REDUCED EXPRESSIONS FOR THE ELEMENTS IN A BRUHAT INTERVAL

Jian-yi Shi

School of Mathematical Sciences
East China Normal University
Shanghai, 200241, P.R.China

Abstract. Let (W,S) be a Coxeter system. We study the relation among the reduced expressions of the elements in the interval $[y,w] := \{z \in W \mid y \leqslant z \leqslant w\}$ for any $y < w$ in W. Fix a reduced expression ξ of w. We find some conditions for various expressions ζ to be reduced, where ζ is obtained from ξ by deleting certain factors in S. We investigate the relations among all such ζ which are reduced expressions for either a certain element in $[y,w]$ or some elements with a certain common property.

§0. Introduction.

Let $\mathbb{Z}$ (resp., $\mathbb{N}$, $\mathbb{P}$) be the set of all integers (resp., non-negative integers, positive integers). For $i \leqslant j$ in $\mathbb{Z}$, denote by $[i,j]$ the set $\{i, i+1, ..., j\}$. Denote $[1,j]$ simply by $[j]$.

Let (W,S) be a Coxeter system with the Bruhat-Chevalley ordering $\leqslant$ and the length function ℓ on W. An expression $w := s_1 s_2 \cdots s_r$ with $s_i \in S$ for $i \in [r]$ is called *reduced* if $r = \ell(w)$. Denote by $\mathrm{Red}(w)$ the set of all reduced expressions of $w \in W$. Let $y < w$ in W with $\ell(w) = \ell(y) + k$ for some $k \in \mathbb{P}$ and fix some $\xi : s_1 s_2 \cdots s_r$ in $\mathrm{Red}(w)$. Denote by $\mathrm{Red}_\xi(y)$ the set of all $\zeta \in \mathrm{Red}(y)$ obtained from ξ by deleting k factors in S. Reduced expressions are basic and frequently occur in the theory of Coxeter systems. Hence it is desirable to give some more detailed investigation for the properties of those expressions.

In this paper, we study the expressions in $\mathrm{Red}_\xi(y)$. We describe explicitly all the expressions in $\mathrm{Red}_\xi(y)$ via a sequence $f(w,y;\xi) : i_1 < i_2 < \cdots < i_t$ in $[\ell(w)]$ when $k = 2$ (Theorem 2.4), and observe how the sequence $f(w,y;\xi)$ changes as ξ varies over

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$\mathrm{Red}(w)$ (Propositions 3.3 and 3.4). Those results are generalized from the case $k = 2$ to arbitrary $k \geqslant 2$. We deduce some necessary and sufficient condition for the relation $sy \leqslant w$ (resp., $y \leqslant sw$), $\forall\ s \in I$ when $y < w$ and $I \subset S - \mathcal{L}(y)$ (resp., $I \subseteq \mathcal{L}(w)$) (Propositions 4.1, 4.3 and Corollary 4.2). We give some criteria for the h-left (resp., h-right) normality of an expression in $\mathrm{Red}_\xi(y)$ for $h \in [k]$ (Proposition 5.6). We show that any two expressions in $\mathrm{Red}_\xi(y)$ can be transformed from one to the other by successively applying some factor-moves (Proposition 6.2). Finally, we consider the number $n(w, y)$ of coatoms in the interval $[y, w]$ for any $y < w$ in W and deduce some criterion for the validity of the equation $n(w, y) = \ell(w)$ (Proposition 7.3).

The contents of the paper are organized as follows. Some known results on a Coxeter system (W, S) are stated in Section 1 for later use. We describe all expressions in $\mathrm{Red}_\xi(y)$ for the case of $k = 2$ in Sections 2–3. Then we extend the results to the case of arbitrary $k \geqslant 2$ in Sections 4–7.

§1. Some known results on a Coxeter system (W, S).

The known results 1.1-1.3 on a Coxeter system (W, S) are basic in subsequent discussion:

Lemma 1.1. (see [5]) *If $y < w$ in W satisfy $\ell(w) = \ell(y) + 2$, then the cardinality of the interval $(y, w) := \{z \in W \mid y < z < w\}$ is 2.*

Let W_I be the parabolic subgroup of W generated by $I \subseteq S$. Denote by $|X|$ the cardinality of a set X. For any $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. For any $w, x, y \in W$, the notation $w = x \cdot y$ means that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$.

Lemma 1.2. (see [4, Theorems 1 and 7]) *Let $I \subseteq S$ be with $|W_I| < \infty$.*

(a) If $x, y \in W$ and $s \in S$ satisfy $xs > x$ and $sy > y$, then $xy < xsy$.

(b) If $x, y \in W$ satisfy $xy = x \cdot y$ and $\mathcal{R}(x) \cap I = \mathcal{L}(y) \cap I = \emptyset$, then $xwy = x \cdot w \cdot y$ for any $w \in W_I$.

Proof. (a) is the assertion of [4, Theorem 1]. (b) follows by [4, Theorem 7]. □

1.3. For an expression $s_1 s_2 \cdots s_r$ in W and a sequence $i_1 < i_2 < \cdots < i_k$ in $[r]$, the notation $s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_k})$ stands for the expression obtained from $s_1 s_2 \cdots s_r$ by deleting the factors $s_{i_1}, s_{i_2}, ..., s_{i_k}$. For any $y < w$ in W and any $\xi : s_1 s_2 \cdots s_r \in \mathrm{Red}(w)$

with $s_i \in S$, we have $\mathrm{Red}_\xi(y) \neq \emptyset$ by [3, Theorem 5.10]. In particular, when $\ell(w)-\ell(y)=1$, there exists a unique $h \in [r]$ such that $\mathrm{Red}_\xi(y) = \{s_1s_2\cdots s_r \setminus (s_h)\}$.

§2. Reduced expressions of y for $y < w$ in W with $\ell(y) = \ell(w) - 2$.

In this section, we always assume $y < w$ in W satisfy $\ell(y) = \ell(w) - 2$.

2.1. Fix

$$\xi:\ s_1s_2\cdots s_r \in \mathrm{Red}(w). \tag{2.1.1}$$

By 1.3, there are some $i < j$ in $[r]$ such that

$$s_1s_2\cdots s_r \setminus (s_i, s_j) \in \mathrm{Red}_\xi(y). \tag{2.1.2}$$

In general, the pair i, j are not uniquely determined by w, y and $\xi \in \mathrm{Red}(w)$. There are exactly two elements (say x, z) in (y, w) by Lemma 1.1. So we have

$$\mathrm{Red}_\xi(x) = \{s_1s_2\cdots s_r \setminus (s_h)\} \quad \text{and} \quad \mathrm{Red}_\xi(z) = \{s_1s_2\cdots s_r \setminus (s_k)\} \tag{2.1.3}$$

for some $h \neq k$ in $[r]$ by 1.3. We may assume $h < k$ for the sake of definiteness. It is not necessary to have $\{h,k\} \cap \{i,j\} \neq \emptyset$ in general. However, there do exist some $l \in [r] - \{h\}$ and $m \in [r] - \{k\}$ such that some expressions in $\mathrm{Red}_\xi(y)$ can be obtained from (2.1.3) by deleting the factors s_l, s_m, respectively.

Lemma 2.2. *Let (2.1.1)-(2.1.3) be reduced expressions of $w, y, x, z \in W$, respectively with $(y, w) = \{x, z\}$, $h < k$ and $i < j$ in $[r]$.*

(a) $h \leqslant i < j \leqslant k$.

(b) If $s_1s_2\cdots s_r \setminus (s_l, s_m) \in \mathrm{Red}_\xi(y)$ satisfies $(i,j) \neq (l,m)$ then either $i < j \leqslant l < m$ or $l < m \leqslant i < j$ holds.

Proof. (a) We claim that neither $i < j < h < k$ nor $i < h < j$ is possible. For, if $i < j < h < k$ then there would be reduced expressions $w' = s_is_{i+1}\cdots s_k$, $x' = s_{i+1}s_{i+2}\cdots s_k$, $z' = s_is_{i+1}\cdots s_{k-1}$, $v' = s_is_{i+1}\cdots s_k \setminus (s_h)$ and $y' = s_{i+1}s_{i+2}\cdots s_k \setminus (s_j)$; if $i < h < j$ then there would be reduced expressions $w' = s_is_{i+1}\cdots s_j$, $x' = s_{i+1}s_{i+2}\cdots s_j$, $z' = s_is_{i+1}\cdots s_{j-1}$, $v' = s_is_{i+1}\cdots s_j \setminus (s_h)$ and $y' = s_{i+1}s_{i+2}\cdots s_{j-1}$. In either case, we have $\ell(w') = \ell(y') + 2$ and pairwise distinct x', z', v' in (y', w'), contradicting Lemma 1.1. Similarly, we can show that neither $h < k < i < j$ nor $i < k < j$ is possible. This proves (a).

(b) We have $i \neq l$ and $j \neq m$ by 1.3 and the assumptions $(i,j) \neq (l,m)$ and $s_1s_2\cdots s_r \setminus (s_i,s_j) = s_1s_2\cdots s_r \setminus (s_l,s_m)$. We may assume $i < l$ for the sake of definiteness. To show our result, we must show that neither $i < l < m < j$ nor $i < l < j < m$ is possible. If $i < l < m < j$ then it would be $s_{i+1}s_{i+2}\cdots s_{j-1} = s_is_{i+1}\cdots s_j \setminus (s_l,s_m)$ and hence $s_is_{i+1}\cdots s_j = s_{i+1}s_{i+2}\cdots s_{j-1}\setminus(s_l,s_m)$, the latter is impossible since $s_is_{i+1}\cdots s_j$ is reduced. If $i < l < j < m$ then it would be $s_{i+1}s_{i+2}\cdots s_m\setminus(s_j) = s_is_{i+1}\cdots s_{m-1}\setminus(s_l)$ and hence $s_is_{i+1}\cdots s_m \setminus (s_j) = s_{i+1}s_{i+2}\cdots s_{m-1} \setminus (s_l)$, so $s_is_{i+1}\cdots s_m \setminus (s_j)$ is not reduced. By the exchanging condition on W, there is some $n \in [i+1,m]-\{j\}$ such that

$$s_is_{i+1}\cdots s_m \setminus (s_j) = \begin{cases} s_{i+1}s_{i+2}\cdots s_m \setminus (s_n,s_j), & \text{if } n < j,\\ s_{i+1}s_{i+2}\cdots s_m \setminus (s_j,s_n), & \text{if } n > j.\end{cases}$$

The case $n < j$ is impossible since $s_is_{i+1}\cdots s_{j-1}$ is reduced. If $n > j$, then it would be $s_{i+1}s_{i+2}\cdots s_m \setminus (s_j,s_n) = s_{i+1}s_{i+2}\cdots s_{m-1} \setminus (s_l)$, hence $s_ls_{l+1}\cdots s_m \setminus (s_j,s_n) = s_{l+1}s_{l+2}\cdots s_{m-1}$. If $n < m$ then $s_ls_{l+1}\cdots s_m = s_{l+1}s_{l+2}\cdots s_{m-1} \setminus (s_j,s_n)$; if $n = m$ then $s_ls_{l+1}\cdots s_{m-1} = s_{l+1}s_{l+2}\cdots s_{m-1} \setminus (s_j)$. In either case, it contradicts the fact of $s_ls_{l+1}\cdots s_m$ being reduced. This proves (b). □

Lemma 2.3. *Let (2.1.1)-(2.1.2) be reduced expressions of $w, y \in W$ for some $i < j$ in $[r]$.*

(a) $s_1s_2\cdots s_r \setminus (s_j)$ is not reduced if and only if $y = s_1s_2\cdots s_r \setminus (s_j,s_n)$ for some $n \in [j+1,r]$.

(b) $s_1s_2\cdots s_r \setminus (s_i)$ is not reduced if and only if $y = s_1s_2\cdots s_r \setminus (s_n,s_i)$ for some $n \in [i-1]$.

Proof. By symmetry, we need only to show (a). The implication "$\Longleftarrow$" follows by Lemma 2.2 (a). Now we consider the implication "$\Longrightarrow$". By the assumption, there is some $m \in [i]$ such that $s_{m+1}s_{m+2}\cdots s_r\setminus(s_j)$ is reduced but $s_ms_{m+1}\cdots s_r\setminus(s_j)$ is not. We claim $m = i$. For otherwise, $m < i$. Since $s_1s_2\cdots s_i$, $s_is_{i+1}\cdots s_r \setminus (s_j)$ and $s_1s_2\cdots s_r \setminus (s_i,s_j)$ are all reduced expressions, the expression $s_1s_2\cdots s_r\setminus(s_j)$ should also be reduced by Lemma 1.2, a contradiction. The claim is proved. By the exchanging condition on W, we see by Lemma 2.2 (b) that there is some $n \in [j+1,r]$ such that $s_is_{i+1}\cdots s_r \setminus (s_j) = s_{i+1}\cdots s_r \setminus (s_j,s_n)$. So $y = s_1s_2\cdots s_r \setminus (s_j,s_n)$. □

For $y < w$ in W with $\ell(w) = \ell(y)+2$, we are now ready to describe the set $\mathrm{Red}_\xi(y)$ and its relation with the elements in the open interval (y,w).

Theorem 2.4. *Let $y < w$ be in W with $\ell(w) = \ell(y) + 2$ and $\xi \in \mathrm{Red}(w)$ in (2.1.1).*

(a) There exists a unique sequence $i_1 < i_2 < \cdots < i_t$ in $[r]$ with some $t \geqslant 2$ such that for any $h < k$ in $[r]$, the equality $y = s_1 s_2 \cdots s_r \setminus (s_h, s_k)$ holds if and only if $(h, k) \in \{(i_l, i_{l+1}) \mid l \in [t-1]\}$.

(b) For any $h \in [t]$, the expression $x_h := s_1 s_2 \cdots s_r \setminus (s_{i_h})$ is reduced if and only if $h \in \{1, t\}$.

(c) $(y, w) = \{x_1, x_t\}$.

Proof. For (a), the desired sequence $i_1 < i_2 < \cdots < i_t$ in $[r]$ can be constructed by repeatedly applying Lemma 2.3. Then (b)-(c) follows by (a) and Lemmas 2.3, 1.1. □

According to Theorem 2.4, we may call $f(w, y; \xi) := (i_1, i_2, \cdots, i_t)$ the *associated sequence* of y with respect to $\xi \in \mathrm{Red}(w)$ in (2.1.1). Denote $d(w, y; \xi) = t$.

§3. The sequence $f(w, y; \xi)$ and the number $d(w, y; \xi)$.

3.1. In the setup of Theorem 2.4, the number $d(w, y; \xi) = t$ and the sequence $f(w, y; \xi) = (i_1, i_2, ..., i_t)$ depend on the the choice of $\xi \in \mathrm{Red}(w)$ in (2.1.1). For example, let $y = s'ss'$ and $w = ss'ss's = s'ss'ss'$ for some $s, s' \in S$ with $o(ss') = 5$. Then $d(w, y; ss'ss's) = 2$, $f(w, y; ss'ss's) = (1, 5)$, $d(w, y; s'ss'ss') = 5$ and $f(w, y; s'ss'ss') = (1, 2, 3, 4, 5)$.

3.2. Let $y < w$ be in W with $\ell(w) = \ell(y) + 2$. Fix $\xi \in \mathrm{Red}(w)$ in (2.1.1) with $f(w, y; \xi) = (i_1, i_2, ..., i_t)$ such that $t := d(w, y; \xi)$ is maximal as ξ ranges over $\mathrm{Red}(w)$.

Proposition 3.3. *Keep the setup of 3.2 with $f(w, y; \xi) = (i_1, i_2, ..., i_t)$. Suppose that there exist some $p < q$ in $[r]$ satisfying the condition below:*

(3.3.1) $s_p s_{p+1} \cdots s_q = \underbrace{aba\cdots}_{c \text{ factors}}$ (alternating in a and b) and $i_k, i_{k+1} \in [p, q]$ for some $k \in [t-1]$ and $a \neq b$ in $S - (\mathcal{R}(s_1 s_2 \cdots s_{p-1}) \cup \mathcal{L}(s_{q+1} s_{q+2} \cdots s_r))$ with $c := q + 1 - p \leqslant o(ab) < \infty$.

Then one of the following cases must occur.

(1) $(i_1, i_2, ..., i_t) = (p, p+1, ..., q)$ (hence $t = q + 1 - p$);

(2) $t = 2$ and $(i_1, i_2) = (p, q)$.

Proof. Since the expression $s_p s_{p+1} \cdots s_q \setminus (s_{i_k}, s_{i_{k+1}})$ is reduced, we have either $p < i_{k+1} = i_k + 1 \leqslant q$ or $(i_k, i_{k+1}) = (p, q)$. If $i_{k+1} = i_k + 1$, then $y = s_1 s_2 \cdots s_r \setminus (s_m, s_{m+1})$ for any $m \in [p, q-1]$. Hence in either case, we see by Lemmas 1.1-1.2 and the condition

(3.3.1) that the expressions $x := s_1s_2\cdots s_r \setminus (s_p)$ and $z := s_1s_2\cdots s_r \setminus (s_q)$ are reduced with $(y,w) = \{x,z\}$. This implies our result by Theorem 2.4. □

Proposition 3.4. *Keep the setup of 3.2 with $f(w,y;\xi) = (i_1, i_2, ..., i_t)$. Then one of the cases (1)-(2) below must occur:*

(1) $d(w,y;\zeta) = 2$ for some $\zeta \in \mathrm{Red}(w)$;

(2) $d(w,y;\zeta) = t$ for any $\zeta \in \mathrm{Red}(w)$.

When $t > 2$ and the case (1) occurs, we have $\zeta : s_1s_2\cdots s_{p-1}(\underbrace{bab\cdots}_{t\text{ factors}})s_{q+1}\cdots s_r$, where $s_ps_{p+1}\cdots s_q = \underbrace{aba\cdots}_{t\text{ factors}}$ (alternating in a and b) for some $a \neq b$ in S, $p < q$ in $[r]$ and $t = q+1-p = o(ab)$.

Proof. We may assume $t > 2$, for otherwise the result is trivial. If $|\mathrm{Red}(w)| = 1$, then there is nothing to do. Now assume that $|\mathrm{Red}(w)| > 1$. It is well known that any two expressions in $\mathrm{Red}(w)$ can be transformed from one to the other by successively applying the braid relations of the form

$$\underbrace{cdc\cdots}_{f\text{ factors}} = \underbrace{dcd\cdots}_{f\text{ factors}} \qquad \text{(alternating in } c \text{ and } d) \tag{3.4.1}$$

for some $c \neq d$ in S with $f := o(cd) < \infty$. So it is enough to show that $d(w,y;\zeta) \in \{2,t\}$, for all ζ which can be obtained from $\xi : s_1s_2\cdots s_r$ by applying a single braid relation of the form in (3.4.1), i.e., there exist some $p < q$ in $[r]$ with $s_ps_{p+1}\cdots s_q = \underbrace{cdc\cdots}_{f\text{ factors}}$ such that $\zeta : s_1s_2\cdots s_{p-1}\cdot(\underbrace{dcd\cdots}_{f\text{ factors}})\cdot s_{q+1}\cdots s_r$ (hence $f := q+1-p = o(cd)$). If $i_j \notin [p,q]$ for any $j \in [t]$ then $f(w,y;\zeta) = (i_1, i_2, ..., i_t)$. If there is exactly one of the i_j's in $[p,q]$, say $i_k \in [p,q]$, then $i_k \in \{p,q\}$ by the fact that either $s_1s_2\cdots s_r \setminus (s_{i_{k-1}}, s_{i_k})$ with some $k \in [2,t]$, or $s_1s_2\cdots s_r \setminus (s_{i_k}, s_{i_{k+1}})$ with some $k \in [t-1]$, is in $\mathrm{Red}_\xi(y)$. Then $f(w,y;\zeta) = (i_1, i_2, ..., i_{k-1}, i'_k, i_{k+1}, ..., i_t)$ with $\{i'_k\} = \{p,q\} - \{i_k\}$. Now assume that there are more than one of the i_j's in $[p,q]$. Then $i_k, i_{k+1} \in [p,q]$ for some $k \in [t-1]$. Since $s_1s_2\cdots s_r \setminus (s_{i_k}, s_{i_{k+1}}) \in \mathrm{Red}_\xi(y)$, we have either $i_{k+1} = i_k + 1$ or $(i_k, i_{k+1}) = (p,q)$. By the assumption $t > 2$, we must be in the former case with $(i_1, ..., i_t) = (p, p+1, ..., q)$ by Proposition 3.3. This implies that $f(w,y;\zeta) = (p,q)$. Our result is proved. □

§4. The criterion for the relation $sy \leqslant w$ (resp., $y \leqslant sw$), $\forall\ s \in I$.

In this section, we give some criterion for the relation $sy \leqslant w$ (resp., $y \leqslant sw$), $\forall\ s \in I$, where $y < w$ in W and $I \subseteq S - \mathcal{L}(y)$ (resp., $I \subseteq \mathcal{L}(w)$).

By a Coxeter element of a Coxeter system (W, S), we mean an element of the form $s_{i_1} s_{i_2} \cdots s_{i_t}$, where $S = \{s_1, s_2, ..., s_t\}$, $|S| = t$, and $i_1, i_2, ..., i_t$ is a permutation of $1, 2, ..., t$.

If $x \in W$ and $I \subseteq S - \mathcal{L}(x)$ then $zx = z \cdot x$ for any $z \in W_I$ by Lemma 1.2 (b).

Proposition 4.1. *Let $y < w$ in W and $\emptyset \neq I \subseteq S - \mathcal{L}(y)$. Then $sy \leqslant w$ for any $s \in I$ if and only if there exists a Coxeter element, say c_I, of W_I such that $c_I y \leqslant w$.*

Proof. The implication " $\Longleftarrow$ " is obvious. It remains to show the implication " $\Longrightarrow$ ". Clearly, $\ell(w), |I| \geqslant 1$. The result is obviously true when either $\ell(w) = 1$ or $|I| = 1$. Now assume that $k := \ell(w) > 1$, $h := |I| > 1$ and that the result has been proved when *either*$|I| < h$, or $|I| = h$ and $\ell(w) < k$.

(a) First assume that there exists some $t \in \mathcal{R}(w) \cap \mathcal{R}(y)$. Write $w = w't$ and $y = y't$ for some $w', y' \in W$. Then $y' < w'$ by the assumption $y < w$. We claim $sy' \leqslant w'$ for any $s \in I$. For otherwise, there would be some $s \in I$ with $sy' \nleqslant w'$. Then $sy' = y't$ by the fact $sy' < sy \leqslant w = w't$. Hence $sy = y' < y$, contradicting our assumption. So by inductive hypothesis, there exists some Coxeter element, say c_I, of W_I with $c_I y' \leqslant w'$. This implies $c_I y \leqslant w$.

(b) Next assume that there exists some $u \in \mathcal{R}(w) - \mathcal{R}(y)$. Write $w = w'u$. Then $y \leqslant w'$. If $u \notin \mathcal{R}(sy)$ for any $s \in I$, then $sy \leqslant w'$ for any $s \in I$. By inductive hypothesis, there exists some Coxeter element, say c_I, of W_I such that $c_I y \leqslant w'$. Hence $c_I y \leqslant w$. If $u \in \mathcal{R}(sy)$ for some $s \in I$, then $sy = yu$. We also have $u \notin \mathcal{R}(ty)$ and hence $ty \leqslant w'$ for any $t \in I' := I - \{s\}$. By inductive hypothesis, there exists some Coxeter element, say $c_{I'}$, of $W_{I'}$ with $c_{I'} y \leqslant w'$. Then the Coxeter element $c_I := c_{I'} s$ of W_I satisfies that $c_I y = c_{I'} yu \leqslant w'u = w$.

So the implication " $\Longrightarrow$ " follows by (a)-(b). $\square$

Corollary 4.2. *Suppose that $y < w$ in W and $s \neq t$ in $S - \mathcal{L}(y)$ satisfy $sy, ty \leqslant w$ and $\ell(w) = \ell(y) + 2$. Then $w \in \{sty, tsy\}$ and $d(w, y; \xi) = 2$ for any $\xi \in \mathrm{Red}(w)$.*

Proof. By Proposition 4.1, we have either $sty \leqslant w$ or $tsy \leqslant w$. This implies $w \in \{sty, tsy\}$ by the fact $\ell(w) = \ell(y) + 2 = \ell(sty) = \ell(tsy)$. Let $\xi \in \mathrm{Red}(w)$ in (2.1.1) with $f(w, y; \xi) =$

$(i_1, i_2, ..., i_u)$. By Lemma 1.1 and Theorem 2.4, we have $s_1s_2\cdots s_r \setminus (s_{i_u}) \in (y, w) = \{sy, ty\}$. We may assume $ty = s_1s_2\cdots s_r \setminus (s_{i_u})$ for the sake of definiteness. Then we see by Theorem 2.4 that

$$y = ts_1s_2\cdots s_r \setminus (s_{i_u}) = s_1s_2\cdots s_r \setminus (s_{i_{u-1}}, s_{i_u}).$$

Hence $ts_1s_2\cdots s_{i_{u-1}} = s_1s_2\cdots s_{i_{u-1}-1}$. So $tw = ts_1s_2\cdots s_r = s_1s_2\cdots s_r \setminus (s_{i_{u-1}})$. Since $s_1s_2\cdots s_r \setminus (s_{i_{u-1}}) \in \text{Red}(tw)$, we have $tw \in (y, w)$. So $u = 2$ by Theorem 2.4 (b). □

Note that the assertion $w \in \{sty, tsy\}$ in Corollary 4.2 can also follow from [2, Theorem 3.2] by the fact that any Coxeter system (W, S) avoids $K_{3,2}$ under Bruhat-Chevalley order $\leqslant$, that is, there are no elements $a_1, a_2, a_3, b_1, b_2 \in W$, all distinct, such that either b_j covers a_i for any $i \in [3]$, $j \in [2]$, or a_i covers b_j for any $i \in [3]$, $j \in [2]$.

Proposition 4.3. *Suppose $w, y \in W$ and $\emptyset \neq I \subseteq \mathcal{L}(w)$. Then $y \leqslant sw$ for any $s \in I$ if and only if there exists some Coxeter element, say c_I, of W_I such that $y \leqslant c_Iw$.*

Proof. We have $w = w_I \cdot w' = c_I^{-1} \cdot c_Iw_I \cdot w' = c_I^{-1} \cdot c_Iw$ for some $w' \in W$, where w_I is the longest element in W_I. It is easily seen that if $x, x', y, y' \in W$ satisfy $x \cdot y = x' \cdot y'$, then $x \leqslant x'$ if and only if $y \geqslant y'$. In particular, $c_Iw \leqslant sw$ for any $s \in I$ by the facts that $c_I^{-1} \cdot c_Iw = s \cdot sw$ and $s \leqslant c_I^{-1}$. Hence the implication " $\Longleftarrow$ " follows since $y \leqslant c_Iw \leqslant sw$ for any $s \in I$. The remaining is to show the implication " $\Longrightarrow$ " .

(a) First assume $|W| < \infty$. Let w_0 be the longest element of W, then $ww_0 < sww_0 \leqslant yw_0$ for any $s \in I$. With ww_0, yw_0 in the places of y, w, respectively, we see by Proposition 4.1 that there exists some Coxeter element, say c_I, of W_I such that $c_Iww_0 \leqslant yw_0$. This implies $y \leqslant c_Iw$ when $|W| < \infty$.

(b) Next consider the general case. The result is obvious if $\ell(w) = 1$. Now assume $\ell(w) > 1$. First assume that there exists some $u \in \mathcal{R}(w) - \mathcal{R}(y)$. Write $w = w' \cdot u$ for some $w' \in W$. Then $y \leqslant w'$. If $sw' < w'$ for any $s \in I$, then $y \leqslant sw'$ by the assumption $y \leqslant sw = sw' \cdot u$ for any $s \in I$, hence by inductive hypothesis there exists some Coxeter element, say c_I, in W_I such that $y \leqslant c_Iw' < c_Iw$. If $tw' > w'$ for some $t \in I$, then $tw' = w'u$ by the assumption $tw < w$. Thus $sw' < w'$ and hence $y \leqslant sw'$ for any $s \in I' := I - \{t\}$. By inductive hypothesis, there exist some Coxeter element, say $c_{I'}$, in $W_{I'}$ such that $y \leqslant c_{I'}w'$. Let $c_I = c_{I'}t$. Then c_I is a Coxeter element in W_I and $y \leqslant c_{I'}w' = c_{I'}tw'u = c_Iw$. So our result is true if $\mathcal{R}(w) \nsubseteq \mathcal{R}(y)$. Now assume $\mathcal{R}(w) \subseteq \mathcal{R}(y)$. Take any $u \in \mathcal{R}(w)$. Write $w = w' \cdot u$ and $y = y' \cdot u$ for some $w', y' \in W$. If $sw' < w'$ for any $s \in I$ then $y' \leqslant sw'$ for any $s \in I$ by the assumption $y \leqslant sw$. By inductive hypothesis, there exists some Coxeter element, say c_I, in W_I such that $y' \leqslant c_I w'$. So $y = y' \cdot u \leqslant c_I w' \cdot u = c_I w$. Now assume that for any $u \in J := \mathcal{R}(w)$ (write $w = w_u \cdot u$ for some $w_u \in W$), there exists some $t_u \in I$ with $t_u w_u > w_u$. Then $t_u w_u = w_u u$ by the assumption $t_u w < w$ (hence t_u is uniquely determined by u). We have an injective map $\phi : J \longrightarrow I$ with $\phi(u) = t_u$. Thus $w = w'' \cdot w_J = w_{\phi(J)} \cdot w''$ for some $w'' \in W$, where w_J, $w_{\phi(J)}$ denote the longest elements of W_J, $W_{\phi(J)}$, respectively. Since $J = \mathcal{R}(w) = J \cup \mathcal{R}(w'')$, this forces w'' to be the identity of W. So $w = w_J$ and our result follows by (a).

This completes our proof. □

§5. The h-right (resp., h-left) normality.

In the present section, we continue to study the set $\mathrm{Red}_\xi(y)$ for $y < w$ in W with $\xi \in \mathrm{Red}(w)$ in (2.1.1) fixed.

5.1. Let $y < w$ in W satisfy $r := \ell(w) = \ell(y) + k$ for some $k \in [2, r]$. Take

$$\zeta : \; s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_k}) \in \mathrm{Red}_\xi(y) \tag{5.1.1}$$

for some $i_1 < i_2 < \cdots < i_k$ in $[r]$.

Proposition 5.2. *Keep the setup of 5.1 for $y < w$ in W. Assume that $(i_1, i_2, ..., i_k)$ (resp., $(i_k, ..., i_2, i_1)$) is the smallest (resp., the largest) possible in the lexicographical order as ζ ranges over* $\mathrm{Red}_\xi(y)$*. Then the expression (5.2.1) below is reduced for any $h \in [k]$:*

$$s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_h}) \quad \textit{(resp., } s_1 s_2 \cdots s_r \setminus (s_{i_h}, s_{i_{h+1}}, ..., s_{i_k})\textit{)}. \tag{5.2.1}$$

Proof. By symmetry, we need only to prove that $s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_h})$ is reduced for any $h \in [k]$ if $(i_1, i_2, \ldots, i_k)$ is the smallest possible in the lexicographical order. Hence the integer i_h is the smallest possible as the element $y_h := s_{i_{h-1}+1} s_{i_{h-1}+2} \cdots s_r \setminus (s_{i_h}, s_{i_{h+1}}, ..., s_{i_k})$ is obtained from the expression $s_{i_{h-1}+1} s_{i_{h-1}+2} \cdots s_r$ by deleting $k-h+1$ factors in S, where we take h to be $1, 2, ..., k$ in turn with the convention that $i_0 = 0$. The result is obvious if either $k = 2$ or $h = k$. Now assume $k > 2$ and $h = m \in [k-1]$. Assume that the result has been proved for any $h \in [m+1, k]$. Suppose that the claim for $h = m$ fails. Then there would exist some $l \in [i_{m+1}, r]$ such that $s_1 s_2 \cdots s_{l-1} \setminus (s_{i_1}, s_{i_2}, ..., s_{i_m})$ is reduced but $s_1 s_2 \cdots s_l \setminus (s_{i_1}, s_{i_2}, ..., s_{i_m})$ is not. In this case, if $l \in [i_{m+1}+1, r]$, then the expression (5.2.1) with $h = m$ is reduced by Lemma 1.2 and the facts that $s_1 s_2 \cdots s_{i_{m+1}} \setminus (s_{i_1}, s_{i_2}, ..., s_{i_m})$, $s_{i_{m+1}} s_{i_{m+1}+1} \cdots s_r$ and (5.2.1) with $h = m+1$ are all reduced, a contradiction. Hence we must have $l = i_{m+1}$. There would exist some $p \in [0, m-1]$ and $q \in [i_p + 1, i_{p+1} - 1]$ such that

$$s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_k}) = s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_p}, s_q, s_{i_{p+1}}, ..., s_{i_m}, s_{i_{m+2}}, ..., s_{i_k}),$$

contradicting the minimality assumption on i_{p+1}. So our result is proved. □

5.3. Fix $\xi \in \mathrm{Red}(w)$ in (2.1.1) and $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) for $y < w$ in W. Let

(5.3.1) $$\zeta_j : \quad s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_{j-1}}, s_{i_{j+1}}, ..., s_{i_k}),$$

(5.3.2) $$\zeta_{[j]} : \quad s_1 s_2 \cdots s_r \setminus (s_{i_{j+1}}, s_{i_{j+2}}, ..., s_{i_k}),$$

(5.3.3) $$\zeta_{[j,k]} : \quad s_1 s_2 \cdots s_r \setminus (s_{i_1}, s_{i_2}, ..., s_{i_{j-1}})$$

for any $j \in [k]$ with the convention that any of $\zeta_{[k]}$, $\zeta_{[1,k]}$ is that in (2.1.1).

Lemma 5.4. *Keep the setup of 5.3 for $y < w$ in W. Let $j \in [k]$.*

(1) ζ_j is reduced if and only if both $\zeta'_j : s_1 s_2 \cdots s_{i_j} \setminus (s_{i_1}, s_{i_2}, ..., s_{i_{j-1}})$ and $\zeta''_j : s_{i_j} s_{i_j+1} \cdots s_r \setminus (s_{i_{j+1}}, s_{i_{j+2}}, ..., s_{i_k})$ are reduced.

(2) $\zeta_{[j]}$ is reduced if $\zeta''_h : s_{i_h} s_{i_h+1} \cdots s_r \setminus (s_{i_{h+1}}, s_{i_{h+2}}, ..., s_{i_k})$ is reduced for any $h \in [j]$.

(3) $\zeta_{[j,k]}$ is reduced if $\zeta'_h : s_1 s_2 \cdots s_{i_h} \setminus (s_{i_1}, s_{i_2}, ..., s_{i_{h-1}})$ is reduced for any $h \in [j, k]$.

Proof. The implication " $\Longrightarrow$ " in (1) is obvious. Then the implication " $\Longleftarrow$ " in (1) follows by Lemma 1.2 and the facts that the expressions ζ'_j, ζ''_j and $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) are all reduced. For (2)-(3), we need only to prove (2) by symmetry. $\zeta_{[1]}$ is reduced by Lemma 1.2 and the facts that the expressions ζ''_1, $s_1 s_2 \cdots s_{i_1}$ and $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) are all reduced. Now assume $j \in [2, k]$ and that $\zeta_{[h]}$ is proved to be reduced for any $h \in [j-1]$. Then we conclude that $\zeta_{[j]}$ is reduced by Lemma 1.2 and the facts that ζ''_j, $s_1 s_2 \cdots s_{i_j}$ and $\zeta_{[j-1]}$ are all reduced. □

5.5. Fix $\xi \in \mathrm{Red}(w)$ in (2.1.1) and $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) for $y < w$ in W. For any $h \in [k]$, we say that ζ is *h-right-normal* (resp., *h-left-normal*) with respect to ξ if $\zeta_{[j]}$ in

(5.3.2) (resp., $\zeta_{[j,k]}$ in (5.3.3)) is reduced for any $j \in [h]$ (resp., $j \in [h,k]$). We simply call ζ *right-normal* (resp., *left-normal*) with respect to ξ if it is k-right-normal (resp., 1-left-normal).

In particular, when $k = 2$, we see by Theorem 2.4 that $s_1 s_2 \cdots s_r \backslash (s_{i_1})$ (resp., $s_1 s_2 \cdots s_r \backslash (s_{i_2})$) is reduced if and only if ζ is left-normal (resp., right-normal) if and only if $s_1 s_2 \cdots s_{i_2} \backslash (s_{i_1})$ (resp., $s_{i_1} s_{i_1+1} \cdots s_r \backslash (s_{i_2})$) is reduced.

Proposition 5.6. *For any $h \in [k]$, the expression $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) is h-right-normal (resp., h-left-normal) if and only if $\zeta''_l : s_{i_l} s_{i_l+1} \cdots s_r \backslash (s_{i_{l+1}}, s_{i_{l+2}}, ..., s_{i_k})$ (resp., $\zeta'_l : s_1 s_2 \cdots s_{i_l} \backslash (s_{i_1}, s_{i_2}, ..., s_{i_{l-1}})$) is reduced for any $l \in [h]$ (resp., $l \in [h,k]$).*

Proof. By symmetry, we need only to deal with the case of $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) being h-right-normal. The implication " $\Longrightarrow$ " is obvious. For the implication " $\Longleftarrow$ " , we see that for any $j \in [h]$, the expression $\zeta_{[j]}$ is reduced by Lemma 5.4 (2) and the condition of ζ''_l being reduced for any $l \in [j]$. So $\zeta \in \mathrm{Red}_\xi(y)$ in (5.1.1) is h-right-normal. □

§6. Factor-moves on the set $\mathrm{Red}_\xi(y)$.

Let $y < w$ in W satisfy $\ell(w) = \ell(y) + k$ and fix $\xi \in \mathrm{Red}(w)$ in (2.1.1). In this section, we shall establish a relation among all the expressions in $\mathrm{Red}_\xi(y)$.

6.1. Let $\zeta \in \mathrm{Red}_\xi(y)$ be in (5.1.1). We say that an expression $s_1 s_2 \cdots s_r \backslash (s_{j_1}, \ldots, s_{j_k})$ in $\mathrm{Red}_\xi(y)$ can be obtained from ζ by a factor-move if $|\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\}| = k - 1$.

Write $(a,b) \preceq (c,d)$ in $\mathbb{N} \times \mathbb{N}$, if $a \leqslant c$ and $b \leqslant d$. This defines a partial order on $\mathbb{N} \times \mathbb{N}$.

Proposition 6.2. *Let $y < w$ be in W and $\xi \in \mathrm{Red}(w)$ in (2.1.1). Then for any $\eta, \zeta \in \mathrm{Red}_\xi(y)$, there is a sequence $\zeta_0 = \eta, \zeta_1, ..., \zeta_t = \zeta$ in $\mathrm{Red}_\xi(y)$ with some $t \geqslant 0$ such that ζ_i is obtained from ζ_{i-1} by some factor-move for any $i \in [t]$.*

Proof. Let $k := \ell(w) - \ell(y)$. By the results in the previous sections, we see that our result is true for $k \leqslant 2$. Now assume $k \in [3, \ell(w)]$. Apply induction on $(k, \ell(w)) \succeq (3,3)$ in $\mathbb{N} \times \mathbb{N}$ with $k \leqslant \ell(w)$. The result is trivially true when $k = \ell(w)$. Assume

$$\eta : s_1 s_2 \cdots s_r \backslash (s_{i_1}, s_{i_2}, ..., s_{i_k}),$$
$$\zeta : s_1 s_2 \cdots s_r \backslash (s_{j_1}, s_{j_2}, ..., s_{j_k})$$

for some $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ in $[\ell(w)]$. When $i_1, j_1 > 1$, let w', y', η',

ζ' be obtained from w, y, η, ζ, respectively by removing the factor s_1 at the left terminal. When $i_1 = j_1 = 1$, let $w' = s_2 s_3 \cdots s_r$. Then

$$\eta' : s_2 s_3 \cdots s_r \setminus (s_{i_2}, s_{i_3}, ..., s_{i_k}),$$
$$\zeta' : s_2 s_3 \cdots s_r \setminus (s_{j_2}, s_{j_3}, ..., s_{j_k})$$

are two reduced expressions of $y' = y$. In either case, we see by inductive hypothesis that ζ' can be obtained from η' by successively applying some factor-moves, and then so does ζ from η.

Now assume $i_1 \neq j_1$ and $\min\{i_1, j_1\} = 1$. By symmetry, we may assume $1 = i_1 < j_1$ for the sake of definiteness. Then $s_1 \in \mathcal{L}(y)$. By the exchanging condition on W, the element $y' := s_1 y$ has a reduced expression

$$\eta' : \ s_2 s_3 \cdots s_r \setminus (s_{i_2}, s_{i_3}, ..., s_{i_m}, s_{i'_1}, s_{i_{m+1}}, ..., s_{i_k})$$

for some $m \in [k]$ and some $i_m < i'_1 < i_{m+1}$, where we stipulate $i_{k+1} = r + 1$. The element y' also has the reduced expression

$$\zeta' : \ s_2 s_3 \cdots s_r \setminus (s_{j_1}, s_{j_2}, ..., s_{j_k}).$$

By inductive hypothesis, ζ' can be obtained from η' by successively applying some factor-moves. Let η'' be obtained from η' by attaching the factor s_1 at the left terminal. Then ζ is obtained from η'' by the corresponding sequence of factor-moves. Since η'' is obtained from η by a factor-move, ζ is obtained from η by a sequence of factor-moves. □

§7. The number $n(w, y)$ for $y < w$ in W with $\ell(w) - \ell(y) \geqslant 2$.

7.1. For any $y < w$ in W with $k := \ell(w) - \ell(y) \geqslant 2$, let $N(w, y) := \{z \in W \mid y < z < w; \ell(z) = \ell(w) - 1\}$ which is the set of all coatoms of the interval $[y, w]$, and let $n(w, y) := |N(w, y)|$. Then $n(w, y) \in [2, \ell(w)]$ by 1.3 and Lemma 1.1. The following examples show that both the lower-bound 2 and the upper-bound $\ell(w)$ on $n(w, y)$ could possibly reach in some case.

Example 7.2: (1) When (W, S) is a dihedral group, we always have $n(w, y) = 2$ for any $y < w$ in W with $\ell(w) - \ell(y) \geqslant 2$.

(2) Let $S = \{s_1, s_2, ..., s_t\}$ be with $|S| = t \geqslant 3$ such that $o(s_i s_j)$ are sufficiently large (say $o(s_i s_j) \geqslant N$ for some $N \gg 0$) for any $i \neq j$ in $[t]$. Let $w_{m,i} = (s_1 s_2 \cdots s_t)^m s_1 s_2 \cdots s_i$

for some $m \geqslant 1$ and $i \in [t]$. Then $n(w_{m,i}, w_{0,j}) = tm + i = \ell(w_{m,i})$ for either $m \in [2, N]$, or $m = 1$ and $i \geqslant j$ in $[t]$.

(3) Björner and Brenti shows in [1, Corollary 2.7.8 and Example 2.7.9] that for an arbitrarily large $m \geqslant 2$, there are some $y < w$ in an infinite Coxeter group W with $\ell(w) = \ell(y) + 3$ and $n(y, w) = m$.

(4) If $w = w_I$ is the longest element in W_I for some $I \subseteq S$ with W_I finite and if $y \in \{e\} \cup \{s \in I \mid st \neq ts \text{ for some } t \in I - \{s\}\}$, then $n(w, y) = |I|$.

We provide a necessary and sufficient condition for $y < w$ to satisfy $n(w, y) = \ell(w)$.

Proposition 7.3. *Keep the setup of 5.1 with $y < w$ in W and $\xi \in \mathrm{Red}(w)$ in (2.1.1). Then $n(w, y) = r := \ell(w)$ if and only if the following two conditions hold:*

(7.3.1) $z_j := s_1 s_2 \cdots s_r \setminus (s_j)$ is reduced for any $j \in [r]$;

(7.3.2) For any $m \in [r]$, there exist some $i_{m1} < i_{m2} < \cdots < i_{mt}$ in $[r] - \{m\}$ such that $y = s_{i_{m1}} s_{i_{m2}} \cdots s_{i_{mt}}$.

Proof. The condition (7.3.1) is necessary for the validity of $n(w, y) = r$ by 1.3. Under the assumption of the condition (7.3.1), we see that the condition (7.3.2) holds if and only if $y \leqslant z_j$ for any $j \in [r]$. This implies our result by 1.3. □

By the subexpression property of Coxeter systems (see [3, Subsection 5.10]), we see that the validity of the condition (7.3.1) depends only on the element w but not on the choice of the expression ξ in $\mathrm{Red}(w)$. So we may say that w satisfies (7.3.1) in the case. However, the condition (7.3.2) depends not only on the pair y, w but also on the expression $\xi \in \mathrm{Red}(w)$ in general. So we should say that $(y, w; \xi)$ satisfies (7.3.2) in the case.

7.4. $(y, w; \xi)$ satisfies (7.3.2) if

(7.4.1) There are two sequences $i_1 < i_2 < \cdots < i_t$ and $j_1 < j_2 < \cdots < j_t$ in $[r]$ with $i_l \neq j_m$ for any $l, m \in [t]$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_t} = s_{j_1} s_{j_2} \cdots s_{j_t}$.

For either $m > 1$, or $m = 1$ and $i \geqslant j$ in $[t]$, we see in Example 7.2 (2) that $w_{m,i}$ satisfies (7.3.1) and that the triple $(w_{0,j}, w_{m,i}; \xi)$ satisfies (7.4.1) with $\xi : (s_1 s_2 \cdots s_t)^m s_1 s_2 \cdots s_i$, so $n(w_{m,i}, w_{0,j}) = \ell(w_{m,i})$ by Proposition 7.3.

When $y = x^m$ and $\ell(y) = m\ell(x)$ for some $x \in W$ and $m > 1$, the triple $(y, w; \xi)$ satisfies (7.3.2) if

(7.4.2) $s_{i_1}s_{i_2}\cdots s_{i_t}\in \mathrm{Red}(x^{m+1})$ satisfies (7.3.1) for some $i_1<i_2<\cdots<i_t$ in $[r]$ with $t=(m+1)\ell(x)$.

Now we prove (7.3.2) by assuming (7.4.2). Let $v=\ell(x)$. Then $t=v(m+1)$. Take any $q_1q_2\cdots q_v\in \mathrm{Red}(x)$ with $q_i\in S$ for $i\in[v]$. Let $q_1'q_2'\cdots q_t'$ satisfy $q_j'=q_i$ for any $j\in[t]$ and $i\in[v]$ with $j\equiv i \pmod v$. Then $q_1'q_2'\cdots q_t'\in \mathrm{Red}(x^{m+1})$ and

$$N(x^{m+1},x^m)=\{q_1'q_2'\cdots q_t'\setminus\{q_j'\}\mid j\in[t]\} \tag{7.4.3}$$

by the condition (7.3.1) on x^{m+1}. We must prove the following assertion:

(7.4.4) For any $p\in[r]$, there exist some $i_{p1}<i_{p2}<\cdots<i_{pu}$ in $[r]-\{p\}$ such that $x^m=s_{i_{p1}}s_{i_{p2}}\cdots s_{i_{pu}}$ with $u=vm$.

There is some subsequence $l_1,l_2,...,l_t$ of $1,2,...,r$ with $x^{m+1}=s_{l_1}s_{l_2}\cdots s_{l_t}$ by (7.4.2). Then $N(x^{m+1},x^m)=\{s_{l_1}s_{l_2}\cdots s_{l_t}\setminus\{s_{l_j}\}\mid j\in[t]\}$ by (7.4.3) and the condition (7.3.1) on x^{m+1}. If $p\in[r]-\{l_j\mid j\in[t]\}$, then there is some subsequence $i_{p1},i_{p2},...,i_{pu}$ of $l_1,l_2,...,l_t$ with $x^m=s_{i_{p1}}s_{i_{p2}}\cdots s_{i_{pu}}$ by the fact $x^m<x^{m+1}$. If $p\in\{l_j\mid j\in[t]\}$, then there is some subsequence $i_{p1},i_{p2},...,i_{pu}$ of $l_1,l_2,...,\widehat{p},...,l_t$ with $x^m=s_{i_{p1}}s_{i_{p2}}\cdots s_{i_{pu}}$ by the fact $s_{l_1}s_{l_2}\cdots s_{l_t}\setminus\{s_p\}\in N(x^{m+1},x^m)$. This implies (7.4.4) in either case.

Example 7.5. (1) Let $S=\{s,r,t,u,v\}$ satisfy $o(sr)=o(rt)=o(st)=4$ and $o(uv)=o(ab)=6$ for any $a\in\{s,r,t\}$ and $b\in\{u,v\}$. Let $w=ruvsuvtuvruvsuvturstv$ and $y=(rst)^2$.

(2) Let $S=\{s,r,t\}$ satisfy $o(sr)=o(st)=o(rt)=5$. Let $y=(srt)^m$ and $w=(srt)^{m+1}$ for any $m>1$.

In any of the cases (1)-(2) above, the element w satisfies (7.3.1), and the triple $(y,w;\xi)$ satisfy (7.4.2) but not (7.4.1), where ξ is $ruvsuvtuvruvsuvturstv$ or $(srt)^{m+1}$ accordingly. So $n(w,y)=\ell(w)$ by Proposition 7.3.

For $w',y'\in W$ and $s\in S$, define $N_s(w',y'):=\{z'\in N(w',y')\mid s\notin\mathcal{L}(z')\}$ and $n_s(w',y'):=|N_s(w',y')|$.

Proposition 7.6. *Let $y'<w'$ in W and $s\in S-\mathcal{L}(w')\cup\mathcal{L}(y')$. Let $w=sw'$ and $y\in\{y',sy'\}$. Then $n(w,y)\leqslant n(w',y')+1$.*

Proof. We have $y<w$ by our assumption. Let $k:=\ell(w)-\ell(y)$. Then $n(w,y)=1,2$ for $k=1,2$, respectively by 1.3 and Lemma 1.1. So the result holds for $k=1,2$. Now assume

$k > 2$. Fix $\xi \in \mathrm{Red}(w)$ in (2.1.1) with $s = s_1$. There exist some $j_1 < j_2 < \cdots < j_t$ in $[r]$ with $N(w,y) = \{s_1 s_2 \cdots s_r \setminus (s_{j_m}) \mid m \in [t]\}$. Hence $n(w,y) = t$ by 1.3. We must prove $t \leqslant n(w',y') + 1$. Let ζ in (5.1.1) be such that i_1 is the smallest possible as ζ ranges over $\mathrm{Red}_\xi(y)$. Then $i_1 = j_1$ by Proposition 5.2. If $i_1 > 1$, then $y' = s_1 y$ and $\{s_2 s_3 \cdots s_t \setminus (s_{j_m}) \mid m \in [t]\} \subseteq N(w',y')$. We have $t \leqslant n(w',y') < n(w',y') + 1$.

Now assume $i_1 = 1$. If $s_1 \notin \mathcal{L}(y)$ then $y' = y$ and $\{s_2 s_3 \cdots s_t \setminus (s_{j_m}) \mid m \in [2,t]\} \subseteq N(w',y')$. We have $t - 1 \leqslant n(w',y')$. If $s_1 \in \mathcal{L}(y)$ then $y' = s_1 y$ and $N_{s_1}(w',y') = \{s_2 s_3 \cdots s_r \setminus (s_{j_h}) \mid h \in [2,t]\}$. So $t - 1 \leqslant n_{s_1}(w',y') \leqslant n(w',y')$. Thus $t \leqslant n(w',y') + 1$ in either case. Our proof is completed. $\square$

Note that the above result can also be proved by a matching argument, since the multiplication by s on the left gives a map from $[y', w]$ to itself. In fact, the result can be generalized to any interval having a special matching by [2, Lemma 2.1].

Remark 7.7. Keep the setup of Proposition 7.6 with $\xi \in \mathrm{Red}(w)$ and $\zeta \in \mathrm{Red}_\xi(y)$ in (2.1.1), (5.1.1), respectively. In particular, $w = s_1 \cdot w'$. We see from the proof of Proposition 7.6 that the equality $n(w,y) = n(w',y') + 1$ holds only if

(i) $i_1 = 1$;

(ii) $y = y'$ if $s_1 \notin \mathcal{L}(y)$;

(iii) $y = s_1 \cdot y'$ and $N(w',y') = N_{s_1}(w',y')$ if $s_1 \in \mathcal{L}(y)$.

Let e be the identity element of W. We see that if $n(w,y) = r := \ell(w)$ then there should exist two sequences $w_1, w_2, ..., w_r = w$ and $y_1, y_2, ..., y_r = y$ in W with $i = \ell(w_i)$ for any $i \in [r]$ such that $n(w_i, y_i) = n(w_{i-1}, y_{i-1}) + 1$ (hence $n(w_i, y_i) = \ell(w_i)$) for any $i \in [2,r]$. This is so if the following conditions (1)-(3) hold:

(1) $w_i = s_{r+1-i} s_{r+2-i} \cdots s_r$ is a reduced expression with $s_l \in S$ for any $i \in [r]$ and $l \in [r+1-i, r]$.

(2) $y_1 = y_2 = e$, $s_{r+1-i} y_{i-1} > y_{i-1}$ and $y_i \in \{s_{r+1-i} y_{i-1}, y_{i-1}\}$ for $i \in [2,r]$.

(3) $N(w_{i-1}, y_{i-1}) = N_{s_{r+1-i}}(w_{i-1}, y_{i-1})$ for any $i \in [2,r]$.

By (1)-(2), we have $y_i < w_i$ for $i \in [r]$. There are some $i_1 < i_2 < \cdots < i_k$ in $[r]$ with $k := \ell(w) - \ell(y)$ such that $y_m = y_{m-1}$ if and only if $m \in \{i_1, i_2, ..., i_k\}$. By (1) and (3), we get

$$N(w_i, y_i) = (s_{r+1-i} \cdot N(w_{i-1}, y_{i-1})) \cup \{w_{i-1}\} \quad \text{for any } i \in [2,r]. \tag{7.7.1}$$

Hence $N(w_i, y_i) = \{\zeta_j : s_{r+1-i}s_{r+2-i}\cdots s_r \setminus (s_j) \mid j \in [r+1-i, r]\}$ with all ζ_j reduced for any $i \in [r]$ by induction on $i \geqslant 2$ and (7.7.1).

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