y$. This can be done by Lemma 4.6 since $\Gamma_y \in \Sigma(\Gamma) = \Sigma(\Gamma')$ and $\mathcal{R}(\Gamma_y) \overset{\not\supset}{\underset{\not\subset}{}} \mathcal{R}(\Gamma')$. But then by Proposition 3.2, we get $w \underset{L}{\sim} w'$. Next assume $\boldsymbol{\lambda} = +\overset{\bigcirc}{\underset{\bigcirc}{\,}}\bigcirc$. Then by Proposition 2.4, we have $w = s_0s_n$ and hence $w' = ws_{n-1}s_n = s_0s_ns_{n-1}s_n$. Clearly, $w \underset{L}{\sim} w'$ by the assumption $n > 3$. Finally assume $\boldsymbol{\lambda} = +\overset{\oplus}{\underset{+}{\,}}+$. Then w is the third term in the $\{s_0, s_1\}$-string containing it. Let $y = ws_0$. Then $\Gamma_y \in \Sigma(\Gamma) = \Sigma(\Gamma')$. We see that the element w' is in an $\{s_0, s_1\}$-string and that some of its neighboring terms, say y', in this $\{s_0, s_1\}$-string satisfies $y' \underset{L}{\sim} y$. Since $w \underset{L}{\not\sim} w'$ by assumption, this implies from Proposition 3.2 that the element $x = ys_1 = ws_0s_1$ satisfies $x \underset{L}{\sim} w'$. In particular, we have $\mathcal{R}(x) = \{s_0, s_n\}$ and hence $0 \leq k(x, \alpha_2) = -k(w, -\alpha_0 - \alpha_1 - \alpha_2) \leq 0$ by (2.2.1), Propositions 2.4 and 2.5, i.e. $k(w, -\alpha_0 - \alpha_1 - \alpha_2) = 0$. But by Proposition 2.4 and by the sign type of w, we get $w = s_0s_1s_0s_n$ and hence $w' = s_0s_1s_0s_ns_{n-1}s_n$. A direct checking shows $w \underset{L}{\sim} w'$ by the assumption $n > 3$. So we get $w \underset{L}{\sim} w'$ in all the cases, which contradicts the assumption $\Gamma \neq \Gamma'$.

6.4 Next assume that W_a has type $\widetilde{G}_2$.

We now show Assertion 4.1 by a direct argument. Note that all the left cells of W_a were described explicitly by Lusztig (see [4]). The possible saturated subsets of S are $\{s_0, s_1\}$ and $\{s_2\}$. Let Ω be the two-sided cell of W_a containing Γ and Γ'. First assume $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = \{s_0, s_1\}$. In this case, the only possible values for $a(\Omega)$ are 3 and 6. There is only one left cell F in Ω with $\mathcal{R}(F) = \{s_0, s_1\}$ in the case of $a(\Omega) = 3$. When $a(\Omega) = 6$, there are two distinct left cells $F,\ F'$ in Ω with $\mathcal{R}(F) = \mathcal{R}(F') = \{s_0, s_1\}$ but $\Sigma(F) \bigcap \Sigma(F') = \emptyset$ as a direct checking shows. This implies $\Gamma = \Gamma'$ since $\Sigma(\Gamma) \neq \emptyset$. Next assume $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = \{s_2\}$. The possible a-values for Ω are 1, 2, 3 and 6. There is only one left cell F with $\mathcal{R}(F) = \{s_2\}$ in Ω if $a(\Omega) = 1$ or 3. When $a(\Omega) = 2$ or 6, there are two distinct left cells $F,\ F'$ with $\mathcal{R}(F) = \mathcal{R}(F') = \{s_2\}$. We can easily show $\Sigma(F) \neq \Sigma(F')$ by only comparing the numbers of left cells P of W_a with $\mathcal{R}(P) = \{s_1\}$ in $\Sigma(F)$ and $\Sigma(F')$ (one is 1 and another is 2). So this again implies $\Gamma = \Gamma'$.

6.5 Finally assume that W_a has type $\widetilde{F}_4$.

In this case, we shall only get some partial results in proving Assertion 4.1. As before, we need only to consider the case when $K = \mathcal{R}(\Gamma) = \mathcal{R}(\Gamma')$ is saturated. The possible saturated subsets of S are $\{s_0, s_1, s_2\}$ and $\{s_3, s_4\}$. When $\mathcal{R}(\Gamma) = \{s_3, s_4\}$ (resp. $\{s_0, s_1, s_2\}$), we have $a(\Gamma) \geq 3$ (resp. $a(\Gamma) \geq 6$). On the other hand, there exists a unique left cell Γ of W_a satisfying the conditions $a(\Gamma) = 3$ and $\mathcal{R}(\Gamma) = \{s_3, s_4\}$ (resp. $a(\Gamma) = 6$ and $\mathcal{R}(\Gamma) = \{s_0, s_1, s_2\}$). In [15], we described all the left cells Γ of W_a with $a(\Gamma) \in \{4, 5\}$. From all these results, it is easily seen that Assertion 4.1 holds in the cases when $a(\Gamma) = a(\Gamma') \leq 5$ and when $a(\Gamma) = a(\Gamma') = 6$ with $\mathcal{R}(\Gamma) = \mathcal{R}(\Gamma') = \{s_0, s_1, s_2\}$.

Now consider the case when both Γ and Γ' are in the two-sided cell Ω with $a(\Omega) = 24$. (i.e. Ω is the lowest two-sided cell of W_a). Note that each left cell of W_a in Ω is a single sign type (see 2.9 and [10][12]). If a left cell Γ of Ω satisfies $|\mathcal{R}(\Gamma)\bigcap\{s_2,s_3\}|=1$, then w is in some $\{s_2,s_3\}$-string with $i(w,s_2,s_3)$ constant as w ranges over Γ (see 3.4). Suppose that $\Gamma,\Gamma'\subseteq\Omega$ satisfy $\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')=\{s_3,s_4\}$ and $\Sigma(\Gamma)=\Sigma(\Gamma')$. If $\Gamma\neq\Gamma'$, then each element w of $\Gamma\bigcup\Gamma'$ is in some $\{s_2,s_3\}$-string ξ with $i(w,s_2,s_3)\in\{1,3\}$ and $|\bar{\xi}\bigcap(\Gamma\bigcup\Gamma')|=2$, where the notation $\bar{\xi}$ stands for the set consisting of all terms of ξ. We may assume

$$i(w,s_2,s_3)=\begin{cases}1, & \text{if } w\in\Gamma;\\ 3, & \text{if } w\in\Gamma'.\end{cases}$$

There exists a bijection from Γ to Γ' by sending w to ws_2s_3. We have $\ell(ws_2s_3)=\ell(w)+2$ for $w\in\Gamma$. Let $x,\ x'$ be the shortest elements of $\Gamma,\ \Gamma'$, respectively (it is known that such an element is unique in Γ or Γ', see 2.9 or [10]). Let $v=x's_2$. Then $\Gamma_v\in\Sigma(\Gamma')$ by 1.3(2)(3) and by the fact $\mathcal{R}(v)=\{s_2,s_3,s_4\}\supsetneqq\{s_3,s_4\}=\mathcal{R}(\Gamma')$. There exists a bijective map from Γ' to Γ_v which sends z to zs_2. We have $\ell(z)+1=\ell(zs_2)$ for $z\in\Gamma'$. This implies that v is the unique shortest element of Γ_v and so $\ell(v')\geq\ell(x)+3$ for $v'\in\Gamma_v$. Since any element w with w—x and $\mathcal{R}(w)\not\subseteq\mathcal{R}(x)$ satisfies $\ell(w)\leq\ell(x)+1$, we have $\Gamma_v\notin\Sigma(\Gamma)$. This implies $\Sigma(\Gamma)\neq\Sigma(\Gamma')$, a contradiction. So we must have $\Gamma=\Gamma'$. By the same argument as above, we can show that if $\mathcal{R}(\Gamma)=\mathcal{R}(\Gamma')=\{s_0,s_1,s_2\}$ and $\Sigma(\Gamma)=\Sigma(\Gamma')$ then $\Gamma=\Gamma'$.

By 1.3(4) and by the knowledge of unipotent classes of the corresponding algebraic group (see [2]), we see that all the possible a-values on elements of W_a of type $\widetilde{F}_4$ are 0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 16 and 24. So we have completed our verification of Assertion 4.1 and hence of Conjecture 1.5 in all the cases listed in 1.6.

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