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Skew Tableaux, Lattice Paths, and Bounded Partitions

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We establish a one to one correspondence between a set of certain bounded partitions and a set of two-rowed standard Young tableaux of skew shape. Then we obtain a formula for a number which enumerates these partitions. We give two proofs for this formula, one by applying the above correspondence, the other by using the reflection principle. Finally, we give another expression for this number in terms of $f(n, k)$'s (see Section 5 for the definition of $f(n, k)$).

1. PRELIMINARIES

1.1. By a partition, we mean a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$, which is denoted by $\lambda = \{\lambda_1 \geqslant \cdots \geqslant \lambda_r\}$, where an empty sequence is allowed and is denoted by $\{0\}$. The sum of the parts is called the weight of λ, denoted by $|\lambda|$: $|\lambda| = \lambda_1 + \cdots + \lambda_r$. The number r of parts of λ, denoted by $l(\lambda)$, is called the length of λ. We say that λ is a partition of n if $|\lambda| = n$. The set of all partitions of n is denoted by Λ_n. Each partition $\lambda = \{\lambda_1 \geqslant \cdots \geqslant \lambda_r\}$ is associated with a Ferrer graph F_λ such that the number of boxes in its ith row is λ_i. For example, the partition $\{5 \geqslant 3 \geqslant 2\}$ is associated with the Ferrer graph shown in Fig. 1.

Two partitions are conjugate if their associated Ferrer graphs are transposed with one another. We denote by λ' the conjugate of a partition λ. Given two partitions $\lambda = \{\lambda_1 \geqslant \cdots \geqslant \lambda_r\}$ and $\mu = \{\mu_1 \geqslant \cdots \geqslant \mu_t\}$, we say that μ is contained in λ, written $\lambda \geqslant \mu$ if $r \geqslant t$ and $\lambda_i \geqslant \mu_i$ for all i, $1 \leqslant i \leqslant t$. Graphically, this amounts to saying that the Ferrer graph associated to μ lies inside the one associated to λ. Clearly, two partitions λ, μ satisfy $\lambda \geqslant \mu$ if and only if $\lambda' \geqslant \mu'$. For $\lambda \geqslant \mu$, we call the set-theoretic difference $\theta = \lambda - \mu$, usually written $\theta = \lambda/\mu$, a skew diagram. The weight of λ/μ is denoted by

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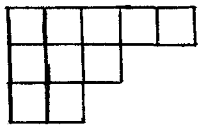

FIGURE 1

$|\lambda/\mu|$: $|\lambda/\mu| = |\lambda| - |\mu|$. For example, if $\lambda = \{5 \geqslant 3 \geqslant 2\}$ and $\mu = \{3 \geqslant 2 \geqslant 1\}$, then the skew diagram λ/μ is the shaded region shown in Fig. 2.

1.2. A quasi-tableau T_λ is a labelling of the boxes of a Ferrer graph F_λ with positive integers which increase weakly from left to right along each row. λ is called the shape of T_λ. Let α_i be the times of occurrence of i in T_λ as a label for $i \geqslant 1$. Then $(\alpha_1, \alpha_2, ...)$ is called the weight of T_λ. A tableau is a quasi-tableau whose labels increase strictly down each column. A standard tableau (resp. a standard quasi-tableau) is a tableau (resp. a quasi-tableau) T_λ in which each number $1, 2, ..., |\lambda|$ occurs exactly once so that its weight is $(1, ..., 1)$. Figure 3a shows a standard tableau and Fig. 3b shows a standard quasi-tableau, both of which have the shape $\{5 \geqslant 3 \geqslant 2\}/\{2 \geqslant 1\}$.

2. LATTICE PATHS AND SKEW TABLEAUX

2.1. Let $\pi = \mathbb{N} \times \mathbb{N}$. By a path of π, we mean a sequence $w = (s_0, s_1, ..., s_n)$ of points $s_i = (x_i, y_i)$ in π such that for each i, $1 \leqslant i \leqslant n$, either $x_{i-1} = x_i$, $y_{i-1} = y_i + 1$ (a vertical step) or $y_{i-1} = y_i$, $x_{i-1} = x_i - 1$ (a horizontal step), where n is called the length of w, s_0 (resp. s_n) is called the initial (resp. terminal) point of w. See Fig. 4 for a path w of π with length 8. Each path w determines a Ferrer diagram and hence a partition denoted by λ_w in a natural way. The partition determined by the path in Fig. 4 is $\{4 \geqslant 3 \geqslant 1 \geqslant 1\}$.

2.2. For any two points $\alpha, \beta \in \pi$, we denote by $P(\alpha, \beta)$ the set of all paths of π whose initial and terminal points are α, β, respectively. Suppose $\alpha = (a_1, b_1)$ and $\beta = (a_2, b_2)$. Then $P(\alpha, \beta) \neq \varnothing$ if and only if $a_1 \leqslant a_2$ and

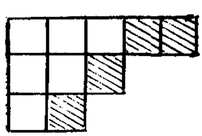

FIGURE 2

FIGURE 3

$b_1 \geqslant b_2$. In particular, in the case when $a_1 = b_2 = 0$, each path in $P(\alpha, \beta)$ determines a partition which is contained in the rectangular partition

$$a_2^{b_1} = \underbrace{\{a_2 \geqslant \cdots \geqslant a_2\}}_{b_1 \text{ times}}.$$

Actually, the map $w \mapsto \lambda_w$ is a bijection from the set $P(\alpha, \beta)$ to the set of partitions contained in $a_2^{b_1}$. Thus by [1], we have $|P(\alpha, \beta)| = \binom{a_2 + b_1}{b_1}$, where $\binom{a}{b} = a!/b!\,(a-b)!$ is a binomial coefficient. More generally, this implies that for any two points $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ of π, the formula

$$|P(\alpha, \beta)| = \binom{a_2 + b_1 - a_1 - b_2}{b_1 - b_2} \tag{2.2.1}$$

holds.

2.3. Fix three integers $a, b > 0$, $s \geqslant 0$ with $a + s \geqslant b$. Then $\{a + s \geqslant b\}/\{s\}$ is a skew diagram. Let $\alpha = (0, a)$ and $\beta = (b, 0)$. For a path $w = (s_0, s_1, ..., s_{a+b}) \in P(\alpha, \beta)$ with $s_i = (x_i, y_i)$, we associate a quasi-tableau $T_w^{(s)} = (d_{\ell m}^{(s)})$ of the skew shape $\{a + s \geqslant b\}/\{s\}$ as follows. Let $j_1 < j_2 < \cdots < j_a$ (resp. $k_1 < k_2 < \cdots < k_b$) be the subsequence of $1, 2, ..., a+b$ satisfying $x_{j_t - 1} = x_{j_t} - 1$ (resp. $y_{k_t - 1} = y_{k_t} + 1$) for all t, $1 \leqslant t \leqslant b$ (resp. $1 \leqslant t \leqslant a$). Then it is clear that the set $\{1, 2, ..., a+b\}$ is a disjoint union of

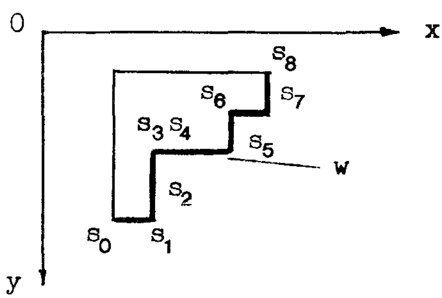

FIGURE 4

FIGURE 5

the sets $\{j_t \mid 1 \leqslant t \leqslant b\}$ and $\{k_r \mid 1 \leqslant r \leqslant a\}$. We define $T_w^{(s)}$ by setting $d_{1,s+r}^{(s)} = k_r$ and $d_{2,t}^{(s)} = j_t$ for $1 \leqslant r \leqslant a$, $1 \leqslant t \leqslant b$. For example, Fig. 5a shows the quasi-tableau $T_w^{(2)}$ of shape $\{6 \geqslant 4\}/\{2\}$ associated to the path w in Fig. 4; Fig. 5b shows the quasi-tableau $T_w^{(0)}$ of shape $\{4 \geqslant 4\}$ associated to the same path. We see that the former is a standard tableau but the latter is not.

2.4. Let $a, b, s \in \mathbb{Z}$ and $\alpha, \beta \in \pi$ be as in 2.3. For the partitions $\lambda = \{a+s \geqslant b\}$ and $\mu = \{s\}$, let $T(\lambda/\mu)$ be the set of all quasi-tableaux of shape λ/μ. It is clear that the map $\varphi_s\colon w \mapsto T_w^{(s)}$ is a bijection from the set $P(\alpha, \beta)$ to the set $T(\lambda/\mu)$. It is also clear that under the map φ_s, $T_w^{(s)}$ is a standard tableau if and only if the partition associated to the path w is contained in the partition

$$\lambda(a, b, s) = \{\underbrace{b \geqslant \cdots \geqslant b}_{a+s-b \text{ times}} \geqslant b-1 \geqslant b-2 \geqslant \cdots \geqslant s\} \tag{2.4.1}$$

with the convention that $\lambda(a, b, s) = b^a$ for $s \geqslant b$.

2.5. Let $\Lambda(a, b, s)$ be the set of all partitions contained in the above partition $\lambda(a, b, s)$ and let $p(a, b, s)$ be the cardinality of $\Lambda(a, b, s)$. Let $f^{\lambda/\mu}$ be the number of all standard tableaux of shape λ/μ. Then it follows from the above discussion that

PROPOSITION. *For any integers $a, b > 0$, $s \geqslant 0$ with $a + s \geqslant b$, we have*

$$f^{\{a+s \geqslant b\}/\{s\}} = p(a, b, s). \tag{2.5.1}$$

3. A FORMULA FOR THE NUMBER $p(a, b, s)$

3.1. We are interested in computing the number $p(a, b, s)$. This is well known in the following two special cases.

(i) $s \geqslant b$. In this case, $p(a, b, s)$ is equal to the number of all partitions contained in the rectangular partition b^a and hence

$$p(a, b, s) = \binom{a+b}{a} \tag{3.1.1}$$

by [1].

(ii) $a=b$ and $s=0$. In this case, $p(a, b, s)$ is equal to the number of all partitions contained in the staircase partition $\{a \geqslant a-1 \geqslant \cdots \geqslant 1\}$ and hence

$$p(a, a, 0)=\frac{1}{a+2}\binom{2a+2}{a+1}, \tag{3.1.2}$$

which is known as a Catalan number [2].

3.2. Now we shall compute the number $p(a, b, s)$ in the general case. To do this, we must introduce the concept of the skew Schur symmetric function $s_{\lambda/\mu}$. Given two partitions λ, μ with $\lambda \geqslant \mu$ and $n=|\lambda/\mu|$, the skew Schur symmetric function $s_{\lambda/\mu}$ is, by definition, an element of the polynomial ring $\mathbb{Z}[x_1, x_2, ..., x_n]$ in n variables $x_1, ..., x_n$ given by

$$s_{\lambda/\mu}=\sum_T x^T \tag{3.2.1}$$

summed over all tableaux T of shape λ/μ, where $x^T=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ with $(\alpha_1, \alpha_2, ..., \alpha_n)$ the weight of T (see 1.1).

3.3. For each $\alpha=(\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Let $\lambda=\{\lambda_1 \geqslant \cdots \geqslant \lambda_r\}$ be any partition of length $\leqslant n$. We denote by m_λ the polynomial $m_\lambda(x_1, ..., x_n)=\sum_\alpha x^\alpha$, summed over all distinct permutations α of $(\lambda_1, ..., \lambda_n)$, with the convention that $\lambda_j=0$ for $j>r$. Then we define, for each $\ell>0$, the polynomial $h_\ell=\sum_{|\lambda|=\ell} m_\lambda$, which is called the ℓth complete symmetric function. The following result, known as the Jacobi–Trudi identity, gives an expression of $s_{\lambda/\mu}$ in terms of the h_ℓ's [4].

Proposition. *For any two partitions λ, μ with $\lambda \geqslant \mu$ and $l(\lambda) \leqslant n$, the identity*

$$s_{\lambda/\mu}=\det(h_{\lambda_i-\mu_j-i+j})_{1 \leqslant i, j \leqslant n} \tag{3.3.1}$$

holds. Here some zero parts are allowed to occur in the expressions $\lambda=\{\lambda_1 \geqslant \cdots \geqslant \lambda_n\}$ and $\mu=\{\mu_1 \geqslant \cdots \geqslant \mu_n\}$, and we stipulate that $h_0=1$ and $h_t=0$ for $t<0$.

3.4. Recall that $f^{\lambda/\mu}$ denotes the number of standard tableaux of shape λ/μ (see 2.5). By (3.2.1), we see that $f^{\lambda/\mu}$ equals the coefficient of $x_1x_2\cdots x_{|\lambda/\mu|}$ in $s_{\lambda/\mu}$. On the other hand, it is easily checked that for any set of integers $\alpha_1, ..., \alpha_m \geqslant 0$, the coefficient of $x_1 x_2 \cdots x_{\alpha_1 + \cdots + \alpha_m}$ in $h_{\alpha_1} \cdots h_{\alpha_m}$ equals the multinomial coefficient

$$\binom{\alpha_1 + \cdots + \alpha_m}{\alpha_1, ..., \alpha_m} = \frac{(\alpha_1 + \cdots + \alpha_m)!}{\alpha_1!\, \alpha_2! \cdots \alpha_m!}.$$

Hence by Proposition 3.3, it implies that

PROPOSITION. *For any two partitions λ, μ with $\lambda \geqslant \mu$ and $l(\lambda) \leqslant n$, we have*

$$f^{\lambda/\mu} = |\lambda/\mu|!\, \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{1 \leqslant i, j \leqslant n}. \tag{3.4.1}$$

3.5. Now we are ready to show

THEOREM. *For any integers $a, b > 0$, $s \geqslant 0$ with $a + s \geqslant b$, we have*

$$p(a, b, s) = \binom{a+b}{a} - \binom{a+b}{a+s+1}. \tag{3.5.1}$$

Proof. By Proposition 2.5, it is enough to show that

$$f^{\{a+s \geqslant b\}/\{s\}} = \binom{a+b}{a} - \binom{a+b}{a+s+1}. \tag{3.5.2}$$

But by Proposition 3.4, we have

$$\begin{aligned} f^{\{a+s \geqslant b\}/\{s\}} &= (a+b)! \begin{vmatrix} \dfrac{1}{a!} & \dfrac{1}{(a+s+1)!} \\ \dfrac{1}{(b-s-1)!} & \dfrac{1}{b!} \end{vmatrix} \\ &= \frac{(a+b)!}{a!\, b!} - \frac{(a+b)!}{(a+s+1)!\,(b-s-1)!} \\ &= \binom{a+b}{a} - \binom{a+b}{a+s+1}. \end{aligned}$$

Our result follows.

4. Another Proof of Theorem 3.5

Theorem 3.5 can also be shown by using the reflection principle [3].

4.1. For $\alpha=(0, a)$, $\beta=(b, 0)$ in π, and $s \geqslant 0$, let $P(\alpha, \beta, s)$ be the set of all paths in $P(\alpha, \beta)$ (see 2.2) whose associated partitions are contained in the partition $\lambda(\alpha, \beta, s)$ (see 2.4). Let $P'(\alpha, \beta, s)$ be the set-theoretic complement of $P(\alpha, \beta, s)$ in $P(\alpha, \beta)$. Then by (3.1.1), we have

$$p(a, b, s)=|P(\alpha, \beta, s)|=|P(\alpha, \beta)|-|P'(\alpha, \beta, s)|$$
$$=\binom{a+b}{a}-|P'(\alpha, \beta, s)|. \tag{4.1.1}$$

So to show Theorem 3.5, it is enough to show that

$$|P'(\alpha, \beta, s)|=\binom{a+b}{a+s+1}. \tag{4.1.2}$$

We may assume that $s<b$ since otherwise $P'(\alpha, \beta, s)=\varnothing$ and the result is obvious. Let $\gamma=(s+1, a+s+1)$. Then by (2.2.1), we have

$$|P(\gamma, \beta)|=\binom{a+b}{a+s+1}. \tag{4.1.3}$$

Thus our proof will be accomplished once we construct a bijection from the set $P'(\alpha, \beta, s)$ to the set $P(\gamma, \beta)$. We see that a path in $P(\alpha, \beta)$ belongs to the set $P'(\alpha, \beta, s)$ if and only if it intersects the line $L\colon x+y=a+s+1$ of $\mathbb{R}^2$ (see Fig. 6, for example).

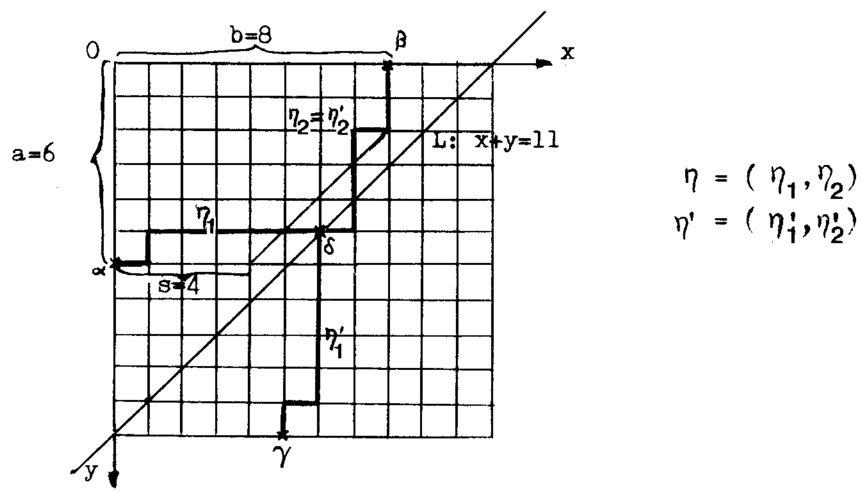

Figure 6

4.2. Given a path η in $P'(\alpha, \beta, s)$, let δ be the lowest common point of the path η and the line L. Then η can be divided into two parts η_1, η_2 with $\eta_1 \in P(\alpha, \delta)$ and $\eta_2 \in P(\delta, \beta)$. There exists a unique path $\eta' \in P(\gamma, \beta)$ passing through the point δ and obtained from η by replacing the part η_1 by its image η'_1 under the reflection of $\mathbb{R}^2$ to the line L (see Fig. 6, for example). Then it is easily checked that the map $\eta \mapsto \eta'$ is a bijection from the set $P'(\alpha, \beta, s)$ to $P(\gamma, \beta)$ and hence Theorem 3.5 follows.

5. The Relation between the Numbers $p(a, b, s)$ and $f(n, k)$

5.1. For $n \geqslant 1$ and $k \geqslant 0$, define

$$f(n, k) = \frac{n-k+1}{n} \binom{n+k-2}{n-1}. \tag{5.1.1}$$

This number has the following combinatorial meaning. Consider the sequences of positive integers $(a_1, a_2, ..., a_n)$ satisfying the conditions:

(i) $1 = a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$;

(ii) $a_i \leqslant i$ $(1 \leqslant i \leqslant n)$.

Then for $n \geqslant 1$ and $0 \leqslant k \leqslant n$, $f(n, k)$ is exactly the number of such sequences with

(iii) $a_n = k$.

(See [2].) It is easily seen that there exists a bijective map from the set of sequences $(a_1, a_2, ..., a_n)$ satisfying conditions (i)–(iii) to the set of partitions contained in the partition $\lambda(n-2, k-1, 1)$ (see (2.4.1)) by sending $(a_1, a_2, ..., a_n)$ to $\{a_{n-1} - 1 \geqslant a_{n-2} - 1 \geqslant \cdots \geqslant a_2 - 1\}$. Thus $f(n, k)$ is also the number of all partitions contained in $\lambda(n-2, k-1, 1)$, i.e.,

$$f(n, k) = p(n-2, k-1, 1). \tag{5.1.2}$$

5.2. How can we do the converse? That is, how can we express the number $p(a, b, s)$ in terms of the $f(n, k)$'s? The following result gives us an answer.

Theorem. *For the integers $a, b > 0$, $s \geqslant 0$ with $a + s \geqslant b$, we have*

$$p(a, b, s) = \sum_{i=0}^{[s/2]} (-1)^i \binom{s-i}{i} f(a+s+1-i, b+1-i), \tag{5.2.1}$$

where $[h]$ denotes the largest integer not greater than h for any rational number h.

5.3. To show this theorem, we need the following result.

LEMMA. *Let a, b, s be as in Theorem* 5.2. *Then the equation*

$$f^{\{a+s\geqslant b\}/\{s\}}=f^{\{a+s\geqslant b\}/\{s-1\}}-f^{\{a+s-1\geqslant b-1\}/\{s-2\}} \tag{5.3.1}$$

holds.

Proof. This can be shown by giving a combinatorial interpretation on Eq. (5.3.1). Consider the set $T[\{a+s\geqslant b\}/\{s-1\}]$ of all standard tableaux of shape $\{a+s\geqslant b\}/\{s-1\}$. Let T_1 (resp. T_2) be the set of all tableaux in $T[\{a+s\geqslant b\}/\{s-1\}]$ with 1 in their first (resp. second) rows. Let $T[\{a+s\geqslant b\}/\{s\}]$ (resp. $T[\{a+s-1\geqslant b-1\}/\{s-2\}]$) be the set of all standard tableaux of shape $\{a+s\geqslant b\}/\{s\}$ (resp. of shape $\{a+s-1\geqslant b-1\}/\{s-2\}$). Then there exists a bijective map from T_1 to $T[\{a+s\geqslant b\}/\{s\}]$ (resp. from T_2 to $T[\{a+s-1\geqslant b-1\}/\{s-2\}]$) by sending T to T', where T' is obtained from T by removing the 1 and by replacing each i, $1<i\leqslant a+b+1$, by $i-1$. For example, when

$$T=\begin{array}{cccccc} & & \boxed{1} & \boxed{4} & \boxed{5} & \boxed{7} \\ \boxed{2} & \boxed{3} & \boxed{6} & & & \end{array} \in T_1[\{6\geqslant 3\}/\{2\}],$$

we have

$$T'=\begin{array}{cccccc} & & & \boxed{3} & \boxed{4} & \boxed{6} \\ \boxed{1} & \boxed{2} & \boxed{5} & & & \end{array} \in T[\{6\geqslant 3\}/\{3\}];$$

when

$$T=\begin{array}{cccccc} & & \boxed{2} & \boxed{3} & \boxed{5} & \boxed{7} \\ \boxed{1} & \boxed{4} & \boxed{6} & & & \end{array} \in T_2[\{6\geqslant 3\}/\{2\}],$$

we have

$$T'=\begin{array}{ccccc} & \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} \\ \boxed{3} & \boxed{5} & & & \end{array} \in T[\{5\geqslant 2\}/\{1\}].$$

Now Eq. (5.3.1) is shown by noting $f^{\{a+s\geqslant b\}/\{s\}}=|T[\{a+s\geqslant b\}/\{s\}]|$, $f^{\{a+s\geqslant b\}/\{s-1\}}=|T[\{a+s\geqslant b\}/\{s-1\}]|$, and $f^{\{a+s-1\geqslant b-1\}/\{s-2\}}=|T[\{a+s-1\geqslant b-1\}/\{s-2\}]|$.

By Proposition 2.5, Eq. (5.3.1) is equivalent to the equation

$$p(a, b, s)=p(a+1, b, s-1)-p(a+1, b-1, s-2). \tag{5.3.2}$$

5.4. *Proof of Theorem* 5.2. Apply induction on $s \geqslant 0$. The result is easily checked in the cases $s = 0, 1$ by Theorem 3.5. Now assume $s \geqslant 2$. Then by Eq. (5.3.2) and the inductive hypothesis, we have

$$\begin{aligned} p(a, b, s) &= p(a+1, b, s-1) - p(a+1, b-1, s-2) \\ &= \sum_{i=0}^{[(s-1)/2]} (-1)^i \binom{s-1-i}{i} f(a+s+1-i, b+1-i) \\ &\quad - \sum_{i=0}^{[s/2]-1} (-1)^i \binom{s-2-i}{i} f(a+s-i, b-i). \end{aligned} \tag{5.4.1}$$

(i) First assume that s is odd. Then by (5.4.1),

$$\begin{aligned} p(a, b, s) &= \sum_{i=0}^{[s/2]} (-1)^i \binom{s-1-i}{i} f(a+s+1-i, b+1-i) \\ &\quad + \sum_{i=1}^{[s/2]} (-1)^i \binom{s-1-i}{i-1} f(a+s+1-i, b+1-i) \\ &= f(a+s+1, b+1) + \sum_{i=1}^{[s/2]} (-1)^i \\ &\quad \times \left[\binom{s-1-i}{i} + \binom{s-1-i}{i-1}\right] f(a+s+1-i, b+1-i) \\ &= f(a+s+1, b+1) + \sum_{i=1}^{[s/2]} (-1)^i \\ &\quad \times \binom{s-i}{i} f(a+s+1-i, b+1-i) \\ &= \sum_{i=0}^{[s/2]} (-1)^i \binom{s-i}{i} f(a+s+1-i, b+1-i). \end{aligned} \tag{5.4.2}$$

(ii) Next assume that $s = 2k$ is even. Then by (5.4.1),

$$\begin{aligned} p(a, b, s) &= \sum_{i=0}^{k-1} (-1)^i \binom{2k-1-i}{i} f(a+2k+1-i, b+1-i) \\ &\quad + \sum_{i=1}^{k} (-1)^i \binom{2k-1-i}{i-1} f(a+2k+1-i, b+1-i) \\ &= f(a+2k+1, b+1) + (-1)^k f(a+k+1, b+1-k) \\ &\quad + \sum_{i=1}^{k-1} (-1)^i \binom{2k-i}{i} f(a+2k+1-i, b+1-i) \\ &= \sum_{i=0}^{k} (-1)^i \binom{2k-i}{i} f(a+2k+1-i, b+1-i). \end{aligned} \tag{5.4.3}$$

In both cases, our result follows.

5.5. Let a, b, s be as in Theorem 5.2. Then the partitions $\lambda(a, b, s)$ and $\lambda(b, a, a+s-b)$ are conjugate. This implies that

$$p(a, b, s) = p(b, a, a+s-b). \tag{5.5.1}$$

So we obtain another expression of $p(a, b, s)$ immediately from Theorem 5.2.

COROLLARY. *Let a, b, s be as in Theorem 5.2. Then we have*

$$p(a, b, s) = \sum_{i=0}^{[(a+s-b)/2]} (-1)^i \binom{a+s-b-i}{i} f(a+s+1-i, a+1-i). \tag{5.5.2}$$

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REFERENCES

1. G. ANDREWS, "The Theory of Partitions," Addison–Wesley, Reading, MA, 1976.
2. L. CARLITZ, Sequences, paths, and ballot numbers, *Fibonacci Quart.* **10** (1972), 531–549.
3. W. FELLER, "An Introduction to Probability Theory and Its Applications," Vol. 1, rev. ed., Wiley, New York, 1958.
4. I. G. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, New York/London, 1979.