KAZHDAN-LUSZTIG CELLS IN SOME WEIGHTED COXETER GROUPS

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Abstract. Let (W,S) be a Coxeter group with $S = I \sqcup J$ such that J consists of all universal elements of S and that I generates a finite parabolic subgroup W_I of W with w_0 the longest element of W_I. We describe all the left cells and two-sided cells of the weighted Coxeter group (W,S,L) that have non-empty intersection with J, where the weight function L of (W,S) is in one of the following cases: (i) $\max\{L(s)|s \in J\} < \min\{L(t)|t \in I\}$; (ii) $\min\{L(s)|s \in J\} \geq L(w_0)$; (iii) There exists some $t \in I$ satisfying $L(t) < L(s)$ for any $s \in I - \{t\}$ and L takes a constant value L_J on J with L_J in some subintervals of $[1, L(w_0) - 1]$. The results in the case (iii) are obtained under a certain assumption on (W, W_I).

Lusztig introduced a weighted Coxeter group $W = (W,S,L)$ and the (left, right and two-sided) cells of W in his book [Lu2], where he expected to extend a number of results in the equal parameter case (i.e., the weight function L is the length function of (W,S)) to the unequal parameter case (i.e., L is not constant on S) by proposing 15 conjectures involving the cells of W. The progresses have been fully achieved in describing cells for some weighted Coxeter groups (W,S,L), such as $I_2(m)$ (m is either even or infinity), F_4, $\widetilde{C}_2$ and $\widetilde{G}_2$ (see [Lu2, Ge, Gu]) and have been partially achieved, such as B_n, $\widetilde{C}_n$ and $\widetilde{B}_n$ ($n > 2$) with certain special weight functions (see [Lu1, Bon, MS, Sh2, Sh3]).

In [SY], we describe all the left cells and two-sided cells of the weighted universal Coxeter group (W,S,L) (i.e., the product st of any $s \neq t$ in S has order $o(st)$ infinite) with L being arbitrary weight function of (W,S). In the present paper, we shall extend our results in [SY] to some more general case: S is a disjoint union of two non-empty subsets I and J, where $J := \{s \in S | o(st) = \infty \text{ for any } t \in S - \{s\}\}$ (call J the *universal part* of S), and the subgroup W_I of W generated by I is finite with w_0 the longest element. To avoid a degenerating case, we shall always assume that the cardinality $|I|$ of I is greater than 1 in the present paper.

We shall describe all the left cells and the two-sided cells of the weighted Coxeter group (W,S,L) that have non-empty intersection with J. We do it for the weight function L of (W,S) first in the cases (i): $\max\{L(s)|s \in J\} < \min\{L(t)|t \in I\}$ and (ii): $\min\{L(s)|s \in J\} \geq L(w_0)$. The main results in these two cases are included in Theorems 2.2 and 2.11, respectively. Then we spend six sections of the paper to consider the case (iii): there exists some $t \in I$ satisfying $L(t) < L(s)$ for any $s \in I - \{t\}$ and L takes a constant value L_J on J with L_J in some particular subintervals of $[1, L(w_0) - 1]$. The main results in this case are Theorems 7.4-7.5.

Key words and phrases. weighted Coxeter group; universal elements; left cells; two-sided cells; the second largest weight element.

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This is the most technical part in the paper. We reach our goal under a certain assumption on (W, W_I) (i.e., (4.1.1)). The second largest weight element w_0' of W_I plays an important role in our discussion. We deduce some properties of w_0', some of which are analogous to those of w_0 and of independent interest.

The contents of the paper are organized as follows. We collect some necessary concepts and known results in Section 1. In Section 2, we describe the left and two-sided cells of (W, S, L) in the cases (i)-(ii). In Section 3, we study the second largest weight element of W_I. We make some degree estimates involving some α- and h'-polynomials in Section 4. Then in Sections 5-6, we investigate the c-basis expansions obtained by left multiplication by c_x with $x \in W_I \cup J$ and study the relations $\sim_L$, $\sim_{LR}$ in W. Finally, in Section 7, we describe the left cells and two-sided cells of (W, S, L) intersecting J nontrivially in the case (iii).

1. Preliminaries

1.1. Let $\mathbb{Z}$ (resp., $\mathbb{N}$, $\mathbb{P}$) denote the set of integers (resp., non-negative integers, positive integers). For any $i \leq j$ in $\mathbb{Z}$, denote by $[i, j]$ the set $\{i, i+1, ..., j\}$ and denote $[1, j]$ simply by $[j]$.

Let W be a Coxeter group with S its Coxeter generator set. Let ℓ be the length function and $\leq$ the Bruhat-Chevalley order on (W, S). Call $L : W \to \mathbb{N}$ a *weight function* on W if $L(xy) = L(x) + L(y)$ for any $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$. Hence $L(s) = L(t)$ for any $s, t \in S$ conjugate in W. Call (W, S, L) a *weighted Coxeter group.*

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. Denote $v_w := v^{L(w)}$ for $w \in W$. We can define the degree of the elements of $\mathcal{A}$ through the following map

$$\deg : \mathcal{A} \to \mathbb{Z} \cup \{-\infty\}.$$

For $\zeta \in \mathcal{A}$, if $\zeta = 0$, then $\deg(\zeta) = -\infty$; otherwise, $\deg(\zeta)$ is defined to be the the integer d_ζ maximal with respect to the condition that $v^{-d_\zeta}\zeta \notin \mathbb{Z}[v^{-1}] - \mathbb{Z}$. For example, $\deg(v^{-3} + v^{-1}) = -1$ under this definition.

The Iwahori-Hecke algebra $\mathcal{H} := \mathcal{H}(W, S, L)$ of (W, S, L) is by definition the associative $\mathcal{A}$-algebra with an $\mathcal{A}$-basis $\{T_w | w \in W\}$ as a free $\mathcal{A}$-module, subject to the multiplication rule:

$$(1.1.1) \quad \begin{aligned} T_s^2 &= (v_s - v_s^{-1})T_s + T_e && \text{for } s \in S, \\ T_x T_y &= T_{xy} && \text{for } x, y \in W \text{ with } \ell(xy) = \ell(x) + \ell(y), \end{aligned}$$

where e is the identity element of W. For any $n \in \mathbb{Z}$, let $\mathcal{A}_{\leq n} := \bigoplus_{m; m \leq n} \mathbb{Z}v^m$, $\mathcal{A}_{<n} := \bigoplus_{m; m<n} \mathbb{Z}v^m$, $\mathcal{H}_{\leq n} := \bigoplus_{w \in W} \mathcal{A}_{\leq n} T_w$ and $\mathcal{H}_{<n} := \bigoplus_{w \in W} \mathcal{A}_{<n} T_w$. Define a ring involution $\bar{\ } : \mathcal{A} \to \mathcal{A}$ by $\overline{\sum a_i v^i} = \sum a_i v^{-i}$ with $i, a_i \in \mathbb{Z}$ and a ring involution $\bar{\ } : \mathcal{H} \to \mathcal{H}$ by $\overline{\sum a_w T_w} = \sum \overline{a_w} T_{w^{-1}}^{-1}$ with $a_w \in \mathcal{A}$.

1.2. For any $w \in W$, there exists a unique $c_w \in \mathcal{H}_{\leq 0}$ satisfying that $\overline{c_w} = c_w$ and $c_w \equiv T_w (\bmod \mathcal{H}_{<0})$. Then $\{c_w \mid w \in W\}$ forms an $\mathcal{A}_{\leq 0}$-basis of $\mathcal{H}_{\leq 0}$ and an $\mathcal{A}$-basis of $\mathcal{H}$ (see [Lu2, Theorem 5.2]).

For any $y, w \in W$, define $p_{y,w} \in \mathcal{A}_{\leq 0}$ by the relation $c_w = \sum_{y \in W} p_{y,w} T_y$. Then $p_{y,w} = 0$ if $y \not\leq w$, $p_{w,w} = 1$ and $p_{y,w} \in \mathcal{A}_{<0}$ if $y < w$.

For any $w \in W$ and $s \in S$, we have

$$(1.2.1)\ c_s c_w = \begin{cases} (v_s + v_s^{-1})c_w, & \text{if } sw < w, \\ c_{sw} + \sum_{y; sy<y} \mu^s_{y,w} c_y & \text{if } sw > w, \end{cases} \qquad \text{([Lu2, Theorem 6.6])}$$

with $\mu^s_{y,w} \in \mathcal{A}$ satisfying $\bar{\mu}^s_{y,w} = \mu^s_{y,w}$.

1.3. For any $w, x, y \in W$, the notation $w = x \cdot y$ means that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. In this case, call w a *left-extension* of y and a *right-extension* of x. Call $w = s_1 s_2 \cdots s_r$ with $s_i \in S$ a *reduced expression* if $r = \ell(w)$. Define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

For $f, g \in \mathcal{A}$ and $n \in \mathbb{Z}$, write $f \overset{\text{mod } \mathcal{A}_{\leq n}}{\equiv} g$ (resp. $f \overset{\text{mod } \mathcal{A}_{<n}}{\equiv} g$) if $f - g \in \mathcal{A}_{\leq n}$ (resp. $f - g \in \mathcal{A}_{<n}$). For $n = 0$, we denote $f \overset{\text{mod } \mathcal{A}_{<n}}{\equiv} g$ simply by $f \equiv g$.

For $z \in W$ and $\alpha \in \mathcal{H}$, we say z *appears* in α if c_z appears with nonzero coefficient when we write α in the c-basis.

The notation δ is the Kronecker delta. In particular, for any $x, y \in W$,

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The following results are known: Let $y, w \in W$ and $s, t \in S$.

(0) If $y \leq w$ and $s \in \mathcal{L}(w) - \mathcal{L}(y)$, then $sy \leq w$. This is called the lifting property.

(1) If $y \leq w$, then $\deg p_{y,w} \geq L(y) - L(w)$.

(2) If a reduced expression of y can be obtained from that of w by deleting a factor s, then $p_{y,w} = v_s^{-1}$ (see [Lu2, Section 6]).

(3) If $y < w$ and $s \in \mathcal{L}(w) - \mathcal{L}(y)$, then $p_{y,w} = v_s^{-1} p_{sy,w}$. This is an easy consequence of [Lu2, Proposition 6.3].

(4) Assume $sy < y < w < sw$. Then $\overline{\mu^s_{y,w}} = \mu^s_{y,w}$ and $\sum_{z; y \leq z < w; sz < z} \mu^s_{z,w} p_{y,z} \equiv v_s p_{y,w}$ (see [Lu2, Proposition 6.3]). This implies that $\deg \mu^s_{y,w} \leq L(s) - \min\{L(t) | t \in S\}$.

(5) $p_{sy,sw} = p_{y,w} + v_s p_{sy,w} - \sum_{z; sy<z<w; sz<z} \mu^s_{z,w} p_{sy,z}$ if $s \notin \mathcal{L}(y) \cup \mathcal{L}(w)$ (see [Lu2, the proof of Theorem 6.6]).

(6) If $sy < y < w < sw$ and $t \in \mathcal{L}(w) - \mathcal{L}(y)$ and $L(t) \geq L(s)$, then $\mu^s_{y,w} \neq 0$ if and only if $L(t) = L(s)$ and $w = ty$ (see [Sh1, Proposition 2.6]).

Let $J \subseteq S$. Denote by W_J the subgroup of W generated by J. Then W_J is also a Coxeter group with J its Coxeter generator set. We can define the weighted Coxeter group $(W_J, J, L|_{W_J})$, the associated Iwahori-Hecke algebra $\mathcal{H}_J := \mathcal{H}(W_J, J, L|_{W_J})$ and the polynomials $p^J_{x,y} \in \mathcal{A}_{\leq 0}$ and $\mu^{s,J}_{x,y} \in \mathcal{A}$ for $x, y \in W_J$ and $s \in J$ accordingly, where $L|_{W_J}$ is the restriction of L to W_J.

Any $w \in W$ can be written uniquely in the form $w = w_J w^J$ with $w_J \in W_J$ and w^J the shortest element in the coset $W_J w$.

(7) If $y, w \in W$ satisfy $w^J = y^J$ then $p_{y,w} = p^J_{y_J, w_J} = p_{y_J, w_J}$ and $\mu^s_{y,w} = \mu^{s,J}_{y_J, w_J} = \mu^s_{y_J, w_J}$ (where $s \in J$) whenever they are defined (see [Lu2, Lemma 9.10]).

1.4. Following Lusztig in [Lu2, Subsections 10.1 and 13.1], we set, for $y, w \in W$,

$$q'_{y,w} = \sum (-1)^n p_{z_0,z_1} p_{z_1,z_2} \cdots p_{z_{n-1},z_n} \in \mathcal{A}$$

summing over all sequences $y = z_0 < z_1 < z_2 \cdots < z_n = w$ in W. Then $q'_{w,w} = 1$, $q'_{y,w} \in \mathcal{A}_{<0}$ if $y \neq w$, and $T_w = \sum_{y \in W} q'_{y,w} c_y$ (see [Lu2, Subsection 10.7]).

For $x,y,z\in W$, define $f_{x,y,z}, f'_{x,y,z}, h_{x,y,z}, h'_{x,y,z}, \alpha_{x,y,z}, \beta_{x,y,z}\in\mathcal{A}$ by

$$\begin{aligned} T_xT_y &= \sum_{z\in W} f_{x,y,z}T_z = \sum_{z\in W} f'_{x,y,z}c_z,\\ c_xc_y &= \sum_{z\in W} h_{x,y,z}c_z = \sum_{z\in W} h'_{x,y,z}T_z.\\ c_xT_y &= \sum_{z\in W} \alpha_{x,y,z}T_z = \sum_{z\in W} \beta_{x,y,z}c_z.\end{aligned}$$

Following [Lu2, Subsection 13.1 (a)-(c)], we have

(1.4.1) $$f_{x,y,z} = \sum_{z'} p_{z,z'}f'_{x,y,z'} = \sum_{x',y'} q'_{x',x}q'_{y',y}h'_{x',y',z},$$

(1.4.2) $$f'_{x,y,z} = \sum_{z'} q'_{z,z'}f_{x,y,z'},$$

(1.4.3) $$h_{x,y,z} = \sum_{x',y'} p_{x',x}p_{y',y}f'_{x',y',z} = \sum_{z'} q'_{z,z'}h'_{x,y,z'},$$

(1.4.4) $$h'_{x,y,z} = \sum_{x',y'} p_{x',x}p_{y',y}f_{x',y',z} = \sum_{z'} p_{z,z'}h_{x,y,z'},$$

(1.4.5) $$h_{x,y,z} = h'_{x,y,z} - \sum_{z'>z} h_{x,y,z'}p_{z,z'},$$

(1.4.6) $$\alpha_{x,y,z} = \sum_{x'} p_{x',x}f_{x',y,z},$$

(1.4.7) $$\beta_{x,y,z} = \sum_{y'} q'_{y',y}h_{x,y',z}.$$

The following result can be obtained from definitions and simple induction:

(1.4.8) If $f_{x,y,z}\neq 0$ (resp., $f'_{x,y,z}\neq 0, h_{x,y,z}\neq 0, h'_{x,y,z}\neq 0, \alpha_{x,y,z}\neq 0, \beta_{x,y,z}\neq 0$) then $\deg f_{x,y,z}$ (resp., $\deg f'_{x,y,z}, \deg h_{x,y,z}, \deg h'_{x,y,z}, \deg\alpha_{x,y,z}, \deg\beta_{x,y,z}$) $\leq \min\{L(x),L(y)\}$, and the equality holds if and only if there exists some $I\subseteq S$ with $|W_I|<\infty$ such that either $x\in W_I$ and $I\subseteq\mathcal{L}(y)$, or $y\in W_I$ and $I\subseteq\mathcal{R}(x)$. When the equivalent conditions hold, we have either $z=y$ or $z=x$.

(1.4.9) $h_{x,y,z}\neq 0\Rightarrow \mathcal{L}(x)\subseteq\mathcal{L}(z), \mathcal{R}(y)\subseteq\mathcal{R}(z), z\leq_R x$ and $z\leq_L y$ (see [Lu2]).

1.5. We say that (W,S,L) is *bounded* if there exists some $N\in\mathbb{N}$ such that $\deg h_{x,y,z}\leq N$ for any $x,y,z\in W$ (see [Lu2, Subsection 13.2]). We shall assume (W,S,L) is bounded throughout the rest of the article. In this case, we may define a function $\mathbf{a}:W\to\mathbb{N}$ such that for any $z\in W$, $h_{x,y,z}\in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}]$ for all $x,y\in W$ and $h_{x',y',z}\notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}]$ for some $x',y'\in W$. For any $x,y,z\in W$, define $\gamma_{x,y,z}\in\mathbb{Z}$ by the condition

(1.5.1)
$$h_{x,y,z} = \gamma_{x,y,z^{-1}}v^{\mathbf{a}(z)} + \text{strictly lower powers of } v \quad \text{(see [Lu2, Subsection 13.6])}.$$

The following facts are well known: Let $x,y,z\in W$.

(1.5.2) $\mathbf{a}(e)=0$ and $\mathbf{a}(z)\geq\min\{L(s)|s\in S\}$ if $z\neq e$ (see [Lu2, Proposition 13.7]).

(1.5.3) $\mathbf{a}(z) = \mathbf{a}(z^{-1})$ and $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$ (see [Lu2, Proposition 13.9]).

(1.5.4) If $h_{x,y,z} \neq 0$, then $\mathcal{L}(x) \subseteq \mathcal{L}(z)$ and $\mathcal{R}(y) \subseteq \mathcal{R}(z)$ (see [Lu2, Lemma 8.6]).

1.6. Let (W, S, L) be a Coxeter system.

Let W be finite with the longest element w_0. It is known that conjugation by w_0 on W preserves the length. If there exists some $s \in S$ such that $L(s) < L(t)$ for any $t \in S - \{s\}$, then $w_0 s = s w_0$, and $w_0 s$ has the second largest weight among the elements of (W, S, L).

In (W, S), call $s \in S$ *universal* if the order of st is infinite for any $t \in S - \{s\}$.

Let $I \subseteq S$. For any function ξ on W, the notation ξ_I stands for the restriction of ξ to W_I. For example, $\mathbf{a}_I(x)$ for $x \in W_I$ is $\mathbf{a}(x)$ computed in terms of W_I. If W_I is finite, then for any $\Gamma \subseteq W_I$, define $\tau_{I,\Gamma} = \max\{\mathbf{a}_I(w) | w \in W_I - \Gamma\}$.

For any $x \in W$, set $\mathcal{C}_x = \{z \in W | z = z' \cdot x \text{ for some } z' \in W\}$, and $\Omega_x = \{z \in W | z = y_1 \cdot x \cdot y_2 \text{ for } y_1, y_2 \in W\}$. For any $X \subseteq W$, set $\Omega_X = \cup_{x \in X} \Omega_x$.

In [Lu2, Chapter 14], Lusztig proposed 15 conjectures (P1)-(P15) on (W, S, L), one of which is

(P7) For any $x, y, z \in W$, $\gamma_{x,y,z} = \gamma_{y,z,x}$.

Denote $x \leftarrow_L y$ in W, if there exists some $s \in S$ such that either $x = s \cdot y$ or $\mu^s_{x,y} \neq 0$. Denote $x \leftarrow_R y$ in W, if $x^{-1} \leftarrow_L y^{-1}$. Denote $x \leftarrow_{LR} y$ in W, if either $x \leftarrow_L y$ or $x \leftarrow_R y$. Through $\leftarrow_L$ (resp. $\leftarrow_R$, $\leftarrow_{LR}$), the preorder $\leq_L$ (resp. $\leq_R$, $\leq_{LR}$) and the corresponding equivalence relation $\sim_L$ (resp. $\sim_R$, $\sim_{LR}$) in W can be defined as in [Lu2, Subsection 8.1]. The equivalence classes of W with respect to $\sim_L, \sim_R, \sim_{LR}$ are called *left cells, right cells, two-sided cells*, respectively.

The following result follows directly by the definition of the relation $\sim_{LR}$.

Lemma 1.7. *Let $I \subseteq S$ and $X \subseteq W_I$. Suppose one of the following conditions is satisfied.*

(1) For any $x \in X$ and $y \in W - X$, $y \not\leftarrow_{LR} x$.

(2) For any $x \in X$ and $y \in W - X$, $x \not\leftarrow_{LR} y$.

The for any $x_1, x_2 \in X$, $x_1 \sim_L x_2$ (resp. $x_1 \sim_{LR} x_2$) if and only if $x_1 \sim_{I,L} x_2$ (resp. $x_1 \sim_{I,LR} x_2$), where $\sim_{I,L}$ and $\sim_{I,LR}$ are the relations $\sim_L$ and $\sim_{LR}$ defined in W_I.

Lemma 1.8. *Let $w \in W$, $s \in S - \mathcal{L}(w)$ and $\mathscr{S}' := \{y \in W | sy < y < w\}$.*

(1) If for any $y' \in \mathscr{S}'$, $\deg v_s p_{y',w} \leq 0$, then $\mu^s_{y',w} \equiv v_s p_{y',w}$.

(2) Suppose that $\mathscr{S}'$ has a unique maximal element y_0, and $\deg(\mu^s_{y_0,w} p_{y,y_0} - v_s p_{y,w}) \leq 0$ for any $y \in \mathscr{S}' - \{y_0\}$. Then for $y' \in \mathscr{S}'$,

$$\mu^s_{y',w} \equiv \begin{cases} v_s p_{y',w} & \text{if } y' = y_0, \\ v_s p_{y',w} - \mu^s_{y_0,w} p_{y',y_0} & \text{if } y' \neq y_0. \end{cases}$$

Proof. This follows by 1.3(4) and the induction on $\ell(w) - \ell(y') \geq 1$. □

2. Cells of W when L is not necessarily constant on W_I

In this section, we assume that $S = I \sqcup J$, and that J consists of universal elements. We shall describe the left cells and two-sided cells of the weighted Coxeter system (W, S, L) intersecting J nontrivially, in the case $\max\{L(s) | s \in J\} < \min\{L(t) | t \in I\}$ (Theorem 2.2) and in the case W_I is finite and $\min\{L(s) | s \in J\} > L(w_0)$, where w_0 is the longest element of W_I (Theorems 2.11).

The following fact is obvious: If $w \in W-\{e\}$, then either $\mathcal{L}(w) \subseteq I$ or $\mathcal{L}(w)=\{s\}$ for some $s \in J$. Thus for $w \in W$, $I \subseteq \mathcal{L}(w)$ implies $\mathcal{L}(w)=I$.

Lemma 2.1. *Suppose* $\max\{L(s)|s \in J\} < \min\{L(t)|t \in I\}$.

(1) Let $u \in W_J$ *and* $z \in W$ *satisfy* $\ell(uz)=\ell(u)+\ell(z)$ *and*

$$\max\{L(s)|s \in J; s \leq u\} < \max\{L(t)|t \in \mathcal{L}(z)\}.$$

Then we have

(a) $c_u c_z = c_{uz}$.

(b) Let $t \in \mathcal{L}(z)$ *and* $s \in J-\mathcal{L}(u)$ *satisfy* $L(s)=L(t)$. *Define* $\mu_1, \mu_2 \in \mathcal{A}$ *by*

$$\mu_1 = \bar{\mu}_1 \equiv \begin{cases} v^{L(s)} p_{e,u} & \text{if } t = s, \\ 0 & \text{if } t \neq s, \end{cases}$$

$$\mu_2 = \begin{cases} 1 & \text{if } s \in \mathcal{L}(utz), \\ 0 & \text{if } s \notin \mathcal{L}(utz). \end{cases}$$

(i) If $u \neq e$ *then*

$$c_s c_{uz} = c_{suz} + \mu_1 c_z + \mu_2 c_{utz}.$$

(ii) If $u = e$ *and* $t \neq s$, *then*

$$c_s c_{uz} = c_{sz} + \mu_2 c_{tz}.$$

(2) If $x \in W_J$ *and* $y \in W - W_J$, *then* $x \not\leftarrow_{LR} y$.

(3) For any $x, y \in W_J$, *we have* $x \sim_L y$ *(resp.* $x \sim_{LR} y$*) if and only if* $x \sim_{J,L} y$ *(resp.* $x \sim_{J,LR} y$*).*

Proof. (1) Apply induction on $n := \ell(u) \geq 0$. When $n = 0$, we only have to check (b)(ii). But this follows by 1.3 (3) and 1.3 (4). Now assume $n > 0$. Write $u = r \cdot u'$ for some $r \in J$ and $u' \in W_J$.

Since $u, u' \in W_J$ and $\ell(uz) = \ell(u) + \ell(z)$, for any $u'' \in W_J$ with $\mu^r_{u'',u} \neq 0$, we have $\ell(u''z) = \ell(u'') + \ell(z)$, thus $c_{u''} c_z = c_{u''z}$ by the inductive hypothesis. Also, $c_{u'} c_z = c_{u'z}$ following by the inductive hypothesis. Consequently, $c_r c_{u'z} = c_r c_{u'} c_z$. So

$$\sum_{z';rz'<z'<u'z} \mu^r_{z',u'z} c_{z'} = \left(\sum_{u'';ru''<u''<u'} \mu^r_{u'',u'} c_{u''} \right) c_z = \sum_{u'';ru''<u''<u'} \mu^r_{u'',u'} c_{u''z}. \tag{2.1.1}$$

Then

(2.1.2)
$$\begin{aligned} & c_u c_z \\ = & \left(c_r c_{u'} - \sum_{u'';ru''<u''<u'} \mu^r_{u'',u'} c_{u''}\right) c_z \text{ (by (1.2.1))} \\ = & c_r c_{u'z} - \sum_{u'';ru''<u''<u'} \mu^r_{u'',u'} c_{u''z} \text{ (by the inductive hypothesis for (a))} \\ = & c_{uz} + \sum_{z';rz'<z'<u'z} \mu^r_{z',u'z} c_{z'} - \sum_{u'';ru''<u''<u'} \mu^r_{u'',u'} c_{u''z} \\ = & c_{uz}. \text{ (by (2.1.1))} \end{aligned}$$

Thus (a) is true.

Let $y \in W$ be with $sy < y < uz$. If $y \neq z$, then $y < z$. This is because $s \in \mathcal{L}(y) \subseteq J$, $s \not\leq u \in W_J$, and every element in J is universal. So if $t = s$, then by 1.3(4), $\mu^s_{z,uz} \equiv v^{L(s)} p_{z,uz} \equiv v^{L(s)} p_{e,u}$. If $t \neq s$, then $sz > z$.

Now suppose $y \neq z$. By comparing the coefficients of T_y on both sides of the equation in (2.1.1), we have

$$\begin{aligned} & v^{L(s)} p_{y,uz} \\ = & v^{L(s)} \sum_{x_1,x_2} p_{x_1,u} p_{x_2,z} f_{x_1,x_2,y} \\ = & v^{L(s)} p_{e,u} p_{y,z} + v^{L(s)} \sum_{x_1,x_2;x_1\neq e} p_{x_1,u} p_{x_2,z} f_{x_1,x_2,y}. \end{aligned}$$

For any $x_1 \leq u$ and $x_2 \leq z$ with $x_2 \neq e$ and $f_{x_1,x_2,y} \neq 0$, we have $tx_2 > x_2$. If $tx_2 < x_2$, then $T_{x_1}T_{x_2} = T_{x_1\cdot x_2}$. We have $y = x_1 \cdot x_2$ as $f_{x_1,x_2,y} \neq 0$. But this is not possible as $t \in \mathcal{L}(y)$, $t \notin \mathcal{L}(x_1 \cdot x_2) = \mathcal{L}(x_1)$. Consequently,

(2.1.3)
$$\begin{aligned} & v^{L(s)} p_{y,uz} \\ = & v^{L(s)} p_{e,u} p_{y,z} + v^{L(s)} \sum_{x_1,x_2;x_1\neq e,tx_2>x_2} p_{x_1,u} p_{x_2,z} f_{x_1,x_2,y} \\ = & v^{L(s)} p_{e,u} p_{y,z} + \sum_{x_1,x_2;x_1\neq e,tx_2>x_2} p_{x_1,u} p_{tx_2,z} f_{x_1,x_2,y} \text{ (by 1.3 (3))} \\ \equiv & v^{L(s)} p_{e,u} p_{y,z} + \begin{cases} 1 & \text{if } z = tu^{-1}y, \\ 0 & \text{if } z \neq tu^{-1}y. \end{cases} \end{aligned}$$

If $t = s$, then by (2.1.3),

(2.1.4)
$$\begin{aligned} & v^{L(s)} p_{y,uz} - \mu^s_{y,uz} p_{y,z} \\ = & v^{L(s)} p_{y,uz} - v^{L(s)} p_{e,u} p_{y,z} \\ \equiv & \begin{cases} 1 & \text{if } z = su^{-1}y, \\ 0 & \text{if } z \neq su^{-1}y. \end{cases} \end{aligned}$$

If $t \neq s$, $h_{s,uz,z} = 0$ by (1.4.9) and the assumption $t \in \mathcal{L}(z)$, $s \in J$, and

$$v^{L(s)} p_{y,uz} \equiv v^{L(s)} p_{e,u} p_{y,z} + \begin{cases} 1 & \text{if } z = tu^{-1}y, \\ 0 & \text{if } z \neq tu^{-1}y, \end{cases} = p_{e,u} p_{ty,z} + \begin{cases} 1 & \text{if } z = tu^{-1}y, \\ 0 & \text{if } z \neq tu^{-1}y, \end{cases} \equiv \begin{cases} 1 & \text{if } z = tu^{-1}y, \\ 0 & \text{if } z \neq tu^{-1}y, \end{cases} \tag{2.1.5}$$

by (2.1.3). So (b) follows by (2.1.4), (2.1.5) and Lemma 1.8.

(2) By (1), we have $\mu^r_{x,y} = 0$ for any $r \in J - \mathcal{L}(y)$. This implies $x \not\leftarrow_L y$. Similarly, we can show that $x \not\leftarrow_R y$.

(3) This follows from (2) and Lemma 1.7.

□

Theorem 2.2. *If* $\max\{L(s)|s \in J\} < \min\{L(t)|t \in I\}$*, then* W_J *is a union of left cells, as well as a union of two-sided cells of* W*. The left (resp. two-sided) cells of* W *in* W_J *are also left (resp. two-sided) cells of* W_J *when considered as a Coxeter group.*

Proof. This is a direct result of Lemma 2.1. □

Since J consists of universal elements, the Coxeter group W_J is a universal Coxeter group. We refer the readers to [SY] for the detailed description for the left and two-sided cells in any weighted universal Coxter groups.

Lemma 2.3. *Suppose that* W_I *is finite with* w_0 *the longest element, and that* $\min\{L(s)|s \in J\} \geq L(w_0)$*. Then for any* $u \in W_I$ *and* $w \in W$ *with* $\mathcal{L}(w) \subseteq J$*,*

$$c_u c_w = \begin{cases} c_{uw} + c_{sw} & \text{if } u = w_0 \text{ and } \mathcal{L}(w) = \{s\} \text{ and } \mathcal{L}(sw) = I \text{ and } L(s) = L(w_0), \\ c_{uw} & \text{otherwise.} \end{cases}$$

Proof. We have $uw = u \cdot w$ by our assumption. When $w = e$, the result is obvious. Now assume $w > e$. Then $\mathcal{L}(w) = \{s\}$ for some $s \in J$ by the assumption $\mathcal{L}(w) \subseteq J$. Write

$$c_u c_w = c_{uw} + \sum_{z < uw} h_{u,w,z} c_z. \tag{2.3.1}$$

For any $z \in W$ with $z < uw$, by comparing the coefficients of T_z on both sides of (2.3.1), we get

$$h_{u,w,z} = \sum_{x_1 \leq u, x_2 \leq w} p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z} - \sum_{z < z'' < uw} h_{u,w,z''} p_{z,z''} - p_{z,uw}.$$

So

$$h_{u,w,z} \equiv \sum_{x_1 \leq u, x_2 \leq w} p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z} - \sum_{z < z'' < uw} h_{u,w,z''} p_{z,z''}. \tag{2.3.2}$$

If for all $z < uw$ in W,

$$\deg \sum_{x_1 \leq u, x_2 \leq w} p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z} \leq 0, \tag{2.3.3}$$

then we will have

$$h_{u,w,z} \equiv \sum_{x_1 \le u, x_2 \le w} p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z}, \tag{2.3.4}$$

for all $z \in W$ with $z < uw$, following by (2.3.3) and induction on $\ell(uw) - \ell(z) > 0$.

Now let $z' \in W$ satisfy $z' < uw$. If $\sum_{x_1 \le u, x_2 \le w} p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} \not\equiv 0$, then there exist some $x_1 \le u$ and $x_2 \le w$ with $p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} \not\equiv 0$. In this case, we claim that $s \notin \mathcal{L}(x_2)$. For otherwise, since $s \in \mathcal{L}(x_2)$ is universal and $x_1 \in W_I$, we have $x_1 x_2 = x_1 \cdot x_2$ and hence $p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} \equiv 0$ by the assumption $z' < uw$, a contradiction. The claim is proved. Then

$$p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} = v^{-L(s)} p_{x_1,u} p_{sx_2,w} f_{x_1,x_2,z'}. \tag{2.3.5}$$

by 1.3 (3). Since $x_1 \le u$ in W_I, we see by (1.4.8) that $\deg f_{x_1,x_2,z'} \le L(w_0)$, and the equality holds if and only if $x_1 = u = w_0$ and $I \subseteq \mathcal{L}(x_2)$ and $x_2 = z'$. This implies by the assumption $L(s) \ge L(w_0)$, (1.4.8) and (2.3.5) that $p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} \not\equiv 0$ if and only if $L(s) = L(w_0)$, $x_1 = u = w_0$, $w = sx_2 = sz'$ and $I \subseteq \mathcal{L}(z')$. When the equivalent conditions hold, we have $p_{x_1,u} p_{x_2,w} f_{x_1,x_2,z'} \equiv 1$ by (2.3.5). So (2.3.3) is proved and (2.3.4) holds. We have

$$h_{u,w,z'} = \begin{cases} 1 & \text{if } z = sw \text{ and } u = w_0 \text{ and } \mathcal{L}(w) = \{s\} \text{ and } \mathcal{L}(sw) = I \text{ and } L(s) = L(w_0), \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. □

Corollary 2.4. *Suppose that W_I is finite with w_0 the longest element and that $\min\{L(s) | s \in J\} \ge L(w_0)$. Let $u \in W_I - \{w_0\}$, $w \in W$, $s \in J$ and $t \in I - \mathcal{L}(u)$ satisfy $\ell(uw) = \ell(u) + \ell(w)$ and $\mathcal{L}(w) \subseteq \{s\}$. Set*

$$\varepsilon = \begin{cases} 1 & \mathcal{L}(w) = \{s\}, tu = w_0, I \subseteq \mathcal{L}(sw) \textit{ and } L(s) = L(w_0), \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$c_t c_{uw} = c_{tuw} + \varepsilon c_{sw} + \sum_{u'; tu' < u} \mu^t_{u',u} c_{u'w}. \tag{2.4.1}$$

Proof. We have

$$\begin{aligned} c_t c_{uw} &= c_t c_u c_w \\ &= \left(c_{tu} + \sum_{u'; tu' < u' < u} \mu^t_{u',u} c_{u'} \right) c_w \\ &= c_{tu} c_w + \sum_{u'; tu' < u' < u} \mu^t_{u',u} c_{u'w} \\ &= c_{tuw} + \varepsilon c_{sw} + \sum_{u'; tu' < u' < u} \mu^t_{u',u} c_{u'w}. \end{aligned}$$

where the last equality follows by Lemma 2.3. □

For any $n > n'$ in $\mathbb{N}$, we state a condition **X(n,n')** condition on (W, S) with $S = I \sqcup J$ and J consisting of universal elements.

X(n,n') W_I is finite with w_0 the longest element, and $\min\{L(s)|s\in J\}\geq L(w_0)$. For any $x\in W$ with $\ell(x)=n$, if $x=x_1\cdot s\cdot x_2$ with $s\in J$, $x_1,x_2\in W$, $\ell(x_1)=n'$ and one of the following conditions (1)-(2) holds:

(1) $x_1\in W_I$ and $L(x_1)<L(s)$,
(2) $x_1\notin W_I$ and $L(s)>\max\{L(s')|s'\in J, s'\leq x_1\}$,

then $c_{x_1}c_{sx_2}=c_x$.

We will prove that the condition **X(n,n')** holds on (W,S) for any $n>n'$ in $\mathbb{N}$. But at the moment, it is only an assumption for the proof of some other facts.

Lemma 2.5. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s\in J\}\geq L(w_0)$. Let $w\in W$ and $s\in J-\mathcal{L}(w)$. Assume $s\leq w$. Write $w=x\cdot s\cdot z$ for some $x,z\in W$ with $s\not\leq x$.*

(1) If $I\subseteq\mathcal{L}(x)$ and $L(w_0)=L(s)$, then

$$c_sc_w=\begin{cases}c_{sw}+c_{sz} & \text{if } x=w_0,\\ c_{sw} & \text{if } x\neq w_0.\end{cases}$$

*(2) Suppose that **X(n,n')** holds on W for any $n'<n$ in $\mathbb{N}$ whenever $n\leq\ell(w)$, and that $x=x_1\cdot s'\cdot x_2$ for some $s'\in J$ with $L(s')\geq L(s)$, where either $x_1\in W_I$ with $L(x_1)<L(s')$ or $x_1\notin W_I$ with $L(s')>\max\{L(s'')|s''\in J, s''\leq x_1\}$. Then*

$$c_sc_w=\begin{cases}c_{sw}+c_{sz} & \text{if } L(s)=L(s'), x_2=x_1^{-1},\\ c_{sw} & \text{otherwise.}\end{cases}$$

Proof. (1) Since $I\subseteq\mathcal{L}(x)$, we have $\mathcal{L}(x)=I$. Write $x=w_0\cdot x'$ for some $x'\in W$. Then $s\not\leq x'$ as $s\not\leq x$. Let $y\in W$ satisfy $sy<y<w$. Then $\mathcal{L}(y)=\{s\}$ as s is universal. By 1.3(3) and the fact $L(s)=L(w_0)$, we have

$$v^{L(s)}p_{y,w}=v^{L(s)-L(w_0)}p_{w_0y,w}\equiv\begin{cases}1 & \text{if } w=w_0y,\\ 0 & \text{if } w\neq w_0y.\end{cases}$$

Since $sy<y=w_0w=x'\cdot s\cdot z$, we have $x'=e$ as s is universal. So by Lemma 1.8, we have $\mu^s_{y,w}=\begin{cases}1 & \text{if } x=w_0,\\ 0 & \text{if } x\neq w_0.\end{cases}$ The result follows.

(2) Since **X(n,n')** holds for any $n'<n$ whenever $n\leq\ell(w)$, we have

$$c_{x_1}c_{s'x_2sz}=c_w. \tag{2.5.1}$$

By comparing the coefficients of T_y on both sides of (2.5.1), we get

$$p_{y,w}=\sum_{y_1\leq x_1,y_2\leq s'x_2sz}p_{y_1,x_1}p_{y_2,s'x_2sz}f_{y_1,y_2,y}, \tag{2.5.2}$$

for any $sy<y<w$ in W. For such y, the relations $y_1\leq x_1$ and $y_2\leq s'x_2sz$ and $f_{y_1,y_2,y}\neq 0$ imply that $y_2=y_1^{-1}\cdot y$ and $f_{y_1,y_2,y}=1$ since $s\not\leq y_1$ and

$s \in \mathcal{L}(y)$ is universal. By (2.5.2) and the assumption $L(s') \geq L(s)$, we get

$$\begin{aligned} v^{L(s)} p_{y,w} &= v^{L(s)} \sum_{y_1 \leq x_1} p_{y_1,x_1} p_{y_1^{-1}y,s'x_2sz} \\ &= v^{L(s)-L(s')} \sum_{y_1 \leq x_1} p_{y_1,x_1} p_{s'y_1^{-1}y,s'x_2sz}. \\ &\equiv \begin{cases} 1 & \text{if } L(s)=L(s'), x_2 = x_1^{-1}, y = sz, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So the result follows from Lemma 1.8.

□

Lemma 2.6. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$. Let $s \in J$ and $w \in W$ satisfy $sw > w \geq s$. Let $n = \ell(w)$. Write $w = x \cdot s \cdot z$ for $x, z \in W$ with $s \not\leq x$. Suppose that **X(n,n')** holds on W for any $n' < n$ in $\mathbb{N}$ whenever $n \leq \ell(w)$. If either $x \in W_I$ with $L(s) > L(x)$ or $x \notin W_I$ with $L(s) > \max\{L(s')|s' \in J, s' < x\}$, then*

$$c_s c_w = \begin{cases} c_{sw} + \mu_1 c_{sz} + c_{xz} & \text{if } \ell(xz) = \ell(z) - \ell(x) \text{ and } s \in \mathcal{L}(xz), \\ c_{sw} + \mu_1 c_{sz} & \text{otherwise,} \end{cases}$$

where $\mu_1 \in \mathcal{A}$ is given by $\mu_1 = \bar{\mu}_1 \equiv v^{L(s)} p_{e,x}$.

Proof. By 1.3 (4) and (7),

$$\mu^s_{sz,xsz} \equiv v^{L(s)} p_{sz,xsz} = v^{L(s)} p_{e,x}. \tag{2.6.1}$$

By $\mathbf{X}(\ell(w), \ell(x))$, we have

$$c_x c_{sz} = c_w. \tag{2.6.2}$$

Let $y \in W$ satisfy $sy < y < w$ and $y \neq sz$. Then $y < sz$ as $s \not\leq x$ and s is universal. By comparing the coefficients of T_y on both sides of (2.6.2), we get

$$p_{y,w} = \sum_{y_1 \leq x, y_2 \leq sz} p_{y_1,x} p_{y_2,sz} f_{y_1,y_2,y}. \tag{2.6.3}$$

So we see by (2.6.1) and (2.6.3) and the equation $w = x \cdot s \cdot z$ that

$$\begin{aligned} v^{L(s)} p_{y,xsz} - \mu^s_{sz,xsz} p_{y,sz} &\equiv \sum_{e < y_1 \leq x, y_2 \leq sz} v^{L(s)} p_{y_1,x} p_{y_2,sz} f_{y_1,y_2,y} \\ &= \sum_{e < y_1 \leq x} v^{L(s)} p_{y_1,x} p_{y_1^{-1}y,sz} \\ &= \sum_{e < y_1 \leq x} p_{y_1,x} p_{sy_1^{-1}y,sz} \\ &\equiv \begin{cases} 1 & \text{if } y = xz, \\ 0 & \text{if } y \neq xz, \end{cases} \end{aligned} \tag{2.6.4}$$

where we use the observation that if $s \in \mathcal{L}(y)$ and $s \not\leq y_1$ and $f_{y_1,y_2,y} \neq 0$ then $y_1 y_2 = y$ and $\ell(y) = \ell(y_2) - \ell(y_1)$ and $f_{y_1,y_2,y} = 1$. So our result follows from (2.6.1), (2.6.4) and Lemma 1.8. □

Lemma 2.7. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$. Then **X(n,n')** holds on W for any $n > n'$ in $\mathbb{N}$.*

Proof. We only have to consider the case when $n' > 0$. Let $x = x_1 \cdot s \cdot x_2$ for some $x_1, x_2 \in W$ and $s \in J$ with $\ell(x) = n$ and $\ell(x_1) = n'$. When $x_1 \in W_I$ and $L(x_1) < L(s)$, we have $c_{x_1} c_{sx_2} = c_x$ by Lemma 2.3 with x_1, sx_2 in the places of u, w respectively. Now assume that $x_1 \notin W_I$ and that $L(s) > \max\{L(s')|s' \in J, s' \leq x_1\}$. We shall prove $c_{x_1} c_{sx_2} = c_x$ by induction first on $n \geq 2$ and then on n', $1 \leq n' < n$. When n = 2, we have $\ell(x_1) = n' = 1$ and $\ell(x_2) = 0$, the result is obvious. Now assume $n > 2$. As $x_1 \notin W_I$, we can write x_1 uniquely as $x_1 = x_{11} \cdot s' \cdot x_{12}$ for $x_{11} \in W_I$, $x_{12} \in W$ and $s' \in J$.

By the inductive hypothesis, we see that **X(m,m')** and **X(n,n")** hold on W for any $m' < m < n$ and $n'' < \min\{n, n'\}$ in $\mathbb{P}$. We are going to prove **X(n,n')** on W.

First assume that $x_{11} = w_0$. Write $x' = s'x_{12}$. By Lemma 2.3, if there exists some $s'' \in J \cap \mathcal{L}(x')$ with $I \subseteq \mathcal{L}(s''x')$ and $L(s'') = L(w_0)$, then $c_{w_0} c_{x'} = c_{w_0 x'} + c_{s''x'}$ and $c_{w_0} c_{x'sx_2} = c_{w_0 x'sx_2} + c_{s''xsx_2}$. This implies that

$$\begin{aligned} c_{x_1} c_{sx_2} &= (c_{w_0} c_{x'} - c_z) c_{sx_2} \\ &= c_{w_0} c_{x'sx_2} - c_{zsx_2} \\ &= c_{w_0 x'sx_2} = c_x, \end{aligned}$$

by the inductive hypothesis. If there is no such s'' in $J \cap \mathcal{L}(x')$, then we see by Lemma 2.3 that $c_{w_0} c_{x'} = c_{w_0 x'}$ and $c_{w_0} c_{x'sx_2} = c_{w_0 x'sx_2}$, so

$$\begin{aligned} & c_{x_1} c_{sx_2} \\ = \; & c_{w_0} c_{x'} c_{sx_2} \\ = \; & c_{w_0} c_{x'sx_2} \\ = \; & c_{w_0 x'sx_2} = c_x, \end{aligned}$$

by the inductive hypothesis.

Now assume that $x_{11} \neq w_0$. Then $L(x_{11}) < L(s')$. We have $c_{x_{11}} c_{s'x_{12}} = c_{x_1}$ and $c_{x_{11}} c_{s'x_{12}sx_2} = c_x$ by Lemma 2.3. If $x_{11} > e$, then $c_{s'x_{12}} c_{sx_2} = c_{s'x_{12}sx_2}$ by the inductive hypothesis, and the result follows. If $x_{11} = e$, then $x_1 = s'x_{12}$ where $s' \in J$ and $L(s') < L(s)$.

Let $E_1 = \{z_1 \in W | s'z_1 < z_1 < x_{12}, \mu^{s'}_{z_1, x_{12}} \neq 0\}$, and $E_2 = \{z_2 \in W | s'z_2 < z_2 < x_{12}sx_2, \mu^{s'}_{z_2, x_{12}sx_2} \neq 0\}$. We claim that

(2.7.1) $z_1 \mapsto z_1 sx_2$ is a bijective map from the set E_1 to E_2 and satisfies $\mu^{s'}_{z_1, x_1} = \mu^{s'}_{z_1 sx_2, x_{12}sx_2}$ and $c_{z_1} c_{sx_2} = c_{z_1 sx_2}$.

If $s' \not\leq x_{12}$, then $c_{s'} c_{x_{12}} = c_{s'x_{12}}$. By Lemma 2.5(2), $c_{s'} c_{x_{12}sx_2} = c_{s'x_{12}sx_2}$. So (2.7.1) follows. If $s' \leq x_{12}$, then we can write $x_{12} = x^{(1)}_{12} \cdot s' \cdot x^{(2)}_{12}$ with $x^{(1)}_{12}, x^{(2)}_{12}$ and $s' \not\leq x^{(1)}_{12}$. If

(1) $\mathcal{L}(x^{(1)}_{12}) = I$, or
(2) $\mathcal{L}(x^{(1)}_{12}) \neq I$ and $\max\{L(s'')|s'' \in J, s'' \leq x^{(1)}_{12}\} \geq L(s')$, or
(3) $\mathcal{L}(x^{(1)}_{12}) \neq I$ and $\max\{L(s'')|s'' \in J, s'' \leq x^{(1)}_{12}\} < L(s')$,

then we can apply

(1) Lemma 2.5(1), or
(2) Lemma 2.5(2), or
(3) Lemma 2.6

respectively, to obtain (2.7.1).

Hence we have

$$\begin{aligned} c_{x_1}c_{sx_2} &= (c_{s'}c_{x_{12}} - \sum_{z_1\in E_1} \mu^{s'}_{z_1,x_{12}} c_{z_1})c_{sx_2} \\ &= c_{s'}c_{x_{12}}c_{sx_2} - \sum_{z_1\in E_1} \mu^{s'}_{z_1,x_{12}} c_{z_1}c_{sx_2} \\ &= c_{s'}c_{x_{12}sx_2} - \sum_{z_1\in E_1} \mu^{s'}_{z_1,x_{12}} c_{z_1sx_2} \\ &= c_x + \sum_{z_2\in E_2} \mu^{s'}_{z_2,x_{12}sx_2} c_{z_2} - \sum_{z_1\in E_1} \mu^{s'}_{z_1,x_{12}} c_{z_1sx_2} \\ &= c_x, \end{aligned}$$

by (2.7.1).

□

Lemmas 2.5-2.6 can be restated as follows.

Lemma 2.5′ . Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$. Let $w \in W$ and $s \in J-\mathcal{L}(w)$. If $s \not\leq w$, then $c_sc_w = c_{sw}$. If $s \leq w$, then $w = x \cdot s \cdot z$ for some $x, z \in W$ with $s \not\leq x$.

(1) If $I \subseteq \mathcal{L}(x)$ and $L(w_0) = L(s)$, then

$$c_sc_w = \begin{cases} c_{sw} + c_{sz} & \text{if } x = w_0, \\ c_{sw} & \text{if } x \neq w_0. \end{cases}$$

(2) Suppose that $x = x_1 \cdot s' \cdot x_2$ for some $s' \in J$ and $x_1, x_2 \in W$ with $L(s') \geq L(s)$, where either $x_1 \in W_I$ with $L(x_1) < L(s')$ or $x_1 \notin W_I$ with $L(s') > \max\{L(s'')|s'' \in J, s'' \leq x_1\}$. Then

$$c_sc_w = \begin{cases} c_{sw} + c_{sz} & \text{if } L(s) = L(s'), x_2 = x_1^{-1}, \\ c_{sw} & \text{otherwise.} \end{cases}$$

(3) If either $x \in W_I$ with $L(s) > L(x)$ or $x \notin W_I$ with $L(s) > \max\{L(s')|s' \in J, s' < x\}$, then

$$c_sc_w = \begin{cases} c_{sw} + \mu_1 c_{sz} + c_{xz} & \text{if } \ell(xz) = \ell(z) - \ell(x), s \in \mathcal{L}(xz), \\ c_{sw} + \mu_1 c_{sz} & \text{otherwise,} \end{cases}$$

where $\mu_1 \in \mathcal{A}$ is given by $\mu_1 = \bar{\mu}_1 \equiv v^{L(s)}p_{e,x}$.

Proof. This is a direct concequence of Lemmas 2.5-2.7. □

Recall the notation Ω_x, $x \in W$, defined in 1.6. For any $i \in \mathbb{P}$, denote

$$W^{(i)} = \begin{cases} W_I - \{w_0\} & \text{if } i < L(w_0), \\ (\cup_{s\in J, L(s)=i}\Omega_s) \cup \Omega_{w_0} & \text{if } i = L(w_0), \\ \cup_{s\in J, L(s)\leq i}\Omega_s & \text{if } i > L(w_0). \end{cases}$$

Lemma 2.8. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$. Let $y, z \in W$, $s \in J$ and $x \in W^{(L(s)-1)}$.*

(1) Suppose $y \leftarrow_L z$.

(a) If $z \in \mathcal{C}_{w_0}$, then $y \in \mathcal{C}_{w_0}$.

(b) If $y \in \mathcal{C}_{w_0}$, then $z \in W^{(L(w_0))}$.

(c) *If $z \in \mathcal{C}_{sx}$, then $y \in \mathcal{C}_{sx}$.*
(d) *If $y \in \mathcal{C}_{sx}$, then $z \in W^{(L(s))}$.*

(2) *Suppose $y \leftarrow_{LR} z$.*
(a) *If $z \in W^{(L(w_0))} - W^{(L(w_0)-1)}$, then $y \in W^{(L(w_0))} - W^{(L(w_0)-1)}$.*
(b) *If $y \in W^{(L(w_0))} - W^{(L(w_0)-1)}$, then $z \in W^{(L(w_0))}$.*
(c) *If $z \in W^{(L(s))} - W^{(L(s)-1)}$, then $y \in W^{(L(s))} - W^{(L(s)-1)}$.*
(d) *If $y \in W^{(L(s))} - W^{(L(s)-1)}$, then $z \in W^{(L(s))}$.*

Proof. We only prove (1). (2) will follow by similar arguments. Let $y, z \in W$ be with $y \leftarrow_L z$, then there exists $s' \in S$ with $h_{s',z,y} \neq 0$. If $y = s'z$, then (1) hold. Now suppose $y \neq s'z$, then $s'y < y < z$ and $\mu^{s'}_{y,z} \neq 0$. Then (1) follows by Lemma Corollary 2.4 (resp. Lemma 2.5′) when $s \in I$ (resp. $s \in J$). □

Lemma 2.9. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$.*

(1) *If $u \in W_I$ and $w \in W$ satisfy $\mathcal{L}(w) = \{s'\} \subseteq J$, then $w \sim_L uw$.*
(2) *If $w \in W$ and $s' \in J$ satisfy $\mathcal{L}(w) \subseteq J$ and $L(s') = L(w_0)$, then $s'w_0w \sim_L w_0w$.*
(3) *Let $s \in J$, $y \in W^{(L(s)-1)}$ and $w \in W$ satisfy $\ell(syw) = \ell(y) + \ell(w) + 1$. If either $\mathcal{L}(w) = I$ and $L(s) = L(w_0)$ or $\mathcal{L}(w) = \{s'\}$ and $L(s') \geq L(s)$, then $syw \sim_L yw$.*

Proof. (1) is trivial if $u = e$. Now assume $u \neq e$. Under the assumption of (1) (resp., (2)), we have $\mu^{s'}_{w,uw} \neq 0$ by Lemma 2.5′ (resp., $h_{w_0,s'w_0w,w_0w} \neq 0$ by Lemma 2.3). This implies (1) (resp., (2)).

It remains to prove (3). If $\mathcal{L}(w) = I$ and $L(s) = L(w_0)$, then the result follows from (2) since $y = e$ in this case. Now suppose $\mathcal{L}(w) = \{s'\}$ and $L(s') \geq L(s)$. If $L(s') = L(s)$, then $\mu^{s'}_{w,y^{-1}syw} \neq 0$ by Lemma 2.5′ . Since $y^{-1}syw = y^{-1} \cdot s \cdot y \cdot w$, we have $yw \sim_L syw$ by the relations $w \leq_L y^{-1}syw \leq_L syw \leq_L yw \leq_L w$. If $L(s') > L(s)$, then $\mu^{s'}_{w,(sy)^{-1}s'(sy)w} \neq 0$ by Lemma 2.5′ . Since $(sy)^{-1}s'(sy)w = (sy)^{-1} \cdot s' \cdot s \cdot y \cdot w$, we again get $yw \sim_L syw$ by the relations $w \leq_L (sy)^{-1}s'syw \leq_L syw \leq_L yw \leq_L w$. □

Corollary 2.10. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s)|s \in J\} \geq L(w_0)$. Let $y \in W$.*

(1) *If $y \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$, then $y \sim_L w_0$.*
(2) *Let $s \in J$ and $x \in W^{(L(s)-1)}$. If $y \in \mathcal{C}_{sx} \cap W^{(L(s))}$, then $y \sim_L sx$.*
(3) *If $y \in W^{(L(w_0))} - W^{(L(w_0)-1)}$, then $y \sim_{LR} w_0$.*
(4) *Let $s \in J$. If $y \in W^{(L(s))} - W^{(L(s)-1)}$, then $y \sim_{LR} s$.*

Proof. (1) Let $y \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$. We show $y \sim_L w_0$ using induction on $n := \ell(y) - \ell(w_0) \geq 0$. When $n = 0$, $y = w_0 \sim_L w_0$. Now suppose $n > 0$ and $y' \sim_L w_0$ for any $y' \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$ with $\ell(y') - \ell(w_0) < n$.

If $\mathcal{L}(y) \subseteq I$, then we can write $y = u \cdot y^0$ where $u = y_I \neq e$ and $y^0 \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$. Then $y^0 \sim_L w_0$ by the inductive hypothesis. Moreover, $y \sim_L y^0$ by Lemma 2.9(1). So $y \sim_L w_0$.

If $\mathcal{L}(y) \not\subseteq I$, then $\mathcal{L}(y) = \{s\} \subseteq J$, and $L(s) = L(w_0)$. Write $y = s \cdot y^1$ for $y^1 \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$. If $\mathcal{L}(y^1) = I$, then $y \sim_L y^1$ by Lemma 2.9(2). If $\mathcal{L}(y^1) \neq I$, then we can write $y^1 = y_1 \cdot s' \cdot y_2$ with $y_1 \in W^{(L(w_0)-1)}$,

$y_2 \in W$ and $s' \in J$ satisfy $L(s') = L(s)$. Then $y \sim_L y^1$ by Lemma 2.9(3). Moreover, $y^1 \sim_L w_0$ by the inductive hypothesis. So $y \sim_L w_0$.

(2) Let $y \in \mathcal{C}_{sx} \cap W^{(L(s))}$. We show $y \sim_L sx$ using induction on $n := \ell(y) - \ell(sx) \geq 0$. When $n = 0$, $y = sx \sim_L sx$. Now suppose $n > 0$ and $y' \sim_L sx$ for any $y' \in \mathcal{C}_{sx} \cap W^{(L(s))}$ with $\ell(y') - \ell(w_0) < n$.

If $\mathcal{L}(y) \subseteq I$, then write $y = u \cdot y^0$ with $u = y_I \neq e$ and $y^0 \in \mathcal{C}_{sx} \cap W^{(L(s))}$. Then $y^0 \sim_L sx$ by the inductive hypothesis. Moreover, $y \sim_L y^0$ by Lemma 2.9(1). So $y \sim_L sx$.

If $\mathcal{L}(y) \not\subseteq I$, then $\mathcal{L}(y) = \{s'\} \subseteq J$, and $L(s') \leq L(s)$. Write $y = s' \cdot y^1$ for $y^1 \in \mathcal{C}_{sx} \cap W^{(L(s))}$. If $\mathcal{L}(y^1) = I$, then $y \sim_L y^1$ by Lemma 2.9(2). If $\mathcal{L}(y^1) \neq I$, then we can write $y^1 = y_1 \cdot s' \cdot y_2$ with $y_1 \in W^{(L(s)-1)}$, $y_2 \in W$ and $s'' \in J$ satisfy $L(s'') \geq L(s')$. Then $y \sim_L y^1$ by Lemma 2.9(3). Moreover, $y^1 \sim_L sx$ by the inductive hypothesis. So $y \sim_L sx$.

(3) Let $y \in W^{(L(w_0))} - W^{(L(w_0)-1)}$. If $y \in \mathcal{C}_{w_0}$, then by (1), $y \sim_L w_0$. If $y \notin \mathcal{C}_{w_0}$, then there exists $s \in J$ and $x \in W^{(L(s)-1)}$ with $L(s) = L(w_0)$ and $y \in \mathcal{C}_{sx} \cap W^{(L(s))}$. So by (2), $y \sim_L sx$. Moreover, $sx \sim_R s \sim_L w_0 s \sim_R w_0$ by Lemma 2.9. So $y \sim_{LR} w_0$.

(4) Let $y \in W^{(L(s))} - W^{(L(s)-1)}$. If $L(s) = L(w_0)$, then by (3), $y \sim_{LR} w_0 \sim_{LR} s$. Now suppose $L(s) > L(w_0)$. Then there exists $s' \in J$, $L(s') = L(s)$ and $x' \in W^{(L(s)-1)}$ with $y \in \mathcal{C}_{s'x'}$. By (2), $y \sim_L s'x'$. Moreover, $s'x' \sim_R s' \sim_{LR} s$ by Lemma 2.9. So $y \sim_{LR} s$.

□

Now we are ready to describe all the left cells and the two-sided cells of (W, S, L) under certain assumptions on I, J, L. Recall that L_I is the weight function of W_I obtained by restriction of L to W_I.

Theorem 2.11. *Suppose that W_I is finite with w_0 the longest element, and that $\min\{L(s) | s \in J\} \geq L(w_0)$.*

(1) The left cells of W are
  - *(a) $\mathcal{C}_{w_0} \cap W^{(L(w_0))}$,*
  - *(b) $\mathcal{C}_{sx} \cap W^{(L(s))}$ for any $s \in J$ and $x \in W^{(L(s)-1)}$,*
  - *(c) Any left cell of the weighted Coxeter group (W_I, I, L_I) in $W_I - \{w_0\}$.*

(2) The two-sided cells of W are
  - *(a) $W^{(L(w_0))}$,*
  - *(b) $W^{(i)} - W^{(i-1)}$ for $i \in \mathbb{P}$, if $i > L(w_0)$ and there exists $s \in J$ with $L(s) = i$.*
  - *(c) Any two-sided cell of the weighted Coxeter group (W_I, I, L_I) in $W_I - \{w_0\}$.*

(3) Let $X = W - (W_I - \{w_0\})$. Then (1a) and (1b) (resp. (2a) and (2b) give a complete and irredundant list of the left cells (resp. two-sided cells) in X.

Proof. It follows by Lemma 2.8 and Corollary 2.10 for the sets in (1a) and (1b) being left cells of W and for those in (2a) and (2b) being two-sided cells of W.

By Lemmas 2.3 and 2.5′ , we have $y_1 \not\leftarrow_{LR} y_2$ for any $y_1 \in W_I - \{w_0\}$ and $y_2 \notin W_I - \{w_0\}$. So it follows by Lemma 1.7 for the sets in (1c) being left cells of W and for those in (2c) being two-sided cells of W.

From the definition of the sets $W^{(i)}$, it is noticed that the sets in (2a) and (2b) give a complete, irredundant list of the two-sided cells in X.

Let $s, s' \in J$, $x \in W^{(L(s)-1)}$ and $x' \in W^{(L(s')-1)}$ with $L(s) \geq L(s')$. We claim that

(i) $(\mathcal{C}_{sx} \cap W^{(L(s))}) \cap (\mathcal{C}_{w_0} \cap W^{(L(w_0))}) = \emptyset$, and

(ii) $(\mathcal{C}_{sx} \cap W^{(L(s))}) \cap (\mathcal{C}_{s'x'} \cap W^{(L(s'))}) = \emptyset$ if $s \neq s'$ or $x \neq x'$.

If $L(s) > L(w_0)$, then $\mathcal{C}_{sx} \cap W^{(L(w_0))} = \emptyset$. So (i) holds. If $L(s) = L(w_0)$ and $\mathcal{C}_{sx} \cap \mathcal{C}_{w_0} \neq \emptyset$, then $\mathcal{R}(x) = I$, contradicting with $x \in W^{(L(s)-1)} = W^{(L(w_0)-1)}$. We also have (i).

If $L(s) > L(s')$, then $\mathcal{C}_{sx} \cap W^{(L(s'))} = \emptyset$, so (ii) holds. If $L(s) = L(s')$ and $\mathcal{C}_{sx} \cap \mathcal{C}_{s'x'} \neq \emptyset$, then $sx = s'x'$, as $x, x' \in W^{(L(s)-1)}$. So (ii) holds.

Let $w \in X$. If there exists $s' \in J$ with $s' \leq w$ and $L(s') > L(w_0)$, then we can write $w = y \cdot s \cdot x$ with $s \in J$, $x \in W^{(L(s)-1)}$, $y \in W$ and $L(s) = \max\{L(s') | s' \in J, s' \leq w\}$. So $w \in \mathcal{C}_{sx} \cap W^{(L(s))}$. If $L(s') = L(w_0)$ for all (if any) $s' \in J$ with $s' \leq w$, then $w \in \mathcal{C}_{w_0} \cap W^{(L(w_0))}$ when $\mathcal{R}(w) = I$; $w \in \mathcal{C}_{sx} \cap W^{(L(s))}$ for some $s \in J$ and $x \in W^{(L(s)-1)}$ with $s \leq w$, when $\mathcal{R}(w) \neq I$.

We conclude that(1a) and (1b) give a complete, irredundant list of the left cells in X.

□

3. On the second largest weight element w'_0

In this section, we temporarily drop the previous assumptions on (W, S, L), and consider Coxeter groups with a finite parabolic subgroup, in which there is a unique simple reflection of minimal weight. We will find some properties of the element of the second largest weight in this finite parabolic subgroup, see Propositions 3.3, 3.5 and 3.8.

In 3.1-3.2, we put no assumption on (W, S, L). From 3.3 until the end of this section, we put the following assumption on (W, S, L).

(3.0.1)
$$\begin{gathered} I \subseteq S, W_I \text{ is a finite parabolic subgroup of } W, \\ \text{and } t \in I \text{ satisfy } L(t) < \min\{L(t') | t' \in I - \{t\}\}. \end{gathered}$$

Under this assumption, we always denote w_0 the longest element of W_I and $w'_0 = tw_0$.

Lemma 3.1. *Let $w \in W$ and $s \in S - \mathcal{L}(w)$. If there exists $r \in \mathcal{L}(w)$ with $L(r) \geq L(s)$, then*

(1)
$$c_s c_w = c_{sw} + \varepsilon c_{rw} + \sum_{z; sz < z < w, z \neq rw} \mu^s_{z,w} c_z,$$
where $\varepsilon = \begin{cases} 1 & \textit{if } L(r) = L(s) \textit{ and } s \in \mathcal{L}(rw), \\ 0 & \textit{otherwise;} \end{cases}$

(2) $rz < z$ for any $z \in W$ with $sz < z < w$, $z \neq rw$ and $\mu^s_{z,w} \neq 0$.

Proof. If $s \in \mathcal{L}(rw)$, then we see by 1.3 (4), (2) that
$$\mu^s_{rw,w} \equiv v^{L(s)} p_{rw,w} = v^{L(s)-L(r)} \equiv \begin{cases} 1 & \text{if } L(r) = L(s), \\ 0 & \text{if } L(r) > L(s). \end{cases}$$

This implies (1) by (1.2.1) and 1.3 (4). Then (2) follows by 1.3 (6). □

Lemma 3.2. *Let W_I be a finite parabolic subgroup of W generated by $I \subseteq S$, and w_0 be the longest element of W_I. For any $x, y \in W$, if $h_{w_0,x,y} \neq 0$, then $I \subseteq \mathcal{L}(y)$.*

Proof. This follows from (1.4.9). □

Proposition 3.3. *Assume (3.0.1) on (W, S, L).*

(1) *$c_t c_{w_0'} = c_{w_0}$, so $\{w_0'\}$ is the second lowest two-sided cell in W_I.*
(2) *If $x, y \in W$ satisfy $h_{w_0',x,y} \neq 0$, then $\ell(w_0'y) = \ell(y) - \ell(w_0')$.*

Proof. If there exists $y^0 \in W$ with $ty^0 < y^0 < w_0'$ and $\mu^t_{y^0,w_0'} \neq 0$, then by Lemma 3.1(2), $I - \{t\} \subseteq \mathcal{L}(y^0)$. So $\mathcal{L}(y^0) = I$, contradicting with $y^0 < w_0'$. We obtain $c_t c_{w_0'} = c_{w_0}$. Consequently, for $y^1 \in W_I$, if $y^1 \leftarrow_{I,L} w_0'$ or $y^1 \leftarrow_{I,R} w_0'$, then $y^1 = w_0$. But $\{w_0\}$ is the lowest two-sided cell in W_I, so $\{w_0'\}$ is the second lowest two-sided cell in W_I.

To prove (2), we only have to show

$$I \subseteq \mathcal{L}(y) \text{ or } I \subseteq \mathcal{L}(ty). \tag{3.3.1}$$

By (1), we have

$$\begin{aligned} c_{w_0}c_x &= c_t c_{w_0'} c_x = \sum_{y'} h_{w_0',x,y'} c_t c_{y'} \\ &= \sum_{y';ty'<y'} h_{w_0',x,y'} \left(v^{L(t)} + v^{-L(t)}\right) c_{y'} + \\ &\quad \sum_{y';ty'>y'} h_{w_0',x,y'} \left(c_{ty'} + \sum_{z;tz<z<y'} \mu^t_{z,y'} c_z\right). \end{aligned} \tag{3.3.2}$$

Write

$$\begin{aligned} A_1 &= \sum_{y';ty'<y'} h_{w_0',x,y'} \left(v^{L(t)} + v^{-L(t)}\right) c_{y'} \\ A_2 &= \sum_{y';ty'>y'} h_{w_0',x,y'} c_{ty'} \\ A_3 &= \sum_{y',z;ty'>y'>z>tz} h_{w_0',x,y'} \mu^t_{z,y'} c_z \end{aligned}$$

For any $y' \in W$ with $h_{w_0',x,y'} \neq 0$, we have

$$\mathcal{L}(y') \supseteq \mathcal{L}(w_0') = I - \{t\}, \tag{3.3.3}$$

by (1.4.9). So only those c_u with $I \subseteq \mathcal{L}(u)$ appears in A_1 with nonzero coefficient. By Lemmas 3.2, only those c_u with $I \subseteq \mathcal{L}(u)$ appears in $A_1 + A_2 + A_3$ with nonzero coefficient. Since $I - \{t\} = \mathcal{L}(w_0') \subseteq \mathcal{L}(y')$ for any $y' \in W$ with $h_{w_0',x,y'} \neq 0$, by Lemma 3.1, only those c_u with $I \subseteq \mathcal{L}(u)$ appears in A_3 with nonzero coefficient. As a result, only those c_u with $I \subseteq \mathcal{L}(u)$ appears in A_2 with nonzero coefficient.

As $h_{w_0',x,y} \neq 0$, if $ty < y$, then $I \subseteq \mathcal{L}(y)$ by (3.3.3). If $ty > y$, then c_{ty} appears in A_2 with nonzero coefficients. So $I \subseteq \mathcal{L}(ty)$. So (3.3.1) is true. □

Corollary 3.4. *Assume (3.0.1) on (W, S, L).*

(1) *If $x, y \in W$ satisfy $\mathcal{L}(x) \cap I = \emptyset$, $ty < y < w_0'x$ and $\mu^t_{y,w_0'x} \neq 0$, then $I \subseteq \mathcal{L}(y)$.*

(2) *If* $x, y, z \in W$ *satisfy* $\ell(w'_0 x) = \ell(x) - \ell(w'_0)$ *and* $h_{x,y,z} \neq 0$, *then* $\ell(w'_0 z) = \ell(z) - \ell(w'_0)$.

Proof. (1) By 1.3 (6), we have $\mathcal{L}(y) \supseteq I - \{t\} = \mathcal{L}(w'_0) \subseteq L(w'_0 x)$ by the assumptions $\mu^t_{y,w'_0 x} \neq 0$ and $L(t) < \min\{L(t')|t' \in I - \{t\}\}$. This implies $I \subseteq \mathcal{L}(y)$ since $ty < y$.

(2) To prove (2), we use induction on $n = \ell(x) - \ell(w'_0) \geq 0$. When $n = 0$, this is shown in Proposition 3.3(2). When $n > 0$, write $x = w'_0 \cdot x^0$. Then

$$\begin{aligned} c_x c_y &= \left(c_{w'_0} c_{x^0} - \sum_{x'<x} h_{w'_0,x^0,x'} c_{x'} \right) c_y \\ &= c_{w'_0} c_{x^0} c_y - \sum_{x'<x} h_{w'_0,x^0,x'} c_{x'} c_y \end{aligned}$$

Since $h_{x,y,z} \neq 0$, z appears in $c_{w'_0} c_{x^0} c_y$ or $c_{x'} c_y$ for some $x' < x$ with $h_{w'_0,x^0,x'} \neq 0$. If z appears in $c_{w'_0} c_{x^0} c_y$, then by Proposition 3.3(2), $\ell(w'_0 z) = \ell(z) - \ell(w'_0)$. For $x' < x$ with $h_{w'_0,x^0,x'} \neq 0$, we have $\ell(x') < \ell(x)$, and $\ell(w'_0 x') = \ell(x') - \ell(w'_0)$ following from Proposition 3.3(2). Consequently, if z appears in $c_{x'} c_y$, then we have $\ell(w'_0 z) = \ell(z) - \ell(w'_0)$ by the inductive hypothesis. So the result follows.

□

Proposition 3.5. *Assume (3.0.1) on* (W, S, L). *Let* $u \in W_I$ *and* $x, y \in W$ *satisfy* $u \leq w'_0$, $(u \cdot x)_I = u$ *and* $(w'_0 \cdot y)_I = w'_0$. *Then* $\deg p_{ux,w'_0 y} \leq \deg p_{u,w'_0}$, *and the equality holds only when* $x = y$.

Proof. If for all $t' \in I - \{t\}$, the order of tt' is 2, then the result is true since

$$\deg p_{ux,w'_0 y} = \deg v^{-L(w'_0)+L(u)} p_{w'_0 x,w'_0 y} \leq -L(w'_0) + L(u) = \deg p_{u,w'_0}.$$

by 1.3 (3). Now assume that we are not in this case. We shall prove our result by induction on $n := \ell(w'_0) - \ell(u) \geq 0$.

When $n = 0$, the result is trivial. Now suppose $n > 0$. If $\mathcal{L}(w'_0) = I - \{t\} \not\subseteq \mathcal{L}(u)$, then by taking $s \in \mathcal{L}(w'_0) - \mathcal{L}(u)$, we get

$$\begin{aligned} p_{ux,w'_0 y} &= v^{-L(s)} p_{sux,w'_0 y}, \\ p_{u,w'_0} &= v^{-L(s)} p_{su,w'_0}, \end{aligned}$$

and $su \leq w'_0$. So the result follows by the inductive hypothesis. Now assume that $\mathcal{L}(u) = I - \{t\}$. By Corollary 3.4, we have

$$c_t c_{w'_0 y} = c_{w_0 y} + \sum_{z; I \subseteq \mathcal{L}(z)} \mu^t_{z,w'_0 y} c_z, \tag{3.5.1}$$

$$c_t c_{w'_0} = c_{w_0}. \tag{3.5.2}$$

So by comparing the coefficients of c_{tux} (resp., c_{tu}) on both sides of (3.5.1) (resp., (3.5.2)), we get

$$p_{ux,w'_0 y} = p_{tux,w_0 y} - v^{L(t)} p_{tux,w'_0 y} + \sum_{z; I \subseteq \mathcal{L}(z)} \mu^t_{z,w'_0 y} p_{tux,z}, \tag{3.5.3}$$

$$p_{u,w'_0} = p_{tu,w_0} - v^{L(t)} p_{tu,w'_0}. \tag{3.5.4}$$

First assume $tu \not\leq w_0'$. Then there exists some $z_1 \in W_{I-\{t\}}$ such that $z_1 \cdot tu = w_0$ and $p_{tux,w_0'y} = v^{-L(z_1)} p_{w_0x,w_0'y}$ by 1.3 (3). So $L(z_1) = L(w_0) - L(tu) = L(w_0') - L(u)$. We see by (3.5.3) that,

$$\begin{aligned}
& v^{L(w_0')-L(u)} p_{ux,w_0'y} \\
= \quad & v^{L(w_0')-L(u)} \left(p_{tux,w_0y} - v^{L(t)} p_{tux,w_0'y} + \sum_{z;I\subseteq\mathcal{L}(z)} \mu^t_{z,w_0'y} p_{tux,z} \right) \\
= \quad & p_{w_0x,w_0y} - v^{L(t)} p_{w_0x,w_0'y} + \sum_{z;I\subseteq\mathcal{L}(z)} \mu^t_{z,w_0'y} p_{w_0x,z} \text{ (by 1.3 (3))} \\
\equiv \quad & p_{w_0x,w_0y} \text{ (by 1.3 (4))} \\
\equiv \quad & \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}
\end{aligned}$$

On the other hand, we see by (3.5.4) that $p_{u,w_0'} = p_{tu,w_0} = v^{-L(w_0')+L(u)}$. So the result follows in this case.

Next assume $tu \leq w_0'$. By Proposition 3.3, $c_t c_{w_0'} = c_{w_0}$, so

$$p_{u,w_0'} = p_{tu,w_0} - v^{L(t)} p_{tu,w_0'} = v^{-L(w_0')+L(u)} - v^{L(t)} p_{tu,w_0'}.$$

Moreover,

$$\begin{aligned}
\deg v^{L(t)} p_{tu,w_0'} &\geq L(t) + L(tu) - L(w_0'), \text{ (by 1.3(1))} \\
\deg p_{tu,w_0} &= L(tu) - L(w_0). \text{ (by 1.3(3))}
\end{aligned}$$

So $\deg p_{tu,w_0} < \deg v^{L(t)} p_{tu,w_0'}$, and

$$\deg p_{u,w_0'} = \deg v^{L(t)} p_{tu,w_0'} \geq L(t) + L(tu) - L(w_0')$$

by (3.5.4).

Also, we have

$$\deg p_{tux,w_0y} \leq L(tu) - L(w_0) < \deg p_{u,w_0'}$$

and

$$\deg \mu^t_{z,w_0'y} p_{tux,z} \leq L(t) + L(tu) - L(w_0) < \deg p_{u,w_0'}$$

for any $z \in W$ with $I \subseteq \mathcal{L}(z)$, by 1.3 (4), (1). Now we must prove the relation $\deg p_{ux,w_0'y} \leq \deg p_{u,w_0'}$. If $\deg p_{ux,w_0'y} \geq \deg p_{u,w_0'}$, then

(3.5.5)

$$\deg p_{ux,w_0'y} = \deg v^{L(t)} p_{tux,w_0'y} \leq \deg v^{L(t)} p_{tu,w_0'} = \deg p_{u,w_0'} \leq \deg p_{ux,w_0'y}$$

by (3.5.3) and the inductive hypothesis. Hence all the equalities in (3.5.5) must hold. But this is the case only when $x = y$ by the equation $\deg p_{tux,w_0'y} = \deg p_{tu,w_0'}$ and the inductive hypothesis. The result is proved. □

3.6. For (W, S, L) under the assumption (3.0.1), we define $n_t : W_I \to \mathbb{N}$ by sending any $x \in W_I$ to the number of occurrence of t in one (thus any) reduced expression of x. Set $n_0' = n_t(w_0')$. So $n_t(w_0) = n_0' + 1$.

Take $x_i \in W_I$, $y_j \in W_{I-\{t\}}$ for $i \in [0, n'_0]$, $j \in [0, n'_0 - 1]$ recursively in the following way: Let x_0 be the longest element in $W_{I-\{t\}}$. Then for $i \in [n'_0]$, suppose that we have defined all x_j and y_{j-1} for $j < i$. Take $y_{i-1} \in W_{I-\{t\}}$ such that $y_{i-1}tx_{i-1}$ is the longest element in the coset $W_{I-\{t\}}tx_{i-1}$. Let $x_i = y_{i-1}tx_{i-1}$. Then $x_i = y_{i-1} \cdot t \cdot x_{i-1}$.

Lemma 3.7. *Assume (3.0.1) on (W, S, L). For any $i \in [0, n'_0]$ and $j \in [0, n'_0 - 1]$, let $x_i \in W_I$, $y_j \in W_{I-\{t\}}$ be given as in 3.6.*

(1) We have
- *(a) $\mathcal{L}(x_i) = \mathcal{R}(x_i) = I - \{t\}$ for any $i \in [0, n'_0]$;*
- *(b) $x_{n'_0} = w'_0$.*

(2) Let $u \leq w'_0$ in W_I.
- *(a) If $z \in W_{I-\{t\}}$ satisfy $\ell(zu) = \ell(z) + \ell(u)$, then $p_{u,w'_0} = v^{-L(z)} p_{zu,w'_0}$.*
- *(b) If $tu > u$, then $p_{u,w'_0} = v^{L(u)-L(w'_0)} - v^{L(t)} p_{tu,w'_0}$.*

(3) For $i \in [0, n'_0 - 1]$, $p_{x_i,w'_0} = v^{L(x_i)-L(w'_0)} - v^{L(t)-L(y_i)} p_{x_{i+1},w'_0}$, with $\deg p_{x_i,w'_0} = L(t) - L(y_i) + \deg p_{x_{i+1},w'_0}$.

(4) $\deg p_{e,w'_0} = -L(w'_0) + 2n'_0 L(t)$.

Proof. (1) Clearly, $\mathcal{L}(x_0) = \mathcal{R}(x_0) = I - \{t\}$. Now let $i \in [n'_0]$. Suppose that the relation $\mathcal{L}(x_j) = \mathcal{R}(x_j) = I - \{t\}$ has been proved for any $j < i$ in $[n'_0]$. Then $\mathcal{L}(x_i) = I - \{t\}$ follows by the construction of x_i. On the other hand, $\mathcal{R}(x_i) \supseteq \mathcal{R}(x_{i-1}) = I - \{t\}$. We have $t \notin \mathcal{R}(x_i)$ since $x_i < w_0$ by the fact $n_t(x_i) < n_t(w_0) = n'_0 + 1$. So $\mathcal{R}(x_i) = I - \{t\}$. This proves (a) by induction.

Since $\mathcal{L}(x_{n'_0}) = I - \{t\}$, we have $tx_{n'_0} > x_{n'_0}$. We claim that $tx_{n'_0} = w_0$. Since $n_t(tx_{n'_0}) = n_t(w_0)$, we have $tx_{n'_0}t < tx_{n'_0}$. So $t \in \mathcal{R}(tx_{n'_0})$. Moreover, $I - \{t\} = \mathcal{R}(x_{n'_0}) \subseteq \mathcal{R}(tx_{n'_0})$. So $\mathcal{R}(tx_{n'_0}) = I$ and $tx_{n'_0} = w_0$. Thus $x_{n'_0} = w'_0$.

(2) (a) follows by 1.3 (3) and the fact $\mathcal{L}(w'_0) = I - \{t\}$. By the equation $c_t c_{w'_0} = c_{w_0}$, we get $p_{tu,w_0} = p_{u,w'_0} + v^{L(t)} p_{tu,w'_0}$. Thus we see by 1.3 (3) that

$$\begin{aligned} p_{u,w'_0} &= p_{tu,w_0} - v^{L(t)} p_{tu,w'_0} \\ &= v^{L(tu)-L(w_0)} - v^{L(t)} p_{tu,w'_0} \\ &= v^{L(u)-L(w'_0)} - v^{L(t)} p_{tu,w'_0}. \end{aligned}$$

This proves (b).

(3) By (2), we see that for any $i \in [0, n'_0 - 1]$,

$$\begin{aligned} p_{x_i,w'_0} &= v^{L(x_i)-L(w'_0)} - v^{L(t)} p_{tx_i,w'_0} \\ &= v^{L(x_i)-L(w'_0)} - v^{L(t)-L(y_i)} p_{x_{i+1},w'_0}, \end{aligned} \tag{3.7.1}$$

with x_i, y_i in the places of u, z, respectively. Since $\deg v^{L(t)-L(y_i)} p_{x_{i+1},w'_0} \geq L(t) - L(y_i) + L(x_{i+1}) - L(w'_0) = 2L(t) + L(x_i) - L(w'_0) > L(x_i) - L(w'_0)$ by 1.3 (1) and the relation $x_{i+1} = y_i \cdot t \cdot x_i$, we have $\deg p_{x_i,w'_0} = \deg v^{L(t)-L(y_i)} p_{x_{i+1},w'_0} = L(t) - L(y_i) + \deg p_{x_{i+1},w'_0}$ by (3.7.1).

(4) By (3) and (1b), we have, for $i \in [0, n_0'-1]$, that

$$\begin{aligned}\deg p_{x_i,w_0'} &= \sum_{j=i}^{n_0'-1}(L(t)-L(y_j)) + \deg p_{x_{n_0'},w_0'}\\ &= (n_0'-i)L(t) - \sum_{j=i}^{n_0'-1} L(y_j).\end{aligned}$$

In particular, $\deg p_{x_0,w_0'} = n_0'L(t) - \sum_{j=0}^{n_0'-1} L(y_j)$. So by (1) and 1.3 (3),

$$\begin{aligned}\deg p_{e,w_0'} &= -L(x_0) + \deg p_{x_0,w_0'}\\ &= n_0'L(t) - \sum_{j=0}^{n_0'-1} L(y_j) - L(x_0)\\ &= 2n_0'L(t) - L(w_0').\end{aligned}$$

□

Proposition 3.8. *Assume (3.0.1) on* (W, S, L). *Then*

$$\deg h_{w_0',w_0',w_0'} = -\deg p_{e,w_0'} = L(w_0') - 2n_0'L(t).$$

Proof. By Lemma 3.3(1), $\{w_0'\}$ is the second lowest two-sided cell of W_I. So we have $h_{w_0',w_0',z} = 0$ for any $z \in W - \{w_0', w_0\}$ by (1.4.9). This implies that

(3.8.1) $$c_{w_0'}c_{w_0'} = h_{w_0',w_0',w_0}c_{w_0} + h_{w_0',w_0',w_0'}c_{w_0'}.$$

By comparing the coefficient of T_e on both sides of (3.8.1), we see by 1.3 (3) that

$$\begin{aligned}h'_{w_0',w_0',e} &= h_{w_0',w_0',w_0}p_{e,w_0} + h_{w_0',w_0',w_0'}p_{e,w_0'}\\ &= v^{-L(w_0)}h_{w_0',w_0',w_0} + h_{w_0',w_0',w_0'}p_{e,w_0'}.\end{aligned}$$

By (1.4.4), we further get

(3.8.2) $$\sum_{x_1,x_2} p_{x_1,w_0'}p_{x_2,w_0'}f_{x_1,x_2,e} = v^{-L(w_0)}h_{w_0',w_0',w_0} + h_{w_0',w_0',w_0'}p_{e,w_0'}.$$

Since the relation $f_{x_1,x_2,e} \neq 0$ implies that $x_2 = x^{-1}$ and $f_{x_1,x_2,e} = 1$ by (1.1.1), we have

(3.8.3)
$$\sum_{x_1,x_2} p_{x_1,w_0'}p_{x_2,w_0'}f_{x_1,x_2,e} = \sum_{x_1} p_{x_1,w_0'}p_{x_1^{-1},w_0'} = 1 + \sum_{x_1<w_0'} p_{x_1,w_0'}p_{x_1^{-1},w_0'}.$$

Since $\deg h_{w_0',w_0',w_0} \leq L(w_0') < L(w_0)$ by (1.4.8), we have $\deg v^{-L(w_0)}h_{w_0',w_0',w_0} < 0$. This, together with (3.8.2)-(3.8.3), implies that $\deg h_{w_0',w_0',w_0'}p_{e,w_0'} = 0$. So $\deg h_{w_0',w_0',w_0'} = \deg p_{e,w_0'} = L(w_0') - 2n_0'L(t)$ by Lemma 3.7. □

4. The degree estimate involving some α- and h'-polynomials

4.1. From now on, we come back to the assumption in Section 2 that $S = I \sqcup J$, where J consists of universal elements. Moreover, we assume that L takes a constant value L_J on J and that W_I is finite, with w_0 its longest element. Since the case when $L_J \geq L(w_0)$ is considered in Section 2, we now assume $L_J < L(w_0)$.

There is a special kind of W_I which will be of interest in the following discussion.

$$(*)\qquad \begin{array}{c}\text{The Coxeter graph of } W_I \text{ is connected and there exists } t\in I\\ \text{with } L(t)<\min\{L(t')|t'\in I-\{t\}\}.\end{array}$$

When this is the case, we always set $w_0' = tw_0$. Note that in this case, W_I must be

(1) a dihedral group (W,S) with $S=\{s,t\}$ such that the order of st is even, or
(2) of type B_k for $k\geq 3$,

and L must only take two values $L_1<L_2$ on I and $L_1=L(t)$.

Moreover, if $(*)$ is satisfied, we will also assume the following to hold.

$$(4.1.1)\qquad \begin{array}{c}\text{For any } x\in W_I-\{w_0',w_0\} \text{ and } y\in W, \mathbf{a}_I(x)<\mathbf{a}_I(w_0'),\\ \text{and } \deg h_{x,y,w}<\mathbf{a}_I(w_0').\end{array}$$

Remark 4.2. It is proved in [Lu2] that if W_I is a finite dihedral group with unequal parameters, then (4.1.1) always holds. Moreover, computation with GAP shows that (4.1.1) also holds for W_I of type B_3. We suspect that (4.1.1) is true for any finite W_I satisfying $(*)$.

If $(*)$ is satisfied, then let $\pi_{w_0'}=\max_{x,y;x\neq w_0'}\deg h_{x,y,w_0'}$. Then $\pi_{w_0'}\leq \mathbf{a}(w_0')$. We further have $\pi_{w_0'}<\mathbf{a}(w_0')$ if (4.1.1) holds for W_I.

Lemma 4.3. *Suppose W_I satisfies $(*)$. For any $u\in W_I$ with $u<w_0'$ and $\mathcal{L}(u)=I-\{t\}$,*

(1) $\deg p_{tu,w_0'}\leq -L_2$,
(2) $\deg p_{u,w_0'}\leq L_1-L_2$.

Proof. (1) We have $\mathcal{L}(w_0')=I-\{t\}\not\subseteq \mathcal{L}(tu)$ since $tu\neq w_0$. Take any $s\in \mathcal{L}(w_0')-\mathcal{L}(tu)$. Then $p_{tu,w_0'}=v^{-L_2}p_{stu,w_0'}$ by 1.3 (3). Thus $\deg p_{tu,w_0'}\leq -L_2$.

(2) By Lemma 3.1, we have

$$\sum_z p_{z,w_0}T_z = c_{w_0}=c_tc_{w_0'}=\left(T_t+v_t^{-1}T_e\right)\sum_z p_{z,w_0'}T_z. \tag{4.3.1}$$

So

$$p_{tu,w_0}=p_{u,w_0'}+v^{L(t)}p_{tu,w_0'}. \tag{4.3.2}$$

By comparing the coefficients of T_{tu} on both sides of (4.3.1). We have $\deg p_{tu,w_0}=L(tu)-L(w_0)\leq -L_2$ by 1.3 (3) and $\deg v^{L(t)}p_{tu,w_0'}\leq L_1-L_2$ by (1). So $\deg p_{u,w_0'}\leq \max\{-L_2,L_1-L_2\}=L_1-L_2$ by (4.3.2). □

Following Lusztig in [Lu2, Subsection 14.1], we define, for any $z\in W$, some $\Delta(z)\in\mathbb{N}$ by

$$p_{e,z}=n_zv^{-\Delta(z)}+\text{ strictly smaller powers of } v,\quad n_z\in\mathbb{Z}-\{0\}.$$

Define $\mathcal{D}=\{z\in W|\mathbf{a}(z)=\Delta(z)\}$. Clearly, $z\in\mathcal{D}$ if and only if $z^{-1}\in\mathcal{D}$. Define Δ_I and $\mathcal{D}_I$ for W_I similarly.

The sets $\mathcal{D}$, $\mathcal{D}_I$ play an important role in the cell representations of W, W_I and the associated Hecke algebras $\mathcal{H}$, $\mathcal{H}_I$, respectively.

Lemma 4.4. *Suppose that W_I satisfies $(*)$ and that (4.1.1) holds in W_I. Then $\mathbf{a}_I(w_0')=\deg h_{w_0',w_0',w_0'}=\Delta_I(w_0')$. In particular, $w_0'\in\mathcal{D}_I$.*

Proof. We see by Proposition 3.8 that $\Delta_I(w_0') = \deg h_{w_0',w_0',w_0'} \leq \mathbf{a}_I(w_0')$. By (4.1.1), we have $\mathbf{a}_I(w_0') = \deg h_{w_0',w_0',w_0'}$. So $\mathbf{a}_I(w_0') = \Delta_I(w_0')$, which implies $w_0' \in \mathcal{D}_I$. □

Note that Lemma 4.4 is proved without assuming (P1).

Define $\tau_{I,X} := \max\{\mathbf{a}_I(w) | w \in W_I - X\}$ for any $X \subseteq W_I$. Recall the notation $h'_{x,y,z}$ for $x, y, z \in W$ defined in 1.4.

Lemma 4.5. *Let $u_1, x \in W_I - \{w_0\}$, $t_1 \in I - \mathcal{L}(x)$ and $u_2 \in W_I$.*

(1) We have

$$\deg\left(h'_{u_1,u_2,x} - v^{-L(t_1)} h'_{u_1,u_2,t_1x}\right) \leq \tau_{I,\{w_0\}}.$$

(2) Suppose that W_I satisfies (∗) and that (4.1.1) holds in W_I. Suppose that $u_1 \neq w_0'$.

(a) If $x = w_0'$, then $t_1 = t$ and

$$\deg\left(h'_{u_1,u_2,x} - v^{-L(t_1)} h'_{u_1,u_2,t_1x}\right) < \mathbf{a}_I(w_0').$$

(b) If $x \neq w_0'$,

$$\deg\left(h'_{u_1,u_2,x} - v^{-L(t_1)} h'_{u_1,u_2,t_1x} - \delta_{t,t_1} h_{u_1,u_2,w_0'}\left(p_{x,w_0'} - v^{-L(t)} p_{tx,w_0'}\right)\right) \leq \tau_{I,\{w_0',w_0\}}.$$

Proof. By (1.4.5), we have

$$\begin{aligned}
(4.5.1) \qquad & h'_{u_1,u_2,x} - v^{-L(t_1)} h'_{u_1,u_2,t_1x} \\
\equiv\ & h_{u_1,u_2,x} + \sum_{x_1 > x} h_{u_1,u_2,x_1} p_{x,x_1} \\
& - v^{-L(t_1)}\left(h_{u_1,u_2,t_1x} + \sum_{x_1 > t_1x} h_{u_1,u_2,x_1} p_{t_1x,x_1}\right) \\
=\ & h_{u_1,u_2,x} + \sum_{x_1; t_1x_1 > x_1 > x} h_{u_1,u_2,x_1} p_{x,x_1} \\
& - v^{-L(t_1)} \sum_{x_1; t_1x_1 > x_1 > x} h_{u_1,u_2,x_1} p_{t_1x,x_1} \text{ (by 1.3 (3))} \\
=\ & h_{u_1,u_2,x} + \sum_{x_1; t_1x_1 > x_1 > x} h_{u_1,u_2,x_1}\left(p_{x,x_1} - v^{-L(t_1)} p_{t_1x,x_1}\right).
\end{aligned}$$

(1) We have $\deg h_{u_1,u_2,x} \leq \mathbf{a}_I(x) \leq \tau_{I,\{w_0\}}$ and $\deg h_{u_1,u_2,x_1} \leq \mathbf{a}_I(x_1) \leq \tau_{I,\{w_0\}}$ for any $x_1 \in W$ with $x_1 > x$ and $t_1x_1 > x_1$. So the result follows by (4.5.1).

(2) (a) By (4.5.1), $h'_{u_1,u_2,x} - v^{-L(t_1)} h'_{u_1,u_2,t_1x} \equiv h_{u_1,u_2,x}$ for $x = w_0'$. Since $\deg h_{u_1,u_2,w_0'} < \mathbf{a}_I(w_0')$ by (4.1.1) and the assumption $u_1 \in W_I - \{w_0, w_0'\}$, the result is true.

(b) Since $x \in W_I - \{w_0', w_0\}$, we have $\deg h_{u_1,u_2,x} \leq \tau_{I,\{w_0',w_0\}}$ the definition of $\tau_{I,\{w_0',w_0\}}$. Let $x_1 \in W$ satisfy $t_1x_1 > x_1 > x$. If $x_1 \neq w_0'$, then $\deg h_{u_1,u_2,x_1} \leq \tau_{I,\{w_0',w_0\}}$ by (4.1.1). If $x_1 = w_0'$, then $t_1 = t$. So the result follows by (4.5.1).

□

Recall the notation $\alpha_{x,y,z}$, $\beta_{x,y,z}$, $q'_{x,y}$ for $x,y,z\in W$ defined in 1.4, and the notation $f \overset{\mathrm{mod}\,\mathcal{A}_{\leq m}}{\equiv} g$ defined in 1.3.

Corollary 4.6. *Let $u_1, x\in W_I-\{w_0\}$, $t_1\in I-\mathcal{L}(x)$ and $u_2\in W_I$.*

(1) We have

$$\deg\left(\alpha_{u_1,u_2,x}-v^{-L(t_1)}\alpha_{u_1,u_2,t_1x}\right)\leq \tau_{I,\{w_0\}}.$$

(2) Suppose that W_I satisfies $(*)$ and that (4.1.1) holds in W_I. Suppose $u_1\neq w'_0$.

(a) If $x=w'_0$, then $t_1=t$ and

$$\deg\left(\alpha_{u_1,u_2,x}-v^{-L(t_1)}\alpha_{u_1,u_2,t_1x}\right)<\mathbf{a}_I(w'_0).$$

(b) If $x\neq w'_0$, then,

$$\deg\left(\alpha_{u_1,u_2,x}-v^{-L(t_1)}\alpha_{u_1,u_2,t_1x}-\delta_{t,t_1}\beta_{u_1,u_2,w'_0}\left(p_{x,w'_0}-v^{-L(t_1)}p_{t_1x,w'_0}\right)\right)\leq\tau_{I,\{w'_0,w_0\}}.$$

Proof. Apply induction on $n:=\ell(u_2)\geq 0$. When $n=0$, the result is obvious. Now suppose $n>0$. We have

$$\begin{aligned}(4.6.1)\qquad c_{u_1}T_{u_2} &= c_{u_1}\left(c_{u_2}-\sum_{y<u_2}p_{y,u_2}T_y\right)\\ &= c_{u_1}c_{u_2}-\sum_{y<u_2}p_{y,u_2}c_{u_1}T_y.\end{aligned}$$

By comparing the coefficients of $T_{z'}$ on both sides of (4.6.1), we get

$$(4.6.2)\qquad \alpha_{u_1,u_2,z'}=h'_{u_1,u_2,z'}-\sum_{y<u_2}p_{y,u_2}\alpha_{u_1,y,z'}$$

for any $z'\in W_I$. So (1) and (2a) follow by induction and Lemma 4.5.

Now assume we are in the case of (2b). Denote $\tau:=\tau_{I,\{w'_0,w_0\}}$ and $\Delta_x:=\delta_{t,t_1}(p_{x,w'_0}-v^{-L(t_1)}p_{t_1x,w'_0})$. Then by (4.6.2), Lemma 4.5 and the inductive hypothesis, we get that

$$\begin{aligned}&\alpha_{u_1,u_2,x}-v^{-L(t_1)}\alpha_{u_1,u_2,t_1x}\\ =\;&\left(h'_{u_1,u_2,x}-v^{-L(t_1)}h'_{u_1,u_2,t_1x}\right)-\sum_{y_1<u_2}p_{y_1,u_2}\left(\alpha_{u_1,y_1,x}-v^{-L(t_1)}\alpha_{u_1,y_1,t_1x}\right)\\ \overset{\mathrm{mod}\,\mathcal{A}_{\leq\tau}}{\equiv}\;&h_{u_1,u_2,w'_0}\Delta_x-\sum_{y_1<u_2}p_{y_1,u_2}\left(\alpha_{u_1,y_1,x}-v^{-L(t_1)}\alpha_{u_1,y_1,t_1x}\right)\ \text{(by Lemma 4.5)}\\ \overset{\mathrm{mod}\,\mathcal{A}_{\leq\tau}}{\equiv}\;&h_{u_1,u_2,w'_0}\Delta_x-\sum_{y_1<u_2}p_{y_1,u_2}\left(h_{u_1,y_1,w'_0}\Delta_x-\sum_{y_2<y_1}p_{y_2,y_1}\left(\alpha_{u_1,y_2,x}-v^{-L(t_1)}\alpha_{u_1,y_2,t_1x}\right)\right)\\ &\text{(by Lemma 4.5)}\\ =\;&h_{u_1,u_2,w'_0}\Delta_x-\sum_{y_1<u_2}p_{y_1,u_2}h_{u_1,y_1,w'_0}\Delta_x+\\ &\sum_{y_2<y_1<u_2}p_{y_1,u_2}p_{y_2,y_1}\left(\alpha_{u_1,y_2,x}-v^{-L(t_1)}\alpha_{u_1,y_2,t_1x}\right).\end{aligned}$$

We repeat the above computation with y_1 in the place of u_2 and so on, then we get

$$\begin{aligned}
& \alpha_{u_1,u_2,x} - v^{-L(t_1)}\alpha_{u_1,u_2,t_1x} \\
\overset{\mathrm{mod}\ \mathcal{A}_{\leq\tau}}{\equiv} \quad & \Delta_x \sum_{e\leq y_k<y_{k-1}<\cdots<y_0=u_2;k\in\mathbb{N}} (-1)^k p_{y_k,y_{k-1}}\cdots p_{y_1,y_0}h_{u_1,y_k,w_0'} \\
= \quad & \Delta_x \sum_{y_k\leq u_2} q'_{y_k,u_2}h_{u_1,y_k,w_0'} \text{ (by 1.4)} \\
= \quad & \Delta_x\beta_{u_1,u_2,w_0'}. \text{ (by (1.4.7))}
\end{aligned}$$

This proves (2b). □

For any $x \in W$, we have $x_I \in W_I$ and $x^I \in W$ defined in 1.3.

Lemma 4.7. *Let $u_1 \in W_I - \{w_0\}$ and $y, z \in W$ satisfy $\mathcal{L}(z) = \{s\} \subseteq J$, $y < u_1z$ and $I \not\subseteq \mathcal{L}(y)$. Write $y = u \cdot y'$ with $u = y_I < w_0$ and $y' = y^I$. Let $t_1 \in I - \mathcal{L}(u)$.*

(1) Suppose that $L(w_0) > L_J > \tau_{I,\{w_0\}}$. Then

$$\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y}\right) < 0.$$

(2) Suppose that W_I satisfies $(*)$ and that (4.1.1) holds in W_I. Suppose that $u_1 \neq w_0'$ and that $\mathbf{a}_I(w_0') \geq L_J > \tau_{I,\{w_0',w_0\}}$.

(a) If $u = w_0'$, then $t_1 = t$ and

$$\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y}\right) < \mathbf{a}_I(w_0') - L_J.$$

(b) If $u \neq w_0'$, then

$$\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y} - \delta_{t,t_1}v^{-L_J}(p_{u,w_0'} - v^{-L(t_1)}p_{t_1u,w_0'})\sum_{x_2\in W_I-\{e\}} p_{sx_2y',z}\beta_{u_1,x_2,w_0'}\right) < 0.$$

Proof. We see by (1.4.4) that

$$\begin{aligned}
& h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y} \\
= \quad & p_{u,u_1}p_{y',z} + \sum_{x_1,x_2';(x_1,x_2')\neq(u,y')} p_{x_1,u_1}p_{x_2',z}f_{x_1,x_2',y} \\
& -v^{-L(t_1)}p_{t_1u,u_1}p_{y',z} - v^{-L(t_1)}\sum_{x_1,x_2';(x_1,x_2')\neq(t_1u,y')} p_{x_1,u_1}p_{x_2',z}f_{x_1,x_2',t_1y}.
\end{aligned}$$

Since $s \in J$, for x_1, x_2' with $x_1 \leq u_1$ and $sx_2' < x_2'$, we have $f_{x_1,x_2',y} \neq 0$ (resp. $f_{x_1,x_2',t_1y} \neq 0$) if and only if $x_1 = u$ and $x_2' = y'$ (resp. $x_1 = t_1u$ and $x_2' = y'$). Consequently,

$$\begin{aligned}
(4.7.1) \qquad & h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y} \\
= \quad & p_{u,u_1}p_{y',z} - v^{-L(t_1)}p_{t_1u,u_1}p_{y',z} \\
& + \sum_{x_1,x_2';sx_2'>x_2'} p_{x_1,u_1}p_{x_2',z}\left(f_{x_1,x_2',y} - v^{-L(t_1)}f_{x_1,x_2',t_1y}\right) \text{ (by (1.4.4))} \\
\equiv \quad & \sum_{x_1,x_2';sx_2'>x_2'} p_{x_1,u_1}p_{x_2',z}\left(f_{x_1,x_2',y} - v^{-L(t_1)}f_{x_1,x_2',t_1y}\right).
\end{aligned}$$

It is easy to notice that for any $x_1 \le u_1$ and $x_2' \le z$ with $sx_2' > x_2'$ and $f_{x_1,x_2',y} \ne 0$ (resp. $f_{x_1,x_2',t_1y} \ne 0$), we have $x_2' = x_2y'$ with $x_2 \in W_I - \{e\}$ and $f_{x_1,x_2',y} = f_{x_1,x_2,u}$ (resp. $f_{x_1,x_2',t_1y} = f_{x_1,x_2,t_1u}$). As a result,

$$\begin{aligned}(4.7.2)\qquad & h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y}\\ = {} & v^{-L_J}\sum_{x_1\in W_I, x_2\in W_I-\{e\}} p_{x_1,u_1}p_{sx_2y',z}\left(f_{x_1,x_2,u} - v^{-L(t_1)}f_{x_1,x_2,t_1u}\right)\\ = {} & v^{-L_J}\sum_{x_2\in W_I-\{e\}} p_{sx_2y',z}\left(\sum_{x_1\in W_I} p_{x_1,u_1}\left(f_{x_1,x_2,u} - v^{-L(t_1)}f_{x_1,x_2,t_1u}\right)\right)\\ = {} & v^{-L_J}\sum_{x_2\in W_I-\{e\}} p_{sx_2y',z}\left(\alpha_{u_1,x_2,u} - v^{-L(t_1)}\alpha_{u_1,x_2,t_1u}\right).\end{aligned}$$

(1) By Corollary 4.6(1) and (4.7.2),
$$\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y}\right) \le \tau_{I,\{w_0\}} - L_J < 0.$$

(2) (a) By Corollary 4.6(2a) and (4.7.2),
$$\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y}\right) < \mathbf{a}_I(w_0') - L_J.$$

(b) By Corollary 4.6(2b) and (4.7.2),
$$\begin{aligned}&\deg\left(h'_{u_1,z,y} - v^{-L(t_1)}h'_{u_1,z,t_1y} - \delta_{t,t_1}v^{-L_J}(p_{u,w_0'} - v^{-L(t_1)}p_{t_1u,w_0'})\sum_{x_2\in W_I-\{e\}} p_{sx_2y',z}\beta_{u_1,x_2,w_0'}\right)\\ \le {} & \tau_{I,\{w_0',w_0\}} - L_J < 0.\end{aligned}$$

□

Lemma 4.8. *Let $u_1 \in W_I$ and $y, z \in W$ satisfy $\mathcal{L}(z) = \{s\} \subseteq J$ and $y < u_1z$. Then $\deg h'_{u_1,z,y} \le L(w_0) - L_J$ and the equality holds only when $u_1 = w_0$.*

Proof. By (1.4.4) and the assumption $\mathcal{L}(z) = \{s\} \subseteq J$ and $y < u_1z$, we have

$$\begin{aligned}(4.8.1)\qquad h'_{u_1,z,y} &= \sum_{x_1,x_2'} p_{x_1,u_1}p_{x_2',z}f_{x_1,x_2',y}\\ &\equiv \sum_{x_1,x_2';sx_2'>x_2'} p_{x_1,u_1}p_{x_2',z}f_{x_1,x_2',y}\\ &= v^{-L_J}\sum_{x_1,x_2';sx_2'>x_2'} p_{x_1,u_1}p_{sx_2',z}f_{x_1,x_2',y}.\end{aligned}$$

We see by (4.8.1) that if $u_1 = w_0$, then $\deg h'_{u_1,z,y} \le L(w_0) - L_J$ and that if $u_1 < w_0$, then $\deg h'_{u_1,z,y} < L(w_0) - L_J$. □

5. Left multiplication by c_u for $u \in W_I$

Keep the assumptions on the weighted Coxeter group (W, S, L) in 4.1.

Lemma 5.1. *Suppose $L(w_0) > L_J > \tau_{I,\{w_0\}}$. Let $u_1 \in W_I - \{w_0\}$ and $y, z \in W$ satisfy $\mathcal{L}(z) \subseteq \{s\} \subseteq J$, $y < u_1z$ and $h_{u_1,z,y} \ne 0$. Then $\deg h_{u_1,z,y} < L(w_0) - L_J$ and $\mathcal{L}(y) = I$.*

Proof. We have $z \neq e$ by the assumptions $y < u_1 z$ and $h_{u_1,z,y} \neq 0$. So $\mathcal{L}(z) = \{s\}$.

By Lemma 4.8, $\deg h'_{u_1,z,y} < L(w_0) - L_J$. So $\deg h_{u_1,z,y} < L(w_0) - L_J$ by (1.4.5) and induction on $\ell(u_1 z) - \ell(y) \geq 1$.

Now assume $\mathcal{L}(y) \neq I$, then $\mathcal{L}(y) \subsetneq I$. Let $Y = \{y' \in W | \mathcal{L}(y') \subsetneq I, y' < u_1 z, h_{u_1,z,y'} \neq 0\}$. If $Y = \emptyset$, then the result follows. Now suppose $Y \neq \emptyset$, and let y^0 be maximal in Y with respect to the partial order $\leq$. Take $t_1 \in I - \mathcal{L}(y^0)$.

By (1.4.5),

$$h_{u_1,z,y^0} = h'_{u_1,z,y^0} - \sum_{x > y^0} h_{u_1,z,x} p_{y^0,x}.$$

Since y^0 is maximal in Y with respect to $\leq$, we have

$$\begin{aligned} h_{u_1,z,y^0} &= h'_{u_1,z,y^0} - \sum_{x>y^0, x_I = w_0} h_{u_1,z,x} p_{y^0,x} \\ &= h'_{u_1,z,y^0} - h_{u_1,z,t_1 y^0} p_{y^0,t_1 y^0} - \sum_{x > t_1 y^0, x_I = w_0} h_{u_1,z,x} p_{y^0,x} \\ &= h'_{u_1,z,y^0} - v^{-L(t_1)} \left(h_{u_1,z,t_1 y^0} + \sum_{x > t_1 y^0, x_I = w_0} h_{u_1,z,x} p_{t_1 y^0,x} \right) \\ &\equiv h'_{u_1,z,y^0} - v^{-L(t_1)} h'_{u_1,z,t_1 y^0} \equiv 0, \end{aligned}$$

by Lemma 4.7(1). Thus $h_{u_1,z,y^0} = 0$, a contradiction. So $Y = \emptyset$ and the proof is completed. □

Recall the notation $\pi_{w'_0}$ preceding Lemma 4.3.

Lemma 5.2. *Suppose that W_I satisfies (∗), and that (4.1.1) holds in W_I, and that $L_J > \max\{\tau_{I,\{w'_0,w_0\}}, \pi_{w'_0} + L_1 - L_2\}$. Let $u_1 \in W_I - \{w'_0, w_0\}$, $y, z \in W$ satisfy $\mathcal{L}(z) \subseteq \{s\} \subseteq J$ and $y < u_1 z$. Let $u = y_I$.*

(1) *If $u = w_0$, then $\deg h_{u_1,z,y} < L(w_0) - L_J$.*
(2) *If $u = w'_0$, then $\deg h_{u_1,z,y} \leq \pi_{w'_0} - L_J$.*
(3) *If $u \notin \{w'_0, w_0\}$, then $h_{u_1,z,y} = 0$.*

Proof. When $z = e$, $h_{u_1,z,y} = 0$ as $y \neq u_1 z$. The result follows. Now assume $z \neq e$. Then $\mathcal{L}(z) = \{s\}$. By Lemma 4.8, $\deg h'_{u_1,z,y} < L(w_0) - L_J$, so $\deg h_{u_1,z,y} < L(w_0) - L_J$ by (1.4.5) and induction on $\ell(u_1 z) - \ell(y) \geq 1$. (1) is just a special case of this conclusion and hence is proved.

Let Y be the set of $y' < u_1 z$ in W satisfying one of the following (i) or (ii),

(i) $y'_I \notin \{w'_0, w_0\}$ and $h_{u_1,z,y'} \neq 0$,
(ii) $y'_I = w'_0$ and $\deg h_{u_1,z,y'} > \pi_{w'_0} - L_J$.

If $Y = \emptyset$, then (2) and (3) follow. Now suppose $Y \neq \emptyset$. Let y^0 be a maximal element in Y with respect to $\leq$. Let $u^0 = y^0_I$.

If $\mathcal{L}(w'_0) \not\subseteq \mathcal{L}(u^0)$, then $u^0 \notin \{w'_0, w_0\}$. Take some $t_1 \in \mathcal{L}(w'_0) - \mathcal{L}(u^0)$. So by 1.3(0), $u^0 \leq w'_0$ if and only if $t_1 u^0 \leq w'_0$. By (1.4.4) and the maximality of y^0 in Y, we have

$$\begin{aligned} h_{u_1,z,y^0} &= h'_{u_1,z,y^0} - \sum_{y''>y^0, y''_I\in\{w'_0,w_0\}} h_{u_1,z,y''}p_{y^0,y''} \\ &= h'_{u_1,z,y^0} - v^{-L(t_1)}h_{u_1,z,t_1y^0} - \sum_{y''>y^0, y''\neq t_1y^0, y''_I\in\{w'_0,w_0\}} v^{-L(t_1)}h_{u_1,z,y''}p_{t_1y^0,y''} \\ &\equiv h'_{u_1,z,y^0} - v^{-L(t_1)}h'_{u_1,z,t_1y^0}. \end{aligned}$$

So $h_{u_1,z,y^0}\equiv 0$ following from Lemma 4.7. We have $h_{u_1,z,y^0}=0$, contradicting with $h_{u_1,z,y^0}\neq 0$.

If $\mathcal{L}(w'_0)\subseteq\mathcal{L}(u^0)$, then $\mathcal{L}(u^0)=\mathcal{L}(w'_0)$ and $u^0\leq w'_0$ by the assumption $u^0\in W_I-\{w_0\}$. Let $y'^0=(u^0)^{-1}y^0$. Then

(5.2.1)

$$\begin{aligned} & h_{u_1,z,y^0} \\ = \; & h'_{u_1,z,y^0} - \sum_{y''>y^0, y''_I\in\{w'_0,w_0\}} h_{u_1,z,y''}p_{y^0,y''} \text{ (by (1.4.4) and the maximality of } y^0) \\ = \; & h'_{u_1,z,y^0} - \delta_{u^0,w'_0}h_{u_1,z,ty^0}p_{y^0,ty^0} - \sum_{y''>y^0, y''_I=w'_0}\left(h_{u_1,z,y''}p_{y^0,y''}+h_{u_1,z,ty''}p_{y^0,ty''}\right) \\ = \; & h'_{u_1,z,y^0} - h_{u_1,z,ty^0}p_{y^0,ty^0} - \\ & \sum_{y''>y^0, y''_I=w'_0}(h_{u_1,z,y''}p_{y^0,y''}+h_{u_1,z,ty''}p_{y^0,ty''}) \quad (h_{u_1,z,ty^0}=0 \text{ if } u^0\neq w'_0) \\ = \; & h'_{u_1,z,y^0} - v^{-L(t)}h'_{u_1,z,ty^0} + v^{-L(t)}\sum_{y''>ty^0, y''_I\in\{w'_0,w_0\}} h_{u_1,z,y''}p_{ty^0,y''} - \\ & \sum_{y''>y^0, y''_I=w'_0}\left(h_{u_1,z,y''}p_{y^0,y''}+v^{-L(t)}h_{u_1,z,ty''}p_{ty^0,ty''}\right) \quad \text{(by 1.3 (3) and (1.4.4))} \\ = \; & h'_{u_1,z,y^0} - v^{-L(t)}h'_{u_1,z,ty^0} + v^{-L(t)}\sum_{y''>ty^0, y''_I=w'_0} h_{u_1,z,y''}p_{ty^0,y''} - \sum_{y''>y^0, y''_I=w'_0} h_{u_1,z,y''}p_{y^0,y''} \\ = \; & h'_{u_1,z,y^0} - v^{-L(t)}h'_{u_1,z,ty^0} + \sum_{y''>y^0, y''_I=w'_0} h_{u_1,z,y''}\left(v^{-L(t)}p_{ty^0,y''}-p_{y^0,y''}\right) \\ \equiv \; & v^{-L_J}(p_{u^0,w'_0}-v^{-L(t)}p_{tu^0,w'_0})\sum_{x_2\in W_I-\{e\}} p_{sx_2y'^0,z}\beta_{u_1,x_2,w'_0} + \\ & \sum_{y''>y^0, y''_I=w'_0} h_{u_1,z,y''}\left(v^{-L(t)}p_{ty^0,y''}-p_{y^0,y''}\right). \text{ (by Lemma 4.7)} \end{aligned}$$

If $u^0=w'_0$, then we see by (5.2.1) that

(5.2.2)

$$h_{u_1,z,y^0}\equiv v^{-L_J}\sum_{x_2\in W_I-\{e\}} p_{sx_2y'^0,z}\beta_{u_1,x_2,w'_0} + \sum_{y''>y^0, y''_I=w'_0} h_{u_1,z,y''}(v^{-L(t)}p_{ty^0,y''}-p_{y^0,y''}).$$

For any $x_2\in W_I-\{e\}$, we see by (1.4.7), (4.1.1) and the assumption on L_J that

$$\deg v^{-L_J}p_{sx_2y'^0,z}\beta_{u_1,x_2,w'_0}\leq \pi_{w'_0}-L_J<L_2-L_1. \quad (5.2.3)$$

By the maximality assumption on y^0, we have

$$\deg h_{u_1,z,y''} \leq \pi_{w_0'} - L_J < L_2 - L_1 \tag{5.2.4}$$

for any $y'' > y^0$ with $y''_I = w_0'$.

So $\deg h_{u_1,z,y^0} \leq \pi_{w_0'} - L_J$ by (5.2.2)-(5.2.3). As $y^0 \in Y$, we have $u^0 \notin \{w_0', w_0\}$ and $h_{u_1,z,y^0} \neq 0$.

Since $u^0 \neq w_0'$,

$$h_{u_1,z,y^0} \equiv A_1 - A_2, \tag{5.2.5}$$

by (5.2.1) and 1.3(3), where

$$\begin{aligned} A_1 &= v^{-L_J}(p_{u^0,w_0'} - v^{-L(t)}p_{tu^0,w_0'}) \sum_{x_2 \in W_I - \{e\}} p_{sx_2y'^0,z}\beta_{u_1,x_2,w_0'} \\ A_2 &= \sum_{y''>y^0, y''_I = w_0'} h_{u_1,z,y''}\left(p_{y^0,y''} - v^{-L(t)}p_{ty^0,y''}\right). \end{aligned}$$

By Proposition 3.5 and Lemma 4.3, we have

$$\begin{aligned} \deg(p_{u^0,w_0'} - v^{-L(t)}p_{tu^0,w_0'}) &\leq L_1 - L_2, \\ \deg(p_{y^0,y''} - v^{-L(t)}p_{ty^0,y''}) &\leq L_1 - L_2, \end{aligned} \tag{5.2.6}$$

for any $y'' > y^0$ with $y''_I = w_0'$. So we see by (5.2.3) and (5.2.6) that $\deg A_1 \leq -L_J + L_1 - L_2 + \pi_{w_0'} < 0$, hence $A_1 \equiv 0$. On the other hand, we get $A_2 \equiv 0$ by (5.2.4) and (5.2.6). Thus $h_{u_1,z,y^0} \equiv 0$ by (5.2.5), and $h_{u_1,z,y^0} = 0$, contradicting with $h_{u_1,z,y^0} \neq 0$. This contradiction shows that $Y = \emptyset$. So (2) and (3) are proved. □

Lemma 5.3. *Let $u \in W_I$, $s \in I - \mathcal{L}(u)$ and $y, z \in W$ satisfy $sy < y < uz$, $\mu^s_{y,uz} \neq 0$ and $\mathcal{L}(z) \subseteq J$. Suppose one of the following conditions (1)-(2) is satisfied,*

(1) $L(w_0) > L_J > \tau_{I,\{w_0\}}$,

(2) W_I satisfies $(*)$ and (4.1.1) holds in W_I, and $L_J = \mathbf{a}_I(w_0')$.

Then either $y_I = w_0$ or $y = u'z$ for some $u' < u$ in W_I.

Proof. If $u = w_0$, then

$$c_u c_z = c_{uz} + \sum_{y' < uz; y'_I = w_0} h_{u,z,y'}c_{y'}. \tag{5.3.1}$$

If $u < w_0$, by Lemmas 5.1-5.2, we also have (5.3.1).

So

(5.3.2)

$$\begin{aligned} c_s c_{uz} &= c_s\left(c_u c_z - \sum_{y' < uz; y'_I = w_0} h_{u,z,y'}c_{y'}\right) \\ &= (c_s c_u)\, c_z - \sum_{y' < uz; y'_I = w_0} h_{u,z,y'}\left(v^{L(s)} + v^{-L(s)}\right)c_{y'} \\ &= \left(c_{su} + \sum_{u'; su' < u' < u} \mu^s_{u',u}c_{u'}\right)c_z - \sum_{y' < uz; y'_I = w_0} h_{u,z,y'}\left(v^{L(s)} + v^{-L(s)}\right)c_{y'}. \end{aligned}$$

Now assume $y_I \neq w_0$. Then we see by (5.3.2) that

$$h_{su,z,y} + \sum_{u';su'<u'<u} \mu^s_{u',u} h_{u',z,y} \neq 0.$$

Similar argument as in showing (5.3.1) gives

$$h_{su,z,y} + \sum_{u';su'<u'<u} \mu^s_{u',u} h_{u',z,y} = \sum_{u';su'<u'<u} \mu^s_{u',u} \delta_{u'z,y},$$

from Lemmas 5.1-5.2. Thus $y = u'z$ for some $u' < u$ in W with $\mu^s_{u',u} \neq 0$. □

Lemma 5.4. *Let $u \in W_I$, $s \in I - \mathcal{L}(u)$ and $y, z \in W$ satisfy $sy < y < uz$, $\mu^s_{y,uz} \neq 0$ and $\mathcal{L}(z) \subseteq J$. Suppose that W_I satisfies $(*)$, and that (4.1.1) holds in W_I, and that $\mathbf{a}_I(w_0') > L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$. Then either $y_I \in \{w_0, w_0'\}$ or $y = u'z$ for some $u' < u$ in W_I.*

Proof. As the condition (3.0.1) in Section 3 is satisfied, by Proposition 3.3, if $u \in \{w_0', w_0\}$, then

$$c_u c_z = c_{uz} + \sum_{y'<uz;y'_I \in \{w_0,w_0'\}} h_{u,z,y'} c_{y'}. \tag{5.4.1}$$

If $u \notin \{w, w_0\}$, we also have (5.4.1) by Lemma 5.2. So

(5.4.2)

$$\begin{aligned}
c_s c_{uz} &= c_s \left(c_u c_z - \sum_{y'<uz;y'_I=w_0} h_{u,z,y'} c_{y'} - \sum_{y'<uz;y'_I=w_0'} h_{u,z,y'} c_{y'} \right) \\
&= (c_s c_u)\, c_z - \sum_{y'<uz;y'_I=w_0} h_{u,z,y'} \left(v^{L(s)} + v^{-L(s)} \right) c_{y'} - \sum_{y'<uz;y'_I=w_0'} h_{u,z,y'} c_s c_{y'} \\
&= \left(c_{su} + \sum_{u';su'<u'<u} \mu^s_{u',u} c_{u'} \right) c_z - \sum_{y'<uz;y'_I=w_0} h_{u,z,y'} \left(v^{L(s)} + v^{-L(s)} \right) c_{y'} \\
&\quad - \sum_{y'<uz;y'_I=w_0'} h_{u,z,y'} c_s c_{y'}.
\end{aligned}$$

Assume $y_I \notin \{w_0, w_0'\}$. Then we see by (5.4.2) that

$$h_{su,z,y} + \sum_{u';su'<u'<u} \mu^s_{u',u} h_{u',z,y} \neq 0.$$

By Lemma 5.2, we have

$$h_{su,z,y} + \sum_{u';su'<u'<u} \mu^s_{u',u} h_{u',z,y} = \sum_{u';su'<u'<u} \mu^s_{u',u} \delta_{u'z,y}.$$

So $y = u'z$ for some $u' < u$ in W_I with $\mu^s_{u',u} \neq 0$. □

Lemma 5.5. *Suppose that W_I satisfies $(*)$, and that (4.1.1) holds in W_I, and that $L_J = \mathbf{a}_I(w_0')$. Let $z \in W$ be with $\mathcal{L}(z) \subseteq \{s\} \subseteq J$. Let $s' \in J$. Then $h_{w_0',s'w_0'z,w_0'z} \equiv v^{-L_J} h_{w_0',w_0',w_0'} \neq 0$.*

Proof. Let

$$\begin{aligned} K &= \{y \in W | y > w_0 z, h_{w_0', s'w_0'z, y} \neq 0\}, \\ K' &= \{y \in W | y > w_0' z, h_{w_0', s'w_0'z, y} \neq 0\}. \end{aligned}$$

For any $y \in K'$, by Proposition 3.3, $y_I \in \{w_0, w_0'\}$. As $w_0'z < y \leq w_0's'w_0'z$, if $y_I = w_0$, then $y = w_0 z$. If $y_I = w_0'$ then y must be of the form $y = w_0' \cdot s' \cdot x \cdot z$ for some $x \in W_I$ with $e < x \leq w_0'$.

We claim that $K' \subseteq \{w_0's'w_0'z, w_0 z\}$. By (1.4.5), to prove this, we only have to show

$$h'_{w_0', s'w_0'z, w_0's'xz} \equiv \begin{cases} 1 & \text{if } x = w_0', \\ 0 & \text{if } e < x < w_0'. \end{cases}$$

This follows by (1.4.4), as (1.4.4) implies that

$$\begin{aligned} h'_{w_0', s'w_0'z, w_0's'xz} &= p_{w_0', w_0'} p_{s'xz, s'w_0'z} f_{w_0', s'xz, w_0's'zx} \\ &= p_{s'xz, s'w_0'z}. \end{aligned}$$

From the fact $K' \subseteq \{w_0's'w_0'z, w_0 z\}$, we have $K = \{w_0's'w_0'z\}$. So

$$\begin{aligned} & h_{w_0', s'w_0'z, w_0 z} \\ \equiv\; & h'_{w_0', s'w_0'z, w_0 z} \text{ (by (1.4.5) and the claim)} \\ =\; & \sum_{x_1, x_2' \in W} p_{x_1, w_0'} p_{x_2', s'w_0'z} f_{x_1, x_2', w_0 z} \text{ (by (1.4.4))} . \end{aligned}$$

For $x_1 \leq w_0'$ and $x_2' \leq s'w_0'z$, if $f_{x_1, x_2', w_0 z} \neq 0$, then $\mathcal{L}(x_2') \subseteq I$, and there exists $x_2 \in W_I$ with $x_2' = x_2 \cdot z$. Consequently,

(5.5.1)
$$\begin{aligned} & h_{w_0', s'w_0'z, w_0 z} \\ =\; & \sum_{x_1, x_2 \in W_I} p_{x_1, w_0'} p_{x_2 z, s'w_0'z} f_{x_1, x_2 z, w_0 z} \\ =\; & v^{-L_J} \sum_{x_1, x_2 \in W_I} p_{x_1, w_0'} p_{s'x_2 z, s'w_0'z} f_{x_1, x_2 z, w_0 z} \text{ (by 1.3 (3))} \\ =\; & v^{-L_J} \sum_{x_1, x_2 \in W_I} p_{x_1, w_0'} p_{s'x_2 z, s'w_0'z} f_{x_1, x_2, w_0} \\ =\; & v^{-L_J} \sum_{x_1, x_2 \in W_I} p_{x_1, w_0'} p_{x_2, w_0'} f_{x_1, x_2, w_0} \text{ (by 1.3 (5), (7))} \\ =\; & v^{-L_J} h'_{w_0', w_0', w_0} \text{ (by (1.4.4))} \\ \equiv\; & v^{-L_J} h_{w_0', w_0', w_0}. \text{ (by (1.4.5))} \end{aligned}$$

As $K' \subseteq \{w_0's'w_0'z, w_0 z\}$, we have

$$\begin{aligned} & h_{w_0', s'w_0'z, w_0'z} \\ \equiv\; & h'_{w_0', s'w_0'z, w_0'z} - h_{w_0', s'w_0'z, w_0 z} p_{w_0'z, w_0 z} \text{ (by (1.4.5))} \\ =\; & \sum_{x_1, x_2' \in W} p_{x_1, w_0'} p_{x_2', s'w_0'z} f_{x_1, x_2', w_0'z} - h_{w_0', s'w_0'z, w_0 z} p_{w_0'z, w_0 z}. \text{ (by (1.4.4))} \end{aligned}$$

For $x_1 \leq w_0'$ and $x_2' \leq s'w_0'z$ with $f_{x_1, x_2', w_0 z} \neq 0$, if $\mathcal{L}(x_2') \not\subseteq I$, then $w_0'z = x_1 \cdot x_2'$, so $x_1 = w_0'$ and $x_2' = z$. If $\mathcal{L}(x') \subseteq I$, then there exists $x_2 \in W_I - \{e\}$ with $x_2 = x_2 \cdot z$. Consequently,

$$\begin{aligned}
& h_{w_0',s'w_0'z,w_0'z} \\
= & \sum_{x_1,x_2\in W_I} p_{x_1,w_0'}p_{x_2z,s'w_0'z}f_{x_1,x_2z,w_0'z} - h_{w_0',s'w_0'z,w_0z}p_{w_0'z,w_0z} \\
\equiv & v^{-L_J}\sum_{x_1,x_2\in W_I} p_{x_1,w_0'}p_{s'x_2z,s'w_0'z}f_{x_1,x_2,w_0'} - v^{-L_J}h_{w_0',w_0',w_0}p_{w_0',w_0} \text{ (by 1.3 (3) and (5.5.1))} \\
= & v^{-L_J}\sum_{x_1,x_2\in W_I} p_{x_1,w_0'}p_{x_2z,w_0'}f_{x_1,x_2,w_0'} - v^{-L_J}h_{w_0',w_0',w_0}p_{w_0',w_0} \text{ (by 1.3 (5), (7))} \\
= & v^{-L_J}h'_{w_0',w_0',w_0'} - v^{-L_J}h_{w_0',w_0',w_0}p_{w_0',w_0} \text{ (by (1.4.4))} \\
\equiv & v^{-L_J}h_{w_0',w_0',w_0'}. \text{ (by (1.4.5))}
\end{aligned}$$

Therefore our proof is complete by Lemma 4.4 and the assumption $\mathbf{a}_I(w_0') = L_J$. □

Recall the notation $\mathcal{C}_x$ $(x \in W)$ defined in 1.6. Note that $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ for any $x, y, z \in W$. Recall the definition of z appearing in α for $z \in W$ and $\alpha \in \mathcal{H}$.

Lemma 5.6. *For $s \in J$ and $u \in W_I$, let $x \in \mathcal{C}_{su}$, $s' \in S$ and $y \in W$ satisfy $s'y < y < x < s'x$ and $\mu^{s'}_{y,x} \neq 0$.*

(1) If $L(w_0) > L_J > \tau_{I,\{w_0\}}$, then $y \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0}$.

(2) Suppose that W_I satisfies $(*)$ and that (4.1.1) holds in W_I. If $\mathbf{a}_I(w_0') \geq L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$, then $y \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0'}$.

Proof. We prove the result using induction on $n = \ell(x) - \ell(su) \geq 0$. When $n = 0$, $x = su$ and $c_{s'}c_{su} = c_{s'su}$ as $s' \not\leq su$. Now suppose $n > 0$. Write $x = x' \cdot su$ for some $x' \in W$. Then

$$\begin{aligned}
(5.6.1) \qquad c_{s'}c_x &= c_{s'}\left(c_{x's}c_u - \sum_{z'<x} h_{x's,u,z'}c_{z'}\right) \\
&= c_{s'}c_{x's}c_u - \sum_{z'<x} h_{x's,u,z'}c_{s'}c_{z'}.
\end{aligned}$$

For any $y'', z'' \in W$ with $\mathcal{R}(z'') = \{s\}$ and $h_{z'',u,y''} \neq 0$ (hence $h_{u^{-1},z''^{-1},y''^{-1}} \neq 0$), we have

(1') $y'' \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0}$ if $L(w_0) > L_J > \tau_{I,\{w_0\}}$, by Lemma 5.1, and

(2') $y'' \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0'}$ if W_I satisfies $(*)$ and (4.1.1) holds in W_I and $\mathbf{a}_I(w_0') \geq L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$, by Lemma 5.2 with u_1^{-1}, z''^{-1}, y''^{-1} in the places of u_1, z, y, respectively.

Write $A = c_{s'}c_{x's}c_u$ and $B = \sum_{z'<x} h_{x's,u,z'}c_{s'}c_{z'}$. Then $c_{s'}c_x = A - B$. Since $\mu^{s'}_{y,x} \neq 0$, y appears in A or B.

If y appears in A, then we have (1) and (2) following from (1') and (2'). Now suppose y does not appear in A. Thus it appears in B, and there exists $z' < x$ with $h_{x's,u,z'} \neq 0$ and $h_{s',z',y} \neq 0$.

As $h_{x's,u,z'} \neq 0$, when $L(w_0) > L_J > \tau_{I,\{w_0\}}$, we have $z' \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0}$ by (1'). Now $h_{s',z',y} \neq 0$. If $z' \in \mathcal{C}_{w_0}$, then $y \in \mathcal{C}_{w_0}$. If $z' \in \mathcal{C}_{su}$, then $y \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0}$ by the inductive hypothesis.

When W_I satisfies $(*)$ and (4.1.1) holds in W_I and $\mathbf{a}_I(w_0') \geq L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$, we have $z' \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0'}$ by (2'). If $z' \in \mathcal{C}_{w_0'}$, then $y \in \mathcal{C}_{w_0'}$ by Corollary

3.4(2). If $z' \in \mathcal{C}_{su}$, then $y \in \mathcal{C}_{su} \cup \mathcal{C}_{w_0'}$ by the inductive hypothesis. The result follows.

□

6. The multiplication by c_s for $s \in J$

Keep the assumptions for the weighted Coxeter group (W, S, L) in 4.1.

Lemma 6.1. *Let $z \in W$ and $\mathcal{L}(z) = \{s'\} \subseteq J$ and $s \in J - \{s'\}$. Then*

$$c_s c_z = c_{sz} + \varepsilon c_{s'z},$$

where $\varepsilon = \begin{cases} 1 & \text{if } s \in \mathcal{L}(s'z), \\ 0 & \text{if } s \notin \mathcal{L}(s'z). \end{cases}$

Proof. From 1.3 (3), we see that for any $y \in W$ with $sy < y < z$,

$$v^{L_J} p_{y,z} = p_{s'y,z} \equiv \delta_{s'y,z} = \delta_{y,s'z}.$$

So the result follows by Lemma 1.8. □

Lemma 6.2. *Let $z = u_1 \cdot z'$ for some $u_1 \in W_I - \{e\}$ and $z' \in W$ with $\mathcal{L}(z') = \{s'\} \subseteq J$. Let $s \in J - \mathcal{L}(z)$. Suppose that one of the following conditions (a)-(b) is satisfied*

(a) $L(w_0) > L_J > \tau_{I,\{w_0\}}$ and $u_1 \neq w_0$;
(b) W_I satisfies $(*)$ and (4.1.1) holds in W_I, $L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$ and $u_1 \notin \{w_0', w_0\}$,

Then

(1) $v^{L_J} h_{u_1,z',z''} p_{y,z''} \equiv 0$ for any $z'', y \in W$ with $sy < y \leq z'' < z$;
(2) $c_s c_z = c_{sz} + \mu_1 c_{z'} + \mu_2 c_{u_1 s' z'}$ where $\mu_1, \mu_2 \in \mathcal{A}$ are defined by $\mu_1 = \bar{\mu}_1 \equiv \begin{cases} v^{L_J} p_{e,u_1} & \text{if } s = s', \\ 0 & \text{if } s \neq s', \end{cases}$ and $\mu_2 = \begin{cases} 1 & \text{if } s \in \mathcal{L}(u_1 s' z'), \\ 0 & \text{if } s \notin \mathcal{L}(u_1 s' z'). \end{cases}$

Proof. (1) In case (a), $h_{u_1,z',z''} \neq 0$ implies $z''_I = w_0$ and $\deg v^{L_J} h_{u_1,z',z''} < L(w_0)$ by Lemma 5.1. So $\deg v^{L_J} h_{u_1,z',z''} p_{y,z''} < L(w_0) - L(w_0) = 0$ by 1.3 (3) and the fact $\mathcal{L}(y) = \{s\} \subseteq J$.

In case (b), we see by Lemma 5.2 that $h_{u_1,z',z''} \neq 0$ implies $z''_I \in \{w_0', w_0\}$ and $\deg h_{u_1,z',z''} < \mathbf{a}_I(z''_I) - L_J$. By 1.3 (3), the fact $\mathcal{L}(y) = \{s\} \subset J$, Propositions 3.5, 3.8 and Lemma s4.4, we see that if $z''_I = w_0$, then $\deg v^{L_J} h_{u_1,z',z''} p_{y,z''} < L_J + (L(w_0) - L_J) - L(w_0) = 0$, and that if $z''_I = w_0'$, then $\deg v^{L_J} h_{u_1,z',z''} p_{y,z''} < L_J + (\mathbf{a}_I(w_0') - L_J) - \mathbf{a}_I(w_0') = 0$.

So the result is proved in either case.

(2) If $s = s'$, then $\mu^s_{z',z} \equiv v^{L_J} p_{e,u_1} = v^{L_J} p_{s,u_1 s'}$ by 1.3 (4), (7). If $s \neq s'$, then $\mu^s_{z',z} = 0$. In this case, $v^{L_J} p_{s,u_1 s'} = 0$.

Let $y \in W$ be with $sy < y < z$ and $y \neq z'$. Then $y < z'$. The result will follow by Lemma 1.8 if

$$v^{L_J} p_{y,z} - v^{L_J} p_{s,u_1 s'} p_{y,z'} \equiv \begin{cases} 1 & \text{if } y = u_1 s' z', \\ 0 & \text{if } y \neq u_1 s' z'. \end{cases} \tag{6.2.1}$$

But

$$\begin{aligned}
& v^{L_J}p_{y,z} - v^{L_J}p_{s,u_1s'}p_{y,z'} \\
= \; & v^{L_J}\sum_{x_1,x_2} p_{x_1,u_1}p_{x_2,z'}f_{x_1,x_2,y} - \sum_{z''<z} v^{L_J}h_{u_1,z',z''}p_{y,z''} - v^{L_J}p_{s,u_1s'}p_{y,z'} \\
\equiv \; & v^{L_J}\sum_{x_1,x_2} p_{x_1,u_1}p_{x_2,z'}f_{x_1,x_2,y} - v^{L_J}p_{s,u_1s'}p_{y,z'} \text{ (by (1))} \\
= \; & \left(v^{L_J}p_{e,u_1}p_{y,z'} - v^{L_J}p_{s,u_1s'}p_{y,z'}\right) + v^{L_J}\sum_{x_1,x_2;x_1\neq e} p_{x_1,u_1}p_{x_2,z'}f_{x_1,x_2,y} \\
\equiv \; & v^{L_J}\sum_{x_1,x_2;x_1\neq e} p_{x_1,u_1}p_{x_2,z'}f_{x_1,x_2,y}.
\end{aligned}$$

We see by (1.1.1) that the conditions $f_{x_1,x_2,y} \neq 0$, $x_1 \in W_I - \{e\}$, $\mathcal{L}(y) = \{s\}$, $\mathcal{L}(z') = \{s'\}$ and $s, s' \in J$ imply $x_2 = x_1^{-1}y$, $f_{x_1,x_2,y} = 1$ and $p_{x_2,z'} = v^{-L_J}p_{s'x_2,z'}$. So

$$v^{L_J}p_{y,z} - v^{L_J}p_{s,u_1s'}p_{y,z'} \equiv \sum_{x_1,x_2;x_1\neq e} p_{x_1,u_1}p_{s'x_1^{-1}y,z'} \equiv \begin{cases} 1 & \text{if } y = u_1s'z', \\ 0 & \text{if } y \neq u_1s'z'. \end{cases}$$

So (6.2.1) is true and the result follows.

□

Lemma 6.3. *Let $z = u_1 \cdot z'$ for some $u_1 \in W_I - \{e\}$ and $z' \in W$ with $\mathcal{L}(z') = \{s'\} \subseteq J$. Let $s \in J - \mathcal{L}(z)$.*

(1) If $L_J < L(w_0)$ and $u_1 = w_0$, then $c_sc_z = c_{sz}$.

(2) Suppose that W_I satisfies (∗) and that (4.1.1) holds in W_I. If $L_J \leq \mathbf{a}_I(w_0')$ and $u_1 = w_0'$, then

$$c_sc_z = c_{sz} + \varepsilon_2 c_{z'},$$

where $\varepsilon \in \mathbb{Z}-\{0\}$ and it is given by $\varepsilon = \bar{\varepsilon} \equiv \begin{cases} v^{L_J}p_{e,w_0'} & \text{if } L_J = \mathbf{a}_I(w_0') \text{ and } s = s', \\ 0 & \text{otherwise.} \end{cases}$

Proof. (1) Let $E := \{y \in W | \mu^s_{y,z} \neq 0; sy < y < z\}$. If $E \neq \emptyset$, take y the maximal in E with respect to $\leq$, then

$$\mu^s_{y,z} \equiv v^{L_J}p_{y,z} \tag{6.3.1}$$

by 1.3 (4). If $L_J < L(w_0)$ and $u_1 = w_0$, then $\mu^s_{y,z} \equiv v^{L_J - L(w_0)}p_{w_0y,z} \equiv 0$ by (6.3.1) and 1.3 (3), a contradiction. This proves (1).

(2) Let $E' = \{y \in W | sy < y < z, v^{L_J}p_{y,w} \not\equiv 0\}$. Take $y \in E'$. Under the assumptions of (2), we have $L_J = \mathbf{a}_I(w_0')$, $y = z'$ and $v^{L_J}p_{y,w} \equiv v^{L_J}p_{e,w_0'}$ and $\deg v^{L_J}p_{y,w} \leq 0$ by 1.3 (7), Propositions 3.5, 3.8 and Lemma 4.4.

So $E' \subseteq \{z'\}$, and $v^{L_J}p_{z',w} \equiv v^{L_J}p_{e,w_0'}$ with $\deg v^{L_J}p_{z',w} \leq 0$. So the result follows by Lemma 1.8.

□

7. Left and two-sided cells Γ of W with $\Gamma \cap J \neq \emptyset$

Keep all the assumptions in 4.1 for (W, S, L). We shall describe all the left cells and two-sided cells (say Γ) of W with $\Gamma \cap J \neq \emptyset$ in the cases

(1) $L(w_0) > L_J > \tau_{I,\{w_0\}}$ (see Proposition 7.4), and

(2) W_I satisfies $(*)$ and (4.1.1) holds in W_I, $\mathbf{a}_I(w_0') \geq L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$ (see Proposition 7.5).

Lemma 7.1. *Let $u_1 \in W_I$. Let $z \in W$ be with $\mathcal{L}(z) = \{s\} \subseteq J$ and $s' \in J - \mathcal{L}(u_1 z)$. Suppose one of the following (1)-(2) is true.*

(1) $L(w_0) > L_J > \tau_{I,\{w_0\}}$ and $u_1 \neq w_0$.
(2) W_I satisfies $(*)$ and (4.1.1) holds in W_I, $L(w_0) > L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$ and $u_1 \notin \{w_0', w_0\}$.

Then $z \sim_L u_1 z \sim_L s' u_1 z$.

Proof. When $u_1 = e$, $z \sim_L s'z$ follows by Lemma 6.1. Now suppose $u_1 > e$. Applying Lemma 6.2 with $u_1^{-1} s' u z$ in the place of z, we get $\mu^s_{z, u_1^{-1} s' u_1 z} = 1$. So $z \leq_L u_1^{-1} s' u_1 z \leq_L s' u_1 z \leq_L u_1 z \leq_L z$, and $z \sim_L u_1 z \sim_L s' u_1 z$. □

Lemma 7.2. *Suppose that W_I satisfies $(*)$, and that (4.1.1) holds in W_I, and that $L_J = \mathbf{a}_I(w_0')$. Let $z \in W - \{e\}$.*

(1) If $\mathcal{L}(z) \subseteq J$, then $z \sim_L w_0' z$.
(2) If $z_I = w_0' < z$, then $z \sim_L sz$ for any $s \in J$.

Proof. (1) Take $s \in \mathcal{L}(z) \subseteq J$. By Lemma 6.3, $\mu^s_{z, w_0' z} \neq 0$. So $z \leq_L w_0' z \leq z$ and $z \sim_L w_0' z$.
(2) By Lemma 5.5, $h_{u_1, sz, z} \neq 0$. So $z \leq_L sz \leq_L z$ and $z \sim_L sz$.
□

Recall the notation $y \leftarrow_L x$, Ω_x and Ω_X for $x, y \in W$ and $X \subseteq W$ defined in 1.6.

Lemma 7.3. *Let $x, y \in W$. Then $y \not\leftarrow_L x$ if one of the following conditions is satisfied.*

(1) $L(w_0) > L_J > \tau_{I,\{w_0\}}$. Either $x \in \Omega_{w_0}$ and $y \notin \Omega_{w_0}$ or $x \in \Omega_J$ and $y \notin \Omega_J \cup \{w_0\}$.
(2) W_I satisfies $(*)$ and (4.1.1) holds in W_I, $L_J = \mathbf{a}_I(w_0')$. Either $x \in \Omega_{w_0}$ and $y \notin \Omega_{w_0}$ or $x \in \Omega_J$ and $y \notin \Omega_J \cup \Omega_{w_0'}$.
(3) W_I satisfies $(*)$ and (4.1.1) holds in W_I, $\mathbf{a}_I(w_0') > L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$. Either $x \in \Omega_{w_0'}$ and $y \notin \Omega_{w_0'}$ or $x \in \Omega_J$ and $y \notin \Omega_J \cup \Omega_{w_0'}$.

Proof. We must prove the equation $h_{s,x,y} = 0$ for any $s \in S$. When $s \in I$, this follows by Lemmas 5.3-5.4.

Now suppose $s \in J$. Then this is obvious if $x \in \Omega_J$ and $y \notin \Omega_J$.

(1) If $L(w_0) > L_J > \tau_{I,\{w_0\}}$, $x \in \Omega_{w_0}$ and $y \notin \Omega_{w_0}$, then $h_{s,x,y} = 0$ follows by Lemmas 6.1, 6.2 (a) and 6.3 (1).
(2) If W_I satisfies $(*)$ and (4.1.1) holds in W_I, $L_J = \mathbf{a}_I(w_0')$, $x \in \Omega_{w_0}$ and $y \notin \Omega_{w_0}$, then $h_{s,x,y} = 0$ follows by Lemmas 6.1, 6.2 (b) and 6.3 (2).
(3) If W_I satisfies $(*)$ and (4.1.1) holds in W_I, $\mathbf{a}_I(w_0') > L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$, $x \in \Omega_{w_0'}$ and $y \notin \Omega_{w_0'}$, then $h_{s,x,y} = 0$ follows by Lemmas 6.1, 6.2 (b) and 6.3.

□

Theorem 7.4. *Suppose $L(w_0) > L_J > \tau_{I,\{w_0\}}$.*

(1) Ω_{w_0} is a union of two-sided cells.
(2) $\Omega_J - \Omega_{w_0}$ is a two-sided cell of W.
(3) $\mathcal{C}_{sx} - \Omega_{w_0}$ is a left cell of W for any $s \in J$ and $x \in W_I - \{w_0\}$.

Proof. (1) follows by Lemma 7.3 (1). Then (3) is a consequence of Lemmas 5.6 (1) and 7.1.

Take $s' \in J$. For any $z \in \Omega_J - \Omega_{w_0}$, there exists $s \in J$ and $x \in W_I - \{w_0\}$ with $z \in \mathcal{C}_{sx} - \Omega_{w_0}$. So by (3), $z \sim_L sx$. Moreover, by Lemma 7.1 and 1.3 (4), $sx \sim_R s \sim_L s's \sim_R s'$. So $z \sim_{LR} s'$ and $\Omega_J - \Omega_{w_0}$ is contained in a single two-sided cell of W. It is known by Lemma 7.3 (1) that $\Omega_J - \Omega_{w_0}$ is a union of two-sided cells of W. So $\Omega_J - \Omega_{w_0}$ itself forms a two-sided cell of W. □

Theorem 7.5. *Suppose that W_I satisfies (∗) and that (4.1.1) holds in W_I.*

(1) If $L_J = \mathbf{a}_I(w_0')$, then
  - *(a) Ω_{w_0} is a union of two-sided cells of W;*
  - *(b) $(\Omega_J \cup \{w_0'\}) - \Omega_{w_0}$ is a two-sided cell of W;*
  - *(c) $\{\mathcal{C}_{w_0'} - \Omega_{w_0}\} \cup \{\mathcal{C}_{sx} - \Omega_{w_0} | s \in J, x \in W_I - \{w_0', w_0\}\}$ is the set of left cells of W in $(\Omega_J \cup \{w_0'\}) - \Omega_{w_0}$;*

(2) If $\mathbf{a}_I(w_0') > L_J > \max\{\tau_{I,\{w_0',w_0\}}, \pi_{w_0'} + L_1 - L_2\}$, then
  - *(a) $\Omega_{w_0'}$ is a union of two-sided cells of W;*
  - *(b) $\Omega_J - \Omega_{w_0'}$ is a two-sided cell of W;*
  - *(c) $\{\mathcal{C}_{sx} - \Omega_{w_0'} | s \in J, x \in W_I - \{w_0', w_0\}\}$ is the set of left cells of W in $\Omega_J - \Omega_{w_0'}$.*

Proof. (1) (a) follows by Lemma 7.3 (2). By Lemmas 5.6, 7.1 and 7.2, $\mathcal{C}_{sx} - \Omega_{w_0}$ is a left cell for any $x \in W_I - \{w_0', w_0\}$. Moreover, $\mathcal{C}_{w_0'} - \Omega_{w_0}$ is a left cell.

Let $z \in \Omega_J - \Omega_{w_0}$. If $z \in \mathcal{C}_{w_0'} - \Omega_{w_0}$, then $z \sim_L w_0'$ as $\mathcal{C}_{w_0'} - \Omega_{w_0}$ is a left cell. If $z \notin \mathcal{C}_{w_0'} - \Omega_{w_0}$, then $z \in \mathcal{C}_{s'x'} - \Omega_{w_0'}$ for some $x' \in W_I - \{w_0, w_0'\}$ and $s' \in J$. We have $s'x' \sim_R s' \sim_L w_0's' \sim_R w_0'$ by Lemma 7.2. So $z \sim_{LR} w_0'$. Thus $(\Omega_J \cup \{w_0'\}) - \Omega_{w_0}$ is contained in a single two-sided cell of W.

By Lemma 7.3 (2), we see that $(\Omega_J \cup \{w_0'\}) - \Omega_{w_0}$ is a union of two-sided cells. So (b) and (c) follow.

(2) (a) follows by Lemma 7.3 (3). By Lemmas 5.6 and 7.1, $\mathcal{C}_{sx} - \Omega_{w_0'}$ is a left cell of W for any $x \in W_I - \{w_0', w_0\}$.

Let $s_1 \in J$ and $z \in \Omega_J - \Omega_{w_0'}$. Then $z \in \mathcal{C}_{s'x'}$ for some $s' \in J$ and $x' \in W_I - \{w_0, w_0'\}$. Since $\mathcal{C}_{s'x'} - \Omega_{w_0'}$ is a left cell, $z \sim_L s'x'$. Moreover, $s'x' \sim_R s'$ by Lemma 7.1, and $s' \sim_L s_1$. So $z \sim_{LR} s_1$. Thus $\Omega_J - \Omega_{w_0'}$ is contained in one two-sided cell.

Moreover, $\Omega_J - \Omega_{w_0'}$ is a union of left cells of W by Lemma 7.3 (3). So (b) and (c) are true.

□

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