THE PARTIAL ORDER ON TWO-SIDED CELLS OF CERTAIN AFFINE WEYL GROUPS

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In their famous paper [6], Kazhdan and Lusztig introduced the concept of equivalence classes such as left cell, right cell and two-sided cell in a Coxeter group W. We inherit the notations $\underset{L}{\leqslant}$, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$, $\underset{L}{\sim}$, $\underset{R}{\sim}$ and $\underset{LR}{\sim}$ in [6]. Thus $w \underset{LR}{\sim} y$ (resp. $w \underset{L}{\sim} y$, resp. $w \underset{R}{\sim} y$) means that the elements $w, y \in W$ are in the same two-sided cell (resp. left cell, resp. right cell) of W, etc. Concerning an affine Weyl group W_a, Lusztig showed that the set $\mathrm{Cell}(W_a)$ of two-sided cells of W_a is in a natural 1-1 correspondence with the set $\mathfrak{U}(G)$ of unipotent classes in the corresponding algebraic group G [11]. We know that $\mathrm{Cell}(W_a)$ is a poset under the relation $\underset{LR}{\leqslant}$. Also, $\mathfrak{U}(G)$ is a poset under the relation: $\mathbf{v} \leq \mathbf{u}$ in $\mathfrak{U}(G) \Longleftrightarrow \mathbf{u} \subset \overline{\mathbf{v}}$, where $\overline{\mathbf{v}}$ is the closure of the conjugacy class $\mathbf{v}$ in the variety of unipotent elements of G. Under the Lusztig's correspondence, the two-sided cell $c = \{\mathbf{1}_{W_a}\} \subset W_a$ is associated to the regular unipotent class of G, and the lowest two-sided cell $W_{(v)}$ (see [11]) of W_a is associated to the trivial class $\{\mathbf{1}_G\} \subset G$. Thus it is natural to formulate the following conjecture which was suggested by Lusztig (See [8, Conjecture D]) .

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Conjecture A. *There exists an order-preserving bijection between the set Cell(W_a) of two-sided cells of W_a and the set $\mathfrak{U}(G)$ of unipotent conjugacy classes of the corresponding algebraic group G.*

The above conjecture has been verified in the cases when W_a has rank ≤ 3 (see [1], [3] or by directly checking). In the present paper, we shall give an affirmative answer of Conjecture A in the cases when W_a is of type $\widetilde{A}_{n-1}$ and when W_a has rank 4. That is, we have

Theorem B. *If W_a is an affine Weyl group which has either type $\widetilde{A}_{n-1}$, $n > 1$, or rank ≤ 4, then Conjecture A holds in W_a.*

The proof of Theorem B is based mainly on two of our results, i.e. Theorem 1.11 and Proposition 1.12 and on the knowledge of the sets $T(\Omega)$ (defined in 1.10) for all the two-sided cells Ω of the concerned affine Weyl groups.

The content of the paper is organized as follows. In section 1, we establish two results concerning the partially-ordered relation on the two-sided cells of an affine Weyl group, i.e. Theorem 1.11 and Proposition 1.12. They are crucial in the proof of our main result, Theorem B. Then we show Theorem B in the type $\widetilde{A}_{n-1}$ $(n > 1)$ case in section 2 and in the rank 4 cases in section 3. Finally, in section 4, we make some comments on the set $T(\Omega)$, where Ω is a two-sided cell of an affine Weyl group.

§1. Some results on the poset Cell(W_a).

1.1. Let $W = (W, S)$ be a Coxeter system, i.e. W is a Coxeter group with S its Coxeter generator set. We denote by ℓ the length function of W. Let $\leq$ be the Bruhat order on W. To any $w \in W$, we associate two subsets of S: $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

Let $A = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. Then the Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ with respect to W is by definition an associative algebra over A with an A-basis $\{T_w \mid w \in W\}$ whose multiplication rule is given by

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'); \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S. \end{cases}$$

1.2. For $w \in W$, define $C_w = \sum_{y \leq w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y$, where the $P_{y,w}$'s are Kazhdan-Lusztig polynomials. It is known that $\{C_w \mid w \in W\}$ also forms an A-basis of $\mathcal{H}$. Thus for any $x, y \in W$, we can write

$$C_x C_y = \sum_z h_{x,y,z} C_z, \qquad \text{with } h_{x,y,z} \in A.$$

In [8], Lusztig defined a function $a : W \longrightarrow \mathbb{N}$ such that for any $z \in W$,

$$u^{a(z)} h_{x,y,z} \in \mathbb{Z}[u], \qquad \text{for any } x, y \in W,$$
$$u^{a(z)-1} h_{x,y,z} \notin \mathbb{Z}[u], \qquad \text{for some } x, y \in W.$$

We state the following property for a Coxeter system (W, S):
$(*)$ There exists an integer $N \geq 0$ such that $u^N h_{x,y,z} \in \mathbb{Z}[u]$ for all $x, y, z \in W$.

Obviously, any finite Coxeter group satisfies the property $(*)$. Lusztig showed that the property $(*)$ also holds for any affine Weyl group (see [8]).

1.3. Kazhdan-Lusztig polynomials $P_{y,w}$, $y, w \in W$, satisfy the properties that $P_{w,w} = 1$, $P_{y,w} = 0$ if $y \not\leq w$, and $\deg P_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$ if $y < w$. We denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$ when $\ell(y) \leq \ell(w)$. We write y—w if $\mu(y, w) \neq 0$.

1.4. The following results on the cells of W are well known: If $x \underset{\mathrm{L}}{\leqslant} y$ (resp. $x \underset{\mathrm{R}}{\leqslant} y$) in W, then $\mathcal{R}(x) \supseteq \mathcal{R}(y)$ (resp. $\mathcal{L}(x) \supseteq \mathcal{L}(y)$). In particular, the relation $x \underset{\mathrm{L}}{\sim} y$ (resp. $x \underset{\mathrm{R}}{\sim} y$) implies $\mathcal{R}(x) = \mathcal{R}(y)$ (resp. $\mathcal{L}(x) = \mathcal{L}(y)$) (see [6]).

1.5. A Coxeter system (W, S) is crystallographic if it arises from some Kac-Moody Lie algebra (or group). In that case, we have that for any $s, t \in S$, the order $o(st)$ of the product st is 1, 2, 3, 4, 6 or ∞. Weyl groups and affine Weyl groups are all crystallographic. The following results were shown by Lusztig and Springer.

Proposition 1.6. *Let (W, S) be a crystallographic Coxeter system. In the following (2) and (3), we further assume (W, S) satisfying the property $(*)$.*
(1) For any $x, y, z \in W$, all the coefficients of the Laurent polynomial $h_{x,y,z}$ are non-negative [8].

(2) The function a is constant on a left (resp. right, resp. two-sided) cell Γ of W (thus we may denote by $a(\Gamma)$ the value $a(w)$ for any $w\in\Gamma$). The relation $y \underset{\mathrm{LR}}{\lnsim} w$ implies the inequality $a(w)<a(y)$ for any $y,w\in W$ [10].
(3) If $x,y\in W$ satisfy $y\underset{\mathrm{L}}{\leqslant} x$ and $a(y)=a(x)$, then $y\underset{\mathrm{L}}{\leqslant} x$ [10].
(4) If $x,y\in W$ satisfy x—y and $x\underset{\mathrm{LR}}{\lnsim} y$, then both relations $x\underset{\mathrm{L}}{\leqslant} y$ and $x\underset{\mathrm{R}}{\leqslant} y$ hold [19].

Lemma 1.7. *Let (W,S) be a crystallographic Coxeter system with the property $(*)$. Let $z,z'\in W$ be such that z—z' and let $s\in S$ be such that $s\in\mathcal{R}(z')\setminus\mathcal{R}(z)$. Let $y_1\in W$ be such that $y_1\underset{\mathrm{L}}{\leqslant} z'$. Then there exists $x_1\in W$ such that $x_1\underset{\mathrm{L}}{\leqslant} z$ and $x_1\underset{\mathrm{R}}{\sim} y_1$.*

Proof. Let $y=y_1^{-1}$. Then $y\underset{\mathrm{R}}{\leqslant}(z')^{-1}$. So C_y appears with non-zero coefficient in $C_{(z')^{-1}}C_x$ and hence in $C_sC_{z^{-1}}C_x$ for some $x\in W$ by Proposition 1.6, (1). Again by Proposition 1.6, (1), this implies that there exists $x'\in W$ such that $C_{x'}$ appears with non-zero coefficient in $C_{z^{-1}}C_x$ and that C_y appears with non-zero coefficient in $C_sC_{x'}$. We have $x'\underset{\mathrm{R}}{\leqslant} z^{-1}$ and hence $(x')^{-1}\underset{\mathrm{L}}{\leqslant} z$.

If $sx'<x'$, then $y=x'$. We can take $x_1=(x')^{-1}=y_1$. Now assume $sx'>x'$. Then y—x' and $y\underset{\mathrm{L}}{\leqslant} x'$. If $y\underset{\mathrm{L}}{\sim} x'$, then $y_1\underset{\mathrm{R}}{\sim}(x')^{-1}\underset{\mathrm{L}}{\leqslant} z$. We can take $x_1=(x')^{-1}$. If $y\underset{\mathrm{L}}{\lnsim} x'$, then by Proposition 1.6, (2) and (3) we have $y\underset{\mathrm{LR}}{\lnsim} x'$. By Proposition 1.6, (4), this implies $y\underset{\mathrm{R}}{\leqslant} x'$ and hence $y_1\underset{\mathrm{L}}{\leqslant}(x')^{-1}\underset{\mathrm{L}}{\leqslant} z$. We can take $x_1=y_1$. □

Theorem 1.8. *Let (W,S) be a crystallographic Coxeter system with the property $(*)$. Let $x,y\in W$ be such that $x\underset{\mathrm{LR}}{\leqslant} y$. Then there exists $z\in W$ such that $z\underset{\mathrm{R}}{\sim} x$ and $z\underset{\mathrm{L}}{\leqslant} y$.*

Proof. Consider all the sequences of elements $x_0=x,x_1,\cdots,x_r=y$ from x to y such that the relation $x_{i-1}\underset{\mathrm{LR}}{\lnsim} x_i$ holds for every i, $1\le i\le r$. Then by Proposition 1.6, (2), we see that the upper-boundary of the lengths r of these sequences is finite, which must be less or equal to $a(x)-a(y)$. So we can assume that $x\underset{\mathrm{LR}}{\not\sim} y$ and that the result is true when x,y are replaced by x,y' with $x\underset{\mathrm{LR}}{\leqslant} y'\underset{\mathrm{LR}}{\leqslant} y$ and $y'\underset{\mathrm{LR}}{\not\sim} y$. From the definition of the relation $x\underset{\mathrm{LR}}{\leqslant} y$, we see that there exist v, v' such that $x \underset{LR}{\leqslant} v \underset{LR}{\lesssim} v' \underset{LR}{\sim} y$ and that either $v \underset{L}{\leqslant} v'$ or $v \underset{R}{\leqslant} v'$ holds. Replacing if necessary v, v' by v^{-1}, $(v')^{-1}$, we can assume that $v \underset{L}{\leqslant} v'$. Let $w \in W_a$ be such that $v' \underset{R}{\sim} w$, $w \underset{L}{\sim} y$ (Such an element w always exists, since in a two-sided cell of W_a, the intersection of a left cell with a right cell is non-empty). Since $w \underset{R}{\sim} v'$, we can find $v' = v'_1, \cdots, v'_r = w$ such that for every i, $1 < i \leq r$, v'_{i-1}—v'_i and $\mathcal{R}(v'_{i-1}) \nsubseteq \mathcal{R}(v'_i)$. Applying repeatedly Lemma 1.7, we find a sequence $v = v_1, \cdots, v_r$ such that $v_i \underset{L}{\leqslant} v'_i$ $(1 \leq i \leq r)$, $v_1 \underset{R}{\sim} v_2 \underset{R}{\sim} \cdots \underset{R}{\sim} v_r$. In particular, we have $v \underset{R}{\sim} v_r \underset{L}{\leqslant} w$. Since $w \underset{L}{\sim} y$, we have $x \underset{LR}{\leqslant} v \underset{R}{\sim} v_r \underset{L}{\leqslant} y$. Since $y \underset{LR}{\nsim} v_r$, the theorem is known to hold for (x, v_r) instead of (x, y). Thus there exists $z \in W_a$ such that $x \underset{R}{\sim} z$, $z \underset{L}{\leqslant} v_r$ and the theorem follows. □

Remark 1.9. Lemma 1.7 and Theorem 1.8 are the counterparts of [12, Lemma 3.1 and Theorem 3.2]. So the proofs of the former are analogous to those of the latter. But in our proof of Theorem 1.8, we don't assume that the number of two-sided cells of (W, S) is finite. So the conclusion of Theorem 1.8 is valid for any crystallographic group with the property $(*)$, not only for an affine Weyl group. By applying the argument analogous to ours, we can show that the conclusion of [12, Theorem 3.2] is also valid for any crystallographic group with the property $(*)$.

1.10. Let $\mathbf{S}$ be the set of all the subsets of S. For any $\mathbf{I}, \mathbf{J} \subseteq \mathbf{S}$, write $\mathbf{I} \ll \mathbf{J}$, if for any $I \in \mathbf{I}$, there exists some $J \in \mathbf{J}$ such that $J \supseteq I$.

For any two-sided cell Ω of W, let $L(\Omega) = \{I \subset S \mid I = \mathcal{L}(w)$ for some $w \in \Omega\}$ and $R(\Omega) = \{I \subset S \mid I = \mathcal{R}(w)$ for some $w \in \Omega\}$. Then by the fact that $x \underset{LR}{\sim} x^{-1}$ for any $x \in W$, we have $L(\Omega) = R(\Omega)$ and hence we can denote this set by $T(\Omega)$.

Theorem 1.11. *Assume that (W_a, S) is an affine Weyl group. Assume that Ω and Ω' are two-sided cells of W_a with $\Omega' \underset{LR}{\leqslant} \Omega$. Then $T(\Omega) \ll T(\Omega')$.*

Proof. This follows from 1.4 and Theorem 1.8. □

This theorem will be useful in checking whether or not some two-sided cells Ω and Ω' of W_a have the relation $\Omega' \underset{LR}{\leqslant} \Omega$. Now we give one more result on this respect. Given $I \subseteq S$. If the subgroup W_I of W_a generated by I is finite, then we denote by w_I the longest element of W_I.

Proposition 1.12. *Let Ω and Ω' be two-sided cells of an affine Weyl group (W_a, S). Assume that $w_I \in \Omega$ for some $I \subseteq S$. Then $\Omega' \underset{\mathrm{LR}}{\leqslant} \Omega$ if and only if there exists some $J \in T(\Omega')$ with $J \supseteq I$.*

Proof. The implication " $\Longrightarrow$ " follows by Theorem 1.11. Now we show the implication " $\Longleftarrow$ ". By the assumption, there exists some element $w \in \Omega'$ with $\mathcal{L}(w) = J$, $J \supseteq I$. Then we have an expression $w = w_I \cdot x$ for some $x \in W$ with $\ell(w) = \ell(w_I) + \ell(x)$. This implies that $w \underset{\mathrm{LR}}{\leqslant} w_I$ and hence $\Omega' \underset{\mathrm{LR}}{\leqslant} \Omega$. $\square$

§2. The proof of Theorem B in type $\widetilde{A}_{n-1}$ case.

In the present section, we always assume that the Coxeter system (W_a, S) has type $\widetilde{A}_{n-1}$.

2.1. A partition of n ($n \in \mathbb{N}$) is any sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ of non-negative integers in decreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^{r} \lambda_i = n$. We shall not distinguish between two such sequences which differ only by a string of zeros at the end. Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part: $\lambda = (1^{m_1} 2^{m_2} \cdots r^{m_r} \cdots)$ means that exactly m_i of the parts of λ equal to i. Let Λ_n be the set of all the partitions of n.

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_t) \in \Lambda_n$, we write $\lambda \leq \mu$, if for any $i \geq 1$, the inequality $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ holds.

We write $\lambda \lessdot \mu$, if $\lambda < \mu$, and there does not exist any $\nu \in \Lambda_n$ satisfying $\lambda \lneqq \nu \lneqq \mu$. It is well known that $\lambda \lessdot \mu$ if and only if there exist two integers i, j, $j > i \geq 1$, satisfying the following conditions.

(1) $\lambda_h = \mu_h$ for any $h \geq 1$ with $h \neq i, j$.

(2) $\lambda_i = \mu_i - 1$, $\lambda_j = \mu_j + 1$.

(3) $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{j-1} \geq \lambda_j$.

For $\lambda, \mu \in \Lambda_n$, we say that $\mu = (\mu_1, \mu_2, \cdots, \mu_t)$ is conjugate to $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, if for any i, $1 \leq i \leq t$, μ_i is equal to the number of parts λ_j with $\lambda_j \geq i$. In this case, we denote μ by λ'. In general, we have that $(\lambda')' = \lambda$ and that $\lambda \leq \mu$ if and only if $\mu' \leq \lambda'$.

2.2. Let $G = \mathrm{GL}\,(n, \mathbb{C})$ be the general linear group over the complex field $\mathbb{C}$ of rank n. Then a unipotent conjugacy class $\mathbf{u}$ of G can be parametrized by a partition $\lambda \in \Lambda_n$ with $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_r)$ such that λ'_1, λ'_2, $\cdots$, λ'_r are the sizes of the Jordan blocks of a standard form of the class $\mathbf{u}$ (counting multiplicities). We denote such a class $\mathbf{u}$ by $\mathbf{u}_\lambda$.

It is well known that for $\lambda, \mu \in \Lambda_n$, we have $\mathbf{u}_\lambda \leq \mathbf{u}_\mu$ if and only if $\lambda \leq \mu$ [18].

2.3. Let $S = \{s_i \mid 0 \leq i < n\}$ be a Coxeter generator set of the affine Weyl group W_a such that the order $o(s_i s_{i+1})$ of the product $s_i s_{i+1}$ is 3 for $0 \leq i < n$ with the convention that $s_{i+n} = s_i$, $i \in \mathbb{Z}$. It is known that the group W_a can be identified with the following permutation group on the integer set $\mathbb{Z}$:

$$\mathcal{A}_n = \left\{ w : \mathbb{Z} \mapsto \mathbb{Z} \,\middle|\, (i+n)w = (i) + n, \quad \text{for all} \quad i \in \mathbb{Z};\ \sum_{i=1}^{n} (i)w = \sum_{i=1}^{n} i \right\}.$$

where the Coxeter generator set $S = \{s_t \mid 0 \leq t \leq n-1\}$ of $\mathcal{A}_n$ is given by

$$(i)s_t = \begin{cases} i, & \text{if } i \not\equiv t, t+1 \pmod{n}, \\ i+1, & \text{if } i \equiv t \pmod{n}, \\ i-1, & \text{if } i \equiv t+1 \pmod{n}. \end{cases}$$

To each element $w \in \mathcal{A}_n$, we associate a sequence of integers $d_1 \leq d_2 \leq \cdots \leq d_t = n$ as follows.

$$d_k = \max \left\{ |X| \,\middle|\, X = \bigcup_{i=1}^{k} X_i \subset \mathbb{Z};\ u \not\equiv v (mod\, n),\ \text{for all } u \neq v \text{ in } X; \right.$$
$$\left. \text{and } u < v \text{ in some } X_i \text{ implies } (u)w > (v)w \right\}$$

where the notation $|X|$ stands for the cardinality of a set X. C. Greene showed that $(d_1, d_2 - d_1, d_3 - d_2, \cdots, d_t - d_{t-1})$ is a partition of n (see [5]). We denote it by $\sigma(w)$. This defines a map $\sigma: \mathcal{A}_n \longrightarrow \Lambda_n$. It is known that the map σ induces a bijection from the set of two-sided cells of $\mathcal{A}_n$ to the set Λ_n (see [13]).

2.4. To each $J \subset S$, we associate a partition $\pi(J)$ of n as below. Decompose the set J into a disjoint union $J = J_1 \cup J_2 \cup \cdots \cup J_r$ satisfying the following conditions.

(1) $|J_1| \geq |J_2| \geq \cdots \geq |J_r| > 0$.

(2) For any i, $1 \leq i \leq r$, we have $J_i = \{s_{k_i+j} \mid 1 \leq j \leq h_i\}$ for some $k_i, h_i \in \mathbb{Z}$ with $h_i \geq 1$ and $s_{k_i}, s_{k_i+h_i+1} \notin J$.

Let $\lambda_i = h_i+1$ for $1 \leq i \leq r$ and $\lambda_k = 1$ for $r < k \leq t = n+r-\sum_{j=1}^{r} \lambda_j$. Then we define $\pi(J) = (\lambda_1, \cdots, \lambda_t)$. Clearly, we have $\pi(J) \in \Lambda_n$. Also, we see that $\pi(J)$ only dependent on J itself, not on the choice of the decomposition of J. So the partition $\pi(J)$ is well defined.

Lemma. *If $J \subseteq I$ in S, then $\pi(J) \leq \pi(I)$.*

Proof. We may assume $J \subsetneq I$ since otherwise there is nothing to prove. We need only to deal with the case of $|J| = |I| - 1$. Then the result can be shown easily by comparing standard decompositions of I and J. □

2.5. Recall the definition of the map $\sigma : \mathcal{A}_n \longrightarrow \Lambda_n$ given above. It is easily seen that the equality $\sigma(w_J) = \pi(J)$ holds for any $J \subset S$. Moreover, we have the following result by [13].

Lemma. *Let $\lambda \in \Lambda_n$.*

(1) There exists some $J \subset S$ such that $\pi(J) = \lambda$.

(2) The two-sided cell $\sigma^{-1}(\lambda)$ of $\mathcal{A}_n$ contains all the elements of the form w_J, $J \subset S$, with $\pi(J) = \lambda$.

(3) For any $I \in T(\sigma^{-1}(\lambda))$, we have $\pi(I) \leq \lambda$.

Lemma 2.6. *Let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be standard decompositions of $I, J \subset S$ such that $I_1 = \{s_h \mid 1 \leq h \leq k_1\}$, $I_2 = \{s_{k_1+1+h} \mid 1 \leq h \leq k_2\}$,*

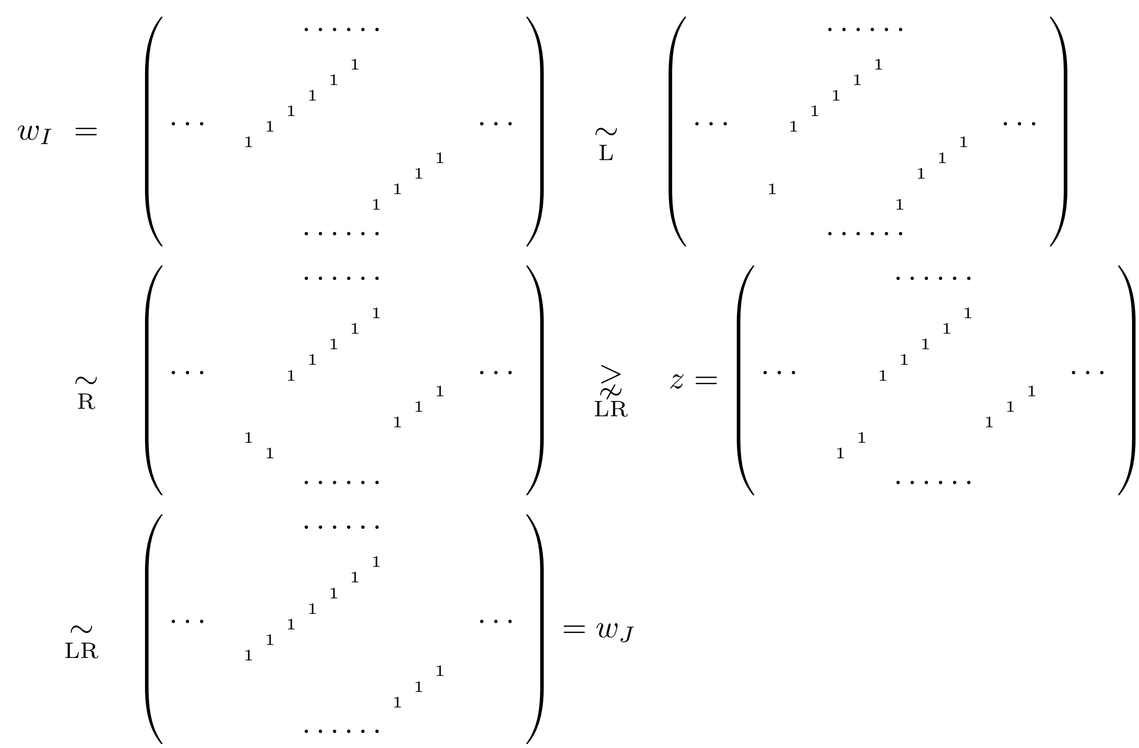
$J_1 = I_1 \cup \{s_{k_1+1}\}$ and $J_2 = I_2 \setminus \{s_{k_1+2}\}$. Then $w_J \underset{\mathrm{LR}}{\lnsim} w_I$.

Proof.

$$\begin{aligned} w_I &\underset{\mathrm{L}}{\sim} s_{k_1+k_2} s_{k_1+k_2-1} \cdots s_{k_1+2} s_{k_1+1} w_I \\ &\underset{\mathrm{R}}{\sim} s_{k_1+k_2} s_{k_1+k_2-1} \cdots s_{k_1+2} s_{k_1+1} w_I s_{k_1+1} s_{k_1} \cdots s_2 \\ &\underset{\mathrm{LR}}{\gnsim} s_{k_1+k_2+1} s_{k_1+k_2} \cdots s_{k_1+2} s_{k_1+1} w_I s_{k_1+1} s_{k_1} \cdots s_2 \\ &= s_{k_1+k_2} s_{k_1+k_2-1} \cdots s_{k_1+2} s_{k_1+1} w_I s_{k_1+1} s_{k_1} \cdots s_2 s_1. \end{aligned}$$

Denote the last element of the above by z. Then we have $\sigma(z) = \sigma(w_J)$ and hence $w_I \underset{\mathrm{LR}}{\gnsim} z \underset{\mathrm{LR}}{\sim} w_J$. □

To understand the above proof, we illustrate it by an example with $k_1 = 5$ and $k_2 = 3$ as below. Notice that an element $w \in \mathcal{A}_n$ can be identified with a $\mathbb{Z} \times \mathbb{Z}$ matrix whose entries are all zero except for those in the positions $\{(i, (i)w) \mid i \in \mathbb{Z}\}$ which are 1 (see [13]).



Corollary 2.7. *Assume that $I, J \subset S$ satisfy $\pi(I) \lessdot \pi(J)$. Then $w_J \underset{\mathrm{LR}}{\lnsim} w_I$.*

Proof. Let $I = I_1 \cup \cdots \cup I_r$ and $J = J_1 \cup \cdots \cup J_r$ be standard decompositions of I and J. By Lemma 2.5, we may assume that there exists some i, j, $1 \leq i < j \leq r$, such that $I_h = J_h$ for any $h \in [1, r]$, $h \neq i, j$, and $I_i = \{s_h \mid 1 \leq h \leq k_1\}$, $I_j = \{s_{k_1+1+h} \mid 1 \leq h \leq k_2\}$, $J_i = I_i \cup \{s_{k_1+1}\}$, $J_j = I_j \setminus \{s_{k_1+2}\}$. Then by Lemma 2.6, we have $w_{I_i \cup I_j} \underset{LR}{\gtrsim} w_{J_i \cup J_j}$. This implies immediately that $w_J \underset{LR}{\lesssim} w_I$. □

Lemma 2.8. *Let Ω and Ω' be two-sided cells of W_a with $\Omega' \underset{LR}{\leqslant} \Omega$. Then $\sigma(\Omega') \geq \sigma(\Omega)$.*

Proof. By Lemma 2.5, there exists some $I \subset S$ such that $w_I \in \Omega$ and $\pi(I) = \sigma(\Omega)$. Also, we have $T(\Omega) \ll T(\Omega')$ by Theorem 1.11. This implies that there exists some $J \in T(\Omega')$ with $J \supseteq I$. By Lemmas 2.4 and 2.5, we have $\sigma(\Omega') \geq \pi(J) \geq \pi(I) = \sigma(\Omega)$. Our proof is completed. □

2.9. *Proof of Theorem B in type $\widetilde{A}_{n-1}$ case.* By Lemma 2.5, it is equivalent to show the following assertions. Let $\lambda, \mu \in \Lambda_n$.

(1) If $\lambda \leq \mu$, then we have $w_J \underset{LR}{\leqslant} w_I$ for some (and hence all) $I \in \pi^{-1}(\lambda)$ and $J \in \pi^{-1}(\mu)$.

(2) If $\sigma^{-1}(\lambda) \underset{LR}{\leqslant} \sigma^{-1}(\mu)$, then $\lambda \geq \mu$.

Assertion (2) is just the conclusion of Lemma 2.8. For assertion (1), we may assume $\lambda \neq \mu$ since otherwise there is nothing to do. On the other hand, for any $\lambda < \mu$ in Λ_n, there exists a sequence of partitions $\nu_0 = \lambda$, ν_2, $\cdots$, $\nu_p = \mu$ in Λ_n such that the relation $\nu_{i-1} \lessdot \nu_i$ holds for every i, $1 \leq i \leq p$. Thus in the proof of this assertion, we may assume $\lambda \lessdot \mu$ without loss of generality. But in this case, assertion (1) follows directly from Lemma 2.5 and Corollary 2.7. □

§3. The proof of Theorem B in rank 4 cases..

Besides $\widetilde{A}_4$, there are four other types for the affine Weyl groups of rank 4, i.e. $\widetilde{B}_4$, $\widetilde{C}_4$, $\widetilde{D}_4$ and $\widetilde{F}_4$. We shall prove Theorem B in these four cases in the present section.

The unipotent classes of a classical algebraic group G can be parametrized by partitions of a certain integer. We denote by $\mathbf{u}_\lambda$ the unipotent class of G parametrized by a partition λ. We have $\mathbf{u}_\lambda \le \mathbf{u}_\mu$ if and only if $\lambda \ge \mu$.

3.1. The unipotent classes of the projective symplectic group $PSp_8(\mathbb{C})$ (which is of type C_4) are parametrized by partitions of 8 in which each odd part occurs with even multiplicity. These partitions of 8 are (8), (62), (61^2), (4^2), (42^2), (421^2), (41^4), (3^22), (3^21^2), (2^4), (2^31^2), (2^21^4), (21^6), (1^8). Thus the unipotent classes of $PSp_8(\mathbb{C})$ have the partial order as in Figure 1, (a), where a partition λ joins with a partition μ from top to bottom by an edge if and only if $\lambda \ge \mu$ if and only if $\mathbf{u}_\lambda \le \mathbf{u}_\mu$ (the same for the subsequent diagrams).

3.2. The unipotent classes of the special orthogonal group $SO_9(\mathbb{C})$ (which has type B_4) are parametrized by partitions of 9 in which each even part occurs with even multiplicity. These partitions of 9 are (9), (71^2), (531),

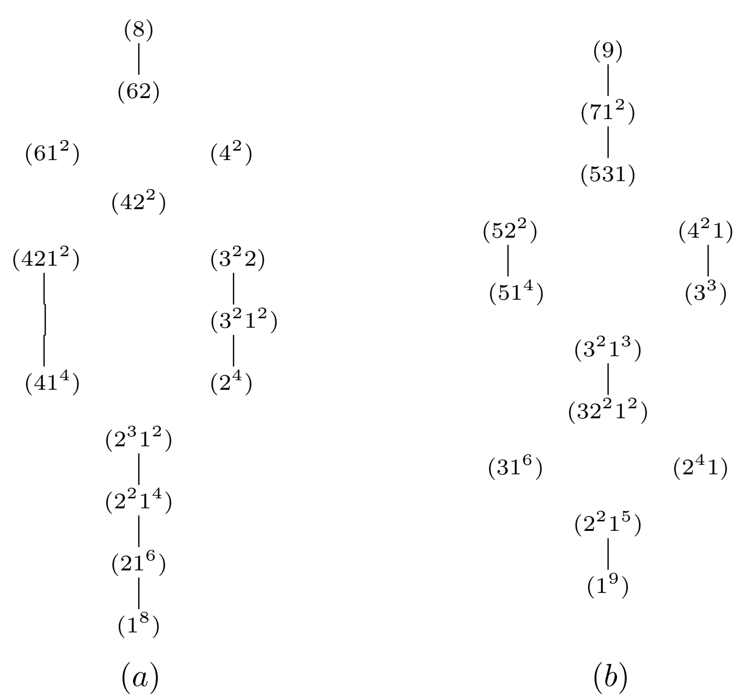


Figure 1.

(52^2), (51^4), (4^21), (3^3), (3^21^3), (32^21^2), (31^6), (2^41), (2^21^5), (1^9). Thus the unipotent classes of $SO_9(\mathbb{C})$ have the partial order as in Figure 1, (b).

3.3. The unipotent classes of the projective special orthogonal group $PSO_8(\mathbb{C})$ (which has type D_4) are parametrized by partitions of 8 in which each even part occurs with even multiplicity, except that each such partition in which all parts are even gives rise to two unipotent classes. These partitions are (71), (53), (51^3), (4^2), (3^21^2), (32^21), (31^5), (2^4), (2^21^4), (1^8). Thus the unipotent classes of $PSO_8(\mathbb{C})$ have the partial order as in Figure 2, (a).

3.4. The partial ordering on the unipotent classes of the reductive complex algebraic group $G(F_4, \mathbb{C})$ of type F_4, using Carter's notation (see [2]), is as in Figure 2, (b).

3.5. According to Lusztig, the unipotent classes of the algebraic group $PSp_8(\mathbb{C})$ (resp. $SO_9(\mathbb{C})$, resp. $PSO_8(\mathbb{C})$, resp. $G(F_4, \mathbb{C})$) are in 1-1 correspondence with the two-sided cells of the affine Weyl group of type $\widetilde{B}_4$

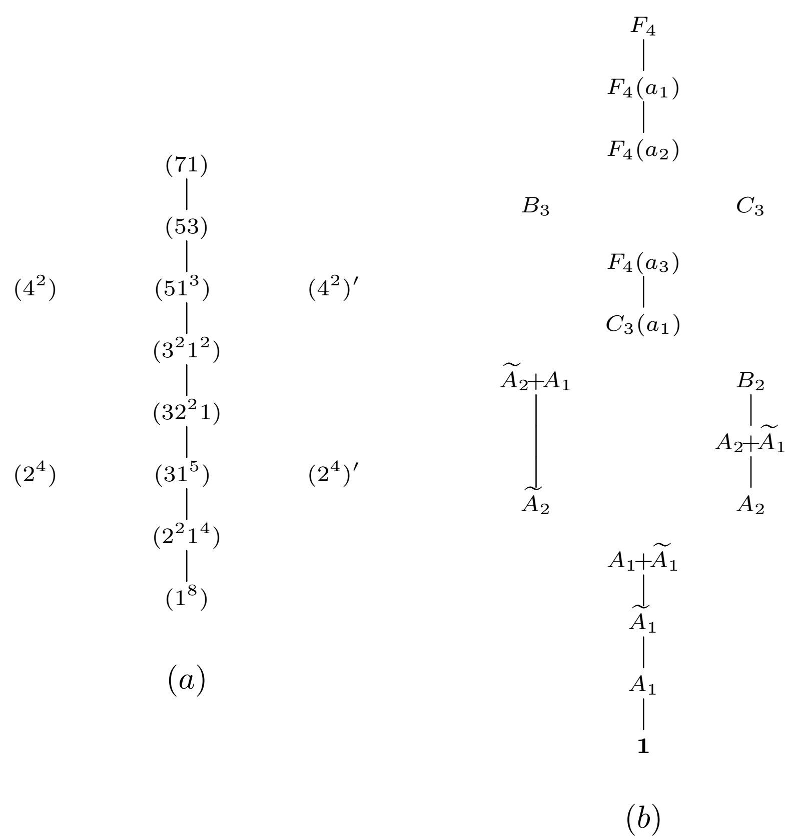


Figure 2.

(resp. $\widetilde{C}_4$, resp. $\widetilde{D}_4$, resp. $\widetilde{F}_4$). The following tables give these correspondences explicitly, where the indexes i of the simple reflections s_i of the affine Weyl groups $W_a(\widetilde{B}_4)$, $W_a(\widetilde{C}_4)$, $W_a(\widetilde{D}_4)$ and $W_a(\widetilde{F}_4)$ are compatible with the respective extended Dynkin diagrams (see Figure 3). We denote a reflection s_i simply by i for brevity. A unipotent class $\mathbf{u}$ of G is represented by its corresponding partition when G is of classical type (i.e. of type B_4, C_4 or D_4), or by a Carter's notation when G has type F_4.The corresponding two-sided cell $c(\mathbf{u})$ is represented by a subset $J \subset S$ such that $w_J \in c(\mathbf{u})$, but with two exceptions: to the two-sided cell Ω in $W_a(\widetilde{D}_4)$ with $a(\Omega) = 7$, we associate an element $w = 121321432 \in \Omega$; to the two-sided cell Ω' in $W_a(\widetilde{F}_4)$ with $a(\Omega') = 13$, we associate an element $z = 1213213234321324321 \in \Omega'$.

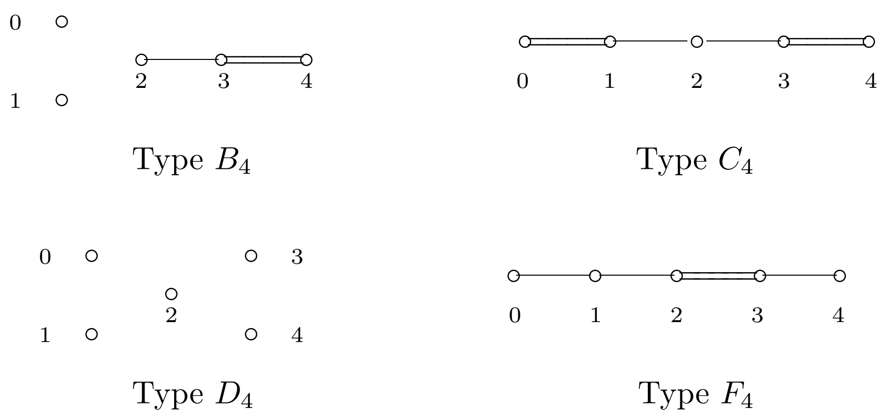


Figure 3.

(1) The correspondence between the unipotent classes of $PSp_8(\mathbb{C})$ and the two-sided cells of $W_a(\widetilde{B}_4)$:

unip. class $\mathbf{u}$	(8)	(62)	(61^2)	(4^2)	(42^2)	(421^2)	(3^22)
$c(\mathbf{u})$	$\emptyset$	1	1,3	0,1	0,1,3	3,4	1,2,4
unip. class $\mathbf{u}$	(3^21^2)	(41^4)	(2^4)	(2^31^2)	(2^21^4)	(21^6)	(1^8)
$c(\mathbf{u})$	1,3,4	0,1,2	0,1,3,4	0,1,2,4	2,3,4	0,1,2,3	1,2,3,4

(2) The correspondence between the unipotent classes of $SO_9(\mathbb{C})$ and the two-sided cells of $W_a(\widetilde{C}_4)$:

unip. class $\mathbf{u}$	(9)	(71^2)	(531)	(52^2)	(4^21)	(51^4)	(3^3)
$c(\mathbf{u})$	$\emptyset$	1	0,4	1,2	0,2,4	0,1	1,2,4
unip. class $\mathbf{u}$	(3^21^3)	(32^21^2)	(31^6)	(2^41)	(2^21^5)	(1^9)	
$c(\mathbf{u})$	0,1,3	1,2,3	0,1,2	0,1,3,4	0,1,2,4	0,1,2,3	

(3) The correspondence between the unipotent classes of $PSO_8(\mathbb{C})$ and the two-sided cells of $W_a(\widetilde{D}_4)$:

unip. class $\mathbf{u}$	(71)	(53)	(4^2)	(51^3)	$(4^2)'$	(3^21^2)
$c(\mathbf{u})$	$\emptyset$	1	0,1	0,3	0,4	0,1,3
unip. class $\mathbf{u}$	(32^21)	(2^4)	(31^5)	$(2^4)'$	(2^21^4)	(1^8)
$c(\mathbf{u})$	0,1,3,4	0,1,2	0,2,3	0,2,4	w	1,2,3,4

(4) The correspondence between the unipotent classes of $G(F_4,\mathbb{C})$ and the two-sided cells of $W_a(\widetilde{F}_4)$:

unip. class $\mathbf{u}$	F_4	$F_4(a_1)$	$F_4(a_2)$	B_3	C_3	$F_4(a_3)$	$C_3(a_1)$	$\widetilde{A}_2{+}A_1$
$c(\mathbf{u})$	$\emptyset$	1	0,2	1,2	0,2,4	2,3	0,2,3	0,1,2
unip. class $\mathbf{u}$	B_2	$A_2{+}\widetilde{A}_1$	$\widetilde{A}_2$	A_2	$A_1{+}\widetilde{A}_1$	$\widetilde{A}_1$	A_1	$\mathbf{1}$
$c(\mathbf{u})$	0,1,3,4	0,1,2,4	2,3,4	1,2,3	0,2,3,4	z	0,1,2,3	1,2,3,4

3.6. *Proof of Theorem B in rank 4 cases.* Let $G \in \{PSp_8(\mathbb{C}), SO_9(\mathbb{C}), PSO_8(\mathbb{C}), G(F_4,\mathbb{C})\}$ and let W_a be the affine Weyl group corresponding to G as above. We need only to prove the following assertions.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are two unipotent classes of G joining by an edge from top to bottom in the above poset diagram, then $c(\mathbf{v}) \underset{LR}{\leqslant} c(\mathbf{u})$.

(2) If $G = PSp_8(\mathbb{C})$, then $c(\mathbf{u}_{41^4}) \underset{LR}{\not\leq} c(\mathbf{u}_{3^22})$.

(3) If $G = SO_9(\mathbb{C})$, then $c(\mathbf{u}_{51^4}) \underset{LR}{\not\leq} c(\mathbf{u}_{4^21})$.

(4) If $G(F_4,\mathbb{C})$, then $c(\mathbf{u}(\widetilde{A}_2)) \underset{LR}{\not\leq} c(\mathbf{u}(B_2))$.

By Theorem 1.11, assertion (2) can be shown by noting that $w_{\{1,2,4\}} \in c(\mathbf{u}_{3^22})$, $\{1,2,4\} \notin T(c(\mathbf{u}_{41^4}))$ and $a(w_I) > 6 = a(c(\mathbf{u}_{41^4}))$ for any $I \subset S$ with $I \supsetneq \{1,2,4\}$ (see [20]); assertion (3) can be shown by noting that $w_{\{0,2,4\}} \in c(\mathbf{u}_{4^21})$, $\{0,2,4\} \notin T(c(\mathbf{u}_{51^4}))$ and $a(w_I) > 4 = a(c(\mathbf{u}_{51^4}))$ for any $I \subset S$ with $I \supsetneq \{0,2,4\}$ (see [16]); assertion (4) follows by noting that $w_{\{0,1,3,4\}} \in c(\mathbf{u}(B_2))$, $\{0,1,3,4\} \notin T(c(\mathbf{u}(\widetilde{A}_2)))$ and that $\{0,1,3,4\}$ is a maximal proper subset of S (see [17]).

On the other hand, assertion (1) follows by Proposition 1.12 and by the results in [15], [16], [17] [20] except for the following cases
(a) W_a has type $\widetilde{D}_4$ and $(\mathbf{u},\mathbf{v}) = (\mathbf{u}_{2^21^4}, \mathbf{u}_{1^8})$.
(b) W_a has type $\widetilde{F}_4$ and $(\mathbf{u},\mathbf{v}) = (\mathbf{u}(\widetilde{A}_1), \mathbf{u}(A_1))$.

In each of the above cases, the two-sided cell $c(\mathbf{u})$ contains no element of the form w_I, $I \subset S$. But in case (a), $c(\mathbf{u}_{1^8})$ is the lowest two-sided cell of $W_a(\widetilde{D}_4)$. So the relation $c(\mathbf{u}_{1^8}) \underset{LR}{\leqslant} c(\mathbf{u}_{2^21^4})$ holds in this case. In case (b), let $x = 1213214321323432132 34$ and $y = x \cdot 4012321$. Then by a result of M. Geck and K. Lux (see [4]), x is an distinguished involution of the two-sided cell $\Omega = c(\mathbf{u}(\widetilde{A}_1))$. We see that the element y can be obtained from x by successively applying right star operations on x (see [13] for the definition of a right star operation) and hence $y \underset{R}{\sim} x$. Let $y' = y \cdot 0$. Then by the fact $\mathcal{R}(y') = \{0,1,2,3\} \not\subseteq \{1,2,3\} = \mathcal{R}(y)$, we see that $a(y') \geq a(w_{\{0,1,2,3\}}) = 16$ and hence $y' \underset{LR}{\lnsim} y \underset{R}{\sim} x$. By a result of Lusztig [11, Theorem 4.8], this implies that the element y' must belong to one of the two-sided cells $\Omega' = c(\mathbf{u}(A_1))$, $\Omega" = c(\mathbf{u}(\mathbf{1}))$ of $W_a(\widetilde{F}_4)$. But we see that there are some zero entries occurring in the alcove form of the element y' and so y' does not belong to the lowest two-sided cell $\Omega"$ of $W_a(\widetilde{F}_4)$ by [14, Theorem 2.4]. This implies $y' \in \Omega'$, i.e. $\Omega' \underset{LR}{\leqslant} \Omega$. The proof is completed. □

§4. Comments.

The proof of Theorem B is heavily relied on the knowledge of the sets $T(\Omega)$ for all the two-sided cells of the concerned affine Weyl groups. Thus it is interesting to give an explicit description of such sets. Comparing with Lemma 2.5, we have an even stronger result which explicitly describes the set $T(\Omega)$ for any two-sided cell Ω of the group $W_a(\widetilde{A}_{n-1})$.

Theorem 4.1. *For any $\lambda \in \Lambda_n$ with $\lambda \neq (1^n)$, we have*

$$T(\sigma^{-1}(\lambda)) = \{I \subset S \mid I \neq \emptyset, \pi(I) \leq \lambda\}.$$

The proof of this result is somewhat lengthy. So we prefer not to include it here and will present it elsewhere. Empiricism encourages us to extend this result to more general cases. Thus we suggest the following

Conjecture 4.2. *Let (W_a, S) be an arbitrary irreducible affine Weyl group. Then for any two-sided cell $\Omega \neq \{\mathbf{1}_{W_a}\}$ of W_a, the set $T(\Omega)$ consists of all the non-empty subset I of S such that the two-sided cell Ω' of W_a containing w_I satisfies the relation $\Omega \underset{\mathrm{LR}}{\leqslant} \Omega'$.*

This conjecture has been supported by all the irreducible affine Weyl groups of ranks not greater than 4. Notice that this conjecture is not valid if the concerned Coxeter system is replaced by a Weyl group W of rank greater than two. The lowest two-sided cell of such a Weyl group provides a counter-example.

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