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Some Results Relating Two Presentations of Certain Affine Weyl Groups

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It is known that an affine Weyl group W_a could be defined in many different ways: as a Coxeter group, as a group of certain affine transformations in a euclidean space, and so on. In particular, when it is of classical type (i.e., $\tilde{A}_m$, $m \geqslant 1$, $\tilde{B}_h$, $h \geqslant 3$, $\tilde{C}_n$, $n \geqslant 2$, or $\tilde{D}_k$, $k \geqslant 4$), the group W_a could also be regarded as a group of certain permutations on the integer set $\mathbb{Z}$. It is interesting to study the properties of the group W_a via various presentations of W_a. In the present paper, we restrict ourselves only to the affine Weyl groups of types $\tilde{A}_m$, $m \geqslant 1$, and $\tilde{C}_n$, $n \geqslant 2$.

First let us consider W_a as a Coxeter group with S its Coxeter generator set. For any proper subset $J \subset S$, Let w_J be the longest element of the parabolic subgroup of W_a generated by J. For $x, y, w \in W_a$, we use the notation $w = x \cdot y$ to indicate the expression $w = xy$ satisfying the condition $l(w) = l(x) + l(y)$, where $l(w)$ is the length function of W_a. Then a question to ask is when an element $w \in W_a$ could be or could not be written as the form $w = x \cdot w_J \cdot y$ for a certain proper subset $J \subset S$. In general, it is not easy to answer such a question by only considering W_a as a Coxeter group.

Next we consider W_a as a group of certain permutations on $\mathbb{Z}$. Then each element w of W_a is determined by a subset $\Gamma_w = \{(a, (a)w) \mid a \in \mathbb{Z}\}$ of $\mathbb{Z}^2$. We can establish certain congruence relations on the set Γ_w such that the number of the corresponding congruence classes in Γ_w is finite. We define a certain partial order on the set of congruence classes of Γ_w and then introduce the concepts of a chain and a k-chain family ($k > 0$) in such a partial ordered set. Let $d_k(w)$ be the cardinality of a maximal k-chain family in the set of congruence classes of Γ_w. Then it is easy to find the numbers $d_k(w_J)$ for any $J \subset S$ and any $k > 0$. Our main results of the present paper (i.e., Theorems 3.1 and 3.2) give a condition for an element

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$w \in W_a$ to be of the form $w = x \cdot w_J \cdot y$ but not of the form $w = x' \cdot w_I \cdot y'$ with some $x, y, x', y' \in W_a$ and certain $J, I \subset S$ in terms of integers $d_k(w)$, $d_k(w_J)$, and $d_k(w_I)$, $k > 0$.

It is known that the cells of the affine Weyl groups defined by D. Kazhdan and G. Lusztig play an important role in the representation theory of these groups and of the corresponding Hecke algebras [6]. Thus it is desirable to give an explicit description of these cells. This has been done for the affine Weyl groups of types $\tilde{A}_m$, $m \geqslant 1$, $\tilde{B}_3$, $\tilde{C}_2$, $\tilde{C}_3$, and $\tilde{G}_2$ [11, 10, 9, 3, 1, 2,] and has partly been done for the one of type $\tilde{D}_4$ [4]. Also, in an irreducible affine Weyl group, the lowest two-sided cell together with all the left cells contained in it, and all the two-sided cells with a-values $\leqslant 3$ have been described explicitly [12, 7]. As an application, we use our main result to give a new description for all the two-sided cells of the affine Weyl group of type $\tilde{A}_m$, $m \geqslant 1$, and also for some two-sided cells of the one of type $\tilde{C}_n$, $n \geqslant 2$.

The contents of this paper are organized as follows. In Section 1, we introduce several presentations of the affine Weyl groups $\mathscr{A}_m$ $(m \geqslant 2)$ and $\mathscr{C}_n$ $(n \geqslant 2)$ which are of types $\tilde{A}_{m-1}$ and $\tilde{C}_n$, respectively. Then in Section 2, the concepts of a chain and a k-chain family for an element of $\mathscr{A}_m$ and of $\mathscr{C}_n$ are defined. We also deduce some simple results concerning the cardinality of a maximal k-chain family of an element in these groups. Then our main results of this paper are stated in Section 3 but their proofs are postponed to Sections 4 and 5. Some interesting applications of these results are made to give a new description of the two-sided cells of $\mathscr{A}_m$ and some of the two-sided cells of $\mathscr{C}_n$ also in Section 3.

1. Several Presentations of the Groups $\mathscr{A}_m$ and $\mathscr{C}_n$

1.1. As a Coxeter group, an affine Weyl group could be defined by generators and relations. The affine Weyl group $\mathscr{A}_m$ $(m > 1)$ of type $\tilde{A}_{m-1}$ has the presentation

$$\tilde{A}_m = \langle r_i \mid r_i^2 = \mathbf{1} \text{ and } (r_i r_j)^{m_{ij}} = \mathbf{1} \text{ for } 0 \leqslant i, j < m \text{ and } i \neq j \rangle,$$

where

$$m_{ij} = \begin{cases} 2 & \text{if } \ i \not\equiv j \pm 1 \pmod{m}, \\ 3 & \text{if } \ i \equiv j \pm 1 \pmod{m}. \end{cases}$$

The affine Weyl group $\mathscr{C}_n$ $(n > 1)$ of type $\tilde{C}_n$ has the presentation

$$\mathscr{C}_n = \langle s_i \mid s_i^2 = \mathbf{1} \text{ and } (s_i s_j)^{n_{ij}} = \mathbf{1} \text{ for } 0 \leqslant i, j \leqslant n \text{ and } i \neq j \rangle,$$

where

$$n_{ij} = \begin{cases} 2 & \text{if } i \neq j \pm 1, \\ 3 & \text{if } i = j \pm 1 \text{ and } 0, n \notin \{i, j\}, \\ 4 & \text{if } \{i, j\} = \{0, 1\} \text{ or } \{n-1, n\}. \end{cases}$$

1.2. It is known that $\mathscr{A}_m$ could also be regarded as a group of certain permutations on the integer set $\mathbb{Z}$:

$$\mathscr{A}_m = \left\{ w : \mathbb{Z} \to \mathbb{Z} \mid (i+m)w = (i)w + m, \forall i \in \mathbb{Z}; \sum_{i=1}^{m} (i)w = \sum_{i=1}^{m} i \right\}.$$

Its Coxeter generator set $S = \{r_0, r_1, ..., r_{m-1}\}$ is given by

$$(i)\, r_t = \begin{cases} i & \text{if } i \not\equiv t, t+1 \ (\text{mod } m), \\ i+1 & \text{if } i \equiv t \ (\text{mod } m), \\ i-1 & \text{if } i \equiv t+1 \ (\text{mod } m), \end{cases}$$

for $i \in \mathbb{Z}$ and $0 \leqslant t \leqslant m-1$. Each element $w \in \mathscr{A}_m$ is determined entirely by an m-tuple $((i+1)w, (i+2)w, ..., (i+m)w)$ for any $i \in \mathbb{Z}$ (see [8, 11]).

1.3. The group $\mathscr{A}_m$ could also be regarded as the group of all $\mathbb{Z} \times \mathbb{Z}$ affine matrices of period m which are defined as follows.

(a) The integer set $\mathbb{Z}$ is the set parametrising its rows (resp. columns). The integers parametrising its rows (resp. columns) are monotonously increasing from top to bottom (resp. from left to right).

(b) The entries in each of its rows (resp. columns) are all zero except for one which is 1.

(c) Let $\{e(u, j_u) \mid u \in \mathbb{Z}\}$ be the set of its non-zero entries, where $e(u, j_u)$ lies in its (u, j_u)-position. Then $j_{u+m} = j_u + m$ (see [11]). Each element $w \in \mathscr{A}_m$ is assigned to such an affine matrix which has the non-zero entry set $\{e(u, (u)w) \mid u \in \mathbb{Z}\}$. Call that matrix the matrix form of w. Note that the matrix form of the inverse w^{-1} of an element w is the transposition of the matrix form of w for $w \in \mathscr{A}_m$.

1.4. The group $\mathscr{C}_n$ could be embedded into the group $\mathscr{A}_{2n+2}$ by sending s_t, $1 \leqslant t < n$, to $r_t r_{2n+1-t}$, s_0 to $r_0 r_{2n+1} r_0$, and s_n to $r_n r_{n+1} r_n$. Thus by identifying it with a subgroup of $\mathscr{A}_{2n+2}$ under such an embedding, the group $\mathscr{C}_n$ could be described as

$$\mathscr{C}_n = \{ w \in \mathscr{A}_{2n+2} \mid (-i)w = -(i)w \text{ for } i \in \mathbb{Z} \}.$$

Then its Coxeter generator set $S' = \{s_0, s_1, ..., s_n\}$ is given as below. For $i \in \mathbb{Z}$,

(a) if $1 \leqslant t < n$, then

$$(i)\, s_t = \begin{cases} i & \text{if } i \not\equiv \pm t,\ \pm(t+1) \pmod{2n+2}, \\ i+1 & \text{if } i \equiv t,\ -t-1 \pmod{2n+2}, \\ i-1 & \text{if } i \equiv t+1,\ -t \pmod{2n+2}, \end{cases}$$

(b)
$$(i)\, s_0 = \begin{cases} i & \text{if } i \not\equiv \pm 1 \pmod{2n+2}, \\ i+2 & \text{if } i \equiv -1 \pmod{2n+2}, \\ i-2 & \text{if } i \equiv 1 \pmod{2n+2}, \end{cases}$$

(c)
$$(i)\, s_n = \begin{cases} i & \text{if } i \not\equiv n,\ n+2 \pmod{2n+2}, \\ i+2 & \text{if } i \equiv n \pmod{2n+2}, \\ i-2 & \text{if } i \equiv n+2 \pmod{2n+2}. \end{cases}$$

1.5. *Remark.* (a) Our definition of $\mathscr{C}_n$ as a group of permutations on $\mathbb{Z}$ is slightly different from that given by R. Bédard (see [1]). According to Bédard, the group $\mathscr{C}_n$ is embedded into the group $\mathscr{A}_{2n+1}$ instead of $\mathscr{A}_{2n+2}$. The advantage of our definition is that the symmetry between the generators s_0 and s_n in $\mathscr{C}_n$ could be revealed more efficiently in form.

(b) As an equivalent description, the group $\mathscr{C}_n$ could also be regarded as the set of all the fixed elements of $\mathscr{A}_{2n+2}$ under an automorphism of $\mathscr{A}_{2n+2}$ which sends r_t to r_{2n+1-t} for $0 \leqslant t \leqslant 2n+1$.

(c) It is easily seen that for any $w \in \mathscr{C}_n$ and any $h, i \in \mathbb{Z}$, the equation

$$(i)w + (2h(n+1) - i)w = 2h(n+1) \tag{1.5.1}$$

holds. In particular, we have

$$(h(n+1))w = h(n+1) \tag{1.5.2}$$

for any $h \in \mathbb{Z}$.

(d) The matrix form of an element w of $\mathscr{C}_n$ could be defined by regarding w as an element of $\mathscr{A}_{2n+2}$.

1.6. Let $l(w)$ be the length function on $\mathscr{A}_m$ and let $l'(y)$ be the length function on $\mathscr{C}_n$. Let $\leqslant$ be the Bruhat order on $\mathscr{A}_m$ (resp. $\mathscr{C}_n$) as a Coxeter group. To each $w \in \mathscr{A}_m$, we assign two subsets of S by

$$\mathscr{L}(w) = \{r \in S \mid rw < w\} \quad \text{and} \quad \mathscr{R}(w) = \{r \in S \mid wr < w\}.$$

Also, to each $y \in \mathscr{C}_n$, we assign two subsets of S' by

$$\mathscr{L}(y) = \{s \in S' \mid sy < y\} \qquad \text{and} \qquad \mathscr{R}(y) = \{s \in S' \mid ys < y\}.$$

The following results are well known.

PROPOSITION. *Let* $w \in \mathscr{A}_m$ *and* $y \in \mathscr{C}_n$. *Then*

(a) $\mathscr{L}(w) = \{r_t \in S \mid (t)w > (t+1)w\}$;

(b) $\mathscr{R}(w) = \{r_t \in S \mid (t)w^{-1} > (t+1)w^{-1}\}$;

(c) $\mathscr{L}(y) = \{s_t \in S' \mid (t)y > (t+1)y\}$;

(d) $\mathscr{R}(y) = \{s_t \in S' \mid (t)y^{-1} > (t+1)y^{-1}\}$.

2. A k-Chain Family of an Element of $\mathscr{A}_m$ or $\mathscr{C}_n$

2.1. Given a positive integer n, by a partition of n, we mean a sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t)$ of non-negative integers in decreasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t$ with $\sum_{i=1}^{t} \lambda_i = n$. We do not distinguish between two such sequences which differ only by a string of zeros at the end. Let Λ_n be the set of all partitions of n. A natural (partial) order on Λ_n is defined as

$$(\lambda_1, \lambda_2, ..., \lambda_t) \geqslant (\mu_1, \mu_2, ..., \mu_r)$$

if

$$\sum_{i=1}^{h} \lambda_i \geqslant \sum_{i=1}^{h} \mu_i \qquad \text{for} \quad 1 \leqslant h \leqslant \min\{t, r\}.$$

2.2. A set $\{(a_i, b_i) \mid 1 \leqslant i \leqslant t\} \subset \mathbb{Z}^2$ is called a chain if there exists a permutation $i_1, i_2, ..., i_t$ of $1, 2, ..., t$ such that $a_{i_1} < a_{i_2} < \cdots < a_{i_t}$ and $b_{i_1} > b_{i_2} > \cdots > b_{i_t}$. We stipulate that an empty set of $\mathbb{Z}^2$ is a chain.

Given $w \in \mathscr{A}_m$ (resp. $w \in \mathscr{C}_n$) and a set $I \subset \mathbb{Z}$, we denote by (I, w) the set $\{(a, (a)w) \mid a \in I\}$. The set (I, w) is called a chain of w if (I, w) is a chain in $\mathbb{Z}^2$. In terms of matrix forms, a chain of $w \in \mathscr{A}_m$ (resp. $w \in \mathscr{C}_n$) corresponds to a set of non-zero entries of the matrix form of w which run from northeast to southwest. Thus for $w \in \mathscr{C}_n$ and $I \subset \mathbb{Z}$, no matter whether w is regarded as an element of $\mathscr{C}_n$ or as an element of $\mathscr{A}_{2n+2}$, the property of whether the set (I, w) is a chain or not is definite.

Two chains (P, w) and (Q, w) of $w \in \mathscr{A}_m$ (resp. of $w \in \mathscr{C}_n$) are congruent if there exists some $h \in \mathbb{Z}$ such that

$$Q = \{a \mid hm + a \in P\} \qquad (\text{resp. } Q = \{a \mid h(2n+2) + a \in P\}).$$

We usually do not distinguish a chain from any of its congruences.

2.3. Fix $w \in \mathscr{A}_m$. A set C of chains of w is called a chain family of w if for any two chains (P, w) and (Q, w) of w in C, we have $a \not\equiv b \pmod m$ for $a \in P$ and $b \in Q$. Let k be a positive integer. A chain family C of w is called a k-chain family of w if it consists of k chains of w. We use the notation $|C|$ to denote the sum $\sum_P |P|$ and call it the size of C, where (P, w) ranges over all the chains of w in C and the notation $|P|$ stands for the cardinality of the set P. A k-chain family C of w is maximal if the inequality $|C'| \leqslant |C|$ holds for any k-chain family C' of w. We denote by $d_k(w)$ the size of a maximal k-chain family of w for $k > 0$. Clearly, we have

$$d_1(w) < d_2(w) < \cdots < d_t(w) = m \tag{2.3.1}$$

for some t, $1 \leqslant t \leqslant m$. By a result of C. Greene [5], we also have the inequality

$$d_1(w) \geqslant d_2(w) - d_1(w) \geqslant d_3(w) - d_2(w) \geqslant \cdots \geqslant d_t(w) - d_{t-1}(w). \tag{2.3.2}$$

Let $\phi(w) = (\lambda_1, \lambda_2, ..., \lambda_t)$ with $\lambda_1 = d_1(w)$ and $\lambda_i = d_i(w) - d_{i-1}(w)$ for $1 < i \leqslant t$. Then $\phi(w) \in \Lambda_m$ and hence we get a map $\phi: \mathscr{A}_m \to \Lambda_m$. It is known that the map ϕ is surjective. The significance of the map ϕ is that the partition of $\mathscr{A}_m$ determined by ϕ, i.e.,

$$\mathscr{A}_m = \bigcup_{\lambda \in \Lambda_m} \phi^{-1}(\lambda), \tag{2.3.3}$$

coincides with the partition of $\mathscr{A}_m$ into two-sided cells (see [11, Theorem 17.4]). A two-sided cell of a Coxeter group W is a certain equivalence class of W which is important in the representation theory of W and of its associated Hecke algebra. The reader can consult a paper [6] by D. Kazhdan and G. Lusztig for its detailed definition. Here we should point out that the set of two-sided cells of W is a partial ordered set, whose partial order is denoted by $\leqslant_{LR}$. In particular, it is known that the map ϕ induces an order-reversing bijection $\lambda \mapsto \phi^{-1}(\lambda)$ from the set Λ_m to the set of two-sided cells of $\mathscr{A}_m$ (see [11]). One of our results of the present paper is to give another description of the two-sided cells of $\mathscr{A}_m$.

2.4. We now turn to define the k-chain families for the group $\mathscr{C}_n$.

For any set $P \subset \mathbb{Z}$ and any integer a, we denote by $P + a$ the set $\{x \mid x - a \in P\}$ and by $-P$ the set $\{x \mid -x \in P\}$.

Fix $w \in \mathscr{C}_n$.

Two chains (P, w) and (P', w) of w form a chain pair of w if $P' = 2n + 2 - P$ and $P \cap P' = \varnothing$. In that case, we denote the set P' by $\bar{P}$ and call $(\bar{P}, w)$ the opposite of (P, w). Clearly, we have $\bar{\bar{P}} = P$.

A chain (P, w) of w is said to be special if one of the following conditions is satisfied.

(a) $(P \cup \bar{P}, w)$ is a chain of w;

(b) $(P \cup (-P), w)$ is a chain of w.

A chain (P, w) of w is n-special (resp. 0-special) if condition (a) (resp. condition (b)) is satisfied.

Note that it is possible for a chain of w to be both n-special and 0-special. Also note that a chain (P, w) is special (resp. 0-special, n-special) if and only if the chain $(\bar{P}, w)$ is special (resp. 0-special, n-special). When this equivalent condition holds, we call $\{(P, w), (\bar{P}, w)\}$ a special (resp. 0-special, n-special) chain pair of w.

An empty set $(\varnothing, w)$ is regarded as a chain and also as a chain pair of w, which is both 0-special and n-special.

A set C of chains of $w \in \mathscr{C}_n$ is called a chain family of w, if the following conditions on C are satisfied.

(i) Each chain (P, w) in C satisfies the condition that

$$a \not\equiv 0, \quad n+1 \pmod{2n+2} \qquad \text{for} \quad a \in P.$$

(ii) If C contains a chain (P, w) then C also contains the chain $(\bar{P}, w)$.

(iii) If (P, w) and (Q, w) are two different non-empty chains of w in C, then $a \not\equiv b \pmod{2n+2}$ for any $a \in P$ and $b \in Q$.

Note that by the above definition, a chain family of w consists of chain pairs of w, some of which may be an empty chain pair of w. Only an empty chain pair of w is allowed to occur in a chain family of w more than once. When this happens, we stipulate that all the empty chains in this chain family of w are pairly distinct. For an integer $k > 0$, a chain family of w is called a k-chain family if it consists of exactly k-chain pairs of w.

2.5. Let C be a chain family of $w \in \mathscr{C}_n$. We define the size of C by

$$|C| = \begin{cases} \sum_P |P| & \text{if none of chain pairs of } w \text{ in } C \text{ is special,} \\ 2 + \sum_P |P| & \text{if there exist two special chain pairs of } w \text{ in } C, \\ & \text{one of which is 0-special and the other is } n\text{-special,} \\ 1 + \sum_P |P| & \text{otherwise,} \end{cases} \tag{2.5.1}$$

where P in the above sums ranges over all the chains of w in C. Clearly, we have $|C| \leqslant 2n+2$ for any chain family C of w. For any $k > 0$, a k-chain family C of w is maximal if the inequality $|C'| \leqslant |C|$ holds for any k-chain family C' of w. We denote by $d'_k(w)$ the size of a maximal k-chain family of $w \in \mathscr{C}_n$ for $k > 0$. The inequalities

$$d'_1(w) < d'_2(w) < \cdots < d'_t(w) = 2n + 2 \tag{2.5.2}$$

obviously hold for some $t > 0$.

2.6. *Remark.* (a) Note that it makes no difference for the definition of a chain of an element $w \in \mathscr{C}_n$ if w is regarded as an element of $\mathscr{A}_{2n+2}$. However, the definition of a chain family of $w \in \mathscr{C}_n$ is quite different when regarding w as an element of $\mathscr{A}_{2n+2}$. Later, unless otherwise specified, for $w \in \mathscr{C}_n$, we always regard w as an element of $\mathscr{C}_n$ rather than of $\mathscr{A}_{2n+2}$ when we mention a chain family of w.

(b) We do not know whether the inequality

$$d'_1(w) \geqslant d'_2(w) - d'_1(w) \geqslant \cdots \geqslant d'_t(w) - d'_{t-1}(w)$$

holds or not for any $w \in \mathscr{C}_n$ and for the number $t > 0$ in (2.5.2) since C. Greene's result (see [5, Theorem 3.14]) is not applicable in our case.

(c) Comparing with that in the case of $\mathscr{A}_m$, it is not always true that the two-sided cells of $\mathscr{C}_n$ containing an element w could be determined by the integers $d'_k(w)$, $k > 0$. But we show later that the integers $d'_k(w)$, $k > 0$, do play some role in the description of certain two-sided cells of $\mathscr{C}_n$.

2.7. Now let us go back to the group $\mathscr{A}_m$. For any proper subset $J \subset \{0, 1, 2, ..., m-1\}$, let w_J be the longest element of the parabolic subgroup of $\mathscr{A}_m$ generated by $\{r_i \mid i \in J\}$. We describe the partition $\phi(w_J)$ (see 2.3 for the map ϕ). Given $J \subset \{0, 1, 2, ..., m-1\}$ with $J \neq \varnothing$, decomposition

$$J = J_1 \cup J_2 \cup \cdots \cup J_t \tag{2.7.1}$$

is standard if the following conditions are satisfied.

(a) Each J_i, $1 \leqslant i \leqslant t$, is not empty and consists of consecutive integers.

(b) For any pair i, j, $1 \leqslant i < j \leqslant t$, we have $J_i \cap J_j = \varnothing$ and that $J_i \cup J_j$ does not consist of consecutive integers.

(c) $|J_1| \geqslant |J_2| \geqslant \cdots \geqslant |J_t|$,

where we stipulate that 0 is the successor of $m-1$. J is indecomposable if $t \leqslant 1$. Note that a standard decomposition for a proper subset of the set $\{0, 1, 2, ..., m-1\}$ always exists but is not necessarily unique, where we stipulate that a standard decomposition of an empty set is the trivial one.

Now assume that $J = J_1 \cup J_2 \cup \cdots \cup J_t$ is a standard decomposition of a proper subset $J \subset \{0, 1, 2, ..., m-1\}$. Then it is easily seen that

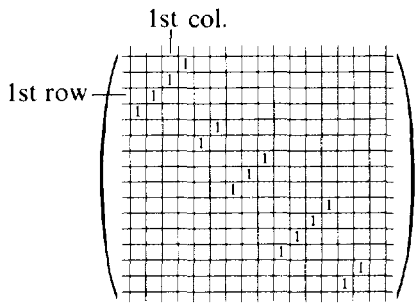
$$d_k(w_J) = \begin{cases} \sum_{i=1}^{k} (J_i| + 1) & \text{for} \quad 1 \leqslant k \leqslant t, \\ k - t + \sum_{j=1}^{t} (|J_j| + 1) & \text{for} \quad t < k \leqslant m + t - \sum_{j=1}^{t} (|J_j| + 1). \end{cases} \tag{2.7.2}$$

This implies the following result immediately.

PROPOSITION. *$\phi(w_J) = (|J_1| + 1, |J_2| + 1, ..., |J_t| + 1, 1, ..., 1) \in \Lambda_m$ for any proper subset $J \subset \{0, 1, 2, ..., m-1\}$ with a standard decomposition $J = J_1 \cup J_2 \cup \cdots \cup J_t$.*

Note that the above partition $\phi(w_J)$ can also be obtained from w_J by applying the Robinson–Schensted map (see [11]).

2.8. EXAMPLE. Let $w = w_J \in \mathscr{A}_9$ with $J = \{0, 1, 3, 5, 6, 8\}$. Then $J = \{8, 0, 1\} \cup \{5, 6\} \cup \{3\}$ is a standard decomposition of J. The matrix form of w is



Let $P = \{-1, 0, 1, 2\}$, $Q = \{3, 4\}$, and $R = \{5, 6, 7\}$. Then $\{(P, w)\}$, $\{(P, w), (R, w)\}$, and $\{(P, w), (R, w), (Q, w)\}$ are a maximal 1-chain family, a maximal 2-chain family, and a maximal 3-chain family of w, respectively. Thus we have $d_1(w) = 4$, $d_2(w) = 7$, and $d_3(w) = 9$. So $\phi(w) = (4, 3, 2)$.

2.9. Next we consider the group $\mathscr{C}_n$. For any proper subset $J \subset \{0, 1, ..., n\}$, let w_J be the longest element of the parabolic subgroup of $\mathscr{C}_n$ generated by $\{s_i \mid i \in J\}$. We describe the integers $d'_k(w_J)$, $k > 0$. Given $J \subset \{0, 1, 2, ..., n\}$ with $J \neq \varnothing$, decomposition

$$J = J_1 \cup J_2 \cup \cdots \cup J_t \tag{2.9.1}$$

is standard if the following conditions are satisfied.

(a) Each J_i, $1 \leqslant i \leqslant t$, is not empty and consists of consecutive integers.

(b) For any pair i, j, $1 \leqslant i < j \leqslant t$, we have $J_i \cap J_j = \varnothing$ and that $J_i \cup J_j$ does not consist of consecutive integers.

(c) Define, for $1 \leqslant i \leqslant t$, the numbers

$$\|J_i\| = \begin{cases} 2(|J_i|+1) & \text{if} \quad 0, n \notin J_i, \\ 2(|J_i|+1)+1 & \text{if either} \quad 0 \in J_i \text{ or } n \in J_i. \end{cases} \tag{2.9.2}$$

Then $\|J_1\| \geqslant \|J_2\| \geqslant \cdots \geqslant \|J_t\|$.

Similar to the case of $\mathscr{A}_m$, for any proper subset J of the set $\{0, 1, 2, ..., n\}$, a standard decomposition of J always exists but is not necessarily unique, where we make the same convention on an empty set as that in 2.7.

Now assume that $J = J_1 \cup J_2 \cup \cdots \cup J_t$ is a standard decomposition of a proper subset $J \subset \{0, 1, 2, ..., n\}$. Then we have

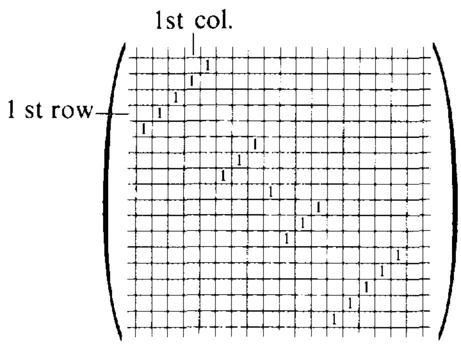
$$d'_k(w_J) = \sum_{i=1}^{k} \|J_i\| \qquad \text{for} \quad 1 \leqslant k \leqslant t \tag{2.9.3}$$

and

$$d'_k(w_J) = \begin{cases} d'_{k-1}(w_J)+2 & \text{if either } d'_{k-1}(w_J) < 2n \text{ or } d'_{k-1}(w_J) = 2n \text{ with} \\ & \varepsilon(J) = 2, \\ d'_{k-1}(w_J)+1 & \text{if either } d'_{k-1}(w_J) = 2\text{n with } \varepsilon(J) = 0 \text{ or} \\ & d'_{k-1}(w_J) = 2n+1 \text{ with } \varepsilon(J) = 0, 1, \end{cases} \tag{2.9.4}$$

where $\varepsilon(J)$ is the number of i, $1 \leqslant i \leqslant t$, with $\|J_i\|$ odd.

2.10. Example. Let $w = w_J \in \mathscr{C}_5$ with $J = \{0, 1, 3, 4\}$. Then the matrix form of w is



Let $P = \{1, 2\}$ and $Q = \{3, 4, 5\}$. Then $\bar{P} = \{10, 11\}$ and $\bar{Q} = \{7, 8, 9\}$. So $\{(P, w), (\bar{P}, w)\}$, $\{(Q, w), (\bar{Q}, w)\}$ are two chain pairs of w. In particular, $\{(P, w), (\bar{P}, w)\}$ is 0-special. $\{(Q, w), (\bar{Q}, w)\}$, $\{(Q, w), (\bar{Q}, w), (P, w), (\bar{P}, w)\}$, and $\{(P, w), (\bar{P}, w), (Q, w), (\bar{Q}, w), (\varnothing, w)\}$ are a maximal 1-chain family, a maximal 2-chain family, and a maximal 3-chain family of w, respectively. Thus by (2.9.2), (2.9.3), and (2.9.4), we have $d'_1(w) = 6$, $d'_2(w) = 11$, and $d'_3(w) = 12$.

Now we show some results about the integers $d_k(w)$ for $w \in \mathscr{A}_m$ and $d'_k(y)$ for $y \in \mathscr{C}_n$ which will be used in subsequent sections.

2.11. Lemma. *Let $w \in \mathscr{A}_m$ with $d_1(w) < d_2(w) < \cdots < d_t(w) = m$. Then*

(a) *$d_k(w) = d_k(w^{-1})$ for $1 \leqslant k \leqslant t$;*

(b) *If $w = rx$ for some $x \in \mathscr{A}_m$ and $r \in \mathscr{L}(w)$ then $d_k(w) \geqslant d_k(x)$ for any $k > 0$.*

Proof. This is an immediate consequence of [11, Lemmas 5.4 and 5.5]. ∎

2.12. Lemma. *Let $y \in \mathscr{C}_n$ with $d'_1(y) < d'_2(y) < \cdots < d'_t(y) = 2n + 2$. Then*

(a) *$d'_k(y) = d'_k(y^{-1})$ for $1 \leqslant k \leqslant t$;*

(b) *If $x \in \mathscr{C}_n$ satisfies the relation $y = sx$ with $s \in \mathscr{L}(y)$ then $d'_k(y) \geqslant d'_k(x)$ for any $k > 0$.*

Proof. In terms of matrix forms, a chain of y is a set of non-zero entries of y which run from northeast to southwest. We know that the matrix form of y^{-1} is the transposition of that of y. Thus assertion (a) follows immediately. For (b), we see that for any chain (P, x) of x, $((P)s, y)$ is a chain of y. In particular, if (P, x) is a 0-special (resp. n-special) chain of x then $((P)s, y)$ is a 0-special (resp. n-special) chain of y. This implies that if C is a k-chain family of x then $C' = \{((R)s, y) \mid (R, x) \in C\}$ is a k-chain family of y with $|C'| \geqslant |C|$. So assertion (b) follows, too. ∎

3. Main Results and Applications

In this section we state two main results of our paper, but postpone the proofs to Sections 4 and 5. We also make some applications of these results to the description of two-sided cells of $\mathscr{A}_m$ and $\mathscr{C}_n$.

3.1. Theorem. *Let $w \in \mathscr{A}_m$ with $d_1(w) < d_2(w) < \cdots < d_t(w) = m$. Then for any k, $1 \leqslant k \leqslant t$, there exists an expression*

$$w = x \cdot w_J \cdot y$$

for some $x, y \in \mathscr{A}_m$ *and* $J \subset \{0, 1, 2, ..., m-1\}$ *with* $d_k(w) = d_k(w_J)$. *On the other hand,* $w \neq x' \cdot w_I \cdot y'$ *for any* $x', y' \in \mathscr{A}_m$ *and* $I \subset \{0, 1, 2, ..., m-1\}$ *with* $d_h(w_I) > d_h(w)$ *for some* h, $1 \leqslant h \leqslant t$.

3.2. THEOREM. *Let* $y \in \mathscr{C}_n$ *with* $d'_1(y) < d'_2(y) < \cdots < d'_t(y) = 2n+2$. *Then for any* k, $1 \leqslant k \leqslant t$, *there exists an expression*

$$y = x \cdot w_J \cdot z$$

for some $x, z \in \mathscr{C}_n$ *and* $J \subset \{0, 1, 2, ..., n\}$ *with* $d'_k(y) = d'_k(w_J)$. *On the other hand,* $w \neq x' \cdot w_I \cdot z'$ *for any* $x', z' \in \mathscr{C}_n$ *and* $I \subset \{0, 1, 2, ..., n\}$ *with* $d'_h(w_I) > d'_h(y)$ *for some* h, $1 \leqslant h \leqslant t$.

3.3. We mentioned in 2.3 that the map $\phi: \mathscr{A}_m \to \Lambda_m$ induces an order-reversing bijection $\lambda \mapsto \phi^{-1}(\lambda)$ from the set Λ_m to the set of two-sided cells of $\mathscr{A}_m$. This is one way to describe the two-sided cells of $\mathscr{A}_m$. Now Theorem 3.1 provides us with a new description of these cells.

3.4. THEOREM. *Let* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t, 1, ..., 1) \in \Lambda_m$ *with* $t \geqslant 0$ *and* $\lambda_t > 1$. *Then an element* w *of* $\mathscr{A}_m$ *is in the two sided cell* $\phi^{-1}(\lambda)$ *of* $\mathscr{A}_m$ *if and only if the following conditions on* w *are satisfied.*

(a) *For any* k, $1 \leqslant k \leqslant t$, *there exists an expression*

$$w = x \cdot w_J \cdot y$$

for some $x, y \in \mathscr{A}_m$ *and some subset* $J \subset \{0, 1, 2, ..., m-1\}$ *such that* J *has a standard decomposition* $J = J_1 \cup \cdots \cup J_k$ *(see* (2.9.1)*) with* $|J| = \sum_{i=1}^{k} (\lambda_i - 1)$.

(b) $w \neq x' \cdot w_I \cdot y'$ *if* $x', y' \in \mathscr{A}_m$ *and if* $I \subset \{0, 1, 2, ..., m-1\}$ *has a standard decomposition* $I = I_1 \cup \cdots \cup I_h$ *with*

$$|I| > \begin{cases} \sum_{i=1}^{h} (\lambda_i - 1) & \text{if } 1 \leqslant h < t, \\ \sum_{i=1}^{t} (\lambda_i - 1) & \text{if } h \geqslant t. \end{cases}$$

Proof. This is an immediate consequence of Theorem 3.1, Proposition 2.7, and Formula (2.7.2). ∎

3.5. It is interesting to consider some special cases of Theorem 3.4. For a proper subset $J \subset \{0, 1, 2, ..., m-1\}$, we see from (2.7.2) that $d_1(w_J) = m$ if and only if $J = \{0, 1, ..., \hat{i}, ..., m-1\}$ for some i, $0 \leqslant i \leqslant m-1$, where the notation $\hat{i}$ means that the number i is omitted from the set $\{0, 1, 2, ..., m-1\}$. Thus by Theorem 3.4, we see that an element $w \in \mathscr{A}_m$ is in $\phi^{-1}((m))$ if and only if there exists an expression $w = x \cdot w_J \cdot y$ with $J = \{0, 1, ..., \hat{i}, ..., m-1\}$ for some $x, y \in \mathscr{A}_m$ and some i, $0 \leqslant i \leqslant m-1$. Note that this is just a special case of a result in my previous paper [12, Theorems 2.3, 2.4, and 5.2].

More generally, for $1 \leqslant h \leqslant m$ and $J \subset \{0, 1, 2, ..., m-1\}$, it is easily seen from (2.7.2) that $d_1(w_J) = h$ if and only if the exists a standard decomposition $J = J_1 \cup J_2 \cup \cdots \cup J_t$ with $|J_1| = h-1$. Thus by Theorem 3.4, we see that for any $w \in \mathscr{A}_m$, the first part of the partition $\phi(w)$ is equal to h if and only if there exists an expression $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{A}_m$ and some indecomposable subset $J \subset \{0, 1, 2, ..., m-1\}$ with $|J| = h-1$. In particular, we have the following result.

PROPOSITION. *Let $\lambda = (h, 1, ..., 1) \in \Lambda_m$ with $1 \leqslant h \leqslant m$. Then the necessary and sufficient conditions for an element $w \in \mathscr{A}_m$ being in the two-sided cell $\phi^{-1}(\lambda)$ of $\mathscr{A}_m$ are as follows.*

(a) *There exists an expression $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{A}_m$ and some indecomposable subset $J \subset \{0, 1, 2, ..., m-1\}$ with $|J| = h-1$.*

(b) *$w \neq x' \cdot w_I \cdot y'$ for any x', $y' \in \mathscr{A}_m$ and any subset $I \subset \{0, 1, 2, ..., m-1\}$ with $|I| \geqslant h$.*

Proof. This follows from Theorem 3.4 by taking $t \leqslant 1$. ∎

3.6. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t, 1, ..., 1) \in \Lambda_m$ with $\lambda_t > 1$. Then for any $w \in \phi^{-1}(\lambda)$, there does not necessarily exist an expression $w = x \cdot w_J \cdot y$ for some element $x, y \in \mathscr{A}_m$ and some subset $J \subset \{0, 1, 2, ..., m-1\}$ such that a standard decomposition $J = J_1 \cup J_2 \cup \cdots \cup J_t$ of J satisfies the equation $|J_i| = \lambda_i - 1$ for $1 \leqslant i \leqslant t$. But it is so in the case when $\lambda_1 = \lambda_2 = \cdots = \lambda_h$, $\lambda_{h+1} = \lambda_{h+2} = \cdots = \lambda_t$, $\lambda_h = \lambda_{h+1} + 1$, and $\sum_{i=1}^{t} \lambda_i = m$ for some h, $1 \leqslant h \leqslant t$. In particular, in that case, let $\mu = (\mu_1, ..., \mu_k) \in \Lambda_m$, then $\mu \nleqslant \lambda$ if and only if either $\mu_1 > \lambda_1$ or $\mu_{h+1} \geqslant \lambda_1$. So we get the following result directly from Theorem 3.4.

PROPOSITION. *Let $\lambda = (a, ..., a, a-1, ..., a-1) \in \Lambda_m$ with h parts equal to a and k parts equal to $a-1$ for some $a > 1$, $h > 0$, and $k \geqslant 0$. Then $w \in \mathscr{A}_m$ is in the two-sided cell $\phi^{-1}(\lambda)$ of $\mathscr{A}_m$ if and only if the following conditions on w are satisfied.*

(a) *There exists an expression $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{A}_m$ and some subset $J \subset \{0, 1, 2, ..., m-1\}$ such that there exists a standard decomposition*

$$J = \begin{cases} J_1 \cup \cdots \cup J_{h+k} & \text{if } a > 2, \\ J_1 \cup \cdots \cup J_h & \text{if } a = 2, \end{cases}$$

with

$$|J_i| = \begin{cases} a-1 & \text{if } 1 \leqslant i \leqslant h, \\ a-2 & \text{if } a > 2 \text{ and } h < i \leqslant h+k. \end{cases}$$

(b) *$w \neq x' \cdot w_I \cdot y'$ for any x', $y' \in \mathscr{A}_m$ and any $I \subset \{0, 1, 2, ..., m-1\}$ such that a standard decomposition of I, say $I = I_1 \cup \cdots \cup I_t$, satisfies either $|I_1| \geqslant a$ or $|I_{h+1}| \geqslant a-1$.*

3.7. Now we consider some applications of Theorem 3.2 to the description of some two-sided cells of $\mathscr{C}_n$. According to a result from my previous paper [12, Theorems 2.3, 2.4 and 5.2], we know that an element $w \in \mathscr{C}_n$ is in the lowest two-sided cell W_v of $\mathscr{C}_n$ with respect to the partial order $\leqslant_{LR}$ (see 2.3) if and only if there exists an expression $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{C}_n$ and some $J \in \{\{0, 1, ..., n-1\}, \{1, 2, ..., n\}\}$. We see from (2.9.2), (2.9.3), and (2.9.4) that if $J = \{0, 1, ..., n-1\}$ or $\{1, 2, ..., n\}$ then $d_1'(w_J) = 2n+1$. We also see from Lemma 2.12 that if $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{C}_n$ and some $J \in \{\{0, 1, ..., n-1\}, \{1, 2, ..., n\}\}$ then $d_1'(w) \geqslant d_1'(w_J)$. But it is clear that $d_1'(w) \leqslant 2n+1$ for any $w \in \mathscr{C}_n$ and that $d_1'(w_I) < 2n+1$ for any $I \subset \{0, 1, ..., n\}$ other than $\{0, 1, ..., n-1\}$ and $\{1, 2, ..., n\}$. Thus by Theorem 3.2, we get a new description of the two-sided cell W_v of $\mathscr{C}_n$.

PROPOSITION. *An element $w \in \mathscr{C}_n$ is in the two-sided cell W_v of $\mathscr{C}_n$ if and only if $d_1'(w) = 2n+1$.*

3.8. Let W_1 be the set of all the non-identity elements of a Coxeter group W each of which has a unique reduced expression. According to a result of G. Lusztig [8, Proposition 3.8], the set W_1 forms a two-sided cell of W. Now let $W = \mathscr{C}_n$. Then by Theorem 3.2, we see that an element w of $\mathscr{C}_n$ is in W_1 if and only if the following condition on w is satisfied.

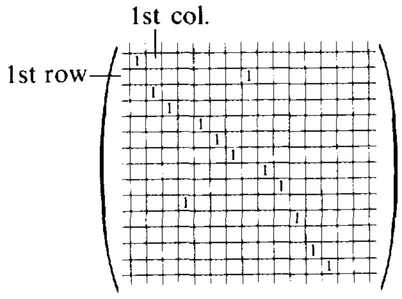
$$(d_1'(w), d_2'(w)) = \begin{cases} (3, 5) \text{ or } (4, 6) & \text{if } n > 2, \\ (3, 5) \text{ or } (4, 5) & \text{if } n = 2, \end{cases}$$

or equivalently, it is so if and only if one of the following conditions is satisfied.

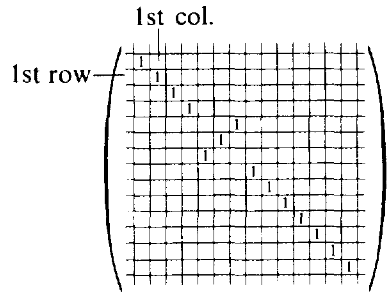
(a) $0 < (1)w < (2)w < \cdots < (i-1)w < (i+1)w < \cdots < (n)w < n+1$ and $(i)w \neq i$ for some i, $1 \leqslant i \leqslant n$;

(b) The n-tuple $((1)w, (2)w, ..., (n)w)$ is equal to either $(1, 2, ..., n-2, n+2, n+3)$ or $(-2, -1, 3, 4, ..., n)$.

The following are examples of some elements of $\mathscr{C}_4$ in W_1.



w



y

We see that $(d_1'(w), d_2'(w)) = (4, 6)$ and $(d_1'(y), d_2'(y)) = (3, 5)$.

3.9. More generally, for $h \geqslant 1$, let W_h be the set of all the elements w of $\mathscr{C}_n$ satisfying the following conditions.

(a) $w = x \cdot w_J \cdot y$ for some $x, y \in \mathscr{C}_n$ and some $J = \{i_1, i_2, ..., i_h\} \subset \mathbb{Z}$ with $0 \leqslant i_1 < i_2 - 1 < i_3 - 2 < \cdots < i_h - h + 1 \leqslant n - h + 1$;

(b) $w \neq x' \cdot w_I \cdot y'$ for any x', $y' \in \mathscr{C}_n$ and any $I \subset \{0, 1, ..., n\}$ with $|I| > h$ or $I = \{i, i+1\}$, $0 \leqslant i < n$.

Clearly, the set W_h is non-empty if and only if $h \leqslant (n+2)/2$. The set W_h, when it is non-empty, forms a two-sided cell of $\mathscr{C}_n$ by a result due to G. H. Lawton [7, Theorem 3.22(i)]. Now assume $h \leqslant (n+2)/2$. Then by Theorem 3.2, we can give another description of the set W_h as follows. An element w of $\mathscr{C}_n$ is in W_h if and only if the inequalities

$$\begin{aligned} 4 &\geqslant d_1'(w) \geqslant d_2'(w) - d_1'(w) \geqslant \cdots \geqslant d_h'(w) - d_{h-1}'(w) \\ &\geqslant 3 > d_{h+1}'(w) - d_h'(w) \end{aligned} \tag{3.9.1}$$

hold. By the property of an element of $\mathscr{C}_n$, it is easily seen that if an element w of $\mathscr{C}_n$ satisfies inequalities (3.9.1) and if $h \geqslant 3$ then $d_i'(w) - d_{i-1}'(w) = 4$ for all i, $1 \leqslant i \leqslant h-2$, with the convention that $d_0'(w) = 0$.

4. Proof of Theorem 3.1

Fix k, $1 \leqslant k \leqslant t$. Let C be a maximal k-chain family of w. Then $|C| = d_k(w)$ and one of the following cases must occur.

Case (i). For each chain (P, w) in C, the set P consists of consecutive integers.

Case (ii). There exists some chain (P, w) in C such that the set P does not consist of consecutive integers.

First assume that we are in Case (i). Then by replacing it by one of its congruences if necessary, we may choose C and an integer h such that for each chain (P, w) in C, we have the inclusion

$$P \subseteq \{h+1, h+2, ..., h+m\}.$$

Let (P_1, w), (P_2, w), ..., (P_k, w) be all the chains of w in C, where

$$P_j = \{a_j, a_j+1, ..., a_j+b_j\} \subseteq \{h+1, h+2, ..., h+m\}$$

with $b_j \geqslant 0$ for $1 \leqslant j \leqslant k$. We define k subsets $J_1, J_2, ..., J_k$ of $\{h+1, h+2, ..., h+m\}$ by

$$J_j = \begin{cases} \{a_j, a_j+1, ..., a_j+b_j-1\} & \text{if } b_j > 0, \\ \varnothing & \text{if } b_j = 0. \end{cases}$$

Let $J = \bigcup_{j=1}^{k} J_j$. Then we have the inclusion $\{r_j \mid j \in J\} \subseteq \mathscr{L}(w)$ by Proposition 1.6(a) with the convention that $r_j = r_{j+m}$ for $j \in \mathbb{Z}$. So $w = w_J \cdot y$ for some $y \in \mathscr{A}_m$. Clearly, we have $d_k(w_J) = d_k(w)$ by Proposition 2.7.

Next assume that we are in Case (ii). Let (P, w) be a chain of w in C with $P = \{a_1, a_2, ..., a_t\}$ such that $a_1 < a_2 < \cdots < a_t$ and $a_u < a_{u+1} - 1$ for some u, $1 \leqslant u < t$.

First assume $(a_u)w > (a_u+1)w$. Let r be the unique element of S which transposes a_u with a_u+1. Then we have $r \in \mathscr{L}(w)$ by Proposition 1.6(a). Let $w_1 = rw$. Then $C_1 = \{((R)r, w_1) \mid (R, w) \in C\}$ is a k-chain family of w_1 with $|C_1| = |C| = d_k(w)$, where the notation $(R)r$ stands for the set $\{(a)r \mid a \in R\}$. Since the inequality $d_k(w_1) \leqslant d_k(w)$ holds by Lemma 2.11(b), this implies the equation $d_k(w_1) = d_k(w)$.

In the case of $(a_{u+1})w < (a_{u+1}-1)w$, we can find an element $w_1 = rw$ for some $r \in \mathscr{L}(w)$ with $d_k(w_1) = d_k(w)$ in a similar way.

Finally, assume that both inequalities $(a_u)w < (a_u+1)w$ and $(a_{u+1})w > (a_{u+1}-1)w$ hold. Then there must exist some integer p, $a_u < p < a_{u+1} - 1$ such that $(p+1)w < (a_u)w < (p)w$. Let r be the unique element of S which transposes p with $p+1$. Then by Proposition 1.6(a), we have $r \in \mathscr{L}(w)$. Let $w_1 = rw$. We define

$$C_1 = \{((R)r, w_1) \mid (R, w) \in C\}$$

if either $(p, (p)w)$ and $(p+1, (p+1)w)$ are not contained simultaneously in any congruence of any chain of w in C, or $(p)w - (p+1)w > m$. On the other hand, assume that both $(p, (p)w)$ and $(p+1, (p+1)w)$ are contained in some congruence of a chain (Q, w) in C with

$(p)w-(p+1)w<m$. We may assume $p, p+1 \in Q$ by replacing (Q, w) by one of its congruences if necessary. Let

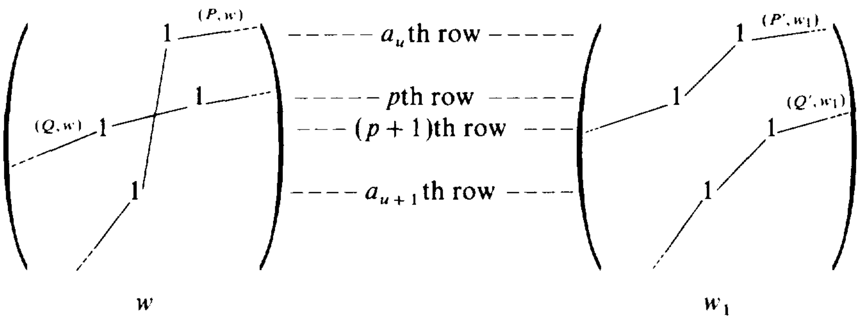
$$Q=\{c_1, \ldots, c_f, p, p+1, d_1, \ldots, d_g\}$$

with $c_1<\cdots<c_f<p<p+1<d_1<\cdots<d_g$. Define two sets

$$P'=\{a_1, \ldots, a_u, p, d_1, \ldots, d_g\}$$

and

$$Q'=\{c_1, \ldots, c_f, p+1, a_{u+1}, \ldots, a_t\}.$$



We define

$$C_1=\{(R, w_1) \mid (R, w) \in C, R \neq P, Q\} \cup \{(P', w_1), (Q', w_1)\}.$$

Then it is easily checked that C_1 is a k-chain family of w_1 with $|C_1|=|C|=d_k(w)$. Again by Lemma 2.11(b), we have $d_k(w_1) \leqslant d_k(w)$. So $d_k(w_1)=d_k(w)$.

Thus in Case (ii), we can always find an element $w_1 \in \mathscr{A}_m$ with $w=rw_1$ and $d_k(w_1)=d_k(w)$ for some $r \in \mathscr{L}(w)$.

If w_1 satisfies the same condition as that for w in Case (i) then as before, we could find a subset $J \subseteq \{h+1, h+2, \ldots, h+m\}$ and an element $y \in \mathscr{A}_m$ for some $h \in \mathbb{Z}$ such that $w_1=w_J \cdot y$ and $d_k(w_1)=d_k(w_J)$. Thus our assertion is shown in that case since $w=r \cdot w_J \cdot y$ and $d_k(w)=d_k(w_1)=d_k(w_J)$. If w satisfies the same condition as that for w in Case (ii), then by the same way as before, we could find $w_2 \in \mathscr{A}_m$ with $w_1=r'w_2$ and $d_k(w_1)=d_k(w_2)$ for some $r' \in \mathscr{L}(w_1)$. Continue this process to obtain a sequence of elements $w_0=w, w_1, w_2, \ldots$, where $w_i w_{i+1}^{-1} \in \mathscr{L}(w_i)$ for $i=0, 1, 2, \ldots$ and $d_k(w_0)=d_k(w_1)=d_k(w_2)=\ldots$. Since $l(w_0)>l(w_1)>l(w_2)>\ldots$, such a process must stop at some stage so that for some $u>0$, we could find a subset $J \subseteq \{h+1, h+2, \ldots, h+m\}$ for some $h \in \mathbb{Z}$ and an element $y \in \mathscr{A}_m$ with

$w_u = w_J \cdot y$ and $d_k(w_u) = d_k(w_J)$. Let $x = ww_u^{-1}$. Then $w = x \cdot w_J \cdot y$ and $d_k(w) = d_k(w_J)$. This verifies the first assertion of the theorem. Finally, the second assertion of the theorem follows immediately from Lemma 2.11.

The proof of Theorem 3.2.

The main idea of the following proof is similar to that for Theorem 3.1. The only difference is that in the present proof, we have to treat some special chains in a given maximal k-chain family of an element of $\mathscr{C}_n$, which will make our proof more complicated.

In this section, when we mention a congruence of an integer, we always mean that it is module $2n+2$.

Let C be a maximal k-chain family of y. Then one of the following cases must occur.

Case (i). In each chain (P, y) in C, the set P either is empty or consists of consecutive integers. If there exist at most one empty chain pair of y and at least one non-empty special chain pair of y in C, then either there exists some non-empty 0-special chain (P, y) in C with the set P containing a congruence of 1 or there exists some non-empty n-special chain (Q, y) in C with the set Q containing a congruence of n. In particular, if there exists no empty chain pair of y in C and if there exist two non-empty chain pairs of y in C, one of which is 0-special and another is n-special, then there exist some 0-special chain (P, y) in C with the set P containing a congruence of 1 and also some n-special chain (Q, y) in C with the set Q containing a congruence of n.

Case (ii). There exists some non-empty chain (P, y) in C such that the set P does not consist of consecutive integers.

Case (iii). There exist exactly one empty chain pair of y and at least one non-empty special chain pair of y in C. But for any non-empty 0-special chain (P, y) in C, the set P contains no congruence of 1. Also, for any non-empty n-special chain (Q, y) in C, the set Q contains no congruence of n.

Case (iv). There exists no empty chain pair of y in C. Either there exist 0-special chains (P, y) in C but none of the sets P contains a congruence of 1, or there exist n-special chains (Q, y) in C but none of the sets Q contains a congruence of n.

(I) First assume that we are in Case (i). Then by replacing it by its congruence if necessary, we may choose exactly one chain (P, y) in each chain pair of y in C such that $P \subseteq \{1, 2, ..., n\}$. Let (P_1, y), (P_2, y), ...,

(P_k, y) be all the chains of y in C such that for every i, $1 \leqslant i \leqslant k$, either $P_i = \varnothing$ or $P_i = \{a_i, a_i+1, ..., a_i+b_i\} \subseteq \{1, 2, ..., n\}$ with $b_i \geqslant 0$. We define k subsets $J_1, J_2, ..., J_k$ of $\{0, 1, ..., n\}$ by

$$J_i = \begin{cases} \{a_i, a_i+1, ..., a_i+b_i-1\} & \text{if } (P_i, y) \text{ is either not special, or} \\ & \text{non-empty 0-special with } a_i \neq 1 \text{ or/and} \\ & \text{non-empty 0-special with } a_i+b_i \neq n, \\ \{0, 1, ..., b_1\} & \text{if } (P_i, y) \text{ is non-empty 0-special} \\ & \text{with } a_i = 1, \\ \{a_i, a_i+1, ..., n\} & \text{if } (P_i, y) \text{ is non-empty } n\text{-special} \\ & \text{with } a_i+b_i = n, \\ \varnothing & \text{if } (P_i, y) \text{ is empty,} \end{cases}$$

where we stipulate that $\{a_i, a_i+1, ..., a_i+b_i-1\} = \varnothing$ if $b_i = 0$. Let $J = \bigcup_{i=1}^{k} J_i$. Then by Proposition 1.6(c), we have $y = w_J \cdot z$ for some $z \in \mathscr{C}_n$. We also have $d'_k(w_J) = d'_k(y)$ by (2.9.3) and (2.9.4).

Next assume that we are not in Case (i). We find an element $y_1 \in \mathscr{C}_n$ from y in Cases (ii)–(iv) such that $y = sy_1$ for some $s \in \mathscr{L}(y)$ and $d'_k(y_1) = d'_k(y)$. Suppose that this could always be done. Then by the same argument as that in the proof of Theorem 3.1, our first assertion follows. The last assertion of the theorem is an immediate consequence of Lemma 2.12.

(II) Now suppose that we are in Case (ii). Then there exists some chain (P, y) in C such that $P = \{a_1, a_2, ..., a_t\}$ with $a_1 < a_2 < \cdots < a_t$ and $a_u < a_{u+1} - 1$ for some u, $1 \leqslant u < t$.

First assume $(a_u)\,y > (a_u+1)\,y$. If $a_u \not\equiv -1$, $n \pmod{2n+2}$ then let s be the unique element of S' which transposes a_u with a_u+1. If $a_u \equiv -1$ (resp. $a_u \equiv n$) $\pmod{2n+2}$ then let $s = s_0$ (resp. $s = s_n$) (note that in this case, we have $a_{u+1} > a_u + 2$ by the definition of a k-chain family). In either case, we have $s \in \mathscr{L}(y)$ by Proposition 1.6(c). Let $y_1 = sy$. Then $C = \{((R)s, y_1) \mid (R, y) \in C\}$ is a k-chain family of y_1 with $|C_1| = |C| = d'_k(y)$. Since the inequality $d'_k(y_1) \leqslant d'_k(y)$ holds by Lemma 2.12(b), this implies that C_1 is a maximal k-chain family of y_1 and hence we have $d'_k(y_1) = d'_k(y)$.

We can find a required element y_1 in the case of $(a_{u+1})\,y < (a_{u+1}-1)\,y$ by a similar way.

Next assume that both inequalities $(a_u)\,y < (a_u+1)\,y$ and $(a_{u+1})\,y > (a_{u+1}-1)\,y$ hold. Then there must exist some integer p, $a_u < p < a_{u+1} - 1$ such that $(p+1)\,y < (a_u)\,y < (p)\,y$.

If $p \not\equiv -1$, $n \pmod{2n+2}$ then let s be the unique element of S' which transposes p with $p+1$. If $p \equiv -1$ (resp. $p \equiv n$) $\pmod{2n+2}$ then let $s = s_0$ (resp. $s = s_n$) (note that in this case, we have $p < a_{u+1} - 2$ by our assumption). In either case, we have $s \in \mathscr{L}(y)$ by Proposition 1.6(c). Let $y_1 = sy$. We define

$$C_1 = \{((R)s, y_1) \mid (R, y) \in C\},$$

if one of the following cases occurs.

(a) $s \neq s_0$, s_n, and either $(p, (p)y)$, $(p+1, (p+1)y)$ are not contained simultaneously in any congruence of any chain of y in C or $(p)y - (p+1)y > 2n+2$.

(b) $s = s_0$ (resp. $s = s_n$), and either p is not contained in any congruence of any 0-special (resp. n-special) chain of y in C or the inequality $(p)y - (p+1)y > 2n+2$ holds.

Now assume that we are not in any of the cases (a) and (b).

First assume that $s \neq s_0$, s_n and that both $(p, (p)y)$ and $(p+1, (p+1)y)$ are contained in some congruence of a chain (Q, y) in C with $(p)y - (p+1)y < 2n+2$. We may assume $p, p+1 \in Q$ by replacing (Q, y) by one of its congruences if necessary. Let $Q = \{b_1, ..., b_f, p, p+1, c_1, ..., c_g\}$ with $b_1 < \cdots < b_f < p < p+1 < c_1 < \cdots < c_g$. Define two sets

$$P' = \{a_1, ..., a_u, p, c_1, ..., c_g\}$$

and

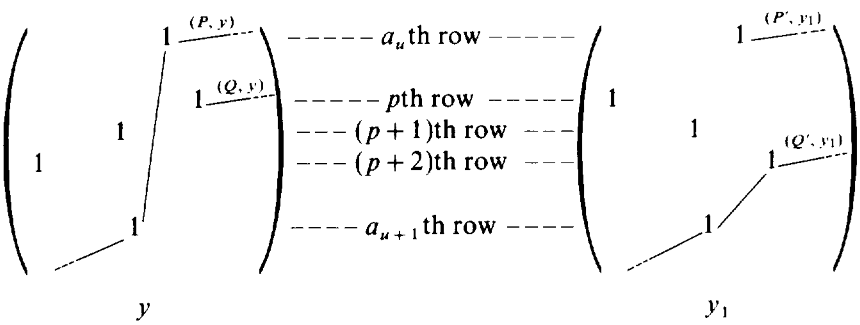
$$Q' = \{b_1, ..., b_f, p+1, a_{u+1}, ..., a_t\}.$$

Next assume that $s = s_0$ (resp. $s = s_n$) and that $(p, (p)y)$ is contained in some congruence of a 0-special (resp. n-special) chain (Q, y) in C with $0 < (p)y - (p+1)y < 2n+2$. Again, we may assume $p \in Q$ by replacing (Q, y) by one of its congruences if necessary. Then $b \leqslant p$ for any $b \in Q$. Define two sets

$$P' = \{a_1, ..., a_u\}$$

and

$$Q' = (Q)s \cup \{a_{u+1}, ..., a_t\}.$$



In either case, we define

$$C_1 = \{(R, y_1) \mid (R, y) \in C, R \neq P, Q, \bar{P}, \bar{Q}\}$$
$$\cup \{(P', y_1), (Q', y_1), (\bar{P}', y_1), (\bar{Q}', y_1)\}.$$

Then it is easily checked that C_1 is a k-chain family of y_1 with $|C_1| = |C| = d'_k(y)$. By Lemma 2.12, we have $d'_k(y_1) = d'_k(y)$.

(III) Now assume that we are in Case (iii). By symmetry, we may assume without loss of generality that there exists a non-empty 0-special chain of y in C. We may choose a non-empty 0-special chain (P, y) in C such that by replacing it by one of its congruences if necessary, the set $P = \{a_1, a_2, ..., a_t\}$ satisfies the conditions $a_1 < a_2 < \cdots < a_t < 0$ and $(a_t)y > 0$.

Having chosen such a 0-special chain (P, y) in C, we first assume that $(a_t)y > (a_t + 1)y$. If $a_t \not\equiv n \pmod{2n+2}$ then let s be the unique element of S' which transposes a_t with $a_t + 1$. If $a_t \equiv n \pmod{2n+2}$ then let $s = s_n$ (note that in this case, we have $a_t + 2 < -1$ since $a_t \leqslant -n-2 \leqslant -4$). In either case, we have $s \in \mathscr{L}(y)$ by Proposition 1.6(c). Let $y_1 = sy$. Then $C_1 = \{((R)s, y_1) \mid (R, y) \in C\}$ is a k-chain family of y with $|C_1| = |C| = d'_k(y)$.

Next assume that $(a_t)y < (a_t + 1)y$. Since $a_t < 0$ and $(a_t)y > 0$, there must exist some p, $a_t < p < 0$, such that $(p)y > (a_t)y > (p+1)y$. If $p \not\equiv -1$, $n \pmod{2n+2}$ then let s be the unique element of S' which transposes p with $p+1$. If $p \equiv -1$ (resp. $p \equiv n$) $\pmod{2n+2}$ then let $s = s_0$ (resp. $s = s_n$). In either case, we have $s \in \mathscr{L}(y)$ by Proposition 1.6(c). Let $y_1 = sy$. We define

$$C_1 = \{((R)s, y_1) \mid (R, y) \in C\}$$

if one of the following cases occurs.

(a) $s \neq s_n$, and either $(p, (p)y)$, $(p+1, (p+1)y)$ are not contained simultaneously in any congruence of any chain of y in C or $(p)y - (p+1)y > 2n+2$.

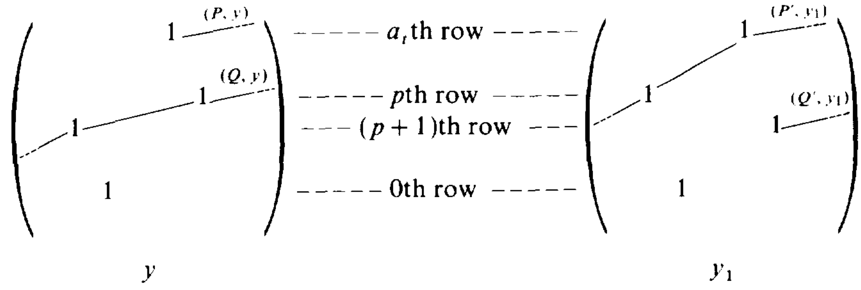
(b) $s = s_n$.

Suppose that $s \neq s_n$ and that both $(p, (p)y)$ and $(p+1, (p+1)y)$ are contained in some congruence of a chain (Q, y) in C with $(p)y - (p+1)y < 2n+2$. We may assume $p, p+1 \in Q$ by replacing (Q, y) by one of its congruences if necessary. Let $Q = \{b_1, ..., b_f, p, p+1, c_1, ..., c_g\}$ with $b_1 < \cdots < b_f < p < p+1 < c_1 < \cdots < c_g$. Define two sets

$$P' = \{a_1, ..., a_t, p, c_1, ..., c_g\}$$

and

$$Q' = \{b_1, ..., b_f, p+1\}.$$



Then the set

$$C_1 = \{(R, y_1) \mid (R, y) \in C,\ R \neq P, Q, \bar{P}, \bar{Q}\}$$
$$\cup \{(P', y_1), (Q', y_1), (\bar{P}', y_1), (\bar{Q}', y_1)\}$$

is a k-chain family of y_1 with $|C_1| = |C| = d'_k(y)$. By the same argument as before, we get $d'_k(y_1) = d'_k(y)$.

(IV) Finally, assume that Case (iv) occurs. Again by symmetry of s_0 and s_n in $\mathscr{C}_n$, we need only consider the case where there exist 0-special chains (P, y) in C but none of the sets P contains a congruence of 1. We choose a 0-special chain (P, y) in C in the following way.

(a) If there exists only one n-special chain pair $\{(Q, y), (\bar{Q}, y)\}$ in C and if this chain pair is also 0-special but it is not unique in C as a 0-special chain pair of y then we choose a 0-special chain (P, y) in C such that (P, y) is not congruent to (Q, y) or $(\bar{Q}, y)$ and such that by replacing it by one of its congruences if necessary, the set $P = \{a_1, ..., a_t\}$ satisfies the conditions $a_1 < a_2 < \cdots < a_t < 0$ and $(a_t)\, y > 0$.

(b) If we are not in case (a) then we choose a 0-special chain (P, y) in C such that $P = \{a_1, ..., a_t\}$ satisfies the conditions $a_1 < a_2 < \cdots < a_t < 0$ and $(a_t)\, y > 0$.

Having chosen such a 0-special chain (P, y) in C, we can find a required element y_1 entirely in the same way as that in (III).

Therefore our proof is completed.

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