THE NUMBER OF $\oplus$-SIGN TYPES

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In [6], the author introduced the concept of a sign type indexed by the set $\{(i,j) \mid 1 \leq i,j \leq n, i \neq j\}$ for the description of the cells in the affine Weyl group of type $\widetilde{A}_\ell$ (in the sense of Kazhdan and Lusztig [3]) . Later, the author generalized it by defining a sign type indexed by any root system [8]. Since then, sign type has played a more and more important role in the study of the cells of any irreducible affine Weyl group [1],[2],[5],[8],[9],[10],[12]. Thus it is worth to study sign type itself. One task is to enumerate various kinds of sign types. In [8], the author verified Carter's conjecture on the number of all sign types indexed by any irreducible root system. In the present paper, we consider some special kind of sign types, called $\oplus$-sign types. We show that these sign types are in one-to-one correspondence with the increasing subsets of the related positive root system as a partially ordered set. Thus we get the number of all the $\oplus$-sign types indexed by any irreducible positive root system Φ^+ by enumerating all the increasing subsets

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of Φ^+. Such a number has a combinatorial interpretation when Φ^+ is of classical type. That is, it is equal to the number of all the subdiagrams of a certain Young diagram. Also, in dealing with Φ^+ of exceptional type, we introduce a method which could be used to enumerate all the increasing subsets of any partially ordered set.

The results in this paper may be helpful in studying the structure of the canonical left cells of an affine Weyl group, (see [4]).

The content of this paper is organized as below. In section 1, we establish a bijection between the set of all the increasing subsets of a positive root system Φ^+ and the set of all the $\oplus$-sign types indexed by the dual positive root system (Theorem 1.4). Then we make certain arrays of Φ^+ of classical types into diagrams in section 2, by which we establish a one-to-one correspondence between the set of all the $\oplus$-sign types indexed by Φ^+ and the set of all the subdiagrams of a certain diagram. Finally, we calculate the number of all the $\oplus$-sign types indexed by any irreducible positive root system in section 3 (Theorems 3.2 and 3.3).

§1. $\oplus$-sign types and increasing subsets of Φ^+.

1.1 Let E be a euclidean space spanned by an irreducible root system Φ with inner product $\langle\ ,\ \rangle$. Let $\Delta = \{\alpha_1, \cdots, \alpha_n\} \subset \Phi^+$ be a choice of simple and positive root systems of Φ, respectively. Denote by $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$ a coroot of α for $\alpha \in \Phi$. For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, define a hyperplane:

$$H_{\alpha,k} = \{v \in E \mid \langle v, \alpha^\vee\rangle = k\}$$

and three regions:

$$\begin{cases} H_{\alpha,+} = \{v \in E \mid \langle v, \alpha^\vee\rangle > 1\}, \\ H_{\alpha,-} = \{v \in E \mid \langle v, \alpha^\vee\rangle < 0\}, \\ H_{\alpha,\bigcirc} = \{v \in E \mid 0 < \langle v, \alpha^\vee\rangle < 1\}. \end{cases}$$

of E. Any connected simplex of

$$E - \bigcup_{\substack{\alpha\in\Phi^+\\ k\in\{0,1\}}} H_{\alpha,k}$$

is called a Φ^+-sign type or a sign type indexed by Φ^+ (or just call it a sign type when no danger of confusion) which has the form

$$S_X = \bigcap_{\alpha\in\Phi^+} H_{\alpha,X_\alpha}$$

for some Φ^+-tuple $X = (X_\alpha)_{\alpha\in\Phi^+}$ with $X_\alpha \in \{+,-,\bigcirc\}$. We may identify S_X with the corresponding Φ^+-tuple $X = (X_\alpha)_{\alpha\in\Phi^+}$ and call the latter a sign type. Note that not any Φ^+-tuple $(X_\alpha)_{\alpha\in\Phi^+}$ over $\{+,-,\bigcirc\}$ gives rise to a sign type as above. The condition for $(X_\alpha)_{\alpha\in\Phi^+}$ to be a sign type is local, that is, a condition on all the irreducible subsystems of Φ^+ of rank 2 (see [5, §2.] for the detailed statement of this condition).

1.2 Call $X = (X_\alpha)_{\alpha\in\Phi^+}$ an $\oplus$-sign type indexed by Φ^+ if it is a Φ^+-sign type with $X_\alpha \in \{+,\bigcirc\}$, for all $\alpha \in \Phi^+$. The following result characterizes an $\oplus$-sign type.

Proposition. *Assume that $X = (X_\alpha)_{\alpha\in\Phi^+}$ is a Φ^+-tuple with $X_\alpha \in \{+,\bigcirc\}$. Then X is an $\oplus$-sign type if and only if the following condition on X holds: if $\alpha,\beta \in \Phi^+$ satisfy $\beta^\vee > \alpha^\vee$ and $X_\alpha = +$, then $X_\beta = +$. Here the notation $\beta^\vee > \alpha^\vee$ means that the difference $\beta^\vee - \alpha^\vee$ is a sum of some coroots in $(\Phi^\vee)^+ = \{\gamma^\vee \mid \gamma \in \Phi^+\}$ (repetition allowed).*

Proof. ($\Longleftarrow$) The condition on X guarantees that all the Ψ^+-tuples $(X_\alpha)_{\alpha\in\Psi^+}$ are Ψ^+-sign types, where Ψ^+ ranges over all the irreducible subsystems of Φ^+ of rank 2. So the assertion follows from the note at the end of 1.1.

($\Longrightarrow$) By [8, Lemma 3.1], there exist a sequence of positive roots $\gamma_0 = \alpha,\ \gamma_1,\ \cdots,\ \gamma_r = \beta$ such that for every $i,\ 1 \leq i \leq r,\ \gamma_i^\vee - \gamma_{i-1}^\vee \in \Delta^\vee$, where $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$. Thus by applying induction on $r \geq 1$, it suffices to show our result in the case when $\beta^\vee - \alpha^\vee = \gamma^\vee \in \Delta^\vee$. There exists a unique subsystem Ψ^+ of Φ^+ of rank 2 which contains roots $\alpha,\ \beta,\ \gamma$. Then $(X_\alpha)_{\alpha\in\Psi^+}$ is an $\oplus$-sign type indexed by Ψ^+. Since all the Ψ^+-sign types with Ψ^+ irreducible and of rank 2 are known (see [7]), our result follows by checking directly. $\square$

Definition 1.3. Let $(P, \leq)$ be a partially ordered set. A subset $K \subseteq P$ is *increasing* if for any $x, y \in P$, $y \geq x$ and $x \in K$ imply $y \in K$.

It is clear that two isomorphic partially ordered sets contain the same number of increasing subsets. A positive root system Φ^+ is a partially ordered set with respect to the relation $\leq$: $\alpha \leq \beta$ in Φ^+ if the difference $\beta - \alpha$ is a sum of some roots in Φ^+ (repetition allowed).

By Proposition 1.2, it is immediate that

Theorem 1.4. *There exists a natural bijective map from the set of all the increasing subsets K of $(\Phi^\vee)^+$ to the set of all the $\oplus$-sign types indexed by Φ^+. This map sends K to $(X_\alpha)_{\alpha\in\Phi^+}$, where*

$$X_\alpha = \begin{cases} \bigcirc, & \text{if } \alpha^\vee \notin K; \\ +, & \text{if } \alpha^\vee \in K. \end{cases}$$

Thus to enumerate all the $\oplus$-sign types indexed by Φ^+, it is enough to compute the number of all the increasing subsets of $(\Phi^\vee)^+$. To do that, we shall begin by constructing certain arrays of roots in Φ^+, where ϕ^+ is of classical type.

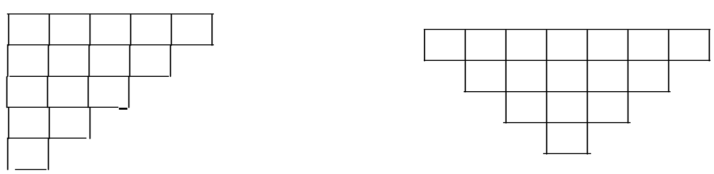
Remark 1.5. One may also define a sign type $X = (X_\alpha)_{\alpha\in\Phi^+}$ to be an $\ominus$-sign type (resp. a $\pm$-sign type) by the requirement $X_\alpha \in \{-, \bigcirc\}$ (resp. $X_\alpha \in \{+, -\}$

) for all $\alpha \in \Phi^+$. It is well known that the set of all the ⊖-sign types (resp. ±-sign types) indexed by Φ^+ is in one-to-one correspondence with the set of all the Weyl chambers in E in a natural way. This implies immediately that the number of all the ⊖-sign types (resp. ±-sign types) indexed by Φ^+ is equal to the order of the Weyl group of the root system Φ. Thus only the number of all the ⊕-sign types indexed by Φ^+ need be computed. This is why we only consider the ⊕-sign types in the present paper.

§2. Arrays of roots into diagrams.

2.1 By a diagram, we mean an array of boxes into rows and columns. We consider the following three kinds of diagrams.

(i) The staircase Young diagram Γ_n of size n (see Figure 1, (a), for Γ_5 as an example).



(a) (b)

Figure 1.

(ii) The staircase skew Young diagram Λ_n of rank n (see Figure 1, (b), for Λ_n as an example).

(iii) The staircase skew Young diagram Λ'_n of rank n which has two leaves in the middle column. Figure 2 is the diagram Λ'_5 as an example.

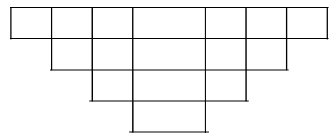


Figure 2.

Generally, in diagram Λ'_n, there is one box, denoted by $B_{i,j}$, at the intersection of the ith row and the jth column for $1 \le i < n$ and $i \le j \le 2n-2-i$ with $j \ne n-1$; there are two boxes at the intersection of the ith row and the $(n-1)$th column for $1 \le i < n$, which are denoted by $B_{i,n-1}$ and $B_{i',n-1}$, respectively.

2.2 Now we define a subdiagram in each of the above diagrams. In these definition, we identifying a diagram with the set of boxes contained in it.

Let X be the diagram Γ_n or Λ_n. A subdiagram λ of X is, by definition, a subset of X satisfying that if a box of X is in λ, then all the boxes of X to the north, the west and the northwest of such a box are also in λ.

The definition of a subdiagram λ' of Λ'_n slightly differ from the above. By definition, λ' is a subset of Λ'_n satisfying the following conditions:

(a) If box B_{ij} is in λ' with $j \ne n-1$, then all the boxes of Λ'_n to the north, the west and the northwest of B_{ij} are also in λ';

(b) If box $B_{i,n-1}$ is in λ' with $i \in \{1, 2, \cdots, n-1\}$, then all the boxes of Λ'_n to the west and the northwest of $B_{i,n-1}$ are in λ', and all the boxes $B_{j,n-1}$ with $j < i$ are also in λ';

(c) If box $B_{i',n-1}$ is in λ' with $i \in \{1, 2, \cdots, n-1\}$, then all the boxes of Λ'_n to the west and the northwest of $B_{i',n-1}$ are in λ', and all the boxes $B_{j',n-1}$ with $j < i$ are also in λ'.

Note that boxes $B_{i,n-1}$ and $B_{i',n-1}$ are regarded to be overlapped for any i, $1 \leq i \leq n-1$. Thus in Figure 3, both (a) and (b) are subdiagrams of Λ'_5 but (c) and (d) are not.

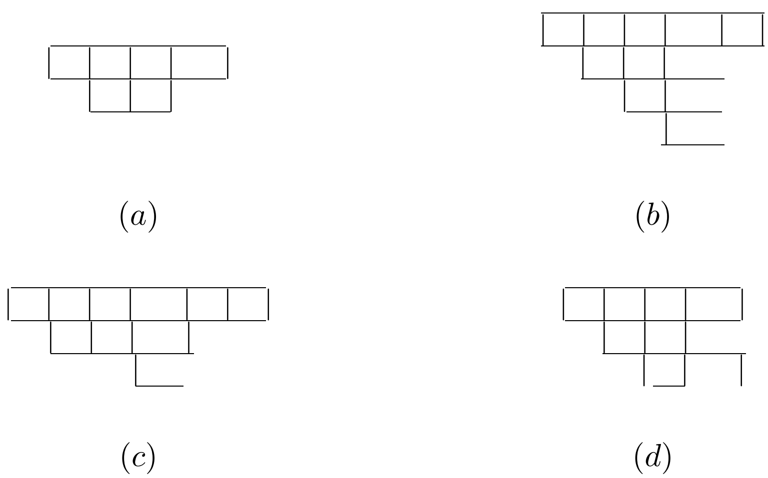


Figure 3.

2.3 In 2.3-2.6, we shall establish a bijection between the set of all the increasing subsets of a positive root system of each classical type and the set of all the subdiagrams of some diagram defined in 2.1.

First assume that Φ^+ has type A_n. Then

$$\Phi^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n+1\},$$

where $\alpha_{ij} = \sum_{t=i}^{j-1} \alpha_t$. We fill the roots of Φ^+ into the diagram Γ_n (see 2.1, (a)), one in each box, such that the root α_{ij} is in the $(n+2-j)$th row and the ith column of Γ_n. Figure 4 is an example with $n = 5$.

$$\begin{array}{lllll}
\alpha_{16} & \alpha_{26} & \alpha_{36} & \alpha_{46} & \alpha_{56} \\
\alpha_{15} & \alpha_{25} & \alpha_{35} & \alpha_{45} & \\
\alpha_{14} & \alpha_{24} & \alpha_{34} & & \\
\alpha_{13} & \alpha_{23} & & & \\
\alpha_{12} & & & &
\end{array}$$

Figure 4.

Then it is easily seen that $K \subseteq \Phi^+$ is increasing if and only if there exists a subdiagram λ of Γ_n such that K is the set of all roots in the boxes of λ.

2.4 Secondly assume that Φ^+ has type B_n. Then

$$\Phi^+ = \{\alpha_{ij},\ \alpha_{i,-j},\ \alpha_r \mid 1 \leq i < j \leq n,\ 1 \leq r \leq n\},$$

where $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$, $\alpha_{i,-j} = \sum_{k=i}^{n} \alpha_k + \sum_{h=j}^{n} \alpha_h$ and $\alpha_r = \sum_{k=r}^{n} \alpha_k$. Denote by B_{ij} the box of the diagram Λ_n (see 2.1, (b)) at the intersection of the ith row and the jth column. We fill the roots of Φ^+ into diagram Λ_n, one in each box, such that the root α_{ij} is in the box $B_{i,2n+1-j}$, that the root $\alpha_{i,-j}$ is in the box $B_{i,j-1}$, and that the root α_r is in the box $B_{r,n}$. Figure 5 is an example with $n = 4$.

$$\begin{array}{lllllll}
\alpha_{1,-2} & \alpha_{1,-3} & \alpha_{1,-4} & \alpha_1 & \alpha_{14} & \alpha_{13} & \alpha_{12} \\
 & \alpha_{2,-3} & \alpha_{2,-4} & \alpha_2 & \alpha_{24} & \alpha_{23} & \\
 & & \alpha_{3,-4} & \alpha_3 & \alpha_{34} & & \\
 & & & \alpha_4 & & &
\end{array}$$

Figure 5.

We can see that $K \subseteq \Phi^+$ is increasing if and only if there exists a subdiagram λ of Λ_n such that K is just the set of all the roots in the boxes of λ.

2.5 Thirdly assume that Φ^+ has type C_n. Then

$$\Phi^+ = \{\alpha_{ij},\ \alpha_{i,-j},\ \alpha_{r,-r} \mid 1 \leq i < j \leq n,\ 1 \leq r \leq n\},$$

where $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$, $\alpha_{r,-r} = 2\sum_{h=r}^{n} \alpha_h - \alpha_n$ and $\alpha_{i,-j} = \sum_{k=i}^{n} \alpha_k + \sum_{h=j}^{n-1} \alpha_h$ with the convention that $\alpha_{i,-n} = \sum_{k=i}^{n} \alpha_k$. We fill the roots of Φ^+ into the diagram Λ_n (see 2.1, (b)), one in each box, such that the root α_{ij} is in the box $B_{i,2n+1-j}$, that the root $\alpha_{i,-j}$ is in the box $B_{i,j}$ and that the root $\alpha_{r,-r}$ is in the box $B_{r,r}$. Figure 6 is an example with $n = 4$.

$\alpha_{1,-1}$	$\alpha_{1,-2}$	$\alpha_{1,-3}$	$\alpha_{1,-4}$	α_{14}	α_{13}	α_{12}
	$\alpha_{2,-2}$	$\alpha_{2,-3}$	$\alpha_{2,-4}$	α_{24}	α_{23}	
		$\alpha_{3,-3}$	$\alpha_{3,-4}$	α_{34}		
			$\alpha_{4,-4}$			

Figure 6.

Again, we see that $K \subseteq \Phi^+$ is increasing if and only if there exists a subdiagram λ of Λ_n such that K is just the set of all the roots in the boxes of λ.

It is known that an irreducible positive root system Φ^+ has the same type as the corresponding coroot system $(\Phi^\vee)^+$ except for the case when Φ^+ has type B_n or C_n. Now the above result together with the result of 2.4 imply that the number of all the increasing subsets in $\Phi^+(B_n)$ is the same as the number of those in $\Phi^+(C_n)$. So it follows from Theorem 1.4 that

Theorem. *The number of all the ⊕-sign types indexed by a positive root system Φ^+ is equal to the number of all the increasing subsets of Φ^+.*

2.6 Finally assume Φ^+ has type D_n. Then

$$\Phi^+ = \{\alpha_{ij},\ \alpha_{i,-j} \mid 1 \leq i < j \leq n\},$$

where $\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k$ and, $\alpha_{i,-j}$ is equal to $\sum_{k=i}^{n} \alpha_k + \sum_{h=j}^{n-2} \alpha_h$ for $j \leq n-2$, to $\sum_{k=i}^{n} \alpha_k$ for $j = n-1$, and to $\sum_{k=i}^{n-2} \alpha_k + \alpha_n$ for $j = n$.

We fill the roots of Φ^+ into the diagram Λ'_n (see 2.1, (c)), one in each box, such that the root α_{ij} ($1 \leq i < j < n$) is in the box $B_{i,2n-1-j}$, that the root $\alpha_{i,-j}$ ($1 \leq i < j < n$) is in the box $B_{i,j-1}$, and the roots $\alpha_{i,-n}$, $\alpha_{i,n}$ ($1 \leq i < n$) are in the boxes $B_{i,n-1}$, $B_{i',n-1}$, respectively. Figure 7 is an example with $n = 5$.

$\alpha_{1,-2}$	$\alpha_{1,-3}$	$\alpha_{1,-4}$	$\alpha_{1,-5}$	α_{15}	α_{14}	α_{13}	α_{12}
	$\alpha_{2,-3}$	$\alpha_{2,-4}$	$\alpha_{2,-5}$	α_{25}	α_{24}	α_{23}	
		$\alpha_{3,-4}$	$\alpha_{3,-5}$	α_{35}	α_{34}		
			$\alpha_{4,-5}$	α_{45}			

Figure 7.

We see that $K \subseteq \Phi^+$ is increasing if and only if there exists a subdiagram λ' of Λ'_n such that K is exactly the set of all the roots in the boxes of λ'.

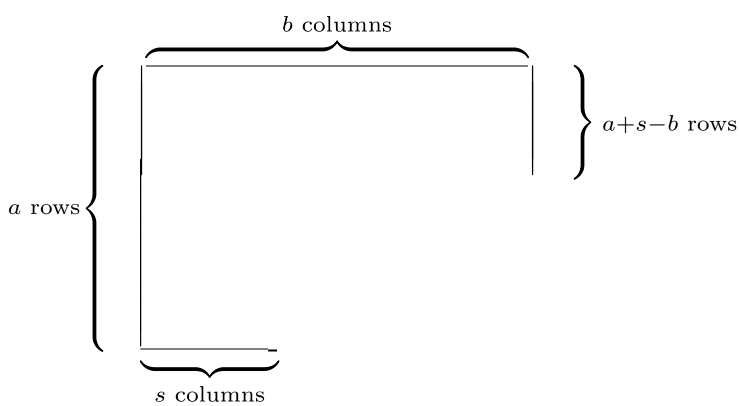


Figure 8.

2.7 For $a, b, s \in \mathbb{N}$ with $0 < s \leq b < a + s$, let $p(a, b, s)$ be the number of all the subdiagrams of the truncated staircase Young diagram $P(a, b, s)$ (see Figure 8.)

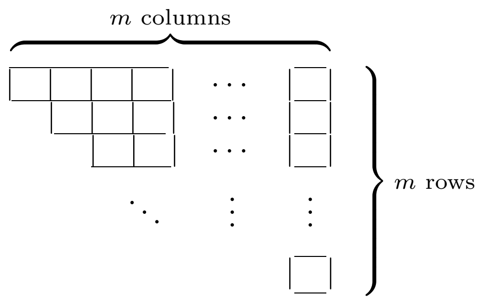


Figure 9.

Also, for $m \in \mathbb{N}$, let $q(m)$ be the number of all the subdiagrams of the skew Young diagram Q_m (see Figure 9.), where a subdiagram of $P(a, b, s)$ or Q_m is defined in the same way as that for Γ_n or Λ_n in 2.2. Then we have the following

Proposition. *For integers a, b, s, m with $0 < s \leq b < a + s$ and $m \geq 0$, we have*

(1) $q(m) = 2^m$;

(2) $p(a, b, s) = \binom{a+b}{a} - \binom{a+b}{a+s+1}$.

where $\binom{x}{y} = \frac{x!}{y!(x-y)!}$ *is a binomial number for* $x, y \in \mathbb{N}$ *with the convention that* $\binom{x}{y} = 0$ *for* $y > x$.

Proof. (1) follows easily by applying induction on $m \geq 0$. (2) is a result in [11]. □

§3. The enumeration of ⊕-sign types.

3.1 We keep the notations and conventions of the previous sections. In this section, we shall enumerate the ⊕-sign types indexed by any irreducible positive root system Φ^+. By Theorems 1.4 and 2.5, it reduces to enumerate the increasing subsets of Φ^+. In particular, when Φ^+ is of classical type, it further reduces to compute the number of subdiagrams of a certain Young diagram.

Let $\mu(X)$ be the number of all the $\oplus$-sign types indexed by the positive root system Φ^+ of type X, where

$$X \in \{A_l,\ B_m,\ C_m,\ D_n \mid l \geq 1, m \geq 2, n \geq 4\} \bigcup \{E_6,\ E_7,\ E_8,\ F_4,\ G_2\}.$$

First we compute the number $\mu(X)$ with X a classical type.

Theorem 3.2. *(a)* $\mu(A_l) = \frac{1}{l+2}\binom{2l+2}{l+1}$ *for* $l \geq 1$*;*
(b) $\mu(B_m) = \mu(C_m) = \binom{2m}{m}$ *for* $m \geq 2$*;*
(c) $\mu(D_n) = \binom{2n-1}{n} + \binom{2n-2}{n}$ *for* $n \geq 4$*.*

Proof.. From Theorems 1.4 and 2.5, one can easily deduce the formulae
(i) $\mu(A_l) = p(l, l, 1)$;
(ii) $\mu(B_m) = \mu(C_m) = p(m-1, m-1, 1) + \sum_{i=1}^{m} q(m-i)p(i-1, m-1, m+1-i)$, where $q(m-i)p(i-1, m-1, m+1-i)$, $1 < i \leq m$, is the number of all the subdiagrams of Λ_m containing box $B_{i-1,m}$ but not containing $B_{i,m}$, $q(m-1)p(0, m-1, m) = q(m-1)$ is the number of all the subdiagrams of Λ_m not containing box $B_{1,m}$, and $p(m-1, m-1, 1)$ is the number of all the subdiagrams of Λ_m containing box $B_{m,m}$.
(iii) $\mu(D_n) = \delta'_n + \delta''_n$, where $\delta'_n = q(n-2) + 2p(n-2, n-1, 2) + p(n-2, n-2, 1)$ and $\delta''_n = \sum_{i=3}^{n} q(n-i)[2p(i-3, n-1, n+3-i) + p(i-2, n-2, n+1-i)]$. More precisely, we have the following interpretations on each of the above terms:

(1) $q(n-2)$ is the number of subdiagrams of Λ'_n not containing boxes $B_{1,n-1}$ and $B_{1',n-1}$;

(2) $2p(n-2,n-1,2)$ subdiagrams of Λ'_n containing exactly one of the boxes $B_{n-1,n-1}$ and $B_{(n-1)',n-1}$;

(3) $p(n-2,n-2,1)$ is the number of subdiagrams of Λ'_n containing both boxes $B_{n-1,n-1}$ and $B_{(n-1)',n-1}$;

(4) $q(n-i)p(i-3,n-1,n+3-i)$ is the number of subdiagrams of Λ'_n containing box $B_{i-2,n-1}$ but not containing $B_{(i-2)',n-1}$ and $B_{i-1,n-1}$ (resp. containing box $B_{(i-2)',n-1}$ but not containing $B_{i-2,n-1}$ and $B_{(i-1)',n-1}$), for $3 \le i \le n$;

(5) $q(n-i)p(i-2,n-2,n+1-i)$ is the number of subdiagrams of Λ'_n containing both boxes $B_{i-2,n-1}$ and $B_{(i-2)',n-1}$ but not containing $B_{i-1,n-1}$ and $B_{(i-1)',n-1}$, for $3 \le i \le n$.

Thus (a) (resp. (b)) follows easily from (i) (resp. (ii)) and Proposition 2.7. For (c), we get from (iii) and Proposition 2.7 that

$$\delta'_n = 2^{n-2} + 2\left[\binom{2n-3}{n-2} - \binom{2n-3}{n+1}\right] + \left[\binom{2n-4}{n-2} - \binom{2n-4}{n}\right] \qquad \text{and}$$
$$\delta''_n = \binom{2n-1}{n+1} + \binom{2n-2}{n+1} - 2^{n-2}.$$

This implies (c) from the equation $\mu(D_n) = \delta'_n + \delta''_n$. □

3.3 To each partially ordered set $(P, \le)$, we associate a digraph $\Gamma(P)$ whose vertex set consists of all the elements of P, two vertices $v,\ w$ are joined by an arrow, say $v \longrightarrow w$, if v covers w, i.e. $v > w$ and there does not exist any $y \in P$ satisfying $v \gneq y \gneq w$. Call $\Gamma(P)$ the Hasse graph of P. Clearly, two partially ordered sets are isomorphic if and only if their Hasse graphs are isomorphic.

3.4 To enumerate all the increasing subsets of a given partially ordered set $(P, \le)$, we need only work with its Hasse graph $\Gamma(P)$. Here we point out a simple fact for later use.

Suppose that the elements $z_1,\ z_2,\ \cdots,\ z_r$ be elements of P satisfy $z_{j-1} \longrightarrow z_j$ for $1 < j \leq r$. Let Z_j ($1 \leq j < r$) be the set of all the increasing subsets K of P such that $z_j \in K$ and $z_{j+1} \notin K$. Let Z_0 (resp. Z_r) be the set of all the increasing subsets K of P with $z_1 \notin K$ (resp. $z_r \in K$). Then the set of all the increasing subsets of P is just a disjoint union of all Z_j, $0 \leq j \leq r$. In particular, when $P = \{z_1,\ z_2,\ \cdots,\ z_r\}$, the number of all the increasing subsets of P is equal to $|P| + 1$.

3.5 Now we want to deal with the exceptional types. The Hasse graph of the positive root system $\Phi^+ = \Phi^+(F_4)$ is isomorphic to the Hasse graph in Figure 10.

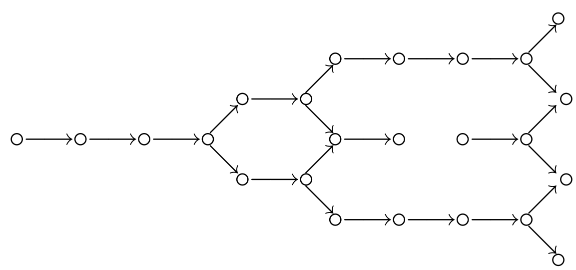


Figure 10.

We shall use this Hasse graph to enumerate all the increasing subsets of Φ^+. It appears to be hard to enumerate all the increasing subsets of Φ^+ directly. Thus we shall simplify the situation. In Figure 10, let Γ_i ($1 \leq i \leq 5$) be the set of all the increasing subsets K of Φ^+ containing vertex i but not vertex $i+1$, and let Γ_0 (resp. Γ_6) be the set of all the increasing subsets K of Φ^+ not containing vertex 1 (resp. containing vertex 6). Then by 3.4, the sets Γ_i, $0 \leq i \leq 6$, are pairwise disjoint whose union forms the whole set of increasing subsets of Φ^+. It is easily seen that there exists a natural bijection from the set Γ_0 to the set of all the increasing subsets of a partially ordered set with the Hasse graph in Figure 11.

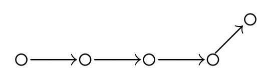


Figure 11.

This implies $|\Gamma_0| = 6$ from 3.4. Similarly, there exist natural bijections from the sets Γ_i, $1 \leq i \leq 6$, to the sets of all the increasing subsets of partially ordered sets with the Hasse graphs in Figure 12, respectively.

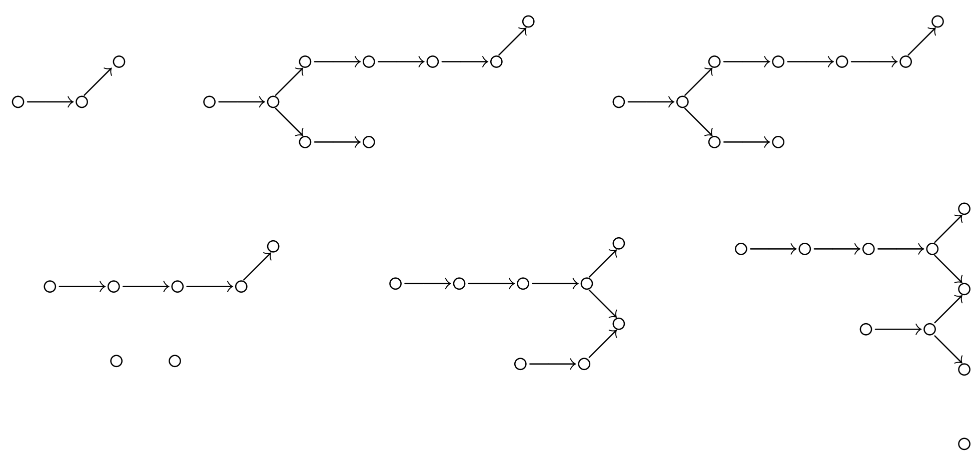


Figure 12.

. Similarly, in enumerating the increasing subsets in each of the above partially ordered sets, we may further simplify the situation if necessary. We eventually get the following result.

i	0	1	2	3	4	5	6
$\|\Gamma_i\|$	6	4	13	13	14	15	40

Thus the total number of increasing subsets of $\Phi^+(F_4)$ is equal to

$$\sum_{i=0}^{6} |\Gamma_i| = 105.$$

This is also the number of all the $\oplus$-sign types indexed by $\Phi^+(F_4)$ by Theorems 1.4 and 2.5.

3.6 By using the same method as above, we can perform a similar enumeration work in the case when Φ^+ has exceptional type. The results are summed up below.

Theorem. *Let $\mu(X)$ be the number of all the $\oplus$-sign types indexed by the positive root system of type X, where $X \in \{E_6,\ E_7,\ E_8,\ F_4,\ G_2\}$. Then we have*

$$\begin{cases} \mu(E_6) = 2^6 \cdot 13; \\ \mu(E_7) = 2^6 \cdot 5 \cdot 13; \\ \mu(E_8) = 2^3 \cdot 3 \cdot 5 \cdot 11 \cdot 19; \\ \mu(F_4) = 105; \\ \mu(G_2) = 8. \end{cases}$$

Remark 3.7. One may ask if there exists any simple relation of the number $\mu(X)$ with some known constants on the corresponding root system in the case where X is any exceptional type. If the answer is affirmative, then could one give a uniform expression of the number $\mu(X)$ for all the types X mentioned in this paper?

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