A SURVEY ON THE CELL THEORY OF AFFINE WEYL GROUPS

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Abstract. This paper makes a survey concerning the contributions of Chinese mathematicians on the cell theory of affine Weyl groups. The main points for their works are as follows. The establishment of the coordinate forms and the sign types for the elements of an irreducible affine Weyl group W_a; the exploration of some relations among the cells of W_a; explicit descriptions for the lowest two-sided cell $W_{(\nu)}$ of W_a and all the left cells of W_a in $W_{(\nu)}$, for all the two-sided cells of the affine Weyl group of type $\widetilde{D}_4$ and for all the cells of the affine Weyl groups of types $\widetilde{A}_l,\ l \geq 1$, and $\widetilde{B}_3$.

§1. Introduction.

1.1 Cells of a Coxeter group W, as defined by Kazhdan and Lusztig [**1**], are certain equivalence classes of W. They are important not only in the representation theory of W and its corresponding Hecke algebra, but also in the representation theory of algebraic groups, finite groups, finite groups of Lie type, Lie algebras and enveloping algebras.

One of the main themes in the cell theory of W is to give an explicit description of the cells of W and to investigate the structural properties of these cells.

This task is far from being completed except for a few cases.

1.2 In this paper, we mainly concentrate the case when W is an affine Weyl group, which is a special kind of infinite Coxeter group. The cell theory of affine Weyl groups is of interest for the representation theory of affine Hecke algebras.

The cell theory of affine Weyl groups was first studied by G. Lusztig. He described explicitly all the cells of the affine Weyl groups of ranks ≤ 2 in l979 [**2**]. By the knowledge of his results, Lusztig raised a number of conjectures on this subject in 1980 and in 1983 [**3**]. After that, he published a series of papers to study the cells in affine Weyl groups and the representations of the corresponding Hecke algebras afforded by them [**2**] [**3**] [**4**] [**5**] [**6**].

1.3 The Chinese mathematicians have been studying the cell theory of affine Weyl groups since 1981 and have made a remarkable contribution to this theory. They are Du Jie, Xi Nanhua and the author. In 1983, the author described all the cells of affine Weyl groups $W_a(\widetilde{A}_{n-1})$ of type $\widetilde{A}_l$ $(l \geq 1)$ explicitly [**7**]. In his book [**7**], the author introduced the coordinate forms and the sign types for the

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elements of $W_a(\widetilde{A}_{n-1})$. These concepts were generalized to the case of a general affine Weyl group by the author in 1985 [**8**] [**9**]. These concepts are very useful in the description of cells of affine Weyl groups. Then in 1986, the author found out the lowest two-sided cell, written $W_{(\nu)}$, of an affine Weyl group of arbitrary type and described all the left cells contained in $W_{(\nu)}$ [**10**] [**11**].

On the other hand, Du gave an explicit description for all the cells of the affine Weyl group of type $\widetilde{B}_3$ in 1986 [**12**] and also for the two-sided cells of the affine Weyl group of type $\widetilde{D}_4$ in 1988 [**13**].

In early 1988, Xi got some nice results on the relations among the cells of affine Weyl groups in his joint works with Lusztig [**14**].

1.4 Besides the above people, T. Springer, R. Bédard and G. M. Lawton also got some results in this field [**5**] [**15**] [**16**].

1.5 In the present paper, we shall mainly introduce the achievements of Chinese mathematicians in the cell theory of affine Weyl groups.

The content of our paper is organized as follows. In §§2-3, we collect some definitions which are applicable for any Coxeter group. Then in the subsequent sections, we shall only consider the case of (irreducible) affine Weyl groups. In §4, we define an affine Weyl group W_a in a geometric way and then associate each element of W_a with its coordinate form, some properties of elements of W_a in terms of their coordinate forms are stated. The concepts of an (admissible) sign type of type Φ and an ST-class of W_a are defined in §5, where Φ is the root system determined by W_a. These two concepts are essentially the same. A result on the number of ST-classes of W_a is stated. Then in §§6-7, we give a detailed description for all the cells of affine Weyl groups of types $\widetilde{A}_l,\ l \geq 1$, and $\widetilde{B}_3$. In §8, we state some partial results on the cells of the affine Weyl groups of arbitrary types or type $\widetilde{D}_4$. These results include the description for the lowest two-sided cell $W_{(\nu)}$ of an affine Weyl group W_a and all the left cells of W_a in $W_{(\nu)}$, the description for all the two-sided cells of the affine Weyl group of type $\widetilde{D}_4$, and some relations among the cells of an affine Weyl group.

§2 The definition of cells.

2.1 Let u be an indeterminate and let $A = \mathbf{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in u.

Let $W = (W, S)$ be a Coxeter group with S the set of Coxeter generators. Let $\leq$ be the Bruhat order on W and let $l(x)$ be the length function of W. We denote by $\mathbf{H}$ the Hecke algebra associated to W: $\mathbf{H}$ is a free A-module with basis $T_w,\ w \in W$, and the multiplication is defined by

$$T_w T_{w'} = T_{ww'},\ \text{if } l(ww') = l(w) + l(w'),$$

$$(T_s + u)(T_s - u^{-1}) = 0,\ \text{if } s \in S.$$

As an A-module, $\mathbf{H}$ also has the basis $(C_w)_{w \in W}$:

$$C_w = \sum_{y \leq w} u^{l(w)-l(y)} P_{y,w}(u^{-2}) T_y,$$

where the $P_{y,w}(u)$ are known as the Kazhdan-Lusztig polynomials in u, which satisfy that $\deg P_{y,w} \leq \frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$ and $P_{w,w} = 1$.

2.2 Given $y, w \in W$, write $y \prec w$ if $y < w$ and $\deg P_{y,w} = \frac{1}{2}(l(w) - l(y) - 1)$. Write y—w if either $y \prec w$ or $w \prec y$

For $w \in W$, define $\mathbf{L}(w) = \{s \in S | sw < w\}$ and $\mathbf{R}(w) = \{s \in S | ws < w\}$. We define a preorder $\leq_L$ on W as follows. Write $y \leq_L w$ if there is a sequence of elements $y_0 = y,\ y_1,\ ...,\ y_r = w$ in W such that for every $i,\ 1 \leq i \leq r$, y_{i-1}—y_i and $\mathbf{L}(y_{i-1}) \not\subset \mathbf{L}(y_i)$. Write $y \sim_L w$, if $y \leq_L w \leq_L y$. The equivalence classes with respect to $\leq_L$ are called left cells of W. Write $y \sim_R w$, if $y^{-1} \sim_L w^{-1}$. The corresponding equivalence classes are called right cells of W. Write $y \leq_{LR} w$, if there is a sequence of elements $y_0 = y,\ y_1,\ ...,\ y_r = w$ in W such that for every $i,\ 1 \leq i \leq r$, either $y_{i-1} \leq_L y_i$ or $y_{i-1}^{-1} \leq_L y_i^{-1}$. Write $y \sim_{LR} w$, if $y \leq_{LR} w \leq_{LR} y$. The corresponding equivalence classes are called two-sided cells of W.

2.3 It is known that each cell of W (i.e. left, right or two-sided cell) affords a representation of W (resp. $\mathbf{H}$) [**1**].

For example, let Γ be a left cell of W and let E_Γ be the free A-module with basis $\{e_w\}_{w \in \Gamma}$. Define, for each $s \in S$, an endomorphism $C_s : E_\Gamma \longrightarrow E_\Gamma$ by

$$C_s(e_w) = \begin{cases} (u + u^{-1})e_w, \text{ if } s \in \mathbf{L}(w), \\ \displaystyle\sum_{\substack{y \text{—} w \\ sy < y \\ y \in \Gamma}} \mu(y, w)e_y, \text{ if } s \notin \mathbf{L}(w) \end{cases}.$$

where $\mu(y, w) = \mu(w, y)$ is the coefficient of $u^{\frac{1}{2}(|l(w)-l(y)|-1)}$ in $P_{y,w}$ if $y \prec w$, or in $P_{w,y}$ if $w \prec y$.

Conversely, any irreducible representation of W (resp. $\mathbf{H}$) occurs as a component of a representation afforded by some cell of W.

§3 SL-classes, PL-classes and generalized τ-invariant.

We still assume that (W, S) is a Coxeter group in this section.

3.1 Fix a pair of elements $s, t \in S$ with $o(st) > 2$, where the notation $o(x)$ stands for the order of x. We define

$$\mathbf{D}_L(s, t) = \{w \in W | \mathbf{L}(w) \cap \{s, t\} \text{ has exactly one element}\}.$$

$$\mathbf{D}_R(s, t) = \{w \in W | \mathbf{R}(w) \cap \{s, t\} \text{ has exactly one element}\}.$$

If $w \in \mathbf{D}_L(s, t)$ then at least one of sw, tw is in $\mathbf{D}_L(s, t)$. We define ${}^*w$ to be the subset $\{sw, tw\}$ if both of sw, tw are in $\mathbf{D}_L(s, t)$, or to be $\{w', w'\}$ if exactly one of sw, tw, say w', is in $\mathbf{D}_L(s, t)$. In particular, the latter case always happens in the case when $o(s, t) = 3$, in that case, we identify $\{w', w'\}$ with the element w'.

On the other hand, when $w \in \mathbf{D}_R(s, t)$, we can defined w^* in the similar way as the above by replacing sw, tw by ws, wt.

Clearly, if ${}^*w = \{w', w''\}$ then $w' \sim_L w \sim_L w''$. The map $w \mapsto {}^*w$, when it makes sense, is called a left star operation on w, Note that the resulting subset ${}^*w$ under a left star operation is dependent of the choice of the pair $s, t \in S$. If $w' \in {}^*w$ then we say that w' can be obtained from w by a left star operation. Clearly, in that case, w can be obtained from w' by a left star operation, also.

3.2 We define an equivalence relation $\sim_{SL}$ on W generated by $x \sim_{SL} tx$ with $t \in S, x \in W$, $\mathbf{L}(x) \not\subset \mathbf{L}(tw)$ and $\mathbf{L}(x) \not\supset \mathbf{L}(tw)$. We call the corresponding equivalence classes by SL-classes.

Clearly, two elements of W are in the same SL-class if and only if one can be obtained from the another by a succession of left star operations. We also see that every SL-class of W is contained in some left cell of W.

3.3 If $y, w \in \mathbf{D}_L(s,t)$ and y—w for some $s, t \in S$ with $o(s,t) > 2$, and if the left star operations $w \mapsto {}^*w$ and $y \mapsto {}^*y$ are both applied in $\mathbf{D}_L(s,t)$, then by a result of Lusztig [**2**], there exist some elements $w' \in {}^*w$ and $y' \in {}^*y$ such that y'—w'. In this case, we say that w' and y' can be obtained from w and y respectively by the same left star operation. More generally, if there exist two sequences of elements $y_0 = y,\ y_1,\ ...,\ y_r = y'$ and $w_0 = w,\ w_1,\ ...,\ w_r = w'$ in W for some $r \geq 0$ such that for every $i,\ 1 \leq i \leq r,\ y_i$ and w_i can be obtained from y_{i-1} and w_{i-1} respectively by the same left star operation, then we say that y' can be obtained from y by the same succession of left star operations as w' from w. By definition, we have y—w and y'—w'. If they satisfy the additional condition that either $\mathbf{L}(y) \not\subset \mathbf{L}(w), \mathbf{L}(w') \not\subset \mathbf{L}(y')$ or $\mathbf{L}(w) \not\subset \mathbf{L}(y), \mathbf{L}(y') \not\subset \mathbf{L}(w')$ hold, then this implies that $y \sim_L w$, in that case, we call $\{y_i, w_i\}$ a left primitive pair for any $i,\ 0 \leq i \leq r$. Clearly, if $\{y, w\}$ forms a left primitive pair, then $y \sim_L w$.

3.4 For $x, y \in W$, write $x \sim_{PL} y$, if there exists a sequence of elements $x_0 = x,\ x_1,\ ...,\ x_r = y$ in W for some $r \geq 0$ such that for every $i,\ 1 \leq i \leq r,\ \{x_{i-1}, x_i\}$ forms a left primitive pair. This is an equivalence relation on W. The corresponding equivalence classes of W are called the PL-classes of W.

By definition, we see that every PL-class of W is a union of SL-classes of W and is contained in some left cell of W.

3.5 For any subset $K \subseteq W$, we call $K^{-1} = \{x \mid x^{-1} \in K\}$ the inverse set of K. Thus we call a subset of W a right primitive pair, an SR-class or a PR-class of W if its inverse set is a left primitive pair, an SL-class or a PL-class of W, respectively.

3.6 We say that $w, y \in W$ are equivalent to order zero, written $w \approx_0 y$, if $\mathbf{R}(w) = \mathbf{R}(y)$. Inductively, we define equivalence to order n for $n \geq 1$. We say that $w,\ y$ are equivalent to order n. written $w \approx_n y$, if $w \approx_{n-1} y$ and for every $s, t \in S$ with $o(st) = 3$ and $y, w \in \mathbf{D}_R(s,t)$, we have $y' \approx_{n-1} w'$, where $y' = y^*$ and $w' = w^*$ with respect to $s,\ t$. We say that $w,\ y$ have the same generalized τ-invariant if $w \approx_n y$ for any $n \geq 0$.

It is well known that if $w \sim_L y$ in W then $w,\ y$ have the same generalized τ-invariant [**1**].

All the elements of W having the same τ-invariant form an equivalence class of W, called a left V-cell of W.

There are the following inclusion relations.

$$\{SL\text{-classes of } W\} \leq \{PL\text{-classes of } W\} \leq \{\text{left cells of } W\} \leq \{\text{left } V\text{-cells of } W\},$$

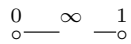
where for two sets $X,\ Y$ of subsets of a set, the expression $X \leq Y$ means that every element of X is contained in some element of Y and that every element of Y contains some element of X also.

§4 Affine Weyl groups.

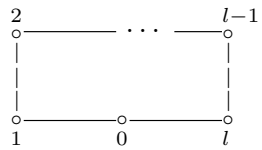
From now on, we concentrate our attention to the affine Weyl group (W_a, S). We assume that W_a is irreducible once and forever.

Theorem 4.1 [7]. *The Coxeter graph of an affine Weyl group (W_a, S) has one of the following forms.*

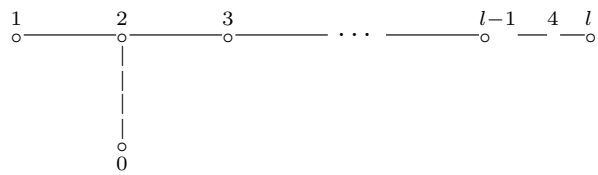
$\widetilde{A}_1$



$\widetilde{A}_l (l \geq 2)$

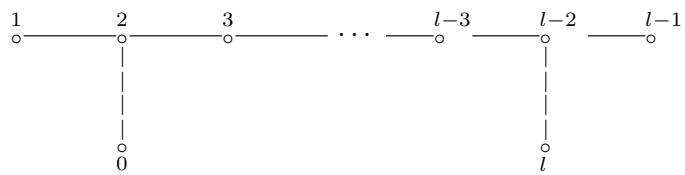


$\widetilde{B}_l (l \geq 3)$

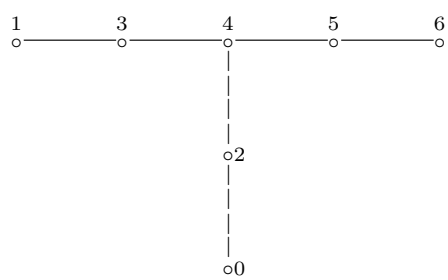


$\widetilde{C}_l (l \geq 2)$

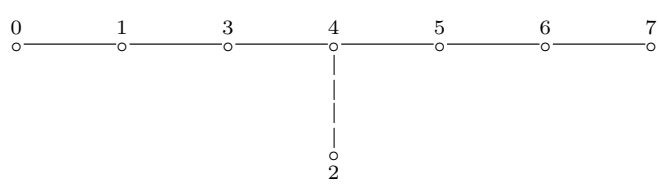
$\widetilde{D}_l (l \geq 4)$



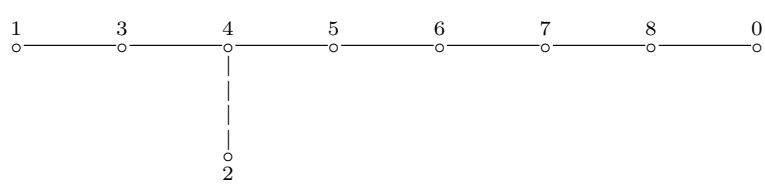
$\widetilde{E}_6$

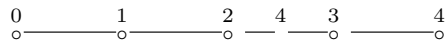


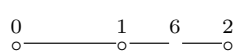
$\widetilde{E}_7$



$\widetilde{E}_8$



$\widetilde{F}_4$ 

$\widetilde{G}_2$ 

where $\widetilde{X}$ denotes the type of the corresponding affine Weyl group W_a (Sometimes we use the notation $W_a(\widetilde{X})$ instead of W_a to indicate the type of W_a). These groups are pairwise non-isomorphic.

4.2 An affine Weyl group W_a of type $\widetilde{X}$ can be defined geometrically as follows [**8**].

Let Φ be the root system of type X and let E be the euclidean space spanned by Φ with inner product $\langle\ ,\ \rangle$ such that $|\alpha|^2 = \langle\alpha,\alpha\rangle = 1$ for any short root $\alpha \in \Phi$. Then the affine Weyl group W_a can be regarded as a group of right isometric transformations on E. More precisely, let W be the Weyl group of Φ generated by the reflections s_α on E for $\alpha \in \Phi$: s_α sends $x \in E$ to $x - \langle x, \alpha^\vee\rangle\,\alpha$, where $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. Let Q denote the root lattice $\mathbf{Z}\Phi$. Let N denote the group consisting of all translations $T_\lambda,\ \lambda \in Q$ on E: T_λ sends x to $x+\lambda$. Then W_a can be regarded as the semi-direct product $W \ltimes N$.

Let $\Pi = \{\alpha_1,\ ...,\ \alpha_l\}$ be a choice of simple root system of Φ and let Φ^+ be the corresponding positive root system. Let $-\alpha_0$ be the highest short root in Φ. We define $s_0 = s_{\alpha_0}\cdot T_{-\alpha_0}$, and $s_i = s_{\alpha_i},\ 1 \le i \le l$. Then $S = \{s_0,\ s_1,\ ...,\ s_l\}$ forms a Coxeter generator set of W_a. By properly numbering these simple roots, we may assume that the subscripts of $s_0,\ s_1,\ ...,\ s_l$ are compatible with the Coxeter graph of W_a (see Theorem 4.1).

4.3 For any $\alpha \in \Phi^+$ and $k \in \mathbf{Z}$, we define a hyperplane

$$H_{\alpha;k} = \{v \in E \mid \langle v, \alpha^\vee\rangle = k\}$$

and a stripe

$$H^1_{\alpha;k} = H^1_{-\alpha;-k} = \{v \in E \mid k < \langle v, \alpha^\vee\rangle < k+1\}$$

We call any non-empty simplex of

$$E - \bigcup_{\substack{\alpha\in\Phi\\ k\in\mathbf{Z}}} H_{\alpha;k}$$

an alcove of E. Each alcove of E has the form $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$ for a Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\mathbf{Z}$. In particular, $A_1 = \bigcap_{\alpha\in\Phi} H^1_{\alpha;0}$ is an alcove of E.

It is well known that the right action of W_a on E induces a simply transitive permutation on the set Ξ of alcoves of E. Hence we have a bijection $w \mapsto (A_1)w = A_w$ from W_a to Ξ. We shall identify w with A_w.

4.4 Since a Φ-tuple over $\mathbf{Z}$ does not always determine an alcove as in the above, it is desirable to give a necessary and sufficient condition on a Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\mathbf{Z}$ such that $\bigcap_{\alpha\in\Phi} H^1_{\alpha,k_\alpha} \in \Xi$.

For this purpose, the author defined a set $E(\Phi)$ consisting of all Φ-tuples $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi}$ which satisfy the following two conditions.

(a) $k_\alpha = -k_\alpha \in \mathbf{Z}$, for $\alpha \in \Phi$.

(b) For any $\alpha, \beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$, the inequality

(**4**.5) $$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta < |\alpha+\beta|^2(k_{\alpha+\beta}+1) < |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2$$

holds. Then the author showed

Theorem 4.6 [8]. *Let W_a be an affine Weyl group of type $\widetilde{X}$ and let Φ be the root system of type X. Then there exists a bijective map*

$$w = \bigcap_{\alpha\in\Phi} H^1_{\alpha;k(w,\alpha)} \mapsto \mathbf{k}^w = (k(w,\alpha))_{\alpha\in\Phi}$$

from W_a to $E(\Phi)$.

The above result characterizes a Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\mathbf{Z}$ which corresponds to an alcove $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha} \in \Xi$.

As in the above theorem, we call $\mathbf{k}^w$ the coordinate form of w. We shall identify $\mathbf{k}^w$ with w. We can see from the following results that the coordinate forms are very useful in the study of the affine Weyl groups.

Proposition 4.7 [8]. *Let $w = (k(w,\alpha))_{\alpha\in\Phi} \in W_a$. Write $w = \overline{w}T_\lambda$ with $\overline{w} \in W$ and $\lambda \in Q$. Then*
(a) for any $\alpha \in \Phi$, we have

$$k(w,\alpha) = \langle \lambda, \alpha^\vee\rangle + k(\overline{w},\alpha) \tag{4.8}$$

where the integer $k(\overline{w},\alpha)$ is given by

$$k(\overline{w},\alpha) = \begin{cases} 0, & \text{if } (\alpha)\overline{w}^{-1} \in \Phi^+, \\ -1, & \text{if } (\alpha)\overline{w}^{-1} \in \Phi^-. \end{cases}$$

(b) $\mathbf{L}(w) = \{s_j \in S \mid k(w,(\alpha_j)\overline{w}) > 0\}$.
(c) $\mathbf{R}(w) = \{s_j \in S \mid k(w,\alpha_j) < 0\}$.
(d) $l(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|$.
(e) Let $w' = s_j w$ with $w \in W_a$ and $0 \le j \le l$. Then for any $\alpha \in \Phi$, we have

$$k(w',\alpha) = k(w,\alpha) + k(s_j,(\alpha)\overline{w}^{-1}).$$

(f) Let $w' = ws_j$ with $w \in W_a$ and $0 \le j \le l$. Then for any $\alpha \in \Phi$, we have

$$k(w',\alpha) = k(w,(\alpha)s_j) + k(s_j,\alpha).$$

(g) $k(w^{-1},\alpha) = k(w,-(\alpha)\overline{w})$ *for any $\alpha \in \Phi$.*

§5 Sign types.

5.1 Let Φ be an irreducible root system of type $\neq G_2$ and let Φ^+ be a choice of positive root system of Φ. In his paper [**9**], the author introduced a sign type of type Φ.

A Φ-tuple $\mathbf{X} = (X_\alpha)_{\alpha\in\Phi}$ is called a sign type of type Φ (or briefly, a sign type), if the set $\{X_\alpha, X_{-\alpha}\}$ is either $\{\mathrm{o},\mathrm{o}\}$ or $\{+,-\}$ for any $\alpha \in \Phi$. Since $\mathbf{X}$ is determined uniquely by the Φ^+-tuple $(X_\alpha)_{\alpha\in\Phi^+}$. We shall identify $(X_\alpha)_{\alpha\in\Phi^+}$ with $\mathbf{X}$. Let $\overline{\Theta} = \overline{\Theta}(\Phi)$ be the set of all sign types of type Φ. We now define the following sets.

$$\Delta_1 = \begin{array}{cccccccccccccccc} + & + & + & \mathrm{o} & - & + & + & \mathrm{o} & \mathrm{o} & - & + & \mathrm{o} & - & \mathrm{o} & - & - \\ ++, & +\mathrm{o}, & +-, & +-, & +-, & \mathrm{o}+, & \mathrm{o}\,\mathrm{o}, & \mathrm{o}\,\mathrm{o}, & \mathrm{o}-, & \mathrm{o}-, & -+, & -+, & -+, & -\mathrm{o}, & -\mathrm{o}, & -- \end{array}.$$

$$\Delta_2 = \left\{ \begin{array}{ccccccccccccccc} \mathrm{o} & \mathrm{o} & \mathrm{o} & - & - & - & + & \mathrm{o} & \mathrm{o} & \mathrm{o} & \mathrm{o} & + & + & - & + \\ \mathrm{o}\,\mathrm{o} & +\mathrm{o} & +\mathrm{o} & \mathrm{o}\,\mathrm{o} & -\mathrm{o} & -\mathrm{o} & +\mathrm{o} & \mathrm{o}- & \mathrm{o}- & +- & -- & \mathrm{o}- & +- & -- & -- \\ \mathrm{o}, & \mathrm{o}, & +, & \mathrm{o}, & \mathrm{o}, & -, & +, & \mathrm{o}, & -, & \mathrm{o}, & -, & -, & \mathrm{o}, & - & - \end{array} \right.$$

$$\left. \begin{array}{ccccccccccc} + & + & + & - & - & - & \mathrm{o} & - & - & - & + \\ -- & +- & +- & \mathrm{o}+ & \mathrm{o}+ & -+ & ++ & ++ & -+ & -+ & ++ \\ -, & -, & +, & \mathrm{o}, & +, & \mathrm{o}, & +, & +, & +, & -, & + \end{array} \right\}.$$

For any subsystem Φ' of Φ, $\Phi'^+ = \Phi^+ \cap \Phi'$ is a positive subsystem of Φ.

Given an irreducible positive subsystem Φ'^+ of Φ of rank 2, we say that a sign type $(X_\alpha)_{\alpha\in\Phi'^+}$ is admissible, if one of the following conditions is satisfied:
(a) Φ'^+ has type A_2, say $\Phi'^+ = \{\alpha, \beta, \alpha+\beta\}$. Then

$$\begin{array}{ccc} & X_{\alpha+\beta} & \\ X_\alpha & & X_\beta \end{array} \in \Delta_1$$

(b) Φ'^+ has type B_2, say $\Phi'^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$. Then

$$\begin{array}{ccc} & X_\beta & \\ X_{\alpha+\beta} & & X_\alpha \\ & X_{2\alpha+\beta} & \end{array} \in \Delta_2.$$

We say that a sign type $(X_\alpha)_{\alpha\in\Phi}$ is admissible if for any irreducible positive subsystem Φ'^+ of Φ of rank 2, the sign type $(X_\alpha)_{\alpha\in\Phi'^+}$ is admissible. Let $\Theta = \Theta(\Phi)$ be the set of all admissible sign types in $\overline{\Theta}(\Phi)$.

Remark 5.2. In the case when Φ is of type G_2, we can also define a sign type (resp. an admissible sign type) of type Φ in the same way as above [**9**]. So in the subsequent discussion involving the sign types, the restriction on the type Φ can be removed.

The author showed

Theorem 5.3 [9]. *Let W_a be an affine Weyl group of type $\widetilde{X}$ and let Φ be the root system of type X. Then there exists a surjective map $\zeta:\ W_a \longrightarrow \Theta$ which sends $x = (k(x,\alpha))_{\alpha\in\Phi}$ to $X = (X_\alpha)_{\alpha\in\Phi}$ such that for any $\alpha \in \Phi$,*

$$k(x,\alpha) > 0 \Longrightarrow X_\alpha = +,$$

$$k(x,\alpha) = 0 \Longrightarrow X_\alpha = \mathrm{o},$$

$$k(x,\alpha) < 0 \Longrightarrow X_\alpha = -.$$

5.4 We call X the sign type of x in the above theorem. We usually denote the sign types of elements $x,\ y,\ \ldots$ of W_a by the corresponding capital latter $X,\ Y,\ \ldots$.

Thus we have a decomposition of W_a:

$$W_a = \bigcup_{X\in\Theta} \zeta^{-1}(X)$$

We call these $\zeta^{-1}(X)$'s the ST-classes of W_a. By Theorem 5.3, we can identify Θ with the set of all ST-classes of W_a.

5.5 There is a geometrical interpretation of admissible sign types. Let $\Upsilon = \{H_{\alpha;k} \mid \alpha \in \Phi^+, k = 0, 1\}$. Then the connected components of $P = E - \bigcup_{H\in\Upsilon} H$ are all open simplices of E. By definition, we see that any alcove of E is contained in some connected component of P and that two alcoves correspond to the same sign type if and only if they are in the same connected component of P. So by Theorem 5.3, the map ζ induces a bijection between the set $\widetilde{P}$ of connected components of P and the set Θ. Thus Θ can also be identified with $\widetilde{P}$ under this correspondence.

5.6 Given a subset K of W_a, we say K is left (resp. right) connected, if for any $x,\ y \in K$, there exists a sequence of elements $x_0 = x,\ x_1,\ ...,\ x_r = y$ in K for some $r \geq 0$ such that $x_{i-1}x_i^{-1} \in S$ (resp. $x_{i-1}^{-1}x_i \in S$) for every $i,\ 1 \leq i \leq r$.

Clearly, any SL-class (resp. SR-class) of W_a is left (resp. right) connected.

The following properties of the ST-classes of W_a were showed by the author.

Theorem 5.7 [9]. *(a) Any ST-class of W_a is left connected and contains a unique shortest element. (b) The number of ST-classes of W_a is equal to $(h+1)^l$, where h and l are the Coxeter number and the rank of Φ, respectively.*

Remark 5.8 The assertion in Theorem 5.7 (b) was conjectured by R. W. Carter.

§6 The cells of $W_a(\widetilde{A}_{n-1}),\ n \geq 2$.

6.1 In the cell theory of affine Weyl groups W_a, the most important thing is to describe the cells of W_a as explicit as possible. But up to now, this has been done for all the cells of W_a only in the cases when W_a is of type $\widetilde{A}_l,\ l \geq 1,\ \widetilde{B}_2,\ \widetilde{B}_3,\ \widetilde{C}_3$ and $\widetilde{G}_2$ [**7**] [**4**] [**2**] [**12**] [**15**]. For other cases, there are some partial results. In the subsequent sections, we shall mainly introduce the results of Du, Xi and the author on this aspect.

6.2 It is well known that $W_a(\widetilde{A}_{n-1})$ can be defined as a permutation group on $\mathbf{Z}$ as follows.

$$W_a(\widetilde{A}_{n-1}) = \left\{ \sigma : \mathbf{Z} \to \mathbf{Z} \mid \sigma(i+n) = \sigma(i) + n, \text{ for any } i, \text{ and } \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i \right\}.$$

Let $s_i \in W_a(\widetilde{A}_{n-1}),\ 0 \leq i < n$, corresponds to the permutation

$$t \mapsto \begin{cases} t & \text{if} \quad \bar{t} \neq \bar{i}, \overline{i+1}, \\ t-1, & \text{if} \quad \bar{t} = \overline{i+1}, \\ t+1, & \text{if} \quad \bar{t} = \bar{i}. \end{cases}$$

where $j \mapsto \bar{j}$ is the natural map from $\mathbf{Z}$ to $\mathbf{Z}/(n)$. Then $S = \{s_0,\ s_1,\ ...,\ s_{n-1}\}$ forms a Coxeter generator set of $W_a(\widetilde{A}_{n-1})$ such that for $0 \leq i, j \leq n$,

$$o(s_i s_j) = \begin{cases} 1, & \text{if} \quad \bar{i} = \bar{j}, \\ 2, & \text{if} \quad \bar{i} \neq \bar{j}, \overline{j \pm 1}, \\ 3, & \text{if} \quad \bar{i} = \overline{j \pm 1} \end{cases}$$

6.3 The relation between the above definition and the previous geometrical definition of the group $W_a(\widetilde{A}_{n-1})$ given before (see §4) is as follows. For any $w \in W_a(\widetilde{A}_{n-1})$, let $(k(w,\alpha))_{\alpha\in\Phi}$ be the coordinate form of w. Then we have

$$w(j) - w(i) = k(w,\alpha_{ij}) \cdot n + r(w,\alpha_{ij}), \text{ if } 1 \le i < j \le n, \tag{6.4}$$

where $\alpha_{ij} = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$ and the integer $r(w,\alpha_{ij})$ satisfies $1 \le r(w,\alpha_{ij}) < n$ [**7**]. The following result can be easily checked

Proposition 6.5 [7]. *For any $w \in W_a(\widetilde{A}_{n-1})$, we have*

(a) $\mathbf{L}(w) = \{s_t \in S \mid w^{-1}(t+1) < w^{-1}(t)\}$.

(b) $\mathbf{R}(w) = \{s_t \in S \mid w(t+1) < w(t)\}$.

(c) $l(w) = \sum\limits_{1\le i<j\le n} \left|\left[\frac{y(j)-y(i)}{n}\right]\right|$,

where $[h]$ is the integer part of h for any rational number h.

6.6 With each $\sigma \in W_a(\widetilde{A}_{n-1})$, Lusztig associated a set $\{d_k \mid 1 \le k \le n\}$ of integers as follows: d_k is the maximal cardinal of a subset of $\mathbf{Z}$ whose elements are non-congruent to each other mod n and which is a disjoint union of k subsets each of which has its natural order reversed by σ. By a result of C. Greene, we have

$$d_1 \ge d_2 - d_1 \ge d_3 - d_2 \ge ... \ge d_n - d_{n-1}.$$

Thus each element σ of $W_a(\widetilde{A}_{n-1})$ determines a partition $\phi(\sigma) = \{d_1 \ge d_2 - d_1 \ge ... \ge d_n - d_{n-1}\}$ of n. Let Λ_n be the set of partitions of n. Then we have a well defined map $\phi:\ W_a(\widetilde{A}_{n-1}) \longrightarrow \Lambda_n$. Clearly, ϕ is surjective. Lusztig had the following

Conjecture 6.7 [3]. *(a) Let $\sigma, \sigma' \in W_a(\widetilde{A}_{n-1})$. Then*

$$\sigma \sim_{LR} \sigma' \Leftrightarrow \phi(\sigma) = \phi(\sigma').$$

(b) Given $\lambda \in \Lambda_n$, let $\mu = \{\mu_1 \ge ... \ge \mu_n\}$ be the dual of λ. Then the number of left cells in $\phi^{-1}(\lambda)$ is equal to

$$\frac{n!}{\prod_{j=1}^m \mu_j!}$$

This conjecture has been proved by Lusztig and the author. The author proved part (b) and the direction " $\Leftarrow$ " of part (a) [**7**]. Then Lusztig proved the remaining part [**4**].

6.8 The proof of the author is purely combinatorial. The main tool of his proof is an affine matrix of period n. Each affine matrix is a generalized permutation matrix, say $\mathbf{M} = (a_{ij})_{i,j\in\mathbf{Z}}$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases}$$

for some $\sigma \in W_a(\widetilde{A}_{n-1})$. Then $W_a(\widetilde{A}_{n-1})$ can be identified with the set of all affine matrices of period n.

6.9 Let $\mathbf{C}_n$ be the set of all Young tableaux X of rank n such that the numbers in each row of X increase from left to right. Then $\mathbf{C}_n$ contains all standard Young tableaux of rank n as its subset.

Example 6.10

$$\begin{array}{ccc} 1 & 3 & 7 \\ 2 & 4 & \\ 5 & 6 & \end{array} \quad \text{and} \quad \begin{array}{cc} 2 & 5 \quad 7 \\ 1 & 4 \\ 3 & 6 \end{array}$$

are two elements of $\mathbf{C}_7$, where the first one is standard but the second one is not.

6.11 Each element X of $\mathbf{C}_n$ determines a unique partition $\psi(X) = \{\lambda_1 \geq ... \geq \lambda_t\}$ of n, where λ_i is the cardinal of the set of numbers in the i-th column of X. Thus $X \mapsto \psi(X)$ gives a surjective map $\psi:\ \mathbf{C}_n \mapsto \Lambda_n$. It is easily seen that for any $\lambda \in \Lambda_n$, the cardinal of the set $\psi^{-1}(\lambda)$ is

$$\frac{n!}{\prod_{j=1}^{m} \mu_j!},$$

where $\{\mu_1 \geq ... \geq \mu_m\} \in \Lambda_n$ is the dual partition of λ.

The author defined a surjective map $T:\ W_a(\widetilde{A}_{n-1}) \mapsto \mathbf{C}_n$ in terms of affine matrices of period n as follows.

6.12 For any $\lambda = \{\lambda_1 \geq ... \geq \lambda_r\} \in \Lambda_n$, let $\mu = \{\mu_1 \geq ... \geq \mu_m\}$ be the dual partition of λ. We call an element $w \in W_a(\widetilde{A}_{n-1})$ a normalized element of type λ if there exists a set of non-zero entries $\{e(i+t, j(t)) \mid 1 \leq t \leq n\}$ of w for some $i \in \mathbf{Z}$, where $e(i,j)$ denotes the entry of w in the position (i,j), satisfying the following conditions: Let

$$\alpha_u = i + \sum_{h=1}^{u} \lambda_h,\ 0 \leq u \leq r.$$

Then (i) $j(\alpha_u - 1) < j(\alpha_u - 2) < ... < j(\alpha_{u-1})$,

(ii) For any t with $1 \leq t \leq \lambda_1$, we have

$$j(\lambda_1 - t) < j(\lambda_2 - t) < ... < j(\lambda_{\mu_t} - t) < j(\lambda_1 - t) + n.$$

In that case, we can check that $\phi(w) = \lambda$.

Example 6.13 Let $n = 9,\ \lambda = \{4 \geq 3 \geq 2\}$. Then $\mu = \{3 \geq 3 \geq 2 \geq 1\}$ is the dual partition of λ. The following element y is a normalized element of type λ:

$$\begin{array}{cc} & \text{1st column} \\ text1strow & \begin{pmatrix} & & & & & & 1 & \\ & & & & 1 & & & \\ & & 1 & & & & & \\ 1 & & & & & & & \\ & & & & & 1 & & \\ & & & 1 & & & & \\ & 1 & & & & & & \\ & & & & & & & 1 \\ & & 1 & & & & & \end{pmatrix} \end{array}$$

+where, y has the non-zero entry set { $e(1,8)$, $e(2,6)$, $e(3,3)$, $e(4,0)$, $e(5,7)$, $e(6,5)$, $e(7,2)$, $e(8,10)$, $e(9,4)$ }, which has the required properties.

Let $\mathbf{N}$ be the set of all normalized elements of $W_a(\widetilde{A}_{n-1})$.

6.14 We defined a map $T: \ \mathbf{N} \mapsto \mathbf{C}_n$ as follows. Let w be an element of $\mathbf{N}$ of type λ with a set of non-zero entries $\{e(i+t, j(t)) \mid 1 \le t \le n\}$ satisfying the conditions as above. We define a Young tableau of $\mathbf{C}_n$ such that the set X_t of numbers in the t-th row of this tableau satisfying that $\overline{X}_t = \{\overline{j(\alpha_h - t)} \mid 1 \le h \le \mu_t\}$ for $1 \le t \le m$, where $i \mapsto \bar{i}$ is the natural map from $\mathbf{Z}$ to $\mathbf{Z}/n\mathbf{Z}$ and for any subset $X \subset \mathbf{Z}$, we denote $\overline{X} = \{\overline{x} \mid x \in X\} \subseteq \mathbf{Z}/n\mathbf{Z}$. We see that such a Young tableau is only dependent on w and is independent on the choice of this entry set of w. Let us denote this tableau by $T(w)$. Then the map T is well defined. Clearly, T is surjective.

In the above example, we have

$$T(y) = \begin{array}{ccc} 2 & 4 & 9 \\ 1 & 3 & 5 \\ 6 & 7 & \\ 8 & & \end{array}$$

The set $\mathbf{N}$ has the following properties.

Proposition 6.15 [7]. *(a) The intersection of the set* $\mathbf{N}$ *with each SL-class of* $W_a(\widetilde{A}_{n-1})$ *is non-empty.*

(b) The intersection of the set $\mathbf{N}$ *with each PL-class of* $W_a(\widetilde{A}_{n-1})$ *is non-empty which has the form* $T^{-1}(X)$ *for some* $X \in \mathbf{C}_n$. *Conversely, any* $T^{-1}(X)$ *with* $X \in \mathbf{C}_n$ *is the intersection of* $\mathbf{N}$ *with some PL-class of* $W_a(\widetilde{A}_{n-1})$. *In particular,* $T^{-1}(X)$ *is contained in some left cell of* $W_a(\widetilde{A}_{n-1})$.

(c) If $X \ne X'$ *in* $\mathbf{C}_n$, *then* $T^{-1}(X)$ *and* $T^{-1}(X')$ *belong to two different left cells of* $W_a(\widetilde{A}_{n-1})$.

6.16 By Proposition 6.15(a), we can extend the map $T: \ \mathbf{N} \mapsto \mathbf{C}_n$ to the whole group $T: W_a(\widetilde{A}_{n-1}) \mapsto \mathbf{C}_n$ in the following way. For any $w \in W_a(\widetilde{A}_{n-1})$, let y be any element of $\mathbf{N}$ with $y \sim_{SL} w$ (There is an algorithm to find such an element y from w). We define $T(w)$ by $T(y)$.

6.17 Robinson defined a map from the symmetric group S_n to the set of standard Young tableaux of the same shape and size n: $\sigma \mapsto (P(\sigma),\ P(\sigma^{-1}))$. Then Schensted proved that this map is bijective. This map is called the Robinson-Schensted map which has the following properties: A left cell of S_n is the set of elements of S_n corresponding to the set of such pairs $(P,\ Q)$ with P fixed; A two-sided cell of S_n is the set of elements of S_n corresponding to the set of such pairs $(P,\ Q)$ with $P,\ Q$ of fixed shape.

Now we are ready to state

Theorem 6.18 [7]. *(a) The diagram*

$$\begin{array}{ccc} W_a(\widetilde{A}_{n-1}) & \xrightarrow{T} & \mathbf{C}_n \\ \phi & & \psi \\ & \Lambda_n & \end{array}$$

is commutative. The maps in this diagram are all surjective.

(b) For $x,\ y \in W_a(\widetilde{A}_{n-1}),\ x \sim_L y \Leftrightarrow T(x) = T(y)$.

(c) For any $\lambda \in \Lambda_n$, $\phi^{-1}(\lambda)$ is in some two-sided cell of $W_a(\widetilde{A}_{n-1})$ which is a union of $\frac{n!}{\prod_{j=1}^{m}\mu_j!}$ left (resp. right) cells of $W_a(\widetilde{A}_{n-1})$, where $\{\mu_1 \geq ... \geq \mu_m\}$ is the dual partition of λ

(d) Let W be the subgroup of $W_a(\widetilde{A}_{n-1})$ generated by $s_1, ..., s_{n-1}$, which is isomorphic to S_n. Then $T(w) = P(w)$ for any $w \in W$. Thus the map T generalizes the Robinson-Schensted map for S_n.

6.19 By using Theorem 6.18, Lusztig proved that for any $\lambda \in \Lambda_n$, the fibre $\phi^{-1}(\lambda)$ is itself a two-sided cell of $W_a(\widetilde{A}_{n-1})$ [**4**]. His proof relies on a deep result of intersection cohomology. He introduced a function a on any affine Weyl group W_a which has the following properties.

Theorem 6.20 [2] [5]. *(a) For any $z \in W_a$, $a(z)$ is an integer satisfying $0 \leq a(z) \leq \frac{1}{2}|\Phi|$, where Φ is the root system determined by W_a (see §4).*
(b) The function a is constant on any two-sided cell of W_a.
(c) If $x, y \in W_a$ satisfy $x \leq_L y$ and $a(x) = a(y)$ then $x \sim_L y$.

By Theorem 6.20 (c), we see that each two-sided cell Ω of W_a is a smallest non-empty set with the property that Ω is both a union of left cells of W_a and a union of right cells of W_a. This, together with Theorem 6.18 (c), implies that for any $\lambda \in \Lambda_n$, $\phi^{-1}(\lambda)$ is a single two-sided cell of $W_a(\widetilde{A}_{n-1})$.

6.21 Recall that in any Coxeter group W, we have

$\{PL$-classes of W $\} \leq \{$left cells of W $\} \leq \{$left V-cells of W $\}$ (see §3)

The author showed that these three families of equivalence classes coincide with each other in the case of $W_a(\widetilde{A}_{n-1})$. That is,

Theorem 6.22 [7]. *Let $K \subset W_a(\widetilde{A}_{n-1})$. Then the following statements are equivalent.*
(a) K is a PL-class of $W_a(\widetilde{A}_{n-1})$.
(b) K is a left cell of $W_a(\widetilde{A}_{n-1})$.
(c) K is a left V-cell of $W_a(\widetilde{A}_{n-1})$.
(d) There exists some $X \in \mathbf{C}_n$ such that $K = T^{-1}(X)$.
(e) There exists a two-sided cell Ω of $W_a(\widetilde{A}_{n-1})$ such that K occurs as a maximal left connected subset of Ω.

Remark 6.23. The above theorem gives five different ways to describe a left cell of $W_a(\widetilde{A}_{n-1})$. However, the statements (a), (c) and (d) will be no larger equivalent to (b) for a general affine Weyl group W_a instead of $W_a(\widetilde{A}_{n-1})$. But one may expect that the statements (b) and (e) would be still equivalent to each other for a general W_a.

6.24 The image of an element $w \in W_a(\widetilde{A}_{n-1})$ under the map ϕ can be interpreted in terms of its coordinate form as follows. Let $\phi(w) = \{\lambda_1 \geq ... \geq \lambda_r\} \in \Lambda_n$. Then for $1 \leq t \leq r$, the integer $\sum_{i=1}^{t}\lambda_i$ is the maximal cardinal of subsets X of $\underline{n} = \{1, 2, ..., n\}$, where $X = \bigcup_{1\leq h\leq t} X_h$ is such that if $i, j \in X_h$, $i < j$ and $1 \leq h \leq t$ then $k(w, \alpha_{ij}) \neq 0$ with $\alpha_{ij} = \alpha_i + \alpha_{i+1} + ... + \alpha_{j-1}$. This interpretation of ϕ acturally gives an alternative description of two-sided cells of $W_a(\widetilde{A}_{n-1})$. By Theorem 6.22 (b), (c), the following result is immediate.

Corollary 6.25 [7]. *Any left cell of $W_a(\widetilde{A}_{n-1})$ is a union of some ST-classes of $W_a(\widetilde{A}_{n-1})$.*

Note that the conclusion of Corollary 6.25 will be no longer valid when $W_a(\widetilde{A}_{n-1})$ is replaced by a general affine Weyl group W_a (e.g. a counterexample can be found in the case of $W_a = W_a(\widetilde{B}_2)$). Nevertheless, we may still expect

Conjecture 6.26. *If the affine Weyl group W_a is of simple laced type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$, then any left cell of W_a is a union of some ST-classes of W_a.*

Finally, there is a formula for computing the a-value of an element $w \in W_a(\widetilde{A}_{n-1})$.

$$a(w) = \sum_{i=1}^{m}(i-1)\mu_i = \sum_{i=1}^{m} i\mu_i - n, \tag{6.27}$$

where $\{\mu_1 \geq ... \geq \mu_m\} \in \Lambda_n$ is the dual partition of $\phi(w)$.

§7 The cells of $W_a(\widetilde{B}_3)$.

7.1 As Lusztig has made explicit description of all cells for the affine Weyl groups W_a of rank ≤ 2, it is natural to ask if one can do the same with W_a of higher rank. Thus the groups W_a of rank 3 are taken into consideration. They are $W_a(\widetilde{A}_3)$, $W_a(\widetilde{B}_3)$ and $W_a(\widetilde{C}_3)$. The case $W_a(\widetilde{A}_3)$ has been included in §6. The cases $W_a(\widetilde{B}_3)$ and $W_a(\widetilde{C}_3)$ were treated by Du Jie and R. Bédard, respectively. Now we shall state the results of Du on the cells of the group $W_a(\widetilde{B}_3)$.

7.2 In his paper [**12**], Du first found out all SR-classes of $W_a(\widetilde{B}_3)$, each of which is represented by an element contained in it. Then he merged these SR-classes into PR-classes of $W_a(\widetilde{B}_3)$ and wrote down a set of representatives for the PR-classes of $W_a(\widetilde{B}_3)$ explicitly. Then he showed that, in $W_a(\widetilde{B}_3)$, the set of PR-classes coincides with the set of right cells. Finally, he merged all the right cells of $W_a(\widetilde{B}_3)$ into eight two-sided cells of $W_a(\widetilde{B}_3)$.

7.3 Suppose that the set $S = \{s_0,\ s_1,\ s_2,\ s_3\}$ of Coxeter generators of $W_a(\widetilde{B}_3)$ satisfies the conditions that $o(s_0s_2) = o(s_1s_2) = 3$ and $o(s_3s_4) = 4$. That is, the subscript i of s_i is compatible with the Coxeter graph

For $J \subseteq \{0,1,2,3\}$, let w_J be the longest element of W_J which is the subgroup of $W_a(\widetilde{B}_3)$ generated by $\{s_i \mid i \in J\}$. Following Du [**12**], we define some elements as below.

$$a_{123} = w_{\{1,2,3\}},\quad \hat{a}_{023} = w_{\{0,2,3\}},\quad b_{012} = w_{\{0,1,2\}},\quad c_{23} = w_{\{2,3\}},$$

$$d_{013} = w_{\{0,1,3\}},\quad e_{13} = w_{\{1,3\}},\quad f_{01} = w_{\{0,1\}},\quad g_0 = s_0,\quad h_\emptyset = \mathbf{1}.$$

Then we define some more elements which occur as vertices in the following diagrams

where two vertices $x,\ y$ are jointed by a labelled segment $\overset{i}{---}$ if $\{x, y\}$ forms a left primitive pair with $xy^{-1} = s_i$. The lengths of the elements in each diagram increase from the top to bottom. The subscripts of these elements w indicate the subsets $\mathbf{L}(w)$ of S, for example, $\mathbf{L}(a_{013}) = \{s_0,\ s_1,\ s_3\}$. Du showed the following results.

Theorem 7.4 [12]. *(a) In $W_a(\widetilde{B}_3)$, the right cells coincide with the PR-classes.*
(b) All the elements occurring as the vertices in the above diagrams form a set of representatives for the right cells of $W_a(\widetilde{B}_3)$. Thus the number of the right cells of $W_a(\widetilde{B}_3)$ is 11^2.
(c) Let $\hat{X}$ be the set of all elements occurring as the vertices in the diagram (X), let $\hat{A} = \hat{A}_1 \cup \hat{A}_2$ and let

$$\Sigma = \{\hat{A},\ \hat{B},\ \hat{C},\ \hat{D},\ \hat{E},\ \hat{F},\ \hat{G},\ \hat{H}\}$$

Then any $\hat{X} \in \Sigma$ is contained in some two-sided cell of $W_a(\widetilde{B}_3)$. If $\hat{X} \neq \hat{Y}$ in Σ then $\hat{X}$ and $\hat{Y}$ belong to two different two-sided cells of $W_a(\widetilde{B}_3)$.

7.5 Denote the two-sided cell of $W_a(\widetilde{B}_3)$ containing $\hat{X}$ by the corresponding bold face $\mathbf{X}$. Then by Theorem 7.4, we see that

$$W_a(\widetilde{B}_3) = \bigcup_{\hat{X}\in\Sigma} \mathbf{X}$$

is the decomposition of $W_a(\widetilde{B}_3)$ into two-sided cells and that any $\hat{X} \in \Sigma$ is a set of representatives of right cells of $W_a(\widetilde{B}_3)$ in $\mathbf{X}$.

To describe the two-sided cells of $W_a(\widetilde{B}_3)$ more precisely, Du defined the following subsets:

$$S(i) = \{I \subset S \mid l(w_I) = i\},\ \text{for } i \geq 0$$

$$S(2)_1 = \{\{s_0, s_1\}\},\quad S(2)_2 = S(2) - S(2)_1.$$

For any i with $i \neq 2$ and $S(i) \neq \emptyset$, define by $C(i)'$ the set of all elements w of $W_a(\widetilde{B}_3)$ which satisfy the following conditions:
(a) $w = x \cdot w_I \cdot z$ for some $x,\ z \in W_a(\widetilde{B}_3)$ and $I \in S(i)$.
(b) $w \neq x \cdot w_J \cdot z$ for any $x,\ z \in W_a(\widetilde{B}_3)$ and $J \subseteq S$ with $J \not\subset \bigcup_{j=0}^{i} S(j)$, where the notation $x = y \cdot z$ means that $x = yz$ and $l(x) = l(y) + l(z)$.

Define $C(2)_j,\ j = 1, 2$, to be the set of all elements $w \in W_a(\widetilde{B}_3)$ such that
(a′) $w = x \cdot w_I \cdot z$ for some $x,\ z \in W_a(\widetilde{B}_3)$ and $I \in S(2)_j$.
(b′) $w \neq x \cdot w_J \cdot z$ for any $x,\ z \in W_a(\widetilde{B}_3)$ and $J \notin S(0) \cup S(1) \cup S(2)_j$.
Then we define the following sets.

$$C(0) = \{\mathbf{1}\},\quad C(i) = C(i)',\quad i = 1,\ 6,\ 9.$$

$$C(3) = C(3)' - C(3)'' \text{ and } C(4) = C(4)' \cup C(3)'',$$

where $C(3)'' = \{x \cdot y \cdot z \mid y \in \{s_1s_2s_1s_3s_2s_1,\ s_0s_2s_0s_3s_2s_0\},\ x,\ z \in W_a(\widetilde{B}_3)\}$. Therefore Du proved

Theorem 7.6 [12]. *(a)* $\mathbf{A} = C(9),\ \mathbf{B} = C(6),\ \mathbf{C} = C(4),\ \mathbf{D} = C(3),\ \mathbf{E} = C(2)_2,\ \mathbf{F} = C(2)_1, \mathbf{G} = C(1),\ \mathbf{H} = C(0)$. *Thus*

$$W_a(\widetilde{B}_3) = C(0) \cup C(1) \cup C(2)_1 \cup C(2)_2 \cup C(3) \cup C(4) \cup C(6) \cup C(9)$$

is the decomposition of $W_a(\widetilde{B}_3)$ *into two-sided cells.*
(b) The a-value on $C(i)$ *(resp.* $C(i)_j$*) is* i*, for* $i = 0,\ 1,\ 2,\ 3,\ 4,\ 6,\ 9$.

Remark 7.7. It seems likely that the number of right cells in $W_a(\widetilde{B}_l)$ is a square number for any $l \geq 2$.

§8 Other results on the cells of affine Weyl groups.

8.1 Although we have known how to describe all the cells of the affine Weyl groups of type $\widetilde{A}_l,\ l \geq 1$, or of rank ≤ 3, it is still open for the same problem with respect to the other affine Weyl groups. However, there are some partial results in this direction. Let (W_a, S) be an affine Weyl group. For any $w \in W_a$, define $\delta(w) = \deg P_{\mathbf{1},w}$. Then Lusztig showed that the inequality

$$a(w) \leq l(w) - 2\delta(w)$$

holds in general. Define

$$\mathbf{D} = \{w \in W_a \mid a(w) = l(w) - 2\delta(w)\}.$$

Lusztig showed that $\mathbf{D}$ is a finite set consisting of involutions which are called the distinguished involutions. The following result is due to Lusztig

Theorem 8.2 [5]. *(a) The number of left cells of* W_a *is finite.*
(b) Each left cell of W_a *contains a unique distinguished involution.*

8.3 Define $W_{(i)} = \{w \in W_a \mid a(w) = i\}$ for $i \geq 0$. Then $W_{(i)}$ is a union of some two-sided cells of W_a. Lusztig showed that $W_{(1)}$ is a single two-sided cell of W_a consisting of all elements $w \neq \mathbf{1}$ of W_a each of which has a unique reduced expression. He also showed that all the left cells of W_a in $W_{(1)}$ are ST-classes of W_a [**3**]. Then Lawton described all the two-sided cells of W_a whose a-values ≤ 3 [**16**].

8.4 In 1986, the author showed that the set $W_{(\nu)}$ is the unique lowest two-sided cell of W_a with respect to the partial order $\leq_{LR}$, where $\nu = \frac{1}{2}|\Phi|$. He gave an explicit description for the set $W_{(\nu)}$ and for all the left cells of W_a in $W_{(\nu)}$ [**10**] [**11**]. Recently, Du found out all the two-sided cells of $W_a(\widetilde{D}_4)$ [**13**]. Also, Xi got some results concerning with the relations among cells of W_a [**14**]. Now we shall introduce the above results of Du, Xi and the author.

I. The lowest two-sided cell of W_a

In his paper [**10**], the author showed that the set $W_{(\nu)}$ itself is a two-sided cell of W_a and he gave the following alternative descriptions for this cell.

Theorem 8.5 [10]. *The following subsets of W_a are all the same.*
(a) $W_{(\nu)}$.
(b) The lowest two-sided cell of W_a with respect to the partial order $\leq_{LR}$.
(c) $W(\mathbf{S}) = \{x \cdot w_J \cdot y \mid x, y \in W_a,\ J \in \mathbf{S}\}$, where $\mathbf{S}$ is the set of all subsets J of S such that W_J is isomorphic to the Weyl group of Φ.
(d) $\mathbf{W} = \{w = (k(w,\alpha))_{\alpha\in\Phi} \in W_a \mid k(w,\alpha) \neq 0$ for $\alpha \in \Phi\}$.

8.6 In his paper [**10**], the author showed the equality (a) = (b) in the above theorem by first showing the equality

$$\mathbf{W} = W(\mathbf{S}) = W_{(\nu)}$$

which is the most hard part in the proof. Here we give a more direct proof for the equality (a) = (b) as follows. Fix a set $J \in \mathbf{S}$. Then $w_J \in W_{(\nu)}$. It is enough to show the relation $x \sim_{LR} w_J$ for any $x \in W_{(\nu)}$. Write $x = w \cdot y$ with $\mathbf{R}(w) \cap J = \emptyset$ and $y \in W_J$. Let $z = y^{-1}w_J$. Then we have

$$x \sim_R w \cdot y \cdot z = w \cdot w_J \sim_L w_J$$

by Theorem 6.20 (c) and by noting that $w \cdot w_J \leq_R x,\ w \cdot w_J \leq_L w_J$ and $a(w \cdot w_J) = a(w_J) = a(x) = \nu$.

8.7 Concerning the left cells of W_a in $W_{(\nu)}$, the author gave the following results.

Theorem 8.8 [11]. *(a) All the left cell of W_a in $W_{(\nu)}$ are ST-classes of W_a and hence they are left connected.*
(b) Let $N = \{y^{-1} \cdot w_J \cdot y \mid J \in \mathbf{S},\ sw_Jy \notin W_{(\nu)}$ for any $s \in J\}$. Then $N = W_{(\nu)} \cap \mathbf{D}$. Hence N is a set of representatives for the left cells of W_a in $W_{(\nu)}$.
(c) $W_{(\nu)}$ consists of $|W|$ left cells of W_a, where W is the Weyl group of Φ.

By Theorem 8.8, the key step for the proof of Theorem 8.8 is to show the equality $N = \mathbf{D} \cap W_{(\nu)}$. The latter can be shown by making used of the structural properties of the corresponding affine Hecke algebra [**11**].

8.9 Since the distinguished involutions of W_a play an important role in the cell theory of W_a, it is desirable to give an explicit description for these elements. Let us make a conjecture for this. For $x,\ y \in W_a$, let

$$C_xC_y = \sum_z h_{x,y,z}C_z, \quad h_{x,y,z} \in \mathbf{Z}[u, u^{-1}].$$

It is known that there exists a unique element, written $\lambda(x,y)$, in the set $\Lambda(x,y) = \{z \in W_a \mid h_{x,y,z} \neq 0\}$ satisfying the condition that

$$\lambda(x,y) \geq w, \text{ for any } w \in \Lambda(x,y) \ [\mathbf{10}].$$

Conjecture 8.10. *Let Γ be a left cell of W_a. Assume that $\mathbf{D} \cap \Gamma = \{d\}$. Then for any shortest element x of Γ, we have $d = \lambda(x^{-1}, x)$.*

II. The two-sided cells of $W_a(\widetilde{D}_4)$

Here we shall explain the description of Du for the two-sided cells of $W_a(\widetilde{D}_4)$ [**13**].

8.11 Let $S = \{s_i \mid 0 \le i \le 4\}$ be the set of Coxeter generators of $W_a(\widetilde{D}_4)$ with $o(s_i s_2) = 3$ for all $i \ne 2$. Let

$$S(i) = \{I \subset S \mid l(w_I) = i\}, \text{ for } i \ge 0.$$

$$S(2)_1 = \{\{s_0, s_1\},\ \{s_3, s_4\}\}, \qquad S(2)_2 = \{\{s_0, s_3\},\ \{s_1, s_4\}\},$$
$$S(2)_3 = \{\{s_0, s_4\},\ \{s_1, s_3\}\}, \qquad S(6)_1 = \{\{s_0, s_2, s_1\},\ \{s_3, s_2, s_4\}\},$$
$$S(6)_2 = \{\{s_0, s_2, s_3\},\ \{s_1, s_2, s_4\}\}, \quad S(6)_3 = \{\{s_0, s_2, s_4\},\ \{s_1, s_2, s_3\}\}.$$

For each i with $S(i) \ne \emptyset$, define by $C(i)$ the set of all elements $w \in W_a(\widetilde{D}_4)$ satisfying the following conditions:

(a) $w = x \cdot w_I \cdot z$ for some $I \in S(i)$ and $x,\ z \in W_a(\widetilde{D}_4)$.

(b) $w \ne x \cdot w_J \cdot z$ for any $J \notin \bigcup_{j=0}^{i} S(j)$ and $x,\ z \in W_a(\widetilde{D}_4)$.

For each $j,\ 1 \le j \le 3$, and each $i \in \{2, 6\}$, define

$$C(i)_j = \left\{ w \in C(i) \mid w \ne x \cdot w_J \cdot z \text{ for any } J \in S(i) - S(i)_j \text{ and } x, z \in W_a(\widetilde{D}_4) \right\}$$

and define $C(7) = C(6) - \bigcup_{j=1}^{3} C(6)_j$.

Then Du showed the following

Theorem 8.12 [13]. *(a)* $W_a(\widetilde{D}_4) = C(0) \cup C(1) \cup (\bigcup_{j=1}^{3} C(2)_j) \cup C(3) \cup C(4) \cup (\bigcup_{j=1}^{3} C(6)_j) \cup C(7) \cup C(12)$ *is the decomposition of* $W_a(\widetilde{D}_4)$ *into 12 two-sided cells.*
(b) The a-value on $C(i)$ *(resp.* $C(i)_j$*) is* i *for any* $i \in \{0, 1, 2, 3, 4, 6, 7, 12\}$ *and* $j = 1, 2, 3$.

Remark 8.13 (a) For the time being, let W'_a denote an affine Weyl group of type $\widetilde{A}_l,\ l \ge 1$, or of rank ≤ 3. Here we shall indicate two points concerning with the properties of cells for $W_a(\widetilde{D}_4)$ which are different from those for W'_a

Firstly, an a-value on W'_a can always be realized on the subset $\{w_J \mid J \subset S\}$ of W'_a. In other words, we have the equation

$$\{a(x) \mid x \in W'_a\} = \{a(w_J) \mid J \subset S\} = \{l(w_J) \mid J \subset S\}.$$

But this equation is no longer valid in the case $W_a(\widetilde{D}_4)$ instead of W'_a since

$$a(s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_2 s_1) = 7 \notin \{a(w_J) \mid J \subset S\} = \{0, 1, 2, 3, 4, 6, 12\}.$$

Secondly, the PL-classes all coincide with the left cells in W'_a. But in the two-sided cell $C(4)$ of $W_a(\widetilde{D}_4)$, we can find a left cell which is a union of more than one PL-classes.

(b) By detailed analysis of Du's result, we see that in $W_a(\widetilde{D}_4)$, any left cell is a union of ST-classes.

III. Some relations among cells of W_a

Recently, Xi Nanhua wrote a joint paper with Lusztig to be concerned with some relations between the left cells and the two-sided cells of an affine Weyl group W_a [**14**].

For each $s \in S$, let $Y_s = \{w \in W_a \mid \mathbf{R}(w) \subset \{s\}\}$.

We say that $s \in S$ is special if the subgroup $W_{S-\{s\}}$ of W_a generated by $S - \{s\}$ is isomorphic to the Weyl group of Φ.

Then the main result of Lusztig and Xi is as follows.

Theorem 8.14 [14]. *Let Ω be a two-sided cell of W_a and let $s \in S$. Then*
(a) $\Omega \cap Y_s$ is non-empty and is a union of some left cells of W_a.
(b) If s is special, then $\Omega \cap Y_s$ is exactly one left cell of W_a.
(c) The number of left cells of W_a in $W_{(\nu)} \cap Y_s$ is $|W_{S-\{s_0\}}|/|W_{S-\{s\}}|$.

The part (a) of the above theorem was showed by Xi. The crucial step in the proof of Xi is to show the following result which is of its own interest.

Theorem 8.15 [14]. *Suppose that Ω_1 and Ω_2 are two two-sided cells of W_a with $\Omega_1 \leq_{LR} \Omega_2$. Then for any left cell Γ_1 in Ω_1, there exists a left cell Γ_2 in Ω_2 with $\Gamma_1 \leq_L \Gamma_2$.*

Remark 8.16 (a) If we are given a left cell Γ_2 in Ω_2 instead of Γ_1 in Ω_1 in Theorem 8.15, then by the similar argument as that in the proof of Xi, we can find a left cell Γ_1 in Ω_1 such that $\Gamma_1 \leq_{LR} \Gamma_2$
(b) Theorem 8.15 tells us that the direct graph associated with the partial ordered set of the two-sided cells of W_a with respect to $\leq_{LR}$ is isomorphic to a subgraph of that associated with the partial ordered set of left cells of W_a with respect to $\leq_L$.
(c) It is well known that if $x \leq_L y$ then $\mathbf{R}(x) \supseteq \mathbf{R}(y)$. In particular, if $x \sim_L y$ then $\mathbf{R}(x) = \mathbf{R}(y)$ [**1**]. Recall the notation $X \leq Y$ for two sets X, Y of subsets of a set defined in §3. Then from Theorem 8.15 and Remark 8.16 (a), we see that if Ω_1 and Ω_2 are two two-sided cells of W_a with $\Omega_1 \leq_{LR} \Omega_2$, then

$$\{\mathbf{R}(x) \mid x \in \Omega_1\} \geq \{\mathbf{R}(y) \mid y \in \Omega_2\}.$$

On the other hand, Xi made a detailed study on the properties of translation elements of W_a in his paper [**17**]. He showed that the a-function of W_a reaches its next largest value on the set of translation elements of W_a. He also showed that in W_a, any left cell consists of at most finitely many left connected components [**17**]. The latter result is a progress in the direction of proving a conjecture of Lusztig which says that any left cell of W_a is left connected [**18**].

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