THE REDUCED EXPRESSIONS IN A COXETER SYSTEM WITH A STRICTLY COMPLETE COXETER GRAPH

Jian-yi Shi

Department of Mathematics
East China Normal University
Shanghai, 200241, P. R. China

Abstract. Let (W, S) be a Coxeter system with a strictly complete Coxeter graph. The present paper is concerned with the set $\mathrm{Red}(z)$ of all reduced expressions for any $z \in W$. By associating each bc-expression to a certain symbol, we describe the set $\mathrm{Red}(z)$ and compute its cardinal $|\mathrm{Red}(z)|$ in terms of symbols. An explicit formula for $|\mathrm{Red}(z)|$ is deduced, where the Fibonacci numbers play a crucial role.

Let (W, S) be a Coxeter system, that is, W is a Coxeter group with S its Coxeter generator set. Let $\mathrm{Red}(z)$ be the set of all reduced expressions of $z \in W$. When W is either a finite or an affine Coxeter group, it is known that the set $\mathrm{Red}(z)$ is closely related with various objects in combinatorics, geometry and representation theory such as Young tableaux, hyperplane arrangements, Schubert functions, symmetric functions, etc (see [1, 3, 5, 6]). The present paper is concerned with the case where the Coxeter graph $\Gamma(W)$ of W is strictly complete, that is, the order m_{st} of the product st is greater than 2 for any $s \neq t$ in S and there does not exist any triple $\{s, r, t\}$ in S with $m_{sr} = m_{st} = 3$ and $m_{tr} < \infty$. The aim of the paper is to describe the set $\mathrm{Red}(z)$ and to compute the cardinal $|\mathrm{Red}(z)|$ for any $z \in W$. To this end, we first reduce ourselves to the case where z has a bc-expression (see 1.5 and Theorem 1.10), then we associate each bc-expression $\zeta \in \mathrm{Red}(z)$ to a certain symbol $S(\zeta)$ (see 3.2) and establish a bijection between the set $\mathrm{Red}(z)$ and the associated symbol set $\mathrm{Symb}(z)$ in the case of $\ell_b(z) > 1$ (see Theorem

Key words and phrases. Coxeter system; strictly complete Coxeter graph; bc-expressions; symbols.
Supported by the NSF of China (11131001 and 11471115) and Program of Shanghai Subject Chief Scientist (11xd1402200) and Shanghai Key Laboratory of PMMP

4.1), by which we reduce ourselves to study the set Symb(z) (see Corollary 4.2). We describe all the symbols associated to bc-expressions of W up to equivalence in Theorem 3.9. To compute $|\mathrm{Red}(z)|$ for any $z \in W$ having a bc-expression, we reduce ourselves to the case where $\alpha_{l_0,n_1,l_1,...,n_r,l_r} \in \mathrm{Symb}(z)$ for some integers $r, n_1, l_1, ..., n_r, l_r \geqslant 1$ and $l_0 \geqslant 0$ (see Proposition 5.3) and deduce an explicit formula of $|\mathrm{Red}(z)|$ for such $z \in W$ (see Theorem 5.7). The Fibonacci numbers play a crucial role in such formulation.

In the study of the set Red(z), there is an interesting phenomenon that the structure and the cardinal of Red(z) only depend on the set $S_{\mathrm{fin}} := \{\{s,t\} \subseteq S \mid s \neq t, m_{st} < \infty\}$, but are independent of the precise values m_{st} for $\{s,t\} \in S_{\mathrm{fin}}$, provided that $\Gamma(W)$ is strictly complete. I wonder if a modified phenomenon occurs in a more general case. We shall make some further investigation concerning this in a forthcoming paper.

The contents of the paper are organized as follows. The concept of a bc-expression is introduced in Section 1. Then the properties of bc-expressions are investigated in Section 2. In Section 3, we associate each bc-expression to a symbol and describe all the symbols associated to bc-expressions of W. The computation of $|\mathrm{Red}(z)|$ is reduced to that of $|\mathrm{Symb}(z)|$ in Section 4. Finally, an explicit formula is deduced for $|\mathrm{Red}(z)|$ in Section 5.

§1. bc-expressions.

In this section, we introduce the concept of a bc-expression in a Coxeter system which will be crucial in the subsequent discussion.

1.1. Let $\mathbb{N}$ (respectively, $\mathbb{P}$) be the set of all non-negative (respectively, positive) integers. For any $i \leqslant j$ in $\mathbb{N}$, denote by $[i,j]$ the set $\{i, i+1, ..., j\}$ and denote $[1,i]$ simply by $[i]$ for any $i \in \mathbb{P}$.

Let (W,S) be a Coxeter system. Each $z \in W$ can be expressed in the form $z = s_1 s_2 \cdots s_r$ with $s_k \in S$ for any $k \in [r]$. Define the *length* $\ell(z)$ of z to be the smallest number r among all such expressions for z and call any expression $z = s_1 s_2 \cdots s_{\ell(z)}$ a *reduced expression* of z. Let Red(z) be the set of all reduced expressions of z. For any $s_1 s_2 \cdots s_r$, $s'_1 s'_2 \cdots s'_r$ in Red(z) with $s_i, s'_j \in S$, we use the notation $s_1 s_2 \cdots s_r \equiv s'_1 s'_2 \cdots s'_r$ to indicate the equations $s_k = s'_k$ hold for all $k \in [r]$.

For any $s \neq t$ in S and any $k \in \mathbb{N}$, denote by $[sts\cdots]_k$, $[\cdots sts]_k$ the expressions $sts\cdots$, $\cdots sts$ (k factors) respectively. For example, $[sts\cdots]_6 \equiv [\cdots tst]_6 \equivststst$. A transformation $s_1 \cdots [sts\cdots]_{m_{st}} \cdots s_r \mapsto s_1 \cdots [tst\cdots]_{m_{st}} \cdots s_r$ is called a *braid-move* if

$s \neq t$ in S satisfy $m_{st} < \infty$. By a result of Tits in [4], we have

Lemma 1.2. *Any two reduced expressions of $z \in W$ can be transformed from one to the other by successively applying some braid-moves.*

1.3. We say that two expressions ζ, ζ' in W are *equivalent*, written $\zeta \sim \zeta'$, if ζ' can be obtained from ζ by successively applying some braid-moves. This defines an equivalence relation on the expressions in W. By Lemma 1.2, we see that two reduced expressions ζ, ζ' in W satisfy $\zeta \sim \zeta'$ if and only if $\zeta, \zeta' \in \mathrm{Red}(z)$ for some $z \in W$. So any equivalence class of reduced expressions in W has the form $\mathrm{Red}(z)$ for some $z \in W$.

From now on, we always assume that the Coxeter graph $\Gamma(W)$ of W is strictly complete. For any $I \subseteq S$, the subgroup W_I of W generated by I is called a *standard parabolic subgroup of rank* $|I|$.

Lemma 1.4. *Any finite standard parabolic subgroup W_I, $I \subseteq S$, of W is of rank $\leqslant 2$.*

Proof. Since $\Gamma(W)$ is a complete graph, any standard parabolic subgroup W_I of W with $I \subseteq S$ and $|I| \geqslant 3$ is infinite by the classification of Coxeter groups (see [2]). □

1.5. Let $\zeta \equiv s_1 s_2 \cdots s_r$ be a reduced expression in W with $s_k \in S$ for $k \in [r]$. By a segment of ζ, we mean a subexpression of ζ of the form $\zeta_{ij} \equiv s_i s_{i+1} \cdots s_j$ for some $i \leqslant j$ in $[r]$. A segment ζ_{ij} of ζ is called *proper*, if $(i, j) \neq (1, r)$.

A segment ζ_{ij} of ζ is called *a braid factor* of ζ, if $\zeta_{ij} \equiv [sts\cdots]_k$ for some $s \neq t$ in S with $m_{st} < \infty$ and $k \in \{m_{st} - c \mid c \in \{0, 1, 2\}\}$ and $\{s_{i-1}, s_{j+1}\} \cap \{s, t\} = \emptyset$. $\{s, t\}$ is called the associated pair in S for ζ_{ij}. We see that the braid factor $\zeta_{ij} \equiv [sts\cdots]_k$ determines $\{s, t\}$ unless $(m_{st}, k) = (3, 1)$. A braid factor $\zeta_{ij} \equiv [sts\cdots]_k$ of ζ is called *full* if $k = m_{st}$. Two braid factors $\zeta_{ij} \equiv [sts\cdots]_{j+1-i}$, $\zeta_{pq} \equiv [s't's'\cdots]_{q+1-p}$ of ζ are called *neighboring* if $i < p$ and $j < q$ and $p \in \{j, j+1\}$, in this case, call ζ_{ij}, ζ_{pq} *intersect* if $j = p$, *disjoint* if $p = j + 1$, and *braid-connected*, if there exists some expression ζ' in W with $\zeta' \sim \zeta$ satisfying one of the following conditions:

(i) $j = p$, and $\zeta'_{i'j}$, $\zeta'_{j,q'}$ are full braid factors of ζ' for some $i', q' \in [r]$ with $i' \leqslant i$ and $q \leqslant q'$;

(ii) $p = j + 1$, and $\zeta'_{i'j'}$, $\zeta'_{j',q'}$ are full braid factors of ζ' for some $i', j', q' \in [r]$ with $i' \leqslant i$ and $j \leqslant j' \leqslant j + 1$ and $q \leqslant q'$.

Clearly, for two braid-connected braid factors ζ_{ij}, ζ_{pq} of ζ, the sum $\ell(\zeta_{ij})+\ell(\zeta_{pq})$ is >3 in the case (i) and >2 in the case (ii). The associated pairs $\{s.t\}$, $\{s',t'\}$ of ζ_{ij}, ζ_{pq} respectively in S satisfy $|\{s.t\}\cap\{s',t'\}|=1$. Later we shall prove that for ζ, the pairs $(\{s,t\},\{s',t'\})$ and (ζ_{ij},ζ_{pq}) are determined each other (see Lemma 2.2).

A result of Xi is reformulated below for the proof of Lemma 1.7.

Lemma 1.6. (see [7, Lemma 2.2]) *Let $r,s,t\in S$ satisfy $m_{rs},m_{rt},m_{st}>2$. Then there is no $w\in W$ such that either $w=rw_1=tsw_2$ or $w=w_1r=w_2st$, where $\ell(w)=\ell(w_1)+1=\ell(w_2)+2$.*

The next result tells us that for a braid factor ξ of a reduced expression ζ in W, the braid factor braid-connected with ξ at a given side is unique whenever it exists.

Lemma 1.7. *Let $\zeta\equiv s_1s_2\cdots s_r$ be a reduced expression in W with $s_k\in S$ for $k\in[r]$. Let ζ_{ij}, ζ_{pq}, ζ_{mn} be three braid factors of ζ with $i<p,m$. If both ζ_{pq} and ζ_{mn} are braid-connected with ζ_{ij} then $(p,q)=(m,n)$.*

Proof. We have $p,m\in\{j,j+1\}$ by the assumption on ζ_{ij}, ζ_{pq}, ζ_{mn}. Suppose $(p,q)\neq(m,n)$. Then ζ_{pq}, ζ_{mn} must have different associated pairs in S, hence $\{p,m\}=\{j,j+1\}$, say $p=j$ and $m=j+1$ for the sake of definiteness. We claim that ζ_{mn} is not a proper segment of ζ_{pq}. For, otherwise, we would have $m=n=j+1$ and $q\geqslant j+1$. Hence there would be a triple $\{s,t,u\}$ in S such that $\zeta_{ij}\equiv[\cdots sus]_h$ and $\zeta_{mn}\equiv t$ and $\zeta_{pq}\equiv[sts\cdots]_k$ for some $h,k\geqslant 2$ with $\{u,t\}$, $\{s,t\}$ the associated pairs of ζ_{mn},ζ_{pq} respectively in S. Write $\zeta\equiv x[\cdots sus]_h ty$ for some $x,y\in W$ with $\ell(\zeta)=\ell(x)+\ell(y)+h+1$. Then $y=[sus\cdots]_{m_{su}}y_1$ for some $y_1\in W$ with $\ell(y)=\ell(y_1)+m_{su}$. Since $m_{su}\geqslant 3$ by the assumption of $\Gamma(W)$ being complete, this would imply that $m_{tu}=m_{st}=3$ and $m_{su}<\infty$ by Lemma 1.6, contradicting the assumption of $\Gamma(W)$ being strictly complete. Our claim is proved. We see by the claim that $i<j$, $q=j+1<n$ and that there exists a triple $\{s,t,u\}$ in S, where $\zeta_{ij}\equiv[\cdots sus]_h$, $\zeta_{j,j+1}\equiv st$ and $\zeta_{j+1,n}\equiv[tut\cdots]_k$ for some $h,k\geqslant 2$. Hence $\zeta\equiv xustuy$ for some $x,y\in W$ with $\ell(\zeta)=\ell(x)+\ell(y)+4$. Since both $\zeta_{j,j+1}$ and $\zeta_{j+1,n}$ are braid-connected with ζ_{ij}, there would be some expressions ζ',ζ'' in W with $\zeta'\sim\zeta\sim\zeta''$ such that $\zeta'_{i'j}$, $\zeta'_{jq'}$ are full braid factors of ζ' with $i'\leqslant i<j<q\leqslant q'$, and that $\zeta''_{i''j''},\zeta''_{j''n''}$ are full braid factors of ζ'' with $i''\leqslant i$ and $j\leqslant j''\leqslant j+1$ and $n\leqslant n''$ and that the associated pairs in S for $\zeta'_{jq'}$, $\zeta''_{j''n''}$ are $\{s,t\}$, $\{t,u\}$, respectively. This would imply that $xus=x_1[\cdots sus]_{m_{su}}$ and $uy=[usu\cdots]_{m_{su}}y_1$ for some $x_1,y_1\in W$

with $\ell(xus) = \ell(x_1) + m_{su}$ and $\ell(uy) = \ell(y_1) + m_{su}$, hence $m_{st} = m_{tu} = 3$ and $m_{su} < \infty$ by Lemma 1.6 and the assumption of $\Gamma(W)$ being complete. The latter contradicts the assumption of $\Gamma(W)$ being strictly complete. Our proof is complete. □

1.8. A reduced expression $\zeta \equiv s_1 s_2 \cdots s_r$ in W with $s_k \in S$ for $k \in [r]$ is called *a braid-connected expression* (or *a bc-expression* in short) if there exists a sequence of braid factors $\tau : \zeta_{i_1 j_1}, \zeta_{i_2 j_2}, ..., \zeta_{i_a j_a}$ of ζ with some $a \in \mathbb{P}$ and $i_1 < i_2 < \cdots < i_a$ and $(i_1, j_a) = (1, r)$ such that either $a = 1$ with $\zeta_{i_1 j_1}$ full, or $a > 1$ with $\zeta_{i_c j_c}, \zeta_{i_{c+1} j_{c+1}}$ being braid-connected for any $c \in [a-1]$. In this case, denote by $\ell_{b,\tau}(\zeta)$ the number a of braid factors in τ, call τ *a braid sequence* of ζ and call $\lambda : \{t_1, t_1'\}, ..., \{t_a, t_a'\}$ *the associated pair sequence in S for ζ, τ*, where $\{t_c, t_c'\}$ is the associated pair in S for the braid factor $\zeta_{i_c j_c}$. For any $c \in [a]$, call $\zeta_{i_c j_c}$ the cth braid factor of ζ, τ, and call $\{t_c, t_c'\}$ the cth associated pair in S for ζ, τ.

The next result is concerning a bc-expression obtained from a given bc-expression by removing or adding a braid factor.

Lemma 1.9. *Let ζ be a bc-expression in W with $\tau : w_1 \equiv [sts\cdots]_k, w_2, ..., w_a$ its braid sequence for some $s \neq t$ in S. Let $\xi \equiv [\cdots uvu]_h$ for some $u \neq v$ in S and $h \in \mathbb{P}$.*

(1) $k \in \{m_{st}, m_{st} - 1\}$.

(2) $\xi\zeta$ is a reduced expression if and only if $u \notin \{s, t\}$ and at least one of three relations $m_{uv} = \infty$, $h < m_{uv} < \infty$ and $v \notin \{s, t\}$ holds.

(3) $\xi\zeta$ is a bc-expression if and only if $u \notin \{s, t\}$, $v \in \{s, t\}$ and $h = m_{uv} - 1 < \infty$.

(4) Assume $a > 1$. If either w_1, w_2 intersect or $k = m_{st} - 1$, let ζ' be obtained from ζ by removing the leftmost segment $[sts\cdots]_{m_{st}-1}$, then ζ' is a bc-expression.

Proof. The results follow directly by the definitions of a bc-expression and a reduced expression in W. □

A segment ξ of a reduced expression ζ is called a *bc-segment*, if ξ itself is a bc-expression. A bc-segment ξ of ζ is called *maximal*, if ξ is not a proper segment of any other bc-segment of ζ.

Theorem 1.10. *Any $z \in W$ can be expressed as a product*

$$z = x_1 z_1 x_2 z_2 \cdots x_a z_a x_{a+1} \tag{1.10.1}$$

of some $x_i, z_j \in W$ such that

(1) $\ell(z) = \sum_{k=1}^{a} \ell(z_k) + \sum_{l=1}^{a+1} \ell(x_l)$;

(2) z_k can be expressed as a maximal bc-segment of some reduced expression of z for any $k \in [a]$;

(3) x_l has a unique reduced expression in W for any $l \in [a+1]$.

The decomposition (1.10.1) is uniquely determined by the element z. We have $|\mathrm{Red}(z)| = \prod_{k=1}^{a} |\mathrm{Red}(z_k)|$.

Proof. Let $\zeta \equiv s_1 s_2 \cdots s_r$ be a reduced expression of z with $s_k \in S$ for $k \in [r]$. By Lemma 1.7, we see that any bc-segment of ζ is a segment of a unique maximal bc-segment of ζ and that any two different maximal bc-segments of ζ are disjoint. Let $z_1, z_2, ..., z_a$ be all maximal bc-segments of ζ arranging from left to right and let x_j, $j \in [a+1]$, be the segment of ζ consisting of all factors in S between z_{j-1} and z_j with the convention that z_0, z_{a+1} are the empty segments located at the left end and the right end of ζ, respectively. As an element of W, each of z_i, x_j remains unchanged under any braid-move on ζ. This implies that the decomposition (1.10.1) of z uniquely exists. Finally, the equation $|\mathrm{Red}(z)| = \prod_{k=1}^{a} |\mathrm{Red}(z_k)|$ follows by Lemma 1.2. □

By Theorem 1.10, to compute $|\mathrm{Red}(z)|$ for any $z \in W$, it is enough to consider the case where z has a bc-expression. Hence in the subsequent discussion of the paper, we always assume that z has a bc-expression unless otherwise specified.

§2. Some properties of a bc-expression.

In the section, we study the properties of a bc-expression in W. Let us first describe the structure of a bc-expression in W.

Lemma 2.1. *Let $\zeta \equiv s_1 s_2 \cdots s_r$ be a reduced expression in W with $r > 1$ and $s_k \in S$ for $k \in [r]$. Then ζ is a bc-expression if and only if there exists a braid sequence $\tau : w_1, ..., w_a$ and a pair sequence $\lambda : \{t_1, t'_1\}, ..., \{t_a, t'_a\}$ in S such that the following conditions (a)-(g) hold:*

(a) $t_d \neq t'_d$, $m_{t_d t'_d} < \infty$ for $d \in [a]$; $|\{t_c, t'_c\} \cap \{t_{c+1}, t'_{c+1}\}| = 1$ for $c \in [a-1]$ if $a > 1$.

(b) For $c \in [a]$, $w_c \equiv s_{p_c+1} s_{p_c+2} \cdots s_{p_c+k_c} \equiv [t_c t'_c t_c \cdots]_{k_c}$ with $k_c \in \{m_{t_c t'_c} - d \mid d \in \{0, 1, 2\}\}$ and $\{s_{p_c}, s_{p_c+k_c+1}\} \cap \{t_c, t'_c\} = \emptyset$.

(c) $(p_1, p_a + k_a) = (0, r)$ and $p_d + k_d \in \{p_{d+1}, p_{d+1} + 1\}$ for $d \in [a-1]$.

Denote by $h_c(\zeta)$ *or* h_c *the number* $k_c - m_{t_c t'_c}$.

(d) $h_1 = 0$ *if either* $a = 1$, *or* w_1, w_2 *intersect;* $h_a = 0$ *if either* $a = 1$, *or* w_{a-1}, w_a *intersect; for* $c \in [2, a-1]$, $h_c = 0$ *if* w_c *intersects with both* w_{c-1} *and* w_{c+1}.

(e) Suppose $h_c = -2$ *for* $c \in [a]$. *Then* $c \in [2, a-1]$*; there exist some* $e < c < f$ *in* $[a]$ *such that* $h_e = h_f = 0$ *and* $h_d = -1$ *for any* $d \in [e+1, f-1] - \{c\}$*;* w_b, w_{b+1} *are disjoint for any* $b \in [e, f-1]$.

(f) Suppose $h_c = -1$ *for* $c \in [a]$. *Then* $a > 1$ *and one of the following cases (f1)-(f2) occurs: (f1) there exists some* $e < c$ *in* $[a]$ *such that* $h_e = 0$ *and* $h_d = -1$ *for any* $d \in [e+1, c-1]$ *and that* w_b, w_{b+1} *are disjoint for any* $b \in [e, c-1]$*; (f2) there exists some* $f > c$ *in* $[a]$ *such that* $h_f = 0$ *and* $h_d = -1$ *for any* $d \in [c+1, f-1]$ *and that* w_b, w_{b+1} *are disjoint for any* $b \in [c, f-1]$.

(g) $\ell(w_c) + \ell(w_{c+1}) > 2$, $h_c + h_{c+1} \geqslant -3$, $w_c \notin W_{t_{c+1} t'_{c+1}}$ *and* $w_{c+1} \notin W_{t_c t'_c}$ *for any* $c \in [a-1]$.

Proof. By the definition of a reduced expression in W being a bc-expression (see 1.8), our result can be proved by induction on $\ell_b(\zeta) := a \geqslant 1$ and by Lemma 1.9. □

In Lemma 2.1, we notice that the conditions (b) and (e), together with the assumption of $\Gamma(W)$ being strictly complete, imply (g) and that the cases (f1), (f2) can't occur simultaneously if $h_c = -1$ since ζ is a reduced expression.

The following result concerns the relation between a braid sequence τ of a bc-expression ζ and the associated pair sequence in S for ζ, τ occurring in Lemma 2.1.

Lemma 2.2. *Let* $\zeta \equiv s_1 s_2 \cdots s_r$ *with* $s_k \in S$ *for* $k \in [r]$ *be a bc-expression in* W *with a braid sequence* $\tau : w_1, ..., w_a$ *and a pair sequence* $\lambda : \{t_1, t'_1\}, ..., \{t_a, t'_a\}$ *in* S *satisfying the conditions (a)-(g) in Lemma 2.1.*

(1) For the bc-expression ζ, λ *and* τ *determine each other.*

(2) λ *and* τ *are uniquely determined by the bc-expression* ζ.

Proof. For a bc-expression ζ, λ clearly determines τ. Now we show that τ determines λ. For, it is obvious if $a = 1$. Now assume $a > 1$ and $c \in [a]$. The cth term $\{t_c, t'_c\}$ of λ is uniquely determined by the cth term $w_c \equiv [t_c t'_c t_c \cdots]_{k_c}$ of τ unless $(m_{t_c t'_c}, k_c) = (3, 1)$. Now assume $(m_{t_c t'_c}, k_c) = (3, 1)$. Then w_c determines t_c. For t'_c, we have $c \in [2, a-1]$ and

$$\{t_{c-1}, t'_{c-1}\} \cap \{t_c, t'_c\} = \{t_{c+1}, t'_{c+1}\} \cap \{t_c, t'_c\} = \{t'_c\} \tag{2.2.1}$$

and $\ell(w_{c-1}), \ell(w_{c+1}) \geqslant 2$ by Lemma 2.1 (a), (g). Since $\{t_{c-1}, t'_{c-1}\}$ and $\{t_{c+1}, t'_{c+1}\}$ are uniquely determined by w_{c-1} and w_{c+1} respectively, we see by (2.2.1) that t'_c is determined uniquely by the sets $\{t_{c-1}, t'_{c-1}\}$ and $\{t_{c+1}, t'_{c+1}\}$ unless that $\{t_{c-1}, t'_{c-1}\} = \{t_{c+1}, t'_{c+1}\}$ and $m_{t_c t_{c-1}} = m_{t_c t'_{c-1}} = 3$, but the latter is impossible by the assumption of $\Gamma(W)$ being strictly complete. (1) is proved.

Let $\tau_1 : w_1, ..., w_a$ and $\tau_2 : x_1, ..., x_b$ be two braid sequences of ζ both satisfying (a)-(g) in Lemma 2.1 for some $a \leqslant b$ in $\mathbb{P}$. To prove (2), we need only to prove that w_c and x_c are the same segment of ζ for any $c \in [a]$ (this implies $a = b$) by (1). The result is obviously true if $a = 1$. Now assume $a > 1$. We have $w_1 \equiv x_1 \equiv [s_1 s_2 s_1 \cdots]_{k_1}$ for some $k_1 \geqslant 2$ by Lemma 1.9 (1). In general, if w_c and x_c are known as the same segment of ζ for some $c \in [a-1]$, then w_{c+1} and x_{c+1} are the same segment of ζ by Lemma 1.7 and the fact that both w_{c+1} and x_{c+1} are braid-connected with w_c and on the same side of w_c in ζ. This proves w_c and x_c are the same segment of ζ for any $c \in [a]$ by induction on $c \geqslant 1$. So (2) is proved. □

2.3. By Lemma 2.2, for a bc-expression ζ, we can call $\tau : w_1, ..., w_a$ and $\lambda : \{t_1, t'_1\}, ..., \{t_a, t'_a\}$ in Lemma 2.1 *the braid sequence* of ζ and *the associated pair sequence in S* for ζ, respectively. For any $c \in [a]$, call w_c *the cth braid factor* of ζ and call $\{t_c, t'_c\}$ *the cth associated pair in S* for ζ. Also, denote $\ell_{b,\tau}(\zeta)$ simply by $\ell_b(\zeta)$ and call it *the b-length* of ζ.

The next result concerns the effect of the braid-moves on a bc-expression in W.

Lemma 2.4. *Let ζ, ζ' be two expressions in W with $\zeta' \sim \zeta$. Assume that ζ is a bc-expression with $\ell_b(\zeta) = a$.*

(1) ζ' is a bc-expression with $\ell_b(\zeta') = a$.

(2) ζ' has the same associated pair sequence as ζ in S.

Proof. In our proof, we may assume that ζ' is obtained from ζ by applying a braid-move at the cth braid factor for some $c \in [a]$. Hence the cth braid factor w_c of ζ is full.

Let $\zeta_{i_1 j_1}, ..., \zeta_{i_a j_a}$ be the braid sequence of ζ. Take the segments $\zeta'_{i'_1 j'_1}, ..., \zeta'_{i'_a j'_a}$ of ζ' as follows: $(i'_d, j'_d) = (i_d, j_d)$ if $d \in [a] - \{c-1, c+1\}$; when $c \in [2, a]$, let (i'_{c-1}, j'_{c-1}) be $(i_{c-1}, j_{c-1} - 1)$ if $j_{c-1} = i_c$ and $(i_{c-1}, j_{c-1} + 1)$ if $j_{c-1} = i_c - 1$; when $c \in [a-1]$, let (i'_{c+1}, j'_{c+1}) be $(i_{c+1} + 1, j_{c+1})$ if $i_{c+1} = j_c$ and $(i_{c+1} - 1, j_{c+1})$ if $i_{c+1} = j_c + 1$. Then it is routine to check that ζ' is a bc-expression with $\zeta'_{i'_1 j'_1}, ..., \zeta'_{i'_a j'_a}$ its braid sequence by Lemma 2.1 (a)-(g) on ζ, (1) is proved.

Now we compare the dth associated pairs $\{t_d, t'_d\}$, $\{u_d, u'_d\}$ in S for ζ, ζ' respectively for any $d \in [a]$. They are the same if $d \in [a] - \{c-1, c+1\}$. Now assume $d \in \{c-1, c+1\}$ (hence $a > 1$). By symmetry, we need only to consider the case of $d = c-1$ (hence $c \in [2, a]$). By the construction of $\zeta'_{i'_{c-1}j'_{c-1}}, \zeta'_{i'_c j'_c}$, we have $\ell(\zeta'_{i'_{c-1}j'_{c-1}}) = \ell(\zeta_{i_{c-1}j_{c-1}}) \pm 1$, $\zeta_{i_c j_c} \equiv [t_c t'_c t_c \cdots]_{m_{t_c t'_c}}$, $\zeta'_{i'_c j'_c} \equiv [t'_c t_c t'_c \cdots]_{m_{t_c t'_c}}$ and $\emptyset \neq \{t_{c-1}, t'_{c-1}\} \cap \{u_{c-1}, u'_{c-1}\} \nsubseteq \{t_c, t'_c\}$ (say $t_{c-1} = u_{c-1} \notin \{t_c, t'_c\}$). If $\{t_{c-1}, t'_{c-1}\} \neq \{u_{c-1}, u'_{c-1}\}$, then $\{t_c, t'_c\} = \{t'_{c-1}, u'_{c-1}\}$, the braid factor pairs $\zeta_{i_{c-1}j_{c-1}}, \zeta_{i_c j_c}$ and $\zeta'_{i'_{c-1}j'_{c-1}}, \zeta'_{i'_c j'_c}$ either both intersect or both are disjoint. This would imply $\ell(\zeta_{i_{c-1}j_{c-1}}) = \ell(\zeta'_{i'_{c-1}j'_{c-1}}) \in \{1, 2\}$, a contradiction. So $\{t_{c-1}, t'_{c-1}\} = \{u_{c-1}, u'_{c-1}\}$, as required. □

2.5. Assume that $z \in W$ has a bc-expression ζ with $\ell_b(\zeta) = a$. Let $\{t_c, t'_c\}$ be the cth associated pair in S for ζ for any $c \in [a]$. Then any $\zeta' \in \text{Red}(z)$ is a bc-expression with $\ell_b(\zeta') = a$ and with $\{t_c, t'_c\}$ the cth associated pair in S for ζ' by Lemmas 1.2 and 2.4. So we can denote $\ell_b(\zeta)$ by $\ell_b(z)$ and call $\{t_c, t'_c\}$ *the cth associated pair in S for z* for any $c \in [a]$. Furthermore, call $\{t_1, t'_1\}, ..., \{t_a, t'_a\}$ *the associated pair sequence in S for z*. By Lemma 2.4, it will cause no confusion if we call a (maximal) bc-segment of ζ *a (maximal) bc-segment of z* for any $z \in W$.

Remark 2.6. Note that the assumption of $\Gamma(W)$ being strictly complete is necessary for the assertions in Lemmas 2.2 and 2.4. When $\Gamma(W)$ is complete but not strictly complete, there is a counter-example to those assertions: one bc-expression in W could possibly have more than one cth braid factor and more than one cth associated pair in S; equivalent bc-expressions could possibly have different cth associated pairs in S. Assume that $S = \{s, r, t\}$ satisfies $m_{st} = 4$ and $m_{tr} = m_{sr} = 3$. Then $\zeta_1 \equiv tstsrstst$, $\zeta_2 \equiv ststrstst$ and $\zeta_3 \equiv stsrtrsts$ are three equivalent bc-expressions. ζ_1 has two 2nd braid factors srs and r with two 2nd associated pairs $\{s, r\}$ and $\{r, t\}$ in S. Also, ζ_2 has two 2nd braid factors tr and rs with two 2nd associated pairs $\{t, r\}$ and $\{r, s\}$ in S. Finally, ζ_3 has only one 2nd braid factor rtr with one 2nd associated pair $\{t, r\}$ in S.

2.7. Keep the notation in Lemma 2.1 for a bc-expression ζ with $\ell_b(\zeta) = a$. Define $\alpha(\zeta; 1) = 0$ and define $\alpha(\zeta; c)$, $c \in [2, a]$, to be the number of all $d \in [c-1]$ such that the dth and the $(d+1)$th braid factors of ζ intersect. By Lemma 2.1 (a)-(g) on ζ, we have

$$\alpha(\zeta; b) - \alpha(\zeta; d) + \sum_{c=b}^{d} k_c \in \left\{ k + \sum_{c=b}^{d} (m_{t_c t'_c} - 1) \,\middle|\, k \in \{0, 1\} \right\} \tag{2.7.1}$$

for any $b \leqslant d$ in $[a]$ with $\{b,d\}\cap\{1,a\}\neq\emptyset$; in particular, $-\alpha(\zeta;a)+\sum_{c=1}^{a}k_c=1+\sum_{c=1}^{a}(m_{t_ct'_c}-1)$.

§3. Symbols associated to a bc-expression in W.

In this section, we associate each bc-expression in W to a symbol and give a description for all the admissible symbols (see Theorem 3.9).

3.1. For any $a\in\mathbb{N}$, define a *symbol* α of *length* $l(\alpha)=a$ to be $\alpha:=i_1j_2i_2j_2\cdots i_aj_a$ with some $i_c\in\{\ [,\ \langle\ \}$ and $j_c\in\{\ ],\ \rangle\ \}$ for any $c\in[a]$. Now assume $a>0$. Call i_kj_k the *kth pair* (or a *pair* in short) of α for any $k\in[a]$. Call $[\ \ ]$ a *full pair*. Next assume $a>1$. When the kth pair i_kj_k of α is full, denote by $\tau_k(\alpha)$ the symbol obtained from α by replacing j_{k-1},i_{k+1} by j'_{k-1},i'_{k+1} respectively if $k\in[2,a-1]$, i_2 by i'_2 if $k=1$, and j_{a-1} by j'_{a-1} if $k=a$, where j'_{k-1},i'_{k+1} are given by the conditions $\{j_{k-1},j'_{k-1}\}=\{\ ],\ \rangle\ \}$ for $k\in[2,a]$ and $\{i_{k+1},i'_{k+1}\}=\{\ [,\ \langle\ \}$ for $k\in[a-1]$. τ_k is called a *pair-reflection* on α at the kth pair. Let $\mathcal{S}_a$ be the set of all symbols of length a and let $\mathcal{S}=\cup_{a\in\mathbb{N}}\mathcal{S}_a$. Then $\mathcal{S}$ forms a monoid with the empty symbol being its identity under the composition by juxtaposition: $\alpha\cdot\beta=\alpha\beta$.

3.2. Keep the notation in Lemma 2.1 for a bc-expression $\zeta\in\mathrm{Red}(z)$ with $\ell_b(z)=a$, we associate ζ to a symbol $S(\zeta)=i_1j_1i_2j_2\cdots i_aj_a\in\mathcal{S}_a$ as follows.

(i) $i_1j_a=[\ \ ]$;

(ii) For any $c\in[2,a]$, we set $i_cj_{c-1}=[\ \ ]$ if $p_{c-1}+k_{c-1}=p_c+1$;

(iii) For any $c\in[2,a]$ with $p_{c-1}+k_{c-1}=p_c$, we set $i_cj_{c-1}=\langle\ \ ]$ if $-\alpha(\zeta;c-1)+\sum_{e=1}^{c-1}k_e=1+\sum_{e=1}^{c-1}(m_{t_et'_e}-1)$ and set $i_cj_{c-1}=[\ \ \rangle$ if $-\alpha(\zeta;c-1)+\sum_{e=1}^{c-1}k_e=\sum_{e=1}^{c-1}(m_{t_et'_e}-1)$ (see 2.7).

3.3. A symbol $\alpha\in\mathcal{S}$ is called *admissible*, if $\alpha=S(\zeta)$ for some bc-expression ζ in W. Denote by $\mathcal{S}_{\mathrm{ad}}$ the set of all admissible symbols in $\mathcal{S}$. By the condition (2.7.1) on a bc-expression, we have $i_cj_{c-1}\neq\langle\ \ \rangle$ for any $i_1j_1i_2j_2\cdots i_aj_a\in\mathcal{S}_{\mathrm{ad}}$ and any $c\in[2,a]$.

$\alpha,\beta\in\mathcal{S}$ are said to be *equivalent*, written $\alpha\sim\beta$, if either $\beta=\alpha$, or β can be obtained from α by successively applying some pair-reflections. An equivalence class in $\mathcal{S}$ containing α is denoted by $\overline{\alpha}$. Clearly, the relation $\alpha\sim\beta$ in $\mathcal{S}$ implies $l(\alpha)=l(\beta)$. If $\alpha,\beta\in\mathcal{S}_{\mathrm{ad}}$, then the relation $\alpha\sim\beta$ holds exactly when there are two bc-expressions ζ,ζ' in W with $\zeta\sim\zeta'$ such that $S(\zeta)=\alpha$ and $S(\zeta')=\beta$. The set $\mathcal{S}_{\mathrm{ad}}$ is a union of some equivalence classes in $\mathcal{S}$.

3.4. A segment ξ of a bc-expression ζ is called *regular* if any braid factor of ζ intersecting with ξ is wholly contained in ξ. Keep the notation in Lemma 2.1 for ζ with $\ell_b(\zeta) = a$. A regular segment of a bc-expression ζ is just a segment of the form $\xi \equiv s_{p_e+1}s_{p_e+2}\cdots s_{p_d+k_d}$ with some $e \leqslant d$ in $[a]$, we define the *associated symbol* of ξ to be $S(\xi) = i_ej_ei_{e+1}j_{e+1}\cdots i_dj_d$ if $S(\zeta) = i_1j_1i_2j_2\cdots i_aj_a$. We also define $\ell_b(\xi)$ to be $d+1-e$. In general, the symbol $S(\xi)$ satisfies the condition 3.2 (ii) but not necessarily 3.2 (i), (iii).

For $X \in \{[\ \ ], [\ \ \rangle, \langle\ \ ]\}$ and $m \in \mathbb{N}$, denote by X_m the symbol $X\cdots X$ (m copies).

Example 3.5. Let (W, S) be a Coxeter system with $S = \{s, r, t, u, v\}$ and defining relations $(sr)^4 = (sv)^7 = (rt)^5 = (rv)^7 = (tu)^6 = (tv)^4 = (uv)^6 = s^2 = r^2 = t^2 = u^2 = v^2 = 1$. Then W has the Coxeter graph $\Gamma(W)$ in Fig. 1.

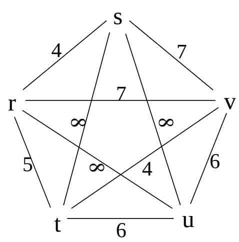

Fig. 1, Coxeter graph $\Gamma(W)$

(1) $\zeta \equiv rsrvsvsvsvtvtuvuvurvrvrv$ is a bc-expression with $\ell_b(\zeta) = 5$, where $w_1 \equiv rsr$, $w_2 \equiv vsvsvsv$, $w_3 \equiv vtvt$, $w_4 \equiv uvuvu$, $w_5 \equiv rvrvrv$ are braid factors of ζ with w_2 and w_3 full. The associated symbol $S(\zeta)$ is $[\ \ \rangle[\ \ ][\ \ ]\langle\ \ ]\langle\ \ ]$. The segment $\xi \equiv uvuvurvrvrv$ of ζ is regular with $S(\xi) = \langle\ \ ]\langle\ \ ]$ the associated symbol.

(2) $\zeta \equiv tvtvrvrvrvuvuvututut rtrt$ is a bc-expression with $\ell_b(\zeta) = 5$, where $w_1 \equiv tvtv$, $w_2 \equiv vrvrvrv$, $w_3 \equiv vuvuvu$, $w_4 \equiv ututut$, $w_5 \equiv trtrt$ are braid factors of ζ, which are all full, any two neighboring braid factors of ζ intersect. Hence $S(\zeta) = [\ \ ][\ \ ][\ \ ][\ \ ][\ \ ]$.

Lemma 3.6. *Let ζ be a bc-expression in W with $S(\zeta) = i_1j_1i_2j_2\cdots i_aj_a$. For any $c \in [a]$, we have*

(1) $h_c(\zeta) = 0$ if and only if $i_cj_c = [\ \ ]$.

(2) $h_c(\zeta) = -2$ if and only if $i_cj_c = \langle\ \ \rangle$.

(3) $h_c(\zeta) = -1$ if and only if $i_cj_c \in \{[\ \ \rangle, \langle\ \ ]\}$.

Proof. This follows directly by the definition of the symbol $S(\zeta)$ of a bc-expression ζ, the relation (2.7.1) and Lemma 2.1 (d)-(f). $\Box$

Examples 3.7. (1) Let ζ be a bc-expression in W with $S(\zeta) = i_1j_1i_2j_2\cdots i_aj_a$ and $h_c(\zeta) = -2$ for some $c \in [2, a-1]$. By Lemmas 2.1 (e) and 3.6, we have

$$i_ej_ei_{e+1}j_{e+1}\cdots i_fj_f = [\ \ ]\langle\ \ ]_{c-e-1}\langle\ \ \rangle[\ \ \rangle_{f-c-1}[\ \ ] \quad \text{for some } e < c < f \text{ in } [a].$$

(2) Let ζ be as in Example 3.5 (2) and let ζ_c, $c \in [5]$, be obtained from ζ by applying a braid-move at the cth braid factor. Then $S(\zeta_1)$, ..., $S(\zeta_5)$ are $[\ \ ]\langle\ \ ][\ \ ][\ \ ][\ \ ]$, $[\ \ \rangle[\ \ ]\langle\ \ ][\ \ ][\ \ ]$, $[\ \ ][\ \ \rangle[\ \ ]\langle\ \ ][\ \ ]$, $[\ \ ][\ \ ][\ \ \rangle[\ \ ]\langle\ \ ]$, $[\ \ ][\ \ ][\ \ ][\ \ \rangle[\ \ ]$, respectively, each of them is obtained from $S(\zeta)$ by some pair-reflection.

3.8. Let $\mathcal{S}_1 = \{[\ \ \rangle_n[\ \ ]_r, [\ \ ]_t\langle\ \ ]_m \mid m, n \in \mathbb{N}, r, t \in \mathbb{P}\}$ and $\mathcal{S}_2 = \{[\ \ \rangle_n[\ \ ]_r\langle\ \ ]_m \mid m, n \in \mathbb{N}, r \in \mathbb{P}\}$. For $k \in [2]$, define $\overline{\mathcal{S}}_k$ to be the subset of $\mathcal{S}$ consisting of all symbols $\alpha_1\alpha_2\cdots\alpha_r$ with $\alpha_i \in \mathcal{S}_k$ for some $r \in \mathbb{N}$ and any $i \in [r]$, then $\overline{\mathcal{S}}_k$ forms a submonoid of $\mathcal{S}$ generated by $\mathcal{S}_k$. Denote $\bar{k} := 3 - k$ for any $k \in [2]$.

The following result describes the subset $\mathcal{S}_{\rm ad}$ of $\mathcal{S}$.

Theorem 3.9. *(1) For any $\alpha \in \overline{\mathcal{S}}_k$, $k \in [2]$, there exists some $\alpha' \in \overline{\mathcal{S}}_{\bar{k}}$ satisfying $\alpha' \sim \alpha$.*

(2) For $\delta \in \overline{\mathcal{S}}_1$ and $\eta \in \{[\ \ ], [\ \ \rangle\}$, there exists some $\kappa \in \overline{\mathcal{S}}_1$ satisfying $\kappa \sim \eta\delta$. If $\delta \sim \delta'$ in $\mathcal{S}$ and $\eta \in \{[\ \ ], [\ \ \rangle\}$, then there exists some $\eta' \in \{[\ \ ], [\ \ \rangle\}$ satisfying $\eta\delta \sim \eta'\delta'$.

(3) For any $\alpha \in \mathcal{S}_{\rm ad}$, there exists some $\delta \in \overline{\mathcal{S}}_1$ with $\alpha \sim \delta$.

(4) Suppose that there is a sequence of pairwise distinct elements $t_1, t_2, ..., t_r$ in S with some $r > 2$ such that $m_{t_ct_{c+1}} < \infty$ for any $c \in [r]$ with the convention that $t_{r+1} = t_1$. Then for any $\delta \in \overline{\mathcal{S}}_1$, there exists some $\alpha \in \mathcal{S}_{\rm ad}$ with $\alpha \sim \delta$.

Proof. We have the inclusion $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and hence $\overline{\mathcal{S}}_1 \subseteq \overline{\mathcal{S}}_2$. On the other hand, $[\ \ \rangle_n[\ \ ]_r\langle\ \ ]_m = [\ \ \rangle_n[\ \ ]\cdot[\ \ ]_{r-1}\langle\ \ ]_m \in \overline{\mathcal{S}}_1$ if $r > 1$ and $[\ \ \rangle_n[\ \ ]\langle\ \ ]_m \sim [\ \ \rangle_{n-1}[\ \ ]_3\langle\ \ ]_{m-1} = [\ \ \rangle_{n-1}[\ \ ]\cdot[\ \ ]_2\langle\ \ ]_{m-1} \in \overline{\mathcal{S}}_1$ if $m, n > 0$ and $[\ \ \rangle_n[\ \ ]\langle\ \ ]_m \in \overline{\mathcal{S}}_1$ if $mn = 0$. This implies (1).

For the first assertion of (2), we need only to deal with the case of $\delta \in \mathcal{S}_1$, or only the case where $\delta = [\ \ ]_m\langle\ \ ]_n$ and $\tau = [\ \ \rangle$ for some $n \in \mathbb{N}$ and $m \in \mathbb{P}$. But the latter can be proved by the same argument as that in (1). For the last assertion of (2), let k be the number of pair-reflections τ_1 (see 3.1) applied in transforming δ to δ'. Take $\eta' = \eta$ if k is even and $\eta' \in \{[\ \ ], [\ \ \rangle\} - \{\eta\}$ if k is odd. Then $\eta\delta \sim \eta'\delta'$.

Next consider (3). We have $\alpha = S(\zeta)$ for some bc-expression ζ in W. Applying induction on $\ell_b(\zeta) \geqslant 1$. If $\ell_b(\zeta) = 1$, then ζ is a full braid expression, hence $S(\zeta) = [\ \ ] \in \overline{\mathcal{S}}_1$ by Lemma 3.6 (1). Now assume $\ell_b(\zeta) > 1$. Write $S(\zeta) = \beta\gamma\alpha'$ with some $\beta, \gamma, \alpha' \in \mathcal{S}$ satisfying $\ell_b(\zeta) = l(\alpha') + 2$ and $l(\beta) = l(\gamma) = 1$. Then $\beta\gamma \in \{[\ ][\ ], [\ ][\ \rangle, [\ ]\langle\ ], [\ ]\langle\ \rangle, [\ \rangle[\ \rangle, [\ \rangle[\ ]\}$. If $\beta\gamma \in \{[\ ]\langle\ ], [\ ]\langle\ \rangle\}$, then a braid-move can be applied on ζ at the 1st braid factor with the resulting bc-expression ζ' satisfying $S(\zeta') \in \{[\ ][\ ]\alpha', [\ ][\ \rangle\alpha'\}$ by 3.1. So we may assume $\beta\gamma \in \{[\ ][\ ], [\ ][\ \rangle, [\ \rangle[\ \rangle, [\ \rangle[\ ]\}$ at the beginning. When β is $[\ ]$ (respectively, $[\ \rangle$), the 1st braid factor of ζ is $[srs\cdots]_{m_{sr}}$ (respectively, $[srs\cdots]_{m_{sr}-1}$) for some $s \neq r$ in S with $m_{sr} < \infty$, and $\beta\gamma$ is in $\{[\ ][\ ], [\ ][\ \rangle\}$ (respectively, $\{[\ \rangle[\ ], [\ \rangle[\ \rangle\}$) by our assumption, so the 1st and the 2nd braid factors of ζ intersect (respectively, disjoint). Let ζ'' be obtained from ζ by removing the leftmost segment $[srs\cdots]_{m_{sr}-1}$. Then ζ'' is a bc-expression with $S(\zeta'') = \gamma\alpha'$ by Lemma 1.9 (4). By inductive hypothesis, there exists some $\delta \in \overline{\mathcal{S}}_1$ with $\delta \sim \gamma\alpha'$. By (2), there exist some $\delta' \in \overline{\mathcal{S}}_1$ and $\eta \in \{[\ \ ], [\ \ \rangle\}$ satisfying $\delta' \sim \eta\delta \sim [\ \ ]\gamma\alpha' = S(\zeta)$ (respectively, $\delta' \sim \eta\delta \sim [\ \ \rangle\gamma\alpha' = S(\zeta)$). (2) is proved.

Finally, consider (4). Let $J = \{t_c \mid c \in [r]\}$. We shall prove a stronger result: For any $\alpha \in \overline{\mathcal{S}}_1$, there exists some bc-expression ζ in W_J with $S(\zeta) \sim \alpha$. Applying induction on $l(\alpha) \geqslant 1$. If $\alpha \in \overline{\mathcal{S}}_1$ satisfies $l(\alpha) = 1$ then $\alpha = [\ \ ]$, any full braid expression ζ in W_J satisfies $\alpha = S(\zeta) \in \mathcal{S}_{\text{ad}}$. Now assume $\alpha \in \overline{\mathcal{S}}_1$ satisfies $l(\alpha) > 1$. Write $\alpha = \beta\gamma\alpha'$ for some $\alpha', \beta, \gamma \in \mathcal{S}$ with $l(\alpha) = l(\alpha') + 2$ and $l(\beta) = l(\gamma) = 1$. Then $\beta\gamma \in \{[\ ][\ ], [\ ][\ \rangle, [\ ]\langle\ ], [\ ]\langle\ \rangle, [\ \rangle[\ \rangle, [\ \rangle[\ ]\}$. Since $[\ ]\langle\ ]\alpha' \sim [\ ][\ ]\alpha'$ and $[\ ]\langle\ \rangle\alpha' \sim [\ ][\ \rangle\alpha'$, we may assume $\beta\gamma \in \{[\ ][\ ], [\ ][\ \rangle, [\ \rangle[\ \rangle, [\ \rangle[\ ]\}$ at the beginning. Hence $\gamma\alpha' \in \overline{\mathcal{S}}_1$. By inductive hypothesis, there exists some bc-expression ζ' in W_J with $S(\zeta') \sim \gamma\alpha'$. The 1st braid factor of ζ' is $[t_it_jt_i\cdots]_{k_1}$ for some $i \neq j$ in $[r]$ with $m_{t_it_j} < \infty$ and $k_1 \in \{m_{t_it_j}, m_{t_it_j} - 1\}$ by Lemma 1.9 (1). Let k be the number of pair-reflections τ_1 applied in transforming $S(\zeta')$ to $\gamma\alpha'$. When either $\beta = [\ \ ]$ with k even, or $\beta = [\ \ \rangle$ with k odd, take $\zeta \equiv [\cdots tt_it]_{m_{tt_i}-1}\zeta'$, where $t \in I - \{t_i, t_j\}$ satisfies $m_{tt_i} < \infty$, the existence of such t is guaranteed by the assumption on J. When either $\beta = [\ \ \rangle$ with k even, or $\beta = [\ \ ]$ with k odd, take $\zeta \equiv [\cdots tt_jt]_{m_{tt_j}-1}\zeta'$, where $t \in I - \{t_i, t_j\}$ satisfies $m_{tt_j} < \infty$, the existence of such t is again guaranteed by the assumption on J. Then ζ is a bc-expression in W_J with $S(\zeta) \sim \alpha$ by (2) and Lemma 1.9 (3). This proves (3). $\square$

§4. The correspondence between the sets Red(z) **and** Symb(z)**.**

To study the structure and the cardinal of the set Red(z) for any $z \in W$, we need only to consider the case where z has a bc-expression by Theorem 1.10. For such an element z, denote Symb(z) $:= \{S(\zeta) \mid \zeta \in \text{Red}(z)\}$. In this section, we establish a bijection between the sets Red(z) and Symb(z) in Theorem 4.1 when $\ell_b(z) > 1$. Two kinds of bc-expressions in W (i.e., simple and ample) are important in the subsequent discussion.

For any $z \in W$, let $\mathcal{L}(z) = \{s \in S \mid \ell(sz) < \ell(z)\}$. Recall a (maximal) bc-segment of z introduced in 2.5.

Theorem 4.1. *Assume that $z \in W$ has a bc-expression with $\ell_b(z) > 1$. For any $\zeta, \zeta' \in$ Red(z), we have $\zeta \equiv \zeta'$ if and only if $S(\zeta) = S(\zeta')$.*

Proof. We know that all expressions in Red(z) are bc-expressions and have the same b-length $\ell_b(z)$ by Lemma 2.4 and by the assumption that z has a bc-expression. The implication " $\Longrightarrow$ " is trivial. Now we prove the implication " $\Longleftarrow$ " by induction on $\ell_b(z) \geqslant 2$. Assume $\zeta, \zeta' \in \text{Red}(z)$ satisfy $S(\zeta) = S(\zeta')$ (denote by α this common symbol). Write $\alpha = \beta\gamma\alpha'$ with some $\beta, \gamma, \alpha' \in \mathcal{S}$ satisfying $l(\alpha) = l(\alpha') + 2$ and $l(\beta) = l(\gamma) = 1$. Then $\beta \in \{[\ \ ], [\ \ \rangle\}$.

(i) First assume $\beta = [\ ]$. Then the 1st braid factors of ζ, ζ' are $[srs\cdots]_{m_{sr}}$, $[s'r's'\cdots]_{m_{s'r'}}$ respectively for some $s \neq r$ and $s' \neq r'$ in S. The first claim is that $\{s, r\} = \{s', r'\}$. For otherwise, we would have $|\mathcal{L}(z)| \geqslant 3$ by the fact $\{s, r, s', r'\} \subseteq \mathcal{L}(z)$, contradicting Lemma 1.4. The second claim is that $(s, r) = (s', r')$. By Lemma 1.2, there is a sequence $\tau_{i_1}, \tau_{i_2}, ..., \tau_{i_b}$ of braid-moves to transform the expression ζ' to ζ, where τ_{i_j} denotes a braid-move at the i_jth braid factor. If $(s, r) = (r', s')$ then the cardinal of the set $\{j \in [b] \mid i_j = 1\}$ should be odd, but this would imply $i_2 \neq i'_2$ in $S(\zeta) = i_1 j_1 i_2 j_2 \cdots$ and $S(\zeta') = i'_1 j'_1 i'_2 j'_2 \cdots$ (see 3.1), contradicting the assumption $S(\zeta) = S(\zeta')$. Now we have $[srs\cdots]_{m_{sr}} \equiv [s'r's'\cdots]_{m_{s'r'}}$. Let ζ_1, ζ'_1 be obtained from ζ, ζ' respectively by removing the leftmost segment $[srs\cdots]_{m_{sr}-1}$ if $\gamma \in \{[\ \ ], [\ \ \rangle\}$, and by applying a braid-move at the 1st braid factor followed by removing the leftmost segment $[rsr\cdots]_{m_{sr}-1}$ if $\gamma \in \{\langle\ \ ], \langle\ \ \rangle\}$. Then ζ_1, ζ'_1 are two bc-expressions of some $z' \in W$ satisfying $S(\zeta_1) = S(\zeta'_1)$ and $\ell_b(z') = \ell_b(z) - 1$ by Lemma 1.9 (4). If $\ell_b(z') = 1$, then both ζ_1 and ζ'_1 are full braid expressions in W with the same leftmost factor in S. This implies $\zeta_1 \equiv \zeta'_1$. If $\ell_b(\zeta') \geqslant 2$, then $\zeta_1 \equiv \zeta'_1$ by inductive hypothesis. So we get $\zeta \equiv \zeta'$ in either case.

(ii) Next assume $\beta = [\ \ \rangle$. Then $\alpha = [\ \ \rangle_m[\ \ ]\alpha'$ for some $\alpha' \in \mathcal{S}$ and $m \in \mathbb{P}$ with $l(\alpha) = l(\alpha') + m + 1$ by the definition of a symbol associated to a bc-expression and Theorem 3.9. Let ζ_1, ζ_1' be the bc-expressions obtained from ζ, ζ', respectively by applying braid-moves $\tau_{m+1}, \tau_m, ..., \tau_2$ in turn. Then $S(\zeta_1) = S(\zeta_1')$. Denote by α_1 this common symbol. Then α_1 can be obtained from α by applying pair-reflections $\tau_{m+1}, \tau_m, ..., \tau_2$ in turn. We have $\alpha_1 = [\ \ ]\beta'$ for some $\beta' \in \mathcal{S}$ with $l(\alpha_1) = l(\beta') + 1$. The relation $\zeta_1 \equiv \zeta_1'$ can be proved by the argument in (i) with ζ_1, ζ_1' in the places of ζ, ζ' respectively. This implies $\zeta \equiv \zeta'$ since ζ, ζ' can be obtained from ζ_1, ζ_1' respectively by the same sequence of braid-moves.

So our result is proved. □

Note that the assumption $\ell_b(z) > 1$ can't be removed for the assertion of Theorem 4.1. For, if $\ell_b(z) = 1$ then z is the longest element in a standard parabolic subgroup W_{sr} of W for some $s \neq r$ in S with $m_{sr} < \infty$, the set $\mathrm{Red}(z)$ contains two different full braid expressions with the same associated symbol $[\ \ ]$.

Corollary 4.2. *Express any $z \in W$ in the form (1.10.1) with $z_1, z_2, ..., z_r$ all maximal bc-segments of some reduced expression of z. Then $|\mathrm{Red}(z)| = \prod_{k=1}^{r} \epsilon_k |\mathrm{Symb}(z_k)|$, where $\epsilon_k = 1$ if $\ell_b(z_k) > 1$ and $\epsilon_k = 2$ if $\ell_b(z_k) = 1$.*

Proof. The result follows by Theorems 1.10, 4.1, Lemma 1.2 and the fact that $|\mathrm{Red}(w)| = 2$ and $|\mathrm{Symb}(w)| = 1$ if $w \in W$ has a bc-expression with $\ell_b(w) = 1$. □

By Corollary 4.2, to compute $|\mathrm{Red}(z)|$ for any $z \in W$, it is enough to consider the case where z has a bc-expression. Hence in the subsequent discussion of the paper, we always assume that z has a bc-expression with $\ell_b(z) > 1$ unless otherwise specified. We need only to compute $|\mathrm{Symb}(z)|$ in order to get $|\mathrm{Red}(z)|$ by Theorem 4.1.

4.3. A bc-expression ζ with $\ell_b(\zeta) = a$ is called *simple*, if the associated symbol $S(\zeta)$ is either $[\ \ ]$, or one of the following symbols with $a \geqslant 2$: $[\ \ ]\langle\ \ ]_{a-1}$, $[\ \ \rangle_{a-1}[\ \ ]$ and $[\ \ \rangle_d\ [\ \ ]_2\langle\ \ ]_{a-d-2}$ for some $d \in [0, a-2]$. Note that those symbols are pairwise different and form a single equivalence class of $\mathcal{S}$ in $\mathcal{S}_{\mathrm{ad}}$ for any given $a \in \mathbb{P}$.

Lemma 4.4. *If $z \in W$ has a simple bc-expression with $\ell_b(z) = a \in \mathbb{P}$ then all expressions in $\mathrm{Red}(z)$ are simple bc-expressions with $L_a := |\mathrm{Red}(z)|$ equal to $a + 1$.*

Proof. The result is obvious if $a = 1$. Now assume $a > 1$. By Theorem 4.1, our result follows directly by the definition of a simple bc-expression and the notice thereafter. $\square$

4.5. A bc-expression ζ with $\ell_b(\zeta) = a$ is called *ample*, if the symbol $S(\zeta) = i_1j_1i_2j_2\cdots i_aj_a$ satisfies the condition (4.5.1) below:

(4.5.1) $i_{c+1}j_{c-1} \in \{[\ \ ], \langle\ \ \rangle\}$ for any $c \in [2, a-1]$. If $i_{c+1}j_{c-1} = \langle\ \ \rangle$ then $i_cj_c = [\ \ ]$; if $i_2 = \langle$, then $i_1j_1 = [\ \ ]$; if $j_{a-1} = \rangle$, then $i_aj_a = [\ \ ]$.

Denote by F_m, $m \in \mathbb{N}$, the Fibonacci numbers defined by the relations

(4.5.2) $$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_{m+2} = F_m + F_{m+1}.$$

The following identities are well known:

(4.5.3) $$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$$

(4.5.4) $$F_{m+n+2} = F_{m+2}F_{n+2} - F_mF_n.$$

for any $m, n \in \mathbb{N}$.

Lemma 4.6. *Assume that $z \in W$ has a bc-expression with $\ell_b(z) = a$.*

(1) $\zeta \in \mathrm{Red}(z)$ is ample if and only if $S(\zeta) \sim [\ \ ]_a$.

(2) If $\mathrm{Red}(z)$ contains an ample bc-expression, then $K_a := |\mathrm{Red}(z)|$ is equal to F_{a+2}.

Proof. The assertion (1) follows by the definition of an ample bc-expression and the fact that a braid-move on a bc-expression, whenever it is applicable, preserves the property of being ample. For (2), apply induction on $a \geqslant 1$. It can be checked directly that $K_1 = F_3$ and $K_2 = F_4$. Now assume $a > 2$. Let E_1 (respectively, E_2) be the set of all $i_1j_1i_2j_2\cdots i_aj_a \in \mathrm{Symb}(z)$ with $j_{a-1} =]$ (respectively, $j_{a-1} = \rangle$). Then E_1 (respectively E_2) consists of all symbols in $\mathrm{Symb}(z)$ which can be obtained from $\underline{[\ \ ]_{a-1}}\cdot[\ \ ]$ (respectively, $\underline{[\ \ ]_{a-2}}\cdot[\ \ \rangle[\ \ ]$) by applying some pair-reflections at the pairs contained in the underlined place. So $|E_1| = K_{a-1}$ and $|E_2| = K_{a-2}$. By Theorem 4.1, the assertion (2) follows by inductive hypothesis, the fact $\mathrm{Symb}(z) = E_1\dot{\cup}E_2$ and the identity (4.5.2). $\square$

By Lemma 4.6 (2) and the fact $(F_0, F_1, F_2) = (0, 1, 1)$, it is reasonable to set $K_0 = K_{-1} = 1$ and $K_a = 0$ for any $a < -1$.

§5. An explicit formula for the cardinal of the set Red(z)**.**

In this section, we always assume $z \in W$ has a bc-expression ζ with $S(\zeta) \in \overline{\mathcal{S}}_1$. Let

$$\alpha_{l_{r+1},n_r,l_r,\dots,n_1,l_1} := [\ \ ]_{l_{r+1}}[\ \ \rangle_{n_r}[\ \ ]_{l_r}\cdots[\ \ \rangle_{n_1}[\ \ ]_{l_1}$$

for some $r, l_1, n_1, ..., l_r, n_r \in \mathbb{P}$ and $l_{r+1} \in \mathbb{N}$. To formulate $|\text{Red}(z)|$, we reduce ourselves to the case where $S(\zeta) = \alpha_{l_{r+1},n_r,l_r,\dots,n_1,l_1}$ by Theorem 3.9 and Proposition 5.3. An explicit formula is given for the number $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\dots,n_1,l_1} := |\text{Red}(z)|$ in Theorem 5.7.

Lemma 5.1. *Assume that $z \in W$ has a bc-expression ζ with $S(\zeta) = [\ \ ]_m\langle\ \ ]_n$ for some $m \in \mathbb{P}$ and $n \in \mathbb{N}$. Then $K^{|<}_{m,n} := |\text{Red}(z)|$ is equal to $F_{m+2} + nF_m$.*

Proof. We have

$$K^{|<}_{1,n} = L_{n+1} = n + 2 = F_3 + nF_1, \tag{5.1.1}$$

$$K^{|<}_{m,0} = K_m = F_{m+2}. \tag{5.1.2}$$

by Lemmas 4.4 and 4.6, the result is true in those cases. Now assume $m > 1$ and $n > 0$. Let E_1 (respectively, E_2) be the set of all $i_1j_1i_2j_2\cdots$ in Symb(z) with $j_m =]$ (respectively, $j_m = \rangle$). Then E_1 consists of all symbols which can be obtained from $\underline{[\ \ ]_m} \cdot \langle\ \ ]_n$ by applying some pair-reflections at the pairs contained in the underlined place. This implies $|E_1| = K_m = F_{m+2}$ by Lemma 4.6. On the other hand, we have

$$[\ \ ]_m\langle\ \ ]_n \sim \underline{[\ \ ]_{m-2}} \cdot [\ \ \rangle_2[\ \ ] \cdot \underline{[\ \ ]\langle\ \ ]_{n-2}} \qquad \text{if } n > 1, \tag{5.1.3}$$

$$[\ \ ]_m\langle\ \ ]_n \sim \underline{[\ \ ]_{m-2}} \cdot [\ \ \rangle_2[\ \ ] \qquad \text{if } n = 1. \tag{5.1.4}$$

Denote by α the symbol on the right-hand side of (5.1.3) or (5.1.4) according to $n > 1$ or $n = 1$. Then E_2 consists of all symbols which can be obtained from α by applying some pair-reflections at the pairs contained in the underlined place. This implies by Lemma 4.6 and (5.1.1) that $|E_2| = K^{|<}_{1,n-2}K_{m-2} = nF_m$ if $n > 1$ and that $|E_2| = K_{m-2} = F_m$ if $n = 1$. So our result follows by Theorem 4.1 and the fact Symb$(z) = E_1\dot{\cup}E_2$. □

Lemma 5.2. *Assume that $z \in W$ has a bc-expression ζ with $S(\zeta) = [\ \ ]_l[\ \ \rangle_m[\ \ ]_p$ for some $l \in \mathbb{N}$ and $m, p \in \mathbb{P}$. Then $K^{|>|}_{l,m,p} := |\text{Red}(z)|$ is equal to $F_{l+p+2} + mF_{l+2}F_p$.*

Proof. When $l = 0$, we have $K^{|>|}_{0,m,p} = K^{|<}_{p,m}$, the result follows by Lemma 5.1. Now assume $l > 0$. Let E_1 (respectively, E_2) be the set of all $i_1j_1i_2j_2\cdots$ in Symb(z) with

$i_{l+1} = [$ (respectively, $i_{l+1} = \langle$). Then E_1 consists of all symbols which can be obtained from $\underline{[\ \ ]_{l-1}} \cdot [\ \ ] \cdot \underline{[\ \ \rangle_m [\ \ ]_p}$ by applying some pair-reflections at the pairs contained in the underlined place. This implies that $|E_1| = K_{l-1}K^{|<}_{p,m}$. On the other hand, we have

$$(5.2.1)\qquad [\ \ ]_l[\ \ \rangle_m[\ \ ]_p \sim \underline{[\ \ ]_{l-2}} \cdot [\ \ \rangle[\ \ ]\langle\ \ \rangle \cdot \underline{[\ \ \rangle_{m-1}[\ \ ]_p} \qquad \text{if } l > 1,$$

$$(5.2.2)\qquad [\ \ ]_l[\ \ \rangle_m[\ \ ]_p \sim [\ \ ]\langle\ \ \rangle \cdot \underline{[\ \ \rangle_{m-1}[\ \ ]_p} \qquad \text{if } l = 1.$$

Denote by α the symbol on the right-hand side of (5.2.1) or (5.2.2) according to $l > 1$ or $l = 1$. Then E_2 consists of all symbols which can be obtained from α by applying some pair-reflections at the pairs contained in the underlined place. This implies that $|E_2| = K_{l-2}K^{|<}_{p,m-1}$ if $l > 1$ and $|E_2| = K^{|<}_{p,m-1}$ if $l = 1$. Our result follows by Theorem 4.1, Lemmas 4.6, 5.1, the identity (4.5.4) and the fact $\mathrm{Symb}(z) = E_1 \dot{\cup} E_2$. □

Proposition 5.3. *If $z \in W$ has a bc-expression, then there exists some $\zeta \in \mathrm{Red}(z)$ with $S(\zeta) = \alpha_{l_{r+1},n_r,l_r,\dots,n_1,l_1}$ for some $r, l_1, n_1, \dots, l_r, n_r \in \mathbb{P}$ and $l_{r+1} \in \mathbb{N}$.*

Proof. The result is trivial when $\ell_b(z) = 1$. Now assume $\ell_b(z) > 1$. By Theorem 3.9, there exists some $\zeta \in \mathrm{Red}(z)$ such that $S(\zeta) = \alpha := \alpha_r\alpha_{r-1}\cdots\alpha_1$ for some $r \in \mathbb{P}$ and some $\alpha_i \in \mathcal{S}_1$, $i \in [r]$. If

$(*)$ $\quad \alpha_i = [\ \ \rangle_{n_i}[\ \ ]_{l_i}$ with some $n_i \in \mathbb{N}$ and $l_i \in \mathbb{P}$ for any $i \in [r]$,

then we are done. Now assume we are not in the case. Then there exists some $j \in [r]$ with $\alpha_j = [\ \ ]_{l_j}\langle\ \ ]_{n_j}$ for some $l_j, n_j \in \mathbb{P}$. Take j the smallest possible with this property and denote it by n_α (take $n_\alpha = r+1$ in the case $(*)$). There are two possible cases as follows.

(i) There exists some $i \in [j+1, r]$ such that $\alpha_i = [\ \ \rangle_{n_i}[\ \ ]_{l_i}$ and $\alpha_k = [\ \ ]_{l_k}\langle\ \ ]_{n_k}$ for any $k \in [j, i-1]$,

(ii) $\alpha_k = [\ \ ]_{l_k}\langle\ \ ]_{n_k}$ for any $k \in [j, r]$,

where $l_i, l_k \in \mathbb{P}$, $n_i, n_k \in \mathbb{N}$. In the case (i), let $\alpha' \in \mathcal{S}$ be obtained from α by replacing the part $\alpha_i\alpha_{i-1}\cdots\alpha_j$ by

$$[\ \ \rangle_{n_i}[\ \ ]_{l_i+l_{i-1}-2}[\ \ \rangle_{n_{i-1}}[\ \ ]_{l_{i-2}}[\ \ \rangle_{n_{i-2}}[\ \ ]_{l_{i-3}}\cdots[\ \ ]_{l_j}[\ \ \rangle_{n_j}[\ \ ]_2.$$

In the case (ii), let $\alpha' \in \mathcal{S}$ be obtained from α by replacing the part $\alpha_r\alpha_{r-1}\cdots\alpha_j$ by

$$[\ \ ]_{l_r-2}[\ \ \rangle_{n_r}[\ \ ]_{l_{r-1}}[\ \ \rangle_{n_{r-1}}[\ \ ]_{l_{r-2}}\cdots[\ \ ]_{l_j}[\ \ \rangle_{n_j}[\ \ ]_2.$$

if $l_r \geqslant 2$ and by

$$[\ \rangle_{n_r-1}[\ ]_{l_{r-1}}[\ \rangle_{n_{r-1}}[\ ]_{l_{r-2}}\cdots[\ ]_{l_j}[\ \rangle_{n_j}[\ ]_2.$$

if $l_r = 1$. Then $\alpha' \sim \alpha$ in either case. Since $n_{\alpha'} > n_\alpha$, our result follows by applying reversing induction on $n_\alpha \leqslant r+1$, 3.3 and Theorem 4.1. □

5.4. Denote $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\ldots,n_1,l_1} := |\mathrm{Red}(z)|$ if $z \in W$ has a bc-expression with $\alpha_{l_{r+1},n_r,l_r,\ldots,n_1,l_1} \in \mathrm{Symb}(z)$.

By Theorem 3.9 and Proposition 5.3, we see that, in order to formulate $|\mathrm{Red}(z)|$ for any $z \in W$ having a bc-expression, it is enough to consider the case of $\alpha_{l_{r+1},n_r,l_r,\ldots,n_1,l_1} \in \mathrm{Symb}(z)$ for some $r, l_1, n_1, \ldots, l_r, n_r \in \mathbb{P}$ and $l_{r+1} \in \mathbb{N}$.

The following result provides a recurrence formula for the number $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\ldots,n_1,l_1}$.

Proposition 5.5. *For any $r \geqslant 2$, the number $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\ldots,n_1,l_1}$ is equal to*

$$\begin{cases} F_{l_{r+1}+2}K^{|(>|)^{r-1}}_{l_r,n_{r-1},l_{r-1},\ldots,n_1,l_1} + (n_rF_{l_{r+1}+2} - F_{l_{r+1}})K^{|(>|)^{r-1}}_{l_r-2,n_{r-1},l_{r-1},\ldots,n_1,l_1}, & \text{if } l_r \geqslant 2, \\ F_{l_{r+1}+2}K^{|(>|)^{r-1}}_{1,n_{r-1},l_{r-1},\ldots,n_1,l_1} + (n_rF_{l_{r+1}+2} - F_{l_{r+1}})K^{|(>|)^{r-1}}_{0,n_{r-1}-1,l_{r-1},\ldots,n_1,l_1}, & \text{if } l_r = 1. \end{cases}$$

Proof. Let E_1 (respectively, E_2) be the set of all $i_1j_1i_2j_2\cdots$ in $\mathrm{Symb}(z)$ with $i_{l_{r+1}+n_r+1} = [$ (respectively, $i_{l_{r+1}+n_r+1} = \langle\ $). Then E_1 consists of all symbols which can be obtained from $\underline{[\ ]_{l_{r+1}}}\cdot[\ \rangle_{n_r}\cdot\underline{[\ ]_{l_r}[\ \rangle_{n_{r-1}}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$ by applying some pair-reflections at the pairs contained in the underlined place. So $|E_1| = K_{l_{r+1}}K^{|(>|)^{r-1}}_{l_r,n_{r-1},l_{r-1},\ldots,n_1,l_1}$. On the other hand, $\alpha_{l_{r+1},n_r,l_r,\ldots,n_1,l_1}$ is equivalent to one of the following symbols:

$\underline{[\ ]_{l_{r+1}}[\ \rangle_{n_r-2}[\ ]}\cdot[\ ]\langle\ ]_2\cdot\underline{[\ ]_{l_r-2}[\ \rangle_{n_{r-1}}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $n_r, l_r \geqslant 2$.
$\underline{[\ ]_{l_{r+1}}[\ \rangle_{n_r-2}[\ ]}\cdot[\ ]\langle\ ]\langle\ \rangle\cdot\underline{[\ \rangle_{n_{r-1}-1}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $n_r > l_r = 1$.
$\underline{[\ ]_{l_{r+1}-1}}\cdot[\ \rangle[\ ]\langle\ ]_2\cdot\underline{[\ ]_{l_r-2}[\ \rangle_{n_{r-1}}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $l_r > n_r = 1 \leqslant l_{r+1}$.
$[\ ]\langle\ ]_2\cdot\underline{[\ ]_{l_r-2}[\ \rangle_{n_{r-1}}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $l_r > n_r = 1,\ l_{r+1} = 0$.
$\underline{[\ ]_{l_{r+1}-1}}\cdot[\ \rangle[\ ]\langle\ ]\langle\ \rangle\cdot\underline{[\ \rangle_{n_{r-1}-1}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $n_r = l_r = 1 \leqslant l_{r+1}$.
$[\ ]\langle\ ]\langle\ \rangle\cdot\underline{[\ \rangle_{n_{r-1}-1}[\ ]_{l_{r-1}}\cdots[\ \rangle_{n_1}[\ ]_{l_1}}$	if $n_r = l_r = 1,\ l_{r+1} = 0$.

Denote by α one of the symbols above according to the values of n_1, l_1, l_0. Then E_2 consists of all symbols which can be obtained from α by applying some pair-reflections at the pairs contained in the underlined place. So $|E_2|$ is equal to

$$\begin{array}{ll} K^{|>|}_{l_{r+1},n_r-2,1}K^{|(>|)^{r-1}}_{l_r-2,n_{r-1},l_{r-1},...,n_1,l_1}, & \text{if } n_r,l_r\geqslant 2,\\ K^{|>|}_{l_{r+1},n_r-2,1}K^{|(>|)^{r-1}}_{0,n_{r-1}-1,l_{r-1},...,n_1,l_1}, & \text{if } n_r>l_r=1,\\ K_{l_{r+1}-1}K^{|(>|)^{r-1}}_{l_r-2,n_{r-1},l_{r-1},...,n_1,l_1}, & \text{if } l_r>n_r=1,\\ K_{l_{r+1}-1}K^{|(>|)^{r-1}}_{0,n_{r-1}-1,l_{r-1},...,n_1,l_1}, & \text{if } n_r=l_r=1.\end{array}$$

Hence our result follows by Theorems 4.1, Lemmas 4.6, 5.2, and the fact $\mathrm{Symb}(z) = E_1\dot{\cup}E_2$. □

5.6. Fix $r, l_1, n_1, ..., l_r, n_r \in \mathbb{P}$ and $l_{r+1} \in \mathbb{N}$. Let $\mathbf{l} = (l_{r+1}, l_r, ..., l_1)$. For any $k \in [r]$, let $I_{k,r} := \{\mathbf{t} := (t_1, t_2, ..., t_k) \in \mathbb{P}^k \mid 1 \leqslant t_1 < t_2 < \cdots < t_k \leqslant r\}$. For any $\mathbf{t} = (t_1, t_2, ..., t_k) \in I_{k,r}$, let $n_{\mathbf{t}} := \prod_{c=1}^k n_{t_c}$ and $F_{\mathbf{t},\mathbf{l}} := F_{(l_{r+1}+2)+l_r+l_{r-1}+\cdots+l_{t_k}+1} \prod_{c=1}^k F_{l_{t_c}+l_{t_c-1}+\cdots+l_{t_{c-1}+1}}$ with the convention that $t_0 = 0$. Then the following is an explicit formula for the number $K^{|(>|)^r}_{l_{r+1},n_r,l_r,...,n_1,l_1}$.

Theorem 5.7. *In the above setup, we have*

$$K^{|(>|)^r}_{l_{r+1},n_r,l_r,...,n_1,l_1} = F_{(l_{r+1}+2)+l_r+\cdots+l_1} + \sum_{k=1}^{r}\sum_{\mathbf{t}\in I_{k,r}} n_{\mathbf{t}}F_{\mathbf{t},\mathbf{l}}. \tag{5.7.1}$$

Proof. Apply induction on $r \geqslant 1$. When $r = 1$, the equation (5.7.1) is just Lemma 5.2. Now assume $r \geqslant 2$. Consider the recurrence formula for $K^{|(>|)^r}_{l_{r+1},n_r,l_r,...,n_1,l_1}$ in Proposition 5.5 and regard it as a polynomial in $n_1, n_2, ..., n_r$. By inductive hypothesis, we can compute the constant term and the coefficients $f_{\mathbf{t}}$ of the term $n_{\mathbf{t}}$ in $K^{|(>|)^r}_{l_{r+1},n_r,l_r,...,n_1,l_1}$ for any $\mathbf{t} = (t_1, t_2, ..., t_k) \in I_{k,r}$ with $k \in [r]$ as follows. We denote $\prod_{c=1}^h F_{l_{t_c}+l_{t_c-1}+\cdots+l_{t_{c-1}+1}}$ simply by $\prod_{c=1}^h$ for any $h \in [r]$ and use the identities (4.5.2)-(4.5.4) and $F_2 = F_1 = 1$ in the following computation. First assume $l_r \geqslant 2$.

$$\begin{aligned} f_{\mathbf{t}} &= (F_{l_{r+1}+2}F_{(l_r+2)+l_{r-1}+\cdots+l_{t_k}+1} - F_{l_{r+1}}F_{((l_r-2)+2)+l_{r-1}+\cdots+l_{t_k}+1}) \cdot \prod_{c=1}^{k} \\ &= F_{(l_{r+1}+2)+l_r+l_{r-1}+\cdots+l_{t_k}+1} \cdot \prod_{c=1}^{k} \qquad \text{if } t_k < r. \end{aligned}$$

$$f_{\mathbf{t}} = F_{l_{r+1}+2}F_{((l_r-2)+2)+l_{r-1}+\cdots+l_{t_{k-1}}+1}\prod_{c=1}^{k-1} = F_{l_{r+1}+2}\prod_{c=1}^{k} \qquad \text{if } t_k = r.$$

The constant term of $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\dots,n_1,l_1}$ is

$$F_{l_{r+1}+2}F_{(l_r+2)+l_{r-1}+\cdots+l_2+l_1} - F_{l_{r+1}}F_{((l_r-2)+2)+l_{r-1}+\cdots+l_2+l_1} = F_{(l_{r+1}+2)+l_r+\cdots+l_2+l_1}.$$

So our result is proved when $l_r \geqslant 2$. Next assume $l_r = 1$. We must consider the following four cases in computing $f_{\mathbf{t}}$ for $\mathbf{t} = (t_1, t_2, ..., t_k)$ with $k \in [r]$: (i) $t_k < r-1$; (ii) $t_k = r$ and $t_{k-1} < r-1$; (iii) $t_k = r-1$; (iv) $(t_{k-1}, t_k) = (r-1, r)$.

$$\begin{aligned} f_{\mathbf{t}} &= F_{l_{r+1}+2}F_{3+l_{r-1}+\cdots+l_{t_k+1}}\prod_{c=1}^{k} - F_{l_{r+1}}(F_{2+l_{r-1}+\cdots+l_{t_k+1}} - F_2F_{l_{r-1}+\cdots+l_{t_k+1}})\prod_{c=1}^{k} \\ &= (F_{l_{r+1}+2}F_{3+l_{r-1}+\cdots+l_{t_k+1}} - F_{l_{r+1}}F_{1+l_{r-1}+\cdots+l_{t_k+1}})\prod_{c=1}^{k} \\ &= F_{(l_{r+1}+2)+l_r+\cdots+l_{t_k+1}}\prod_{c=1}^{k} \qquad \text{in the case (i).} \end{aligned}$$

$$\begin{aligned} f_{\mathbf{t}} &= F_{l_{r+1}+2}(F_{2+l_{r-1}+\cdots+l_{t_{k-1}+1}} - F_2F_{l_{r-1}+\cdots+l_{t_{k-1}+1}})\prod_{c=1}^{k-1} \\ &= F_{l_{r+1}+2}F_{l_r+l_{r-1}+\cdots+l_{t_{k-1}+1}}\prod_{c=1}^{k-1} \qquad \text{in the case (ii).} \end{aligned}$$

$f_{\mathbf{t}} = F_{l_{r+1}+2}F_3\prod_{c=1}^{k} - F_{l_{r+1}}F_2\prod_{c=1}^{k} = (F_{l_{r+1}+2}F_3 - F_{l_{r+1}}F_1)\prod_{c=1}^{k} = F_{(l_{r+1}+2)+l_r}\prod_{c=1}^{k}$ in the case (iii) and $f_{\mathbf{t}} = F_{l_{r+1}+2}F_2\prod_{c=1}^{k-1} = F_{l_{r+1}+2}F_{l_r}\prod_{c=1}^{k-1}$ in the case (iv). Finally, the constant term of $K^{|(>|)^r}_{l_{r+1},n_r,l_r,\dots,n_1,l_1}$ is

$$F_{l_{r+1}+2}F_{3+l_{r-1}+\cdots+l_1} - F_{l_{r+1}}(F_{2+l_{r-1}+\cdots+l_1} - F_2F_{l_{r-1}+\cdots+l_1}) = F_{(l_{r+1}+2)+l_r+l_{r-1}+\cdots+l_1}.$$

So our result is also proved when $l_r = 1$. □

Example 5.8. By (5.7.1), we have

(1) $K^{|>|>|}_{l_3,n_2,l_2,n_1,l_1} = F_{(l_3+2)+l_2+l_1} + n_2F_{l_3+2}F_{l_2+l_1} + n_1F_{(l_3+2)+l_2}F_{l_1} + n_2n_1F_{l_3+2}F_{l_2}F_{l_1}$.

(2) $K^{|>|>|>|}_{l_4,n_3,l_3,n_2,l_2,n_1,l_1} = F_{(l_4+2)+l_3+l_2+l_1} + n_3F_{l_4+2}F_{l_3+l_2+l_1} + n_2F_{(l_4+2)+l_3}F_{l_2+l_1} + n_1F_{(l_4+2)+l_3+l_2}F_{l_1} + n_3n_2F_{l_4+2}F_{l_3}F_{l_2+l_1} + n_3n_1F_{l_4+2}F_{l_3+l_2}F_{l_1} + n_2n_1F_{(l_4+2)+l_3}F_{l_2}F_{l_1} + n_3n_2n_1F_{l_4+2}F_{l_3}F_{l_2}F_{l_1}$.

References

1. K. Eriksson, *Reduced words in affine Coxeter groups*, Discrete Math. **157 (1-3)** (1996), 127–146.
2. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol 29, 1992.
3. R. P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5 (4)** (1984), 359–372.
4. J. Tits, *Le problème des mots dans les groupes de Coxeter, in: Symposia Mathematica, vol 1 INDAM, Rome, 1967/1968, Academic Press, London* (1969), 175–185.
5. R. Winkel, *A combinatorial bijection between standard Young tableaux and reduced words of Grassmannian permutations*, Sém. Lothar. Combin. **36** (1996), 1–24.
6. R. Winkel, *Schubert functions and the number of reduced words of permutations*, Sém. Lothar. Combin. **39** (1997), 1–28.
7. N. H. Xi, *Lusztig's a-function for Coxeter groups with complete graphs*, Bull. Inst. Math. Acad. Sin. **7 (1)** (2012), 71–90.