LEFT CELLS IN CERTAIN COXETER GROUPS

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Dedicated to Professor Hsiao-Fu Duan on His Eightieth Birthday

Abstract. A survey is given on the achievements of KL-cell theory of certain Coxeter groups. Some techniques applied in that theory are introduced. Also, we propose several open problems for further study.

In order to construct the representations of a Coxeter group W and the associative Hecke algebra, D. Kazhdan and G. Lusztig defined certain equivalence classes of W called left, right and two-sided cells. Thus the description of cells of W and the structural study of these cells become interesting and also important in the representation theory of groups and algebras. In the present paper, we shall make a survey on the achievements of studying left cells of W. According to the definition, the description of left cells might involve complicated computation of Kazhdan-Lusztig polynomials and is hard even by a computer when the order of W is getting larger. Thus we shall introduce some methods to simplify our work. They reduce the computation in significant rate so that sometimes we can reach our goal only by hand even when W is in some infinite case. We shall see that the study of cells of W involves some combinatorial techniques and has to invoke some other mathematical theory. We also propose some related open problems for further study.

The content of this paper is organized as below. In section 1, we make a historical review on the cells of a Coxeter group, introduce the definition of cells by Kazhdan and Lusztig and some related concepts. Then we state some results of Lusztig concerning cells of Coxeter groups with properties 2.3, (a), (b), mainly of affine Weyl groups. A survey is given in section 3 on the achievements for the description of left cells of Weyl groups,

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affine Weyl groups and some other finite Coxeter groups. We construct an algorithm in section 4 for finding a representative set of left cells in any given two-sided cell of Coxeter groups with properties 2.3, (a), (b). Several conjectures concerning left cells of W are scattered in sections 2 to 4. Finally in section 5, we state some more conjectures on left cells of affine Weyl groups.

§1. Kazhdan-Lusztig cells.

1.1 The concept of cells originally came from combinatorial theory. Robinson [26] defined a map from the symmetric group S_n to the set of pairs of standard Young tableaux of the same shape and of rank n: $\sigma \longrightarrow (\phi(\sigma), \phi(\sigma^{-1}))$. Then Schensted [27] proved that this map is bijective. This map is called the Robinson-Schensted map. Hence a left cell of S_n is, by definition, the set of elements of S_n corresponding to the set of such pairs (P, Q) with P fixed. A two-sided cell of S_n is defined to be the set of elements of S_n corresponding to the set of such pairs (P, Q) with P, Q of fixed shape. Thus there is 1-1 correspondence between the set of two-sided cells of S_n and the set of partitions of n. This is the prototype of cells which applied to any Coxeter group.

Then Vogan and Joseph defined the concept of left cells in the Weyl group W in terms of primitive ideals in the enveloping algebra of a complex semisimple Lie algebra. For $w \in W$, let J_w be the annihilator of the irreducible module of the enveloping algebra with highest weight $-w\rho - \rho$, where ρ is half the sum of positive roots. Then w, w' are said to be in the same left cell precisely when $J_w = J_{w'}$. This definition of left cells and the corresponding Weyl group representations involves some deep results about the multiplicities of the composition factors of the Verma modules with highest weight $-w\rho-\rho$, [44], [14].

In 1979, Kazhdan and Lusztig [15] gave the definition of cells for an arbitrary Coxeter group. Their definition is elementary but it gives rise not only to the representations of the Coxeter group, but also to that of the corresponding Hecke algebra. This makes possible applications of the results on cells to more general representation theory. On the other hand, the definition of Kazhdan-Lusztig cells coincides with that of Vogan and Joseph in the case of Weyl groups.

We adopt Kazhdan-Lusztig's definition of cells.

1.2 Let (W, S) be a Coxeter group with S its simple reflection set. Let $\leq$ be the Bruhat order of W and let $\ell(x)$ be the length of an element $x \in W$. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u with integer coefficients. There exists an associative algebra $\mathcal{H} = \mathcal{H}(W)$ over $\mathcal{A}$ with $\{T_w \mid w \in W\}$ and $\{C_w \mid w \in W\}$ its two free $\mathcal{A}$-bases. Its multiplication rule in terms of T_w's is given by

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'); \\ (T_s - u^{-1})(T_s + u) = 0, & \text{for } s \in S. \end{cases}$$

The relation between these two bases is as below.

$$C_w = \sum_{y \leq w} u^{\ell(w)-\ell(y)} P_{y,w}(u^{-2}) T_y, \quad \text{for } w \in W.$$

where the $P_{y,w}$'s are Kazhdan-Lusztig polynomials in $\mathbb{Z}[u]$, which satisfy the conditions: $P_{y,w} = 0$ if $y \not\leq w$; $P_{w,w} = 1$ and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$.

1.3 For $y, w \in W$ with $\ell(y) \leq \ell(w)$, we denote by $\mu(y, w)$ or $\mu(w, y)$ the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We say that y and w are joint, written y—w, if $\mu(y, w) \neq 0$. To any $x \in W$, we associate two subsets of S:

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

We have the following relations: for any $x \in W$ and $s \in S$,

$$C_s C_x = \begin{cases} (u^{-1} + u) C_x, & \text{if } s \in \mathcal{L}(x); \\ \sum_{\substack{y \text{---} x \\ sy < y}} \mu(x, y) C_y, & \text{if } s \notin \mathcal{L}(x); \end{cases} \tag{1.3.1}$$

1.4 For any $x, y \in W$, we denote $x \underset{L}{\leq} y$ (resp. $x \underset{R}{\leq} y$), if there exist a sequence of elements $x_0 = x,\ x_1,\ \cdots,\ x_r = y$ in W with some $r \geq 0$ such that for every $i,\ 1 \leq i \leq r$, x_{i-1}—x_i and $\mathcal{L}(x_{i-1}) \not\subseteq \mathcal{L}(x_i)$ (resp. $\mathcal{R}(x_{i-1}) \not\subseteq \mathcal{R}(x_i)$). We denote $x \underset{LR}{\leq} y$, if there exist elements $x_0 = x,\ x_1,\ \cdots,\ x_r = y$ in W such that either $x_{i-1} \underset{L}{\leq} x_i$ or $x_{i-1} \underset{R}{\leq} x_i$ holds for $1 \leq i \leq r$. We write $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$, resp. $x \underset{LR}{\sim} y$), if the relation $x \underset{L}{\leq} y \underset{L}{\leq} x$ (resp. $x \underset{R}{\leq} y \underset{R}{\leq} x$, resp. $x \underset{LR}{\leq} y \underset{LR}{\leq} x$) holds. These are equivalence relations on W, and the equivalence classes of W with respect to $\underset{L}{\sim}$ (resp. $\underset{R}{\sim}$, resp. $\underset{LR}{\sim}$) are called the left (resp. right, resp. two-sided) cells of W. The preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$ and $\underset{LR}{\leq}$ on elements of W induce partial orders on the corresponding cells of W.

1.5 Each cell of W provides a representation of W and of the associated Hecke algebra. Suppose that Γ is a left cell of W. Let $I_{\leq\Gamma}$ (resp. $I_{<\Gamma}$) be the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{C_w \mid w \underset{L}{\leq} x, \text{ for some } x \in \Gamma\}$ (resp. $\{C_w \mid w \underset{L}{\leq} x, \text{ for some } x \in \Gamma \text{ but } w \notin \Gamma\}$). Then $I_{\leq\Gamma}$ (resp. $I_{<\Gamma}$) is a left ideal of $\mathcal{H}$ by 1.2.1. Thus the quotient $I_\Gamma = I_{\leq\Gamma}/I_{<\Gamma}$ is a left $\mathcal{H}$-module. It becomes a W-module if we specialize $u = 1$ for $\mathcal{A}$. So studying cells of W is of significance in the representation theory of groups and algebras. In principle, when W is a finite Coxeter group, all the cells of W could be described explicitly by a finite step of computation. For example, let $W = D_m = \langle s, t \mid o(s) = o(t) = 2,\ o(st) = m\rangle$ be the dihedral group of order $2m$, $m > 1$. Then we have $P_{x,y} = 1$ iff $x \leq y$. This implies that there are three two-sided cells $\{e\}$, $\{w_0\}$ and $\Gamma_s \bigcup \Gamma_t$, where e, w_0 are the identity element and the longest element of D_m, respectively, and

$$\Gamma_s = \{s,\ ts,\ sts,\ \cdots,\ \overbrace{\cdots sts}^{m-1 \text{ factors}}\ \}$$
$$\Gamma_t = \{t,\ ts,\ tst,\ \cdots,\ \overbrace{\cdots tst}^{m-1 \text{ factors}}\ \}.$$

The first two two-sided cells themselves are left cells. The third one is a union of two left cells Γ_s and Γ_t.

Kazhdan and Lusztig gave explicit description of cells for some Coxeter groups of lower ranks [15], D. Alvis described all the left cells of the Coxeter group of type H_4 [1], and K. Takahashi did the same thing for the Weyl group of type F_4 [42]. The most of their results were obtained by making direct computation of the related KL-polynomials. In Alvis's case, more than one million KL-polynomials were calculated by a computer. So their methods are not effective for the Coxeter groups of higher orders or of infinite orders. We must search some new methods, directly or indirectly.

1.6 Given an element $x \in W$, We define the set $M(x)$ of all the elements y such that there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in W with some $r \geq 0$, where for every i, $1 \leq i \leq r$, the conditions $x_{i-1}^{-1}x_i \in S$ and $\mathcal{R}(x_{i-1}) \underset{\not\subseteq}{\not\supseteq} \mathcal{R}(x_i)$ are satisfied. To any $x \in W$, we associate a graph $\mathfrak{M}(x)$ as follows. Its vertex set is $M(x)$. Its edge set consists of all two-element subsets $\{y, z\} \subset M(x)$ with $y^{-1}z \in S$ and $\mathcal{R}(y) \underset{\not\subseteq}{\not\supseteq} \mathcal{R}(z)$. By a path in the graph $\mathfrak{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_t$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any i, $1 \leq i \leq t$. Two elements $x, x' \in W$ are said to have the same *generalized τ-invariant* if for any path $z_0 = x, z_1, \ldots, z_t$ in the graph $\mathfrak{M}(x)$, there exists a path $z'_0 = x', z'_1, \ldots, z'_t$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every i, $0 \leq i \leq t$, and if the same condition holds when interchanging the roles of x and x'.

Note that our definition of a generalized τ-invariant is slightly different from the one given by D. Vogan (see [44]). It is known that if two elements of W are in the same left cell of W then they have the same generalized τ-invariant and that the converse is not true in general, i.e. it might happen that two elements having the same generalized τ-invariant belong to different left cells of W. However, when $W = S_n$, we have

Proposition (see [29]). *$x \underset{L}{\sim} y$ in S_n iff x, y have the same generalized τ-invariant.*

Remark 1.7 Given a Coxeter group W, define a V-left cell of W to be the set of all the elements of W having the same generalized τ-invariant. We use the terminology " a V-left cell " because when W is a Weyl group, it is precisely a left cell defined by D. Vogan (see [44]). As pointed out in the above, a V-left cell of W is a union of some left cells of W defined by Kazhdan and Lusztig. R. Bédard described the V-left cells of all the crystallographic, compact, hyperbolic groups of rank 3 [5].

§2. Some results of Lusztig on cells in certain Coxeter groups.

All the results in the present section are due to Lusztig unless otherwise specified. These results are important not only on the cell theory itself but also on its application to the other mathematical fields. Their proofs are based on the very deep theory of the intersection cohomology, the algebraic K-equivariance and the character sheaves.

2.1 For any $x, y, z \in W$, we define $h_{x,y,z} \in \mathcal{A}$ by

$$C_x C_y = \sum_z h_{x,y,z} C_z. \tag{2.1.1}$$

Define a function $a\colon W \longrightarrow \mathbb{N}$ by setting $a(z)$ to be the smallest integer k satisfying the condition

$$u^k h_{x,y,z} \in \mathbb{Z}[u], \quad \text{for all } x, y \in W.$$

for $z \in W$, where we stipulate $a(z) = \infty$ if no such an integer k exists.

2.2 Two families of Coxeter groups are more interesting to us: Weyl groups and affine Weyl groups. This is because the cell theory of these groups is closely related with the representations of connected reductive algebraic groups over the complex field and over a p-adic field. These Coxeter groups have nice properties: their a-functions are upper-bounded, and the coefficients of their KL-polynomials and their Laurent polynomials $h_{x,y,z}$ are non-negative. These two properties strongly effect the cell structure of these groups.

2.3 Let W be a Coxeter group satisfying the following two properties:

(a) Its a-function is upper bounded;

(b) The coefficients of the KL-polynomials and the Laurent polynomials $h_{x,y,z}$ associating with its elements are non-negative.

Then the following results on W hold.

(1) The function a is constant on each two-sided cell of W. So we may define the a-value $a(\Gamma)$ on a (left, right or two-sided) cell Γ of W by $a(x)$ for any $x \in \Gamma$ [20].

(2) Let $\delta(z) = \deg P_{e,z}$ for $z \in W$, where e is the identity of the group W. Then the inequality $i(z) := \ell(z) - 2\delta(z) - a(z) \geq 0$ holds for any $z \in W$. The set $\mathcal{D}_0 = \{w \in W \mid i(w) = 0\}$ consists of involutions (called distinguished involutions by Lusztig). Each left (resp. right) cell of W contains a unique element of $\mathcal{D}_0$ [21]. Let W' be a standard parabolic subgroup of W. Then a consequence of the above result is that the intersection of any left cell of W with W' is either empty or a single left cell of W'.

2.4 Let W_a be an irreducible affine Weyl group and let G be the connected reductive algebraic group over $\mathbb{C}$ associated to W_a. Then the following result of Lusztig establishes some connection between the set of two-sided cells of W_a and the set of unipotent conjugacy classes of G.

Theorem (see [23] **).** *There exists a bijection* $\mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ *from the set of unipotent conjugacy classes in* G *to the set of two-sided cells in* W_a*. This bijection satisfies the equation* $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$, *where* u *is any element in* $\mathbf{u}$, *and* $\mathfrak{B}_u$ *is the variety of Borel subgroups of* G *containing* u.

This is a result concerning two-sided cells of an affine Weyl group W_a. We also have a result concerning left cells of W_a: $\mathcal{D}_0$ is a finite set of involutions in W_a [21]. This particularly implies that the number of left cells of an affine Weyl group is finite.

2.5 We know the number of two-sided cells of an affine Weyl group but not the number of left cells of such a group in general. This is because we know the number of unipotent conjugacy classes of G (see [6]) but we don't know the number of distinguished involutions of W_a in general. However, Lusztig proposed the following

Conjecture (see [2]). The number of left cells of W_a contained in the two-sided cell corresponding to a unipotent element $u \in G$ is equal to $\sum_i (-1)^i \dim H^i(\mathfrak{B}_u)^{A(u)}$, where $A(u)$ is the group of connected components of the centralizer of u in G, and $H^i(X)$ is the ith étale cohomology space of the variety X with values in the constant sheaf.

Now we assume the above conjecture. Suppose that $\mathbf{u}$ is a unipotent class of G containing a regular unipotent element of a Levi subgroup L. Then the number of left cells in the corresponding two-sided cell $\mathbf{c}(\mathbf{u})$ is equal to the number of left cosets of W with respect to W_L, provided that the centralizer $C_G(u)$ of u in G is connected, where u is any element in $\mathbf{u}$, and W_L is the standard parabolic subgroup of W determined by L.

The above conjecture is supported in all the cases where the numbers of left cells of W_a have been calculated. For example, the author showed it for W_a of type $\widetilde{A}_{n-1}$. In that case, the group $A(u)$ is trivial for any unipotent element $u \in G$, and any unipotent class of G contains a regular unipotent element of some Levi subgroup L (see [6]). On the other hand, the two-sided cells of $W_a(\widetilde{A}_{n-1})$ are in one-to-one correspondence with the partitions of n, where a partition of n is by definition a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ with $\sum_{i=1}^{r} \lambda_i = n$. The number of left cells of $W_a(\widetilde{A}_{n-1})$ in the two-sided cell corresponding to the partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ is equal to $\frac{n!}{\prod_{j=1}^{m} \mu_j!}$, where $\mu_j = \#\{i \mid \lambda_i \geq j\}$ (see [29]).

§3. A survey on the achievements for describing left cells.

3.1 Although it has taken about fifteen years to study, the left cells for the most Coxeter groups are still far from being known. However, some significent progresses have been made on the explicit description of all left cells of certain Coxeter groups satisfying the properties 2.3,(a), (b) , which are listed as follows.

(i) Weyl groups of types A (Kazhdan & Lusztig, see [15]), B, C, D (Barbasch-Vogan, see [3], Garfinkle, Devra, see [13]), E_6 (Tong, see [43]), F_4 (K. Takahashi, see [42]) and G_2 ($\cong D_6$);

(ii) Affine Weyl groups of types $\widetilde{A}$ (Shi, see [29]), $\widetilde{B}_i$ ($i = 3, 4$) (Du and Zhang, [10] [46]), $\widetilde{C}_j$ ($j = 2, 3, 4$) (Lusztig, Bédard and Shi, see [20] [4] [40]), $\widetilde{D}_4$ (Chen and Shi, see [8] [39]) and $\widetilde{G_2}$ (Lusztig, see [20]).

(iii) The Coxeter groups of types H_3 and H_4 (D. Alvis, see [1]), and all the dihedral groups D_m, $m \geq 1$, (see 1.4).

3.2 The description of the left cells of the affine Weyl groups $W_a(\widetilde{A}_{n-1})$ is the most successful work among all in the above list.

A Young tableau is called *quasi-standard* if the numbers in each of its columns increase downwards, and is *standard* if it is quasi-standard and the numbers in each of its rows increase from left to right. Let $\mathfrak{C}_n$ (resp. $\mathfrak{G}_n$) be the set of all quasi-standard (resp. standard) Young tableaux of rank n.

Recall the Robinson-Schensted map ϕ from the symmetric group S_n to the set $\mathfrak{G}_n$ (see 1.1). The author constructed a surjective map from the affine Weyl group $W_a(\widetilde{A}_{n-1})$ to the set $\mathfrak{C}_n$. This map, when restricted on S_n (note that S_n could be regarded as a maximal standard parabolic subgroup of $W_a(\widetilde{A}_{n-1})$), is exactly the Robinson-Schensted map ϕ. So we call this map a generalized Robinson-Schensted map. The significance of this map is that it induces a bijection from the set of all left cells of $W_a(\widetilde{A}_{n-1})$ to the set $\mathfrak{C}_n$. Two left cells are in the same two-sided cell iff the shapes of the corresponding Young tableaux are the same (see [29] [36]).

3.3 Left cells of the group $W_a(\widetilde{A}_{n-1})$ could also be characterized by the generalized τ-invariant (see 1.6): two elements of $W_a(\widetilde{A}_{n-1})$ are in the same left cell iff they have the same generalized τ-invariant (see [29]). This result generalizes Proposition 1.6. Left cells of $W_a(\widetilde{A}_{n-1})$ have even more nice properties (see 3.6).

3.4 Garfinkle defined another kind of generalized Robinson-Schensted map. She associated each signed permutation to a domino Young tableau, by which she got a surjective map from the set of all the elements in the Weyl group of type B_n or D_n to a set of certain standard domino Young tableaux. Then she concluded that the fibres of such a map should be exactly all the left cells of the corresponding Weyl group (see [13]).

3.5 Let W_a be an affine Weyl group. Let Φ be the associated root system with $\{\alpha_j \mid 1 \leq j \leq \ell\}$ a choice of simple root system and $-\alpha_0$ its highest short root. Denote $s_i = s_{\alpha_i}$, $1 \leq i \leq \ell$, the reflection with respect to α_i, and denote $s_0 = T_{-\alpha_0} s_{\alpha_0}$, where $T_{-\alpha_0}$ is the translation $\lambda \mapsto \lambda - \alpha_0$ in the euclidean space spanned by Φ. Let $w \mapsto \bar{w}$ be the natural map $W_a = N \rtimes W \longrightarrow W_a/N \cong W$, where N is the maximal normal abelian subgroup of W_a consisting of all translations and W is the Weyl group of Φ. To each element $w \in W_a$, we can associate a unique Φ-tuple $(k(w,\alpha))_{\alpha\in\Phi}$ over $\mathbb{Z}$ subject to the following conditions:

$$\begin{cases} k(e,\alpha) = 0, & \text{for all } \ \alpha \in \Phi, \\ k(ws_i,\alpha) = k(w,(\alpha)\bar{w}) + k(s_i,\alpha), & \text{for all } \ s_i \in S, \end{cases}$$

where e is the identity of W_a and $k(s_i,\alpha)$ is defined by

$$k(s_i,\alpha) = \begin{cases} 0, & \text{if } \alpha \neq \pm\alpha_i, \\ -1, & \text{if } \alpha = \alpha_i, \\ 1, & \text{if } \alpha = -\alpha_i. \end{cases}$$

$(k(w,\alpha))_{\alpha\in\Phi}$ is called the alcove form of an element $w \in W_a$ (see [30]).

3.6 A Φ-sign type is a Φ-tuple $(X_\alpha)_{\alpha\in\Phi}$ with $X_\alpha \in \{+,-,\bigcirc\}$. We can associate an element $w \in W_a$ to a Φ-sign type $(X_\alpha)_{\alpha\in\Phi}$ by

$$X_\alpha = \begin{cases} +, & \text{if } k(w,\alpha) > 0, \\ -, & \text{if } k(w,\alpha) < 0, \\ \bigcirc, & \text{if } k(w,\alpha) = 0. \end{cases}$$

By abuse of terminology, we call the set of all elements of W_a corresponding to any given Φ-sign type also by " a sign type of W_a " provided that this set is non-empty [31]. Our result asserts

Theorem (see [29] **).** *Each left cell of the group $W_a(\widetilde{A}_{n-1})$ is a unoin of some sign types.*

Since any Weyl group could be regarded as a standard parabolic subgroup of some affine Weyl group, one can also define the alcove form and the sign type of an element of a Weyl group. Note that the above theorem is trivially valid in the case when $W_a(\widetilde{A}_{n-1})$ is replaced by a Weyl group since each sign type of a Weyl group consists of exactly one element. But this result does not hold for an arbitrary affine Weyl group. A counter-example could be found in the case of $W_a(\widetilde{B}_2)$ (see [20]). However, one may expect the truth of the following

Conjecture (see [34] **)** If W is an irreducible affine Weyl group with simply-laced Dynkin diagram (i.e. W has type $\widetilde{A}$, $\widetilde{D}$ or $\widetilde{E}$), then each left cell of W is a union of some sign types of W.

The results on the left cells of the affine Weyl group $W_a(\widetilde{D}_4)$ support this conjecture.

3.7 Sometimes we are unable to describe all the left cells of certain Coxeter groups, but we can describe all the left cells of those groups in certain two-sided cells. This is the case in the following two-sided cells Ω of affine Weyl groups W_a:

(1) Ω is the lowest two-sided cell W_ν of W_a with respect to the partial order $\underset{LR}{\leq}$ (see [32] [33]);

(2) $a(\Omega) \leq 3$ (see [17] [18] [7] [28]);

(3) W_a has type $\widetilde{B}_n$, $\widetilde{C}_n$, or $\widetilde{D}_n$ $(n \geq 5)$, and, $a(\Omega) = 4$ (see [9]).

3.8 Let us explain the results of 3.7,(1). The lowest two-sided cell W_ν of W_a could be described as below.

$$\begin{aligned} W_\nu &= \{w \in W_a \mid k(w,\alpha) \neq 0, \quad \text{for all} \quad \alpha \in \Phi\} \\ &= \{w \in W_a \mid w = x w_J y, \ \text{for some } x, y \in W_a \text{ and } J \subseteq S \text{ with} \\ &\qquad \ell(w) = \ell(x) + \ell(y) + \frac{1}{2}|\Phi|\}. \end{aligned}$$

where w_J is the longest element in the subgroup of W_a generated by J. Let

$$M = \{w \in W_\nu \mid sw \notin W_\nu \quad \text{for all} \quad s \in \mathcal{L}(w)\}.$$

Then each element $w \in M$ could be written uniquely in the form $w = w_J x$ for some $x \in W_a$ and $J \subseteq S$ with $\ell(w) = \ell(x) + \frac{1}{2}|\Phi|$ (call it the standard expression of w). There exists a bijective map from the set M to the set of all the left cells of W_a in W_ν by sending $w \in M$ to the set

$$\Gamma_w = \{yw \mid y \in W_a,\ \ell(yw) = \ell(y) + \ell(w)\}.$$

The last set forms a left cell of W_a, which is also a sign type of W_a (see [32] [33]).

The distinguished involutions of W_a in W_ν could be described as below (see [33]).

$$\mathcal{D}_0(W_\nu) = \{x^{-1} w_J x \mid w_J x \in M \text{ standard expression }\}. \tag{3.8.1}$$

Now we want to give another expression of this set. Given $x,\ y \in W_a$. In the product $T_x T_y = \sum_z f_{x,y,z} T_z$ ($f_{x,y,z} \in \mathcal{A}$), there exists a unique element $w \in W_a$ with $f_{x,y,w} \neq 0$ such that if z satisfies $f_{x,y,z} \neq 0$ then $z \leq w$ (see [33]). Denote this element w by $\lambda(x, y)$. Then the set (3.8.1) could be reformulated as below.

$$\mathcal{D}_0(W_\nu) = \{\lambda(x^{-1}, x) \mid x \in M\}.$$

From this, one may expect the following more general result.

Conjecture 3.9. (see [34]) *Let $x \in W_a$ be a shortest element in the left cell of W_a containing it. Then $\lambda(x^{-1}, x) \in \mathcal{D}_0$. Conversely, any element $d \in \mathcal{D}_0$ has the form $d = \lambda(x^{-1}, x)$ for any shortest element x in the left cell of W_a containing d.*

Note that there may exist more than one shortest elements in a left cell of W_a. But this fact does not conflict with the above conjecture. For example, in the affine Weyl group

$$W_a(\widetilde{A}_3) = \langle s_i \mid 0 \leq i \leq 3\rangle \quad \text{with} \quad o(s_i s_j) = 3 \quad \text{for} \quad j \equiv i \pm 1 (\mathrm{mod}\, 4),$$

let $x = s_1 s_2 s_1 s_3$ and $y = s_2 s_3 s_2 s_1$. Then $x \underset{L}{\sim} y$ and, both x and y are shortest elements in the left cell containing them. We have

$$\lambda(x^{-1}, x) = \lambda(y^{-1}, y) = s_1 s_2 s_3 s_2 s_1 \in \mathcal{D}_0.$$

The above conjecture has been verified in all the left cells Γ of W_a with $a(\Gamma) \leq 3$ (see [7] [28]) and in all the left cells Γ of W_a of classical types with $a(\Gamma) = 4$ (see [8] [9]).

3.10 The canonical left cells of an affine Weyl group, which will be defined shortly, are closely related to the spherical representations of the corresponding Hecke algebra [23, §9.]. The description of these left cells were given by Lusztig and Xi [24].

It is well known that for any x, y in a Coxeter group W,

$$x \underset{L}{\sim} y \Longrightarrow \mathcal{R}(x) = \mathcal{R}(y) \qquad \text{(see [15])}. \tag{3.10.1}$$

Thus we can use the notation $\mathcal{R}(\Gamma)$ for a left cell Γ of W, which is $\mathcal{R}(x)$ for any $x \in \Gamma$. But the converse is not true in general:

$$\mathcal{R}(x) = \mathcal{R}(y) \not\Rightarrow x \underset{L}{\sim} y.$$

The weaker one is still not true in general:

$$x \underset{LR}{\sim} y \text{ and } \mathcal{R}(x) = \mathcal{R}(y) \not\Rightarrow x \underset{L}{\sim} y.$$

Call a subset $I \subseteq S$ to be special if $|I| = 1$ and $|W_{S-I}| = \max\{|W_{S-\{t\}}| \mid t \in S\}$, where the notation $|X|$ stands for the cardinality of a set X. For $J \subseteq S$, let Y_J be the set of all the elements $x \in W_a$ satisfying $\mathcal{R}(x) = J$. Then the fact (3.10.1) tells us that for any subset $J \subseteq S$, the intersection of a two-sided cell of W_a with the set Y_J is either empty or a union of some left cells of W_a. Lusztig and Xi showed the following stronger result in certain circumstance.

Proposition (see [24] **).** *For any special $I \subseteq S$, the intersection of a two-sided cell $\Omega \neq \{e\}$ of W_a with the set Y_I is exactly a single left cell of W_a (i.e. it is neither empty nor a union of more than one left cells).*

This tells us that for x, y in an affine Weyl group,

$$x \underset{LR}{\sim} y \text{ with } \mathcal{R}(x) = \mathcal{R}(y) \text{ special } \Longrightarrow x \underset{L}{\sim} y.$$

A left cell Γ of W_a is *canonical* if either $\Gamma = \{e\}$ or that $\mathcal{R}(\Gamma)$ is special. An easy consequence is that the number $\kappa(W_a)$ of the canonical left cells of W_a is equal to $gh+1-g$, where g is the number of special subsets in S, and h is the number of two-sided cells of W_a, both of which are relatively well known. For example, when W_a is of type $\widetilde{A}_{n-1}$, we have $g = n$ and $h = p_n$, the number of partitions of n. So $\kappa(W_a) = n \cdot p_n + 1 - n$.

§4. Algorithm for finding a representative set of left cells.

Although we have succeeded in describing so many left cells, the whole figure about left cells of Coxeter groups is still far from being exposed. Thus we must describe more left cells of Coxeter groups whenever it is possible. To do so, it is desirable to design some algorithms. Here we introduce an algorithm which is effective for any Coxeter group with properties 2.3, (a), (b). [38]

In the present section, we always assume W to be a Coxeter group with properties 2.3, (a), (b).

4.1 For $x \in W$, let $\Sigma(x)$ be the set of all left cells Γ satisfying: there exists some $y \in \Gamma$ with y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(x) = a(y)$. Then the author showed.

Theorem (see [38]). *If $x \underset{L}{\sim} y$ in W, then $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$.*

Remark 4.2 (1) The author conjectured that the converse of the above result should also be true (see [38]), i.e.

$$x \underset{L}{\sim} y \Longleftrightarrow \mathcal{R}(x) = \mathcal{R}(y) \text{ and } \Sigma(x) = \Sigma(y). \tag{4.2.1}$$

The truth of the conjecture would characterize the left cells of W. This conjecture has been verified in the cases when W is a Weyl group and when W is an irreducible affine Weyl group with few cases excepted in $W_a(\widetilde{F}_4)$ (see [41]).

(2) It is easily seen that if $x, y \in W$ satisfy $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$ then x, y have the same generalized τ-invariant (see 1.6 for the definition).

4.3 A subset $K \subset W$ is called *a representative set of left cells* (or *an l.c.r. set* for brevity) of W (resp. of W in a two-sided cell Ω), if $|K \bigcap \Gamma| = 1$ for any left cell Γ of W (resp. of W in Ω). The following is a criterion for an l.c.r. set of W in a given two-sided cell.

Theorem (see [38]). *Let Ω be a two-sided cell of W. Assume that a non-empty subset $M \subset \Omega$ satisfies the following conditions.*

(1) $x \underset{L}{\not\sim} y$ for any $x \neq y$ in M;

(2) Given an element $y \in W$. Suppose that there exists some element $x \in M$ such that y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$. Then there exists some $z \in M$ with $y \underset{L}{\sim} z$.

Then M is an l.c.r. set of W in Ω.

In principle, this theorem provides us a method to find an l.c.r. set of W in any given two-sided cell Ω from a non-empty subset of Ω, and hence of W itself provided that one could find at least one element from each two-sided cell of W.

4.4 A subset $P \subset W$ is said to be *distinguished* if $P \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in P.

Now assume that P is a non-empty subset of a two-sided cell Ω of W. We introduce the following two processes.

(A). *Take a subset Q of the largest possible cardinality from the set $\bigcup_{x\in P} M(x)$ with Q distinguished.*

(B). *For each $x \in P$, find elements $y \in W$ such that y—x, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.*

Note that the resulting sets in both Processes (**A**) and (**B**) are automatically in the two-sided cell Ω.

Now we are ready to introduce an algorithm to find an l.c.r. set of W in a given two-sided cell Ω.

Algorithm 4.5. (see [38] **)**

(1) Find a non-empty subset P of Ω (usually we take P to be distinguished for avoiding unnecessary complication if possible);

(2) Perform Processes (**A**) and (**B**) alternately on P until the resulting distinguished set can not be further enlarged by both processes.

Remark 4.6 The above algorithm has been applied by several person to classify the left cells of affine Weyl groups W in the following cases.

(1) For W of type $\widetilde{D}_4$, by the author [39] (The author understand that Chen Chengdong also did this but his method is different from the author [8]).

(2) For W of type $\widetilde{C}_4$, by the author [40].

(3) For W of type $\widetilde{B}_4$, by Zhang [46].

(4) For all the left cells with their a-values equal to 3 in any irreducible affine Weyl group W, by Rui [28].

(5) For all the left cells with their a-values ≤ 5 in W of type $\widetilde{F}_4$, by the author [38].

4.7 As mentioned in 2.3,(2), the set $\mathcal{D}_0$ forms an l.c.r. set of W. Thus one can also classify the left cells of W by first finding the set $\mathcal{D}_0$. Chen made this approach in his papers [7], [8] and [9]. In general, it is more difficult to find the set $\mathcal{D}_0$ directly than to find an arbitrary l.c.r. set of W by algorithm 4.5. On the other hand, suppose that one has got an l.c.r. set of W by algorithm 4.5. Then by applying the result [35, Proposition 5.12], one can find the set $\mathcal{D}_0$ considerably easier.

§5. Some more open problems.

In this section, we assume that W is either a Weyl group or an affine Weyl group. We want to state two more conjectures proposed by Lusztig. One is concerning the connectedness of left cells of W. The validity of this conjecture would ensure certain good behavior for the left cell representations of the corresponding Hecke algebra, such as cyclicity. The other is concerning the cells of the affine Weyl groups of classical types. That conjecture is based on the belief that there should exist certain very strong combinatorial background for the cells of these groups.

5.1 Connectedness of left cells of W.

A subset $K \subseteq W$ is *connected*, if, for all $x, y \in K$, there exist a sequence of elements $x_0 = x,\ x_1,\ \cdots,\ x_r = y$ in K with some $r \geq 0$ such that for all $i,\ 1 \leq i \leq r$, we have $x_{i-1}x_i^{-1} \in S$. Lusztig proposed:

Conjecture (see [2] **).** *When W is either a Weyl group or an affine Weyl group, each left cell of W is connected.*

This conjecture is supported in all the cases where the left cells of W have been described explicitly (see 3.1). On the other hand, Xi and Du made some progress in the proof of this conjecture by showing the following result individually: It is finite for the number of connected components in each left cell of an affine Weyl group (see [45] [12]). Comparing with the result of Xi-Du, the above conjecture asserts that this number is one.

By 2.3, it is sufficient to verify the above conjecture in the cases of affine Weyl groups. Note that the truth of Conjecture 3.8 might be helpful in the verification of the present conjecture.

5.2 Left cells in the affine Weyl group of type $\widetilde{B}_n$, $\widetilde{C}_n$ or $\widetilde{D}_n$.

Define a permutation group on the integer set $\mathbb{Z}$ as follows.

$$\mathcal{A}_n = \left\{ \sigma : \mathbb{Z} \mapsto \mathbb{Z} \,\middle|\, (i+n)\sigma = (i)\sigma + n, \quad \text{for all} \quad i \in \mathbb{Z};\ \sum_{i=1}^{n}(i)\sigma = \sum_{i=1}^{n} i \right\}.$$

Then it is known that $\mathcal{A}_n$ is isomorphic to the affine Weyl group of type $\widetilde{A}_{n-1}$ with its simple reflection set $S = \{s_t \mid 0 \leq t \leq n-1\}$, where

$$s_t(i) = \begin{cases} i, & \text{if } i \not\equiv t, t+1 \ (\bmod\ n), \\ i+1, & \text{if } i \equiv t \ (\bmod\ n), \\ i-1, & \text{if } i \equiv t+1 \ (\bmod\ n). \end{cases}$$

The symmetric group S_n could be described as a subgroup of $\mathcal{A}_n$:

$$S_n = \{\sigma \in \mathcal{A}_n \mid 1 \leq (i)\sigma \leq n, \quad \text{for all} \quad 1 \leq i \leq n\}.$$

The affine Weyl groups of types $\widetilde{B}_\ell$, $\widetilde{C}_\ell$ and $\widetilde{D}_\ell$ could also be described as permutation groups on $\mathbb{Z}$.

(a) Let $\sigma:\ s_i \mapsto s_{2\ell+1-i}$, for all $0 \le i \le 2\ell$, be the involutive automorphism in $\mathcal{A}_{2\ell+1}$ with the convention that $s_{2\ell+1+i} = s_i$ for $i \in \mathbb{Z}$. Then we have $W_a(\widetilde{B}_\ell) \cong \mathcal{A}^\sigma_{2\ell+1}$, the latter is the subgroup of $\mathcal{A}_{2\ell+1}$ consisting of all the σ-fixed elements (similar notations will be used in (b) and (c)).

(b) Let $\tau:\ s_i \mapsto s_{2\ell+1-i}$, for all $0 \le i \le 2\ell+1$, be the involutive automorphism in $\mathcal{A}_{2\ell+2}$ with the convention that $s_{2\ell+2+i} = s_i$ for $i \in \mathbb{Z}$. Then we have $W_a(\widetilde{C}_\ell) \cong \mathcal{A}^\tau_{2\ell+2}$.

(c) Let $\eta:\ s_i \mapsto s_{2\ell-i}$, for all $0 \le i \le 2\ell-1$, be the involutive automorphism in $\mathcal{A}_{2\ell}$ with the convention that $s_{2\ell+i} = s_i$ for $i \in \mathbb{Z}$. Then $W_a(\widetilde{D}_\ell) \cong \mathcal{A}^\eta_{2\ell}$.

We have even simplier description for $W_a(\widetilde{C}_\ell)$ as follows.

$$W_a(\widetilde{C}_\ell) = \{w \in \mathcal{A}_{2\ell+2} \mid (-i)w = -(i)w, \quad \text{for all} \;\; i \in \mathbb{Z}\}.$$

Note that this description is slightly defferent from the one by R. Bédard (see [4]). Our description of $W_a(\widetilde{C}_\ell)$ has the advantage of exposing more group-theoretic symmetry in element form.

We can also give some similar descriptions for $W_a(\widetilde{B}_\ell)$ and $W_a(\widetilde{D}_\ell)$, but they are slightly complicated.

Lusztig suggested the following

Conjecture. *Each left cell of* $W_a(\widetilde{X})$ *(* $X \in \{B_\ell,\ C_\ell,\ D_\ell\}$ *) has the form*

$$\Gamma = W_a(\widetilde{X}) \bigcap (\bigcup_{i\in I} \Gamma_i),$$

where the Γ_i *(* $i \in I$ *) are some left cells of* $\mathcal{A}_m$ *(* m *is determined by* X *as above).*

Remark 5.3 The author would like to take this opportunity to thank Lusztig for telling me the above conjecture by private communication. This conjecture has been verified by the author in the case of $\ell \le 3$ (unpublished).

5.4 Description of cells in terms of partitions.

To each element $w \in \mathcal{A}_n$, we associate a sequence of integers $d_1 \le d_2 \le \cdots \le d_t = n$ as follows.

$$d_k = \max\{|X| \mid X = \bigcup_{i=1}^{k} X_i \subset \mathbb{Z};\ u \not\equiv v (mod\, n),\ \text{for all } u \ne v \text{ in } X; \\ \text{and } u < v \text{ in some } X_i \text{ implies } (u)w > (v)w\}$$

Then we have $d_1 \geq d_2 - d_1 \geq d_3 - d_2 \geq \cdots \geq d_t - d_{t-1}$, which is a partition of n. We denote it by $\psi(w)$. This defines a map $\psi: \mathcal{A}_n \longrightarrow \Lambda_n$, where Λ_n is the set of all partitions of n. This map is compatible with the map mentioned in 3.2, and so it induces a bijection from the set of two-sided cells of $\mathcal{A}_n$ to the set Λ_n. It would be interesting to define an analogous map for the group $W_a(\widetilde{X})$ ($X \in \{B_\ell, C_\ell, D_\ell\}$) such that one can describe all the two-sided cells (and hence all the left cells) in the element level. That is, to associate any element of $W_a(\widetilde{X})$ to a pair of partitions of some fixed integer with certain properties such that the fibres of the corresponding map coincide with two-sided cells of $W_a(\widetilde{X})$.

Note that some progress in this direction has been made on the group $W_a(\widetilde{C}_n)$ by the author (see [37]). We omit the detail.

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