A RESULT ON THE BRUHAT ORDER OF A COXETER GROUPS

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Let $W = (W, S)$ be a Coxeter group with S the set of its Coxeter generators. Let $\leq$ be the Bruhat order of W. That is, we denote $y \leq w$ for $y, w \in W$ if there exist reduced expressions $w = s_1 s_2 \cdots s_r$ and $y = s_{i_1} s_{i_2} \cdots s_{i_t}$ with $s_{i_j} \in S$, $1 \leq j \leq r$, and $i_1, i_2, \cdots, i_t$ a subsequence of $1, 2, ..., r$. Let $\ell(x)$ be the length function on W. To each $w \in W$, we associate two subsets of S:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Now we can state our main result as follows.

Theorem 1. *Suppose that $x, y \in W$ and $s \in S$ satisfy the condition $s \notin \mathcal{L}(y) \cup \mathcal{R}(x)$. Then $xy < xsy$.*

The content of this paper is organized as follows. In Section 1, we shall provide a proof of Theorem 1 and also a generalization of the theorem. Then in Section 2, we shall apply Theorem 1 to investigate some properties of the Hecke algebra associated to W and also to verify a conjecture of L. K. Jones [1].

§1 THE PROOF OF THEOREM 1.

Fix a Coxeter group (W, S). The terminolog " an expression of W "means an element of W being written as a product of elements of S. We say an expression w of W is reduced if w is a product of $\ell(w)$ elements of S. Given two expressions $s_1 s_2 \cdots s_k$ and $t_1 t_2 \cdots t_m$ of W with $s_i, t_j \in S$, we write $s_1 s_2 \cdots s_k \equiv t_1 t_2 \cdots t_m$ if otherwise. We say that two expressions g and g' are congruent if $g \equiv g'$. Thus we have $g \equiv g' \Longrightarrow g = g'$. We say g' is a subexpression of an expression g if $g \equiv xg'y$ for some expressions $x, y \in W$.

By a Coxeter transformation on an expression $s_1 s_2 \cdots s_t$ with $s_i \in S$, we mean that it is one of the following transformations.

(A) If there exist some $a, b \in S$ and $i, j \in \mathbb{Z}$ with $a \neq b$ and $1 \leq i < j \leq t$ such that

$$s_i s_{i+1} \cdots s_j \equiv aba \cdots \quad \text{and} \quad o(ab) = j - i + 1.$$

then we define a transformation

$$s_1 s_2 \cdots s_t \mapsto s_1 s_2 \cdots s_{i-1} \underbrace{(bab \cdots)}_{o(ab) \text{ factors}} s_{j+1} \cdots s_t,$$

where the notation $o(x)$ stands for the order of the element x.

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(B) If there exists some $i \in \mathbb{Z}$, $1 \leq i < t$, such that $s_i = s_{i+1}$, then we define a transformation

$$s_1 s_2 \cdots s_t \mapsto s_1 s_2 \cdots s_{i-1} s_{i+2} \cdots s_t.$$

(C) For any i, $0 \leq i \leq t$, and $a \in S$, we define a transformation

$$s_1 s_2 \cdots s_t \mapsto s_1 s_2 \cdots s_i (aa) s_{i+1} \cdots s_t.$$

Remark 2. Given any two expressions $s_1 s_2 \cdots s_k$ and $t_1 t_2 \cdots t_m$ of W with $s_i, t_j \in S$ and $s_1 s_2 \cdots s_k \not\equiv t_1 t_2 \cdots t_m$, it is well known that the expression $s_1 s_2 \cdots s_k$ can be passed to $t_1 t_2 \cdots t_m$ by a succession of Coxeter transformations. In particular, in the case when $t_1 t_2 \cdots t_m$ is a reduced expression, $s_1 s_2 \cdots s_k$ can be passed to $t_1 t_2 \cdots t_m$ by only performing the Coxeter transformations of kinds (A) and (B)

Define a set of triples

$$T = \{(x, s, y) \mid x, y \in W, s \in S, s \notin \mathcal{L}(y) \cup \mathcal{R}(x)\}.$$

For $i \in \mathbb{Z}$, define

$$T_i = \{(x, s, y) \in T \mid \ell(x) + \ell(y) + 1 - \ell(xsy) = i\}.$$

Clearly, if $T_i \neq \emptyset$ then $i \geq 0$ and $i \in 2\mathbb{Z}$. Thus we have a decomposition:

$$T = \bigcup_{j \geq 0} T_{2j}.$$

If $(x, s, y) \in T_i$ then we define $p(x.s.y) = i$. Then Theorem 1 can be reformulated as follows.

Theorem 3. *If $(x, s, y) \in T$ then $xy < xsy$.*

It is obvious that

Lemma 4. *If $(x, s, y) \in T_0$ then $xy < xsy$.*

Let $g \equiv s_1 s_2 \cdots s_k s t_1 t_2 \cdots t_m$ be an expression of W with $s_i, s, t_j \in S$ satisfying the following conditions:

(a) Both $x \equiv s_1 s_2 \cdots s_k$ and $y \equiv t1 t_2 \cdots t_m$ are reduced expressions.

(b) $(x, s, y) \in T_{2n}$ for some $n > 0$.

Let g' be an expression obtained from the expression g by a Coxeter transformation f of kind $\neq (C)$. Then f must have kind (A). Suppose that f does not involve the factor s. Let $s' = s$, $x' \equiv s'_1 s'_2 \cdots s'_k$, and $y' \equiv t'_1 t'_2 \cdots t'_m$ be such that $g' \equiv x' s' y'$. Then x' and y' are also reduced expressions with $x' = x$ and $y' = y$.

Now suppose that f involves the factor s. Then we have

$$g \equiv g_1 \underbrace{(aba...)}_{r \text{ factors}} g_2 \quad \text{and } g' \equiv g_1 \underbrace{(bab...)}_{r \text{ factors}} g_2$$

for some subexpressions g_1, g_2 of g, where $a, b \in S$ satisfy $a \neq b$ and $r = o(ab)$, and s is the ith factor in the parentheses of the expression g with $1 \leq i \leq r$. Then by choosing s' to be the $(r + 1 - i)$th factor in the parentheses of the expression g', we have the expression

$$g' \equiv g_1 (bab \cdots) g_2 \equiv s'_1 s'_2 \cdots s'_k s' t'_1 t'_2 \cdots t'_m$$

with $s'_i, t'_j \in S$ satisfying the following conditions:

(i) $k' = k + r + 1 - 2i$, $m' = m + 2i - r - 1$.

(ii) Let $x \equiv s'_1 s'_2 \cdots s'_k$ and $y' \equiv t'_1 t'_2 \cdots t'_m$. Then $x'y' = xy$ and $(x', s', y') \in T_{2h}$ for some $h \leq n$. The equality $h = n$ holds if and only if both the expressions x' and y' are reduced.

Sum up, we have the following result

Lemma 5. *Let $g \equiv s_1 s_2 \cdots s_k s t_1 t_2 \cdots t_m$, x, y, n be as above. Let g' be an expression obtained from the expression g by a Coxeter transformation f of kind $\neq (C)$. Then we can choose a factor s' in the expression g' such that*

(a) $g' = s'_1 s'_2 \cdots s'_k s' t'_1 t'_2 \cdots t'_m$ with $s'_i, s', t'_j \in S$.

(b) Let $x' \equiv s'_1 s'_2 \cdots s'_k$ and $y' \equiv t'_1 t'_2 \cdots t'_m$. Then $x'y' = xy$ and $(x', s', y') \in T_{2h}$ for some $h \leq n$. The equality $h = n$ holds if and only if both the expressions x' and y' are reduced.

Again assume that $g \equiv s_1 s_2 \cdots s_k s t_1 t_2 \cdots t_m$, x, y, n are as above. If $n > 0$, then by Remark 2, there exists a sequence of expressions $g_0 \equiv g, g_1, \cdots, g_h$ of xsy for some $h > 0$ such that for every i, $1 \leq i \leq h$, g_i is obtained from g_{i-1} by a Coxeter transformation of kind $\neq (C)$ and g_h is a reduced expression. By Lemma 5, we see that there must exist some integer u with $1 \leq u < h$ such that (i) $g_i \equiv s(i,1)...s(i,k_i)s(i)t(i,1)...t(i,m_i)$ with $s(i,j), s(i), t(i,j) \in S$ for any i, $0 \leq i < u$.

(ii) The expressions $s(i,1)...s(i,k_i)$ and $t(i,1)...t(i,m_i)$ are reduced for all i, $0 \leq i < u$.

(iii) Either $s(u,1)...s(u,k_u)$ or $t(u,1)...t(u,m_u)$ is not a reduced expression.

(iv) Let $x_i = s(i,1)...s(i,k_i)$ and $y_i = t(i,1)...t(i,m_i)$ for $0 \leq i \leq u$. Then $x_i y_i = xy$ and $(x_i, s(i), y_i) \in T_{2n}$ with $n_i = n$ for $0 \leq i < u$ and $n_u < n$.

Given $(x, s, y) \in T$, we call a sequence of expressions, say $g_0, g_1, \cdots, g_u$ for some $u \geq 0$, of xsy a declining sequence with respect to (x, s, y). if the following conditions are satisfied.

(a) For every i, $1 \leq i \leq u$, g_i is obtained from g_{i-1} by a Coxeter transformation of kind $\neq (C)$.

(b) The above conditions (i)-(iv) are satisfied, where $n = p(x, s, y)$ and $(x, s, y) = (x_0, s(0), y_0)$.

Thus the above discussion shows that

Lemma 6. *For any $(x, s, y) \in T$ with $p(x, s, y) > 0$, there exists some declining sequence with respect to (x, s, y).*

Proof of Theorem 3:. Apply induction on $p(x, s, y) \geq 0$. If $p(x, s, y) = 0$ then it is just the assertion of Lemma 4. Now assume that $p(x, s, y) > 0$ and that the result has been shown for any $(x', s', y') \in T$ with $p(x', s', y') < p(x, s, y)$. By Lemma 6, there exists a declining sequence $g_0, g_1, \cdots, g_u$ with respect to (x, s, y) described as above. Thus $p(x_u, s(u), y_u) < p(x, s, y)$. By inductive hypothesis, we have $xsy = x_u s(u) y_u > x_u y_u = xy$. So our result is shown. □

Thus Theorem 1 follows as it is equivalent to Theorem 3.

By noting $\ell(xsy) \equiv \ell(xy) (\bmod 2)$, we see that in Theorem 1 we have

$$\ell(xsy) > \ell(xy) + 1. \tag{1}$$

For any subset J of S, let W_J be the subgroup of W generated by J.

Theorem 7. *Given $x, y \in W$, $J \subseteq S - \mathcal{R}(x)$, and $w \in W_J$, if $\ell(wy) = \ell(w) + \ell(y)$, then $xy \leq xwy$ and $\ell(xwy) \geq \ell(xy) + \ell(w)$. In particular, in the case where $\ell(xy) = \ell(x) + \ell(y)$, we have $\ell(xwy) = \ell(x) + \ell(y) + \ell(w)$.*

§2. SOME APPLICATIONS OF THEOREM 1.

We shall make some applications of Theorem 1 in the present section. The first one is concerned with some multiplication properties of a Hecke algebra. Let $\mathcal{H}$ be the Hecke algebra over $A = \mathbb{Z}[u, u^{-1}]$ associated to a Coxeter group (W, S) as below. $\mathcal{H}$ is a free A-module with a basis $\{T_w \mid w \in W\}$ and its multiplication satisfies the rule:

$$T_wT_{w'} = T_{ww'}, \qquad \text{if } w, w' \in W \text{ with } \ell(ww') = \ell(w) + \ell(w'),$$
$$(T_s - u^{-1})(T_s + u) = 0, \qquad \text{for } s \in S.$$

The rule (1) are equivalent to the rules

$$T_sT_w = \begin{cases} (u^{-1} - u)T_w + T_{sw}, & \text{if } s \in \mathcal{L}(w), \\ T_{sw}, & \text{if } s \notin \mathcal{L}(w). \end{cases} \tag{2}$$

For any $x, y, z \in W$, define an element $f_{x,y,z} \in A$ by

$$T_xT_y = \sum_z f_{x,y,z}T_z. \tag{3}$$

It is easily seen from (2) that $f_{x,y,z}$ is a polynomial in $v = u^{-1} - u$ of positive coefficients.

Define a subset

$$\Lambda(x, y) = \{z \in W \mid f_{x,y,z} \neq 0\}$$

of W for any $x, y \in W$. It is well known that there exists a unique maximal element, written $\lambda(x, y)$ in $\Lambda(x, y)$ with respect to the Bruhat order $\leq$ (see [2]). That is,

$$\lambda(x, y) \geq z, \qquad \forall z \in \Lambda(x, y). \tag{4}$$

Here we shall apply Theorem 1 to show that there also exists a unique minimal element in $\Lambda(x, y)$ with respect to the same partial order.

Theorem 8. *For any $x, y \in W$, we have $xy \in \Lambda(x, y)$ and*

$$xy \leq z, \qquad \forall z \in \Lambda(x, y). \tag{5}$$

Proof. Apply induction on $\ell(x) \geq 0$. It is trivial in case $\ell(x) = 0$. Now assume $\ell(x) > 0$. Let $s \in \mathcal{R}(x)$ and $x' = xs$. Then

$$T_xT_y = T_{x'}T_sT_y.$$

If $s \notin \mathcal{L}(y)$, then $T_xT_y = T_{x'}T_{sy}$ and so $\Lambda(x, y) = \Lambda(x', sy)$. Since $\ell(x') < \ell(x)$, this implies by the inductive hypothesis that

$$x' \cdot sy \leq z, \qquad \forall z \in \Lambda(x', sy).$$

So we get (5) in this case. If $s \in \mathcal{R}(y)$ then

$$T_xT_y = T_{x'}(v \cdot T_y + T_{sy}) = v \cdot T_{x'}T_y + T_{x'}T_{sy}.$$

By the positivity of the coefficients of the $f_{x,y,z}$'s, we have

$$\Lambda(x, y) = \Lambda(x', y) \cup \Lambda(x', sy).$$

By the inductive hypothesis and the fact that $\ell(x') < \ell(x)$, we get

$$x'y \leq z, \qquad \forall z \in \Lambda(x', y)$$

and

$$x'sy \leq z', \qquad \forall z' \in \Lambda(x', sy).$$

But $s \notin \mathcal{R}(x') \cup \mathcal{L}(sy)$. This implies from Theorem 1 that

$$xy = x'(sy) < x's(sy) = x'y.$$

Therefore we again get (5). $\square$

Let $\deg f_{x,y,z}$ be the degree of the polynomial $f_{x,y,z}$ in $v = u^{-1} - u$. Here we state a property of f

Corollary 9. *For any $x, y \in W$ and $z \in \Lambda(x, y)$, we have* $\deg f_{x,y,z} \geq 0$. *The equality holds if and only if $z = xy$. The constant of $f_{x,y,z}$ in v is equal to zero if $z \neq xy$ and is equal to 1 if $z = xy$.*

Proof. This follows directly from the multiplication rule of the Hecke algebra $\mathcal{H}$ and Theorem 8. □

The second application of Theorem 1 is to verify a conjecture of L. K. Jones. Let g be an expression of W. For any$s \in S$, let $n_s(g)$ be the number of the factor s occurring in g. For any $w \in W$, let $I(w)$ be the set of all reduced expressions of w. Define a number

$$N_s(w) = \min\{n_s(g) \mid g \in I(w)\}.$$

Then Jones made the following conjecture which plays a crucial role in his paper.

Conjecture 10. *Let (W, S) be a symmetric group. Suppose that $x, y \in W$ and $s \in S$ satisfy the conditions $s \notin \mathcal{L}(y) \cup \mathcal{R}(x)$. Then for any $t \in S - \{s\}$, we have $N_t(xy) \leq N_t(xsy)$.*

To show the above conjecture, we shall prove the following result which includes this conjecture as a special case.

Proposition 11. *Let (W, S) be a Coxeter group. Let $x, y \in W$ and $s \in S$ be such that $s \notin \mathcal{L}(y) \cup \mathcal{R}(x)$. Then for any $t \in S$, we have $N_t(xy) \leq N_(xsy)$.*

Proof. It is well known that for any reduced expression $s_1 s_2 \cdots s_r$ of w with $s_i \in S$, there exists some subsequence $i_1, i_2, \cdots, i_t$ of $1, 2, \cdots, r$ such that $s_{i_1} s_{i_2} \cdots s_{i_t}$ is a reduced expression of y. This implies that

$$N_s(y) \leq N_s(w), \qquad \text{for any } y \leq w \text{ in } W \text{ and } s \in S.$$

So by Theorem 1, our result follows. □

References

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