THE CELLS OF THE AFFINE WEYL GROUP $\widetilde{C}_n$ IN A CERTAIN QUASI-SPLIT CASE

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Dedicated to Professor Roger W. Carter on his 80th birthday

ABSTRACT. The affine Weyl group $(\widetilde{C}_n, S)$ can be realized as the fixed point set of the affine Weyl group $(\widetilde{A}_{2n-1}, \widetilde{S})$ under a certain group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$. Let $\widetilde{\ell}$ be the length function of $\widetilde{A}_{2n-1}$. We study the cells of the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell})$. The main results of the paper are to give an explicit description for all the cells of $(\widetilde{C}_n, \widetilde{\ell})$ corresponding to the partitionss

§0. Introduction.

The cells of a weighted Coxeter group (W, L) are discussed in [7]. A particular interesting case is that W is the fixed point set of a finite or affine Coxeter system $(\widetilde{W}, \widetilde{S})$ under a group automorphism α with $\alpha(\widetilde{S}) = \widetilde{S}$ and L is the restriction of the length function of $\widetilde{W}$ (see [7, Chapter 16], [5], [1], [3]). In this paper we discuss the case that $\widetilde{W}$ is of type $\widetilde{A}_{2n-1}$ and W is of type $\widetilde{C}_n$.

For any $i \leqslant j$ in the integer set $\mathbb{Z}$, denote by $[i, j]$ the set $\{i, i+1, ..., j\}$. Denote $[1, j]$ simply by $[j]$. There is a natural bijection between the set of two-sided cells of $\widetilde{A}_{2n-1}$ and the set Λ_{2n} of partitions of $2n$ (see [8], [6]).

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Let Ω_λ be the two-sided cell of $\widetilde{A}_{2n-1}$ corresponding to $\lambda \in \Lambda_{2n}$. We are interested in the sets $E_\lambda = \Omega_\lambda \cap \widetilde{C}_n$. We describe all the cells in the sets $E_{\mathbf{k1^{2n-k}}}$ (Theorem 4.9) and $E_{\mathbf{h21^{2n-h-2}}}$ (Theorem 5.1) for all $k \in [2n]$ and $h \in [2, 2n-2]$ and also all the cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$ (Theorem 6.1).

For all considered $\lambda \in \Lambda_{2n}$, we prove that all left (respectively, two-sided) cells in E_λ are left- (respectively, two-sided-) connected (see 2.15 and Theorems 4.9, 5.1 and 6.1). I conjecture that this should be true for the left cells and two-sided cells of any weighted Coxeter group.

The paper is organized as follows. Section 1 is devoted to collect some basic concepts and known facts concerning cells of weighted Coxeter groups. In Sections 2-3, we focus on the weighted Coxeter group $(\widetilde{C}_n, \widetilde{\ell}_{2n-1})$. Then we discuss the sets $E_{\mathbf{k1^{2n-k}}}$ and $E_{\mathbf{h21^{2n-h-2}}}$ for all $k \in [2n]$ and $h \in [2, 2n-2]$ in Sections 4 and 5, respectively. Finally, we discuss cells of the weighted Coxeter group $(\widetilde{C}_3, \widetilde{\ell}_5)$.

§1. Cells in Coxeter groups.

In this section, we collect some concepts and results concerning cells of a weighted Coxeter group, all but Lemma 1.4 follow Lusztig in [7].

1.1. Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. A *weight function* on W is a function $L : W \longrightarrow \mathbb{Z}$ such that $L(wu) = L(w) + L(u)$ if $\ell(wu) = \ell(w) + \ell(u)$ for $w, u \in W$. Call (W, L) a *weighted Coxeter group.* Call (W, L) in the *split* case if $L = \ell$.

When α is a group automorphism of W with $\alpha(S) = S$, let $W^\alpha = \{w \in W \mid \alpha(w) = w\}$. For any α-orbit J on S, let $w_J \in W^\alpha$ be the longest element in the subgroup W_J of W generated by J. Let S_α be the set of elements w_J with J ranging over all α-orbits on S. Then (W^α, S_α) is a Coxeter group and the restriction to W^α of the length function ℓ is a weight function on W^α. The weighted Coxeter group (W^α, ℓ) is called in the *quasi-split* case.

1.2. Let $\underset{L}{\leqslant}$, $\underset{R}{\leqslant}$, $\underset{LR}{\leqslant}$ be the preorders on (W, L) defined in [7, Chapter 8]. The corresponding equivalence classes in W are called *left cells, right cells, two-sided cells* of W, respectively. For $w \in W$, define $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ and $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. If $y, w \in W$ satisfy $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$), then $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) \supseteq \mathcal{L}(w)$). In particular, if $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$), then $\mathcal{R}(y) = \mathcal{R}(w)$ (respectively, $\mathcal{L}(y) = \mathcal{L}(w)$) (see [7, Lemma 8.6]).

1.3. In [7, Chapter 13], Lusztig defined a function $a : W \longrightarrow \mathbb{N} \cup \{\infty\}$ for a weighted Coxeter group (W, L), he proved the following results when W is either a finite or an affine Coxeter group and when (W, L) is either in the split case or in the quasi-split case.

(1) $y \underset{LR}{\leqslant} w$ in W implies $a(w) \leqslant a(y)$. Hence $y \underset{LR}{\sim} w$ in W implies $a(w) = a(y)$.

(2) If $w, y \in W$ satisfy $a(w) = a(y)$ and $y \underset{L}{\leqslant} w$ (respectively, $y \underset{R}{\leqslant} w$, $y \underset{LR}{\leqslant} w$) then $y \underset{L}{\sim} w$ (respectively, $y \underset{R}{\sim} w$, $y \underset{LR}{\sim} w$).

For any $X \subset W$, write $X^{-1} := \{x^{-1} \mid x \in X\}$.

Lemma 1.4. *Suppose that W is either a finite or an affine Coxeter group and that (W, L) is either in the split case or in the quasi-split case.*

Let E be a non-empty subset of W satisfying the following conditions:

(a) There exists some $k \in \mathbb{N}$ with $a(x) = k$ for any $x \in E$;

(b) E is a union of some left cells of W;

(c) $E^{-1} = E$.

Then E is a union of some two-sided cells of W.

Proof. By (b)-(c), E is also a union of some right cells of W. The set $W_{(k)} := \{w \in W \mid a(w) = k\}$ is a union of some two-sided cells of W by 1.3 (1). If the result is false, then by (c), there must exist some $x \in E$ and $y \in W_{(k)} \setminus E$ such that either $x \underset{L}{\leqslant} y$ or $y \underset{L}{\leqslant} x$. In either case, we would have $x \underset{L}{\sim} y$ by 1.3 (2), contradicting (b). This proves our result. $\square$

§2. The affine Weyl groups $\widetilde{A}_{2n-1}$ and $\widetilde{C}_n$.

From now on, we focus on the weighted Coxeter groups $(\widetilde{A}_{2n-1},\widetilde{\ell})$ and $(\widetilde{C}_n,\widetilde{\ell})$, where $\widetilde{\ell}$ is the length function of the affine Weyl group $\widetilde{A}_{2n-1}$.

2.1. The affine Weyl group $\widetilde{A}_{2n-1}$ can be realized as the following permutation group on the set $\mathbb{Z}$ (see [4, Subsection 3.6] and [8, Subsection 4.1)]:

$$\widetilde{A}_{2n-1}=\left\{w:\mathbb{Z}\longrightarrow\mathbb{Z}\,\middle|\,(i+2n)w=(i)w+2n,\sum_{i=1}^{2n}(i)w=\sum_{i=1}^{2n}i\right\}.$$

The Coxeter generator set $\widetilde{S}=\{s_i\mid i\in[0,2n-1]\}$ of $\widetilde{A}_{2n-1}$ is given by

$$(t)s_i=\begin{cases}t, & \text{if } t\not\equiv i,i+1 \pmod{2n},\\ t+1, & \text{if } t\equiv i \pmod{2n},\\ t-1, & \text{if } t\equiv i+1 \pmod{2n},\end{cases}$$

for $t\in\mathbb{Z}$ and $i\in[0,2n-1]$. Any $w\in\widetilde{A}_{2n-1}$ can be realized as a $\mathbb{Z}\times\mathbb{Z}$ monomial matrix $A_w=(a_{ij})_{i,j\in\mathbb{Z}}$, where a_{ij} is 1 if $j=(i)w$ and 0 if otherwise. The row (respectively, column) indices of A_w are increasing from top to bottom (respectively, from left to right).

Let α be the group automorphism of $\widetilde{A}_{2n-1}$ determined by $\alpha(s_i)=s_{2n-i}$ for $i\in[0,2n-1]$. Then the affine Weyl group $\widetilde{C}_n$ can be realized as the fixed point set of $\widetilde{A}_{2n-1}$ under α. As a permutation group on $\mathbb{Z}$, we have

$$\widetilde{C}_n=\{w:\mathbb{Z}\longrightarrow\mathbb{Z}\mid(i+2n)w=(i)w+2n,(i)w+(1-i)w=1,\ \forall\ i\in\mathbb{Z}\}$$

with the Coxeter generator set $S=\{t_i\mid i\in[0,n]\}$, where $t_i=s_is_{2n-i}$ for $i\in[n-1]$, $t_0=s_0$ and $t_n=s_n$. For the sake of convenience, we define s_i and t_j for any $i,j\in\mathbb{Z}$ by setting $s_{2qn+b}=s_b$ and $t_{2pn\pm a}=t_a$ for any $p,q\in\mathbb{Z}$, $b\in[0,2n-1]$, $a\in[0,n]$.

2.2. By a partition of an integer $n>0$, we mean an r-tuple $\lambda:=(\lambda_1,\lambda_2,...,\lambda_r)$ of integers $\lambda_1\geqslant\cdots\geqslant\lambda_r>0$ with $\sum_{k=1}^r\lambda_k=n$ for some $r\geqslant 1$. Call λ_i a *part* of λ. We sometimes denote λ in the form $\mathbf{j_1^{k_1}j_2^{k_2}}\cdots\mathbf{j_m^{k_m}}$ (boldfaced) with

$j_1 > j_2 > \cdots > j_m$ if j_i is a part of λ with multiplicity $k_i \geqslant 1$ for $i \geqslant 1$. Let Λ_n be the set of all partitions of n.

Fix $w \in \widetilde{A}_{2n-1}$. For any $i \neq j$ in $[2n]$, we write $i \prec_w j$, if there exist some $p, q \in \mathbb{Z}$ such that $2pn + i > 2qn + j$ and $(2pn + i)w < (2qn + j)w$. In the matrix of w, this means that the position $(2qn + j, (2qn + j)w)$ is located at the northeastern of the position $(2pn + i, (2pn + i)w)$. This defines a partial order $\preceq_w$ on the set $[2n]$.

A sequence $a_1, a_2, ..., a_r$ in $[2n]$ is called a *w-chain*, if $a_1 \prec_w a_2 \prec_w \cdots \prec_w a_r$. Sometimes we identify a w-chain $a_1, a_2, ..., a_r$ with the corresponding set $\{a_1, a_2, ..., a_r\}$. For any $k \geqslant 1$, a k-w-chain-family is by definition a disjoint union $X = \dot{\bigcup}_{i=1}^k X_i$ of k w-chains $X_1, ..., X_k$ in $[2n]$. Let d_k be the maximally possible cardinal of a k-w-chain-family for any $k \geqslant 1$. Then there exists some $r \geqslant 1$ such that $d_1 < d_2 < \cdots < d_r = 2n$. Let $\lambda_1 = d_1$ and $\lambda_{k+1} = d_{k+1} - d_k$ for $k \in [r-1]$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ by a result of Greene in [2]. Hence $w \mapsto \psi(w) := (\lambda_1, ..., \lambda_r)$ defines a map $\psi : \widetilde{A}_{2n-1} \longrightarrow \Lambda_{2n}$.

2.3. Let $\widetilde{\ell}$, ℓ be the length functions on $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. By 1.1, we see that the weighted Coxeter group $(\widetilde{A}_{2n-1}, \widetilde{\ell})$ is in the split case, while $(\widetilde{C}_n, \widetilde{\ell})$ is in the quasi-split case (see [7, Lemma 16.2]).

For any $x \in \widetilde{A}_{2n-1}$ and $k \in \mathbb{Z}$, let $m_k(x) = {}^{\#}\{i \in \mathbb{Z} \mid i < k \ \text{ and } \ (i)x > (k)x\}$. Then the formulae for the functions $\widetilde{\ell}$ and ℓ are as follows.

Proposition 2.4. *For any $w \in \widetilde{A}_{2n-1}$ and $x \in \widetilde{C}_n$, we have*

(1) $\widetilde{\ell}(w) = \sum_{1 \leqslant i < j \leqslant 2n} \left| \left\lfloor \dfrac{(j)w - (i)w}{2n} \right\rfloor \right| = \sum_{k=1}^{2n} m_k(w)$;

(2) $\ell(x) = \frac{1}{2}(\widetilde{\ell}(x) + m_1(x) + m_{n+1}(x))$,

where $\lfloor a \rfloor$ is the largest integer not larger than a, and $|a|$ is the absolute value of a for any $a \in \mathbb{Q}$.

Proof. The first equality of (1) is just [8, Lemma 4.2.2], while the second equality of (1) follows by the facts that for any $i < j$ in $[2n]$, at most one of $m_{ij}(w) := {}^{\#}\{k \in \mathbb{Z} \mid k \equiv i \pmod{2n}; k < j; (k)w > (j)w\}$ and $m_{ji}(w) := {}^{\#}\{k \in \mathbb{Z} \mid k \equiv j \pmod{2n}; k < i; (k)w > (i)w\}$ is positive, that $\left|\left\lfloor \frac{(j)w-(i)w}{2n} \right\rfloor\right| = \max\{m_{ij}(w), m_{ji}(w)\}$ and that $m_k(w) = \sum_{i\in[2n]\setminus\{k\}} m_{ik}(w)$. (2) follows by the definition of t_i's in terms of s_j's. $\square$

2.5. Let $\leqslant$, $\leqslant_C$ be the Bruhat-Chevalley orders on $(\widetilde{A}_{2n-1}, \widetilde{S})$, $(\widetilde{C}_n, S)$, respectively. Since the condition $x \leqslant_C y$ is equivalent to $x \leqslant y$ for any $x, y \in \widetilde{C}_n$, we may use $\leqslant$ for both $\leqslant$ and $\leqslant_C$ from now on.

Let $\widetilde{\mathcal{L}}(x) = \{s \in \widetilde{S} \mid sx < x\}$ and $\widetilde{\mathcal{R}}(x) = \{s \in \widetilde{S} \mid xs < x\}$ for $x \in \widetilde{A}_{2n-1}$ and let $\mathcal{L}(y) = \{t \in S \mid ty < y\}$ and $\mathcal{R}(y) = \{t \in S \mid yt < y\}$ for $y \in \widetilde{C}_n$.

Corollary 2.6. *For any $x \in \widetilde{C}_n$ and $i \in [0, n]$,*

$$\begin{aligned}
s_i \in \widetilde{\mathcal{L}}(x) &\iff s_{2n-i} \in \widetilde{\mathcal{L}}(x) \iff t_i \in \mathcal{L}(x) \\
&\iff (i)x > (i+1)x \iff (2n+1-i)x < (2n-i)x, \\
s_i \in \widetilde{\mathcal{R}}(x) &\iff s_{2n-i} \in \widetilde{\mathcal{R}}(x) \iff t_i \in \mathcal{R}(x) \\
&\iff (i)x^{-1} > (i+1)x^{-1} \iff (2n+1-i)x^{-1} < (2n-i)x^{-1}
\end{aligned}$$

Proof. The equivalent conditions involving the s_i's hold by [8, Lemma 4.2.4], while those involving the t_j's hold by the expression of t_j in terms of s_i's and by Proposition 2.4. $\square$

2.7. Any $w \in \widetilde{C}_n$ is determined uniquely by the n-tuple $((1)w, (2)w, ..., (n)w)$. Hence we may denote w by $[(1)w, (2)w, ..., (n)w]$. For any $a \in \mathbb{Z}$, denote by $\langle a\rangle$ the unique integer in $[2n]$ satisfying $a \equiv \langle a\rangle \pmod{2n}$. Let η be the group automorphism of $\widetilde{C}_n$ determined by $\eta(t_i) = t_{n-i}$ for any $i \in [0, n]$. The following results are related to the expression $w = [a_1, a_2, ..., a_n] \in \widetilde{C}_n$.

Proposition 2.8. *Let $w = [a_1, a_2, ..., a_n]$ and $w' = \eta(w) = [a'_1, a'_2, ..., a'_n]$ be in $\widetilde{C}_n$. Let $k \in [0, n]$. Then*

(1) $t_k \in \mathcal{L}(w)$ if and only if $a_k > a_{k+1}$, with the convention that $a_0 = 1$ and $a_{n+1} = n$.

(2) Let $\langle a_i\rangle, \langle a_j\rangle \in \{k, k+1, 2n-k, 2n+1-k\}$ for some $i \neq j$ in $[n]$. Then $t_k \in \mathcal{R}(w)$ if one of the following conditions holds:

(i) $(\langle a_i\rangle,\langle a_j\rangle)\in\{(k,k+1),(2n-k,2n+1-k)\}$. *Either* $a_j-a_i>2n$, *or* $i>j$ *and* $a_j>a_i$.

(ii) $(\langle a_i\rangle,\langle a_j\rangle)=(k,2n-k)$ *and* $a_i+a_j<1$.

(iii) $(\langle a_i\rangle,\langle a_j\rangle)=(2n+1-k,k+1)$ *and* $a_i+a_j>2n+1$.

(3) $a'_i=n+1-a_{n+1-i}$ *for any* $i\in[n]$.

Proof. (1)-(2) follow by Corollary 2.6. For (3), apply induction on $\ell(w)\geqslant 0$. It is trivial when $\ell(w)=0$. If $\ell(w)>0$, write $w=t_iy$ for some $t_i\in\mathcal{L}(w)$, then $\eta(w)=t_{n-i}\eta(y)$. We have $\eta(y)=[n+1-b_n,n+1-b_{n-1},...,n+1-b_1]$ for $y=[b_1,...,b_n]$ by inductive hypothesis. So $\eta(w)=[n+1-a_n,n+1-a_{n-1},...,n+1-a_1]$ by the relations $\eta(w)=t_{n-i}\eta(y)$ and $w=t_iy$. □

2.9. For any $i\in[0,2n-1]$, let $\widetilde{D}_R(i)=\{w\in\widetilde{A}_{2n-1}\mid|\{s_i,s_{i+1}\}\cap\widetilde{\mathcal{R}}(w)|=1\}$. When $w\in\widetilde{D}_R(i)$, define w^* by the condition $w^*\in\{ws_i,ws_{i+1}\}\cap\widetilde{D}_R(i)$, call the transformation $w\mapsto w^*$ a *right* $\{s_i,s_{i+1}\}$*-star operation* (or a *right star operation* in short) on w. For any $w\in\widetilde{A}_{2n-1}$, let $\widetilde{M}(w)$ be the set of all $y\in\widetilde{A}_{2n-1}$ which is either w or obtained from w by successively applying right star operations. Define a graph $\widetilde{\mathcal{M}}(w)$: its vertex set is $\widetilde{M}(w)$; each $x\in\widetilde{M}(w)$ is labeled by $\widetilde{\mathcal{R}}(x)$; $x,y\in\widetilde{M}(w)$ are joined by a solid edge if y can be obtained from x by a right star operation. By a *path* in $\widetilde{\mathcal{M}}(w)$, we mean a sequence $x_0,x_1,...,x_r$ in $\widetilde{M}(w)$ with some $r\geqslant 0$ such that x_{i-1} and x_i are joined by a solid edge for every $i\in[r]$. Say $w,y\in\widetilde{A}_{2n-1}$ have *the same generalized* τ*-invariants*, if for any path $w_1=w,w_2,...,w_r$ in $\widetilde{\mathcal{M}}(w)$, there exists a path $y_1=y,y_2,...,y_r$ in $\widetilde{\mathcal{M}}(y)$ such that $\widetilde{\mathcal{R}}(w_i)=\widetilde{\mathcal{R}}(y_i)$ for every $i\in[r]$ and if this condition still holds when the roles of w and y are interchanged.

For any $i\in[0,n-1]$, let $D_R(i)=\{w\in\widetilde{C}_n\mid|\{t_i,t_{i+1}\}\cap\mathcal{R}(w)|=1\}$. Regarding $\widetilde{C}_n$ as a subset of $\widetilde{A}_{2n-1}$, we have $D_R(i)=\widetilde{C}_n\cap\widetilde{D}_R(i)=\widetilde{C}_n\cap\widetilde{D}_R(2n-i-1)$. When $w\in D_R(i)$, we have $|\{wt_i,wt_{i+1}\}\cap D_R(i)|=1$ unless that $i\in\{0,n-1\}$ and $w\in\{xt_it_{i+1},xt_{i+1}t_i\}$ for some $x\in\widetilde{C}_n$ with $\mathcal{R}(x)\cap\{t_i,t_{i+1}\}=\emptyset$. In this excepted case, both wt_i and wt_{i+1} are in

$D_R(i)$. When $|\{wt_i, wt_{i+1}\} \cap D_R(i)| = 1$, define w^* by the condition $w^* \in \{wt_i, wt_{i+1}\} \cap D_R(i)$, then w^* can be obtained from w by a pair of right star operations if $i \in [n-2]$ and, by a single right star operation if $w^* \in \{wt_0, wt_n\}$ with $i \in \{0, n-1\}$ and, by none of the above two ways if $w^* \in \{wt_1, wt_{n-1}\}$ with $i \in \{0, n-1\}$. When $\{wt_i, wt_{i+1}\} \subset D_R(i)$, define w_1^*, w_2^* by the conditions $\{w_1^*, w_2^*\} = \{wt_i, wt_{i+1}\}$ and $w_1^* < w_2^*$, then $x \in \{w_1^*, w_2^*\}$ can be obtained from w by one right star operation if $x \in \{wt_0, wt_n\}$ and, not by one or two right star operation if $x \in \{wt_1, wt_{n-1}\}$.

In the remaining part of the paper, when we mention a right star operation and the generalized τ-invariants on $w \in \widetilde{C}_n$, we always regard w as an element of $\widetilde{A}_{2n-1}$.

2.10. For any $w \in \widetilde{C}_n$, define $M(w)$ to be the set of all $y \in \widetilde{C}_n$, where there exists $x_0 = w, x_1, ..., x_r = y$ with some $r \geqslant 0$ such that for every $i \in [r]$, $x_i^{-1}x_{i-1} \in S$ and x_i can be obtained from x_{i-1} by one or two right star operations. Define a graph $\mathcal{M}(w)$: its vertex set is $M(w)$; label each $x \in M(w)$ by $\mathcal{R}(x)$; join $x, y \in M(w)$ by a solid edge if $x^{-1}y \in S$ and x can be obtained from y by one or two right star operations.

It is easy to see that if $y, w \in \widetilde{C}_n$ have the same generalized τ-invariants, then for any path $w_1 = w, w_2, ..., w_r$ in $\mathcal{M}(w)$, there exists a path $y_1 = y, y_2, ..., y_r$ in $\mathcal{M}(y)$ such that $\mathcal{R}(w_i) = \mathcal{R}(y_i)$ for every $i \in [r]$ and the above condition still holds when interchanging the roles of w, y. In Section 6, the graphs $\mathcal{M}(w)$ with $w \in \widetilde{C}_3$ will be used to confirm that two elements of $\widetilde{C}_3$ have different generalized τ-invariants.

2.11. For any $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and $\mu = (\mu_1, \mu_2, ..., \mu_t)$ in Λ_{2n}, we write $\lambda \leqslant \mu$ if $\lambda_1 + \cdots + \lambda_k \leqslant \mu_1 + \cdots + \mu_k$ for any $1 \leqslant k \leqslant \min\{r, t\}$. This defines a partial order on Λ_{2n}. If $x \in \widetilde{A}_{2n-1}$ and $s \in \widetilde{\mathcal{L}}(x)$ and $t \in \widetilde{\mathcal{R}}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$ (see [8, Lemma 5.5 and Corollary 5.6]). This implies by Corollary 2.6 that if $x \in \widetilde{C}_n$ and $s \in \mathcal{L}(x)$ and $t \in \mathcal{R}(x)$ then $\psi(sx), \psi(xt) \leqslant \psi(x)$.

Let $\widetilde{a}$, a be the a-functions of the weighted Coxeter groups $(\widetilde{A}_{2n-1}, \widetilde{\ell})$,

$(\widetilde{C}_n, \widetilde{\ell})$, respectively (see 2.3 and 1.3).

Lemma 2.12. (see [7, Lemma 16.5]) *$a(z) = \widetilde{a}(z)$ for any $z \in \widetilde{C}_n$.*

Lemma 2.13. (see [7, Lemma 16.14]) *Let $x, y \in \widetilde{C}_n$. Then $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{C}_n$ if and only if $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) in $\widetilde{A}_{2n-1}$.*

By Lemma 2.13, we can just use the notation $x \underset{L}{\sim} y$ (respectively, $x \underset{R}{\sim} y$) for $x, y \in \widetilde{C}_n$ without indicating whether the relation refers to the group $\widetilde{A}_{2n-1}$ or $\widetilde{C}_n$.

For any $\lambda = (\lambda_1, ..., \lambda_r) \in \Lambda_{2n}$, define $\mu = (\mu_1, ..., \mu_t) \in \Lambda_{2n}$ by setting $\mu_j = {}^{\#}\{k \in [r] \mid \lambda_k \geqslant j\}$ for any $j \geqslant 1$, call μ the *dual partition* of λ.

Lemma 2.14. *Let $x, y \in \widetilde{A}_{2n-1}$.*

(1) $x \underset{L}{\sim} y$ if and only if x, y have the same generalized τ-invariants (see [8, Theorem 16.1.2]).

(2) $x \underset{LR}{\leqslant} y$ if and only if $\psi(y) \leqslant \psi(x)$. The set $\psi^{-1}(\lambda)$ forms a two-sided cell of $\widetilde{A}_{2n-1}$ for any $\lambda \in \Lambda_{2n}$ (see [6, Theorem 6] and [8, Theorem 17.4] and [10, Theorem B]).

(3) $\widetilde{a}(x) = \sum_{i=1}^{t} (i-1)\mu_i$, where $(\mu_1, ..., \mu_t)$ is the dual partition of $\psi(x)$ (see [9, Subsection 6.27]).

2.15. A non-empty subset E of a Coxeter group $W = (W, S)$ is said *left-connected*, (respectively, *right-connected*) if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that $x_{i-1}x_i^{-1} \in S$ (respectively, $x_i^{-1}x_{i-1} \in S$) for every $i \in [r]$. E is said *two-sided-connected* if for any $x, y \in E$, there exists a sequence $x_0 = x, x_1, ..., x_r = y$ in E such that either $x_{i-1}x_i^{-1}$ or $x_i^{-1}x_{i-1}$ is in S for every $i \in [r]$.

Let $F \subseteq E$ in W. Call F a *left-connected component* (or *lcc* in short) of E, if F is a maximal left-connected subset of E. One can define a right-connected component and a two-sided-connected component (or *rcc* and *tcc* in short) of E similarly.

For any $\lambda \in \Lambda_{2n}$, denote $E_\lambda := \widetilde{C}_n \cap \psi^{-1}(\lambda)$.

Lemma 2.16. *Let $\lambda \in \Lambda_{2n}$.*

(1) Any lcc (respectively, rcc, tcc) of $\psi^{-1}(\lambda)$ is contained in some left (respectively, right, two-sided) cell of $\widetilde{A}_{2n-1}$.

(2) Any lcc (respectively, rcc, tcc) of E_λ is contained in some left (respectively, right, two-sided) cell of $\widetilde{C}_n$.

(3) The set E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$.

Proof. (1)-(2) follow by 1.3 (1)-(2), Lemmas 2.12 and 2.14. By Lemmas 2.13-2.14, we see that E_λ is either empty or a union of some left cells of $\widetilde{C}_n$ with $E_\lambda^{-1} = E_\lambda$ for any $\lambda \in \Lambda_{2n}$. So (3) follows by Lemmas 2.12 and 1.4. □

Corollary 2.17. *Let $x, y \in \widetilde{A}_{2n-1}$ satisfy $x, y \in \psi^{-1}(\lambda)$ for some $\lambda \in \Lambda_{2n}$.*

(1) If $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(yx^{-1})$ then x, y are in the same lcc of $\psi^{-1}(\lambda)$ and hence $x \underset{L}{\sim} y$.

(2) If $\widetilde{\ell}(y) = \widetilde{\ell}(x) + \widetilde{\ell}(x^{-1}y)$ then x, y are in the same rcc of $\psi^{-1}(\lambda)$ and hence $x \underset{R}{\sim} y$.

Let $x, y \in \widetilde{C}_n$ be in E_λ for some $\lambda \in \Lambda_{2n}$.

(3) If $\ell(y) = \ell(x) + \ell(yx^{-1})$ then x, y are in the same lcc of E_λ and hence $x \underset{L}{\sim} y$.

(4) If $\ell(y) = \ell(x) + \ell(x^{-1}y)$ then x, y are in the same rcc of E_λ and hence $x \underset{R}{\sim} y$.

Proof. By symmetry, we need only to show (1) and (3).

(1) Let $yx^{-1} = s_{i_r} s_{i_{r-1}} \cdots s_{i_2} s_{i_1}$ be a reduced expression of yx^{-1} with $s_{i_j} \in \widetilde{S}$. Let $x_k = s_{i_k} s_{i_{k-1}} \cdots s_{i_2} s_{i_1} x$ for $k \in [0, r]$, where we stipulate $x_0 = x$. Then $\widetilde{\ell}(x_k) = \widetilde{\ell}(x_{k-1}) + 1$ for any $k \in [r]$. Hence $\psi(x) = \psi(x_0) \leqslant \psi(x_1) \leqslant \cdots \leqslant \psi(x_r) = \psi(y) = \psi(x)$ by 2.11. This implies that x, y are in the same lcc of $\psi^{-1}(\lambda)$. Hence $x \underset{L}{\sim} y$ by Lemma 2.16.

(3) Let $yx^{-1} = t_{i_r} t_{i_{r-1}} \cdots t_{i_1}$ be a reduced expression of yx^{-1} with $t_{i_j} \in S$. Let $x_k = t_{i_k} t_{i_{k-1}} \cdots t_{i_1} x$ for $k \in [0, r]$, where we stipulate $x_0 = x$. Then $\ell(x_k) = \ell(x_{k-1}) + 1$ for any $k \in [r]$. Hence $\psi(x) = \psi(x_0) \leqslant \psi(x_1) \leqslant \cdots \leqslant$

$\psi(x_r) = \psi(y) = \psi(x)$ by 2.11. This implies that x, y are in the same lcc of E_λ. Hence $x \underset{L}{\sim} y$ by Lemmas 2.13 and 2.16. □

§3. Partial order $\preceq_w$ on $[2n]$ determined by an element w.

In this section, we introduce two technical tools. One is a transformation on an element in 3.3, which is a crucial step in proving the left-connectedness of a left cell and in finding a representative set for the left cells of $\widetilde{C}_n$ in E_λ, $\lambda \in \Lambda_{2n}$. The other is a generalized tabloid in 3.5, by which we can check if two elements of $\widetilde{C}_n$ are in the same left cell.

3.1. $i, j \in [2n]$ are said $2n$-*dual*, if $i + j = 2n + 1$; in this case, denote $j = \bar{i}$. Further, denote $\overline{E} := \{\bar{i} \mid i \in E\}$ for $E \subseteq [2n]$. Recall the relation $\preceq_w$ on $[2n]$ defined in 2.2 for $w \in \widetilde{A}_{2n-1}$ and that $\widetilde{C}_n$ is regarded as a subset of $\widetilde{A}_{2n-1}$ (see 2.1). Fix $w \in \widetilde{A}_{2n-1}$. Say $i \neq j$ in $[2n]$ w-*comparable* if either $i \prec_w j$ or $j \prec_w i$, and w-*uncomparable* if otherwise. When $w \in \widetilde{C}_n$, say $i \in [2n]$ w-*wild* if i, $\bar{i}$ are w-comparable and w-*tame* if otherwise. Say $i \in [2n]$ *a* w-*wild head* (respectively, *a* w-*tame head*), if i is w-wild (respectively, w-tame) with $(\bar{i})w < (i)w$.

$i < j$ in $[2n]$ are w-uncomparable if and only if $(i)w < (j)w < (i)w + 2n$.

Call $E \subseteq [2n]$ a w-*chain*, if $E = \{i_1, i_2, ..., i_r\}$ and $i_1 \prec_w i_2 \prec_w \cdots \prec_w i_r$.

Lemma 3.2. *Fix $w \in \widetilde{C}_n$. Let $i, j, k \in [2n]$.*

(i) $j \prec_w k$ if and only if $\bar{k} \prec_w \bar{j}$;

Now suppose that $j \neq k$ are w-wild heads and i is w-tame.

(ii) $\bar{j} \prec_w k$ if and only if $\bar{j}, k$ are w-comparable.

(iii) If $\bar{j}, k$ are w-uncomparable then so are j, k (respectively, $\bar{j}, \bar{k}$);

(iv) i and k are w-comparable if and only if $i \prec_w k$.

(v) $\{j, i, \bar{j}\}$ is a w-chain if and only if j is w-comparable with both i and $\bar{i}$;

(vi) $\{j, k, \bar{j}, \bar{k}\}$ is a w-chain if and only if j, k are w-comparable.

Proof. (i)-(iv) can be checked directly. Then (v) follows by (i) and (iv). Finally, (vi) is a simple consequence of (i)-(iii). □

3.3. Let

$$\text{(3.3.1)}\qquad \begin{aligned} t_{i,j} &= t_{i+j-1}t_{i+j-2}\cdots t_{i+1}t_i,\\ d_{i,j} &= t_{i-j+1}t_{i-j+2}\cdots t_{i-1}t_i. \end{aligned}$$

for any $i, j \in \mathbb{Z}$ with $j > 0$. Suppose that $x \in \widetilde{C}_n$ and $i \in \mathbb{Z}$ satisfy $(i)x - 2n > (j)x$ for any $i < j \leqslant i + a$ with some $a \in [2n-1]$. Let $x' = t_{i,a}x$. Then $\ell(x') = \ell(x) - a$ and $\psi(x) = \psi(x')$. Moreover, if $(i)x - 2n > (j)x$ for any $i < j < i + 2n$, let $x'' = t_{i,2n}x$, then

$$(k)x'' = \begin{cases} (k)x - 2n, & \text{if } k \equiv i \pmod{2n},\\ (k)x + 2n, & \text{if } k \equiv 2n - i \pmod{2n},\\ (k)x, & \text{if otherwise.} \end{cases}$$

for any $k \in \mathbb{Z}$, where x'' satisfies $\ell(x'') = \ell(x) - 2n$ and $\psi(x) = \psi(x'')$.

Fix $w \in \widetilde{C}_n$. Suppose that $E_1 = \{i_1, i_2, ..., i_a\}$ and $E_2 = \{j_1, j_2, ..., j_b\}$ are two subsets of $[2n]$ satisfying that

(i) $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$ with $a > 0$ and $b \geqslant 0$ and $a + b = n$;

(ii) the elements of $E_1 \cup E_2$ are pairwise not $2n$-dual;

(iii) $(\bar{k})w < (k)w$ for any $k \in E_1 \cup E_2$;

(iv) If $b > 0$ then $(i)w - (j)w > 2ln$ for any $i \in E_1$ and $j \in E_2$; if $b = 0$ then $(i)w > (2l+1)n$ for any $i \in E_1$, where l is some positive integer.

By repeatedly left multiplying various elements of the form $t_{i,j}$ on w, we can obtain some $w' \in \widetilde{C}_n$ such that there are some $1 \leqslant k_1 < k_2 < \cdots < k_b \leqslant 2b$ (the latter is an empty condition if $b = 0$) satisfying that

(1) $\ell(w') = \ell(w) - \ell(ww'^{-1})$;

(2) If $b > 0$ then $[2b] = \{k_1, k_2, ..., k_b, 2b+1-k_1, 2b+1-k_2, ..., 2b+1-k_b\}$ and the map $\phi : \{j_1, j_2, ..., j_b, \bar{j}_1, \bar{j}_2, ..., \bar{j}_b\} \longrightarrow [2b]$ given by $\phi(j_m) = k_m$ and $\phi(\bar{j}_m) = 2b + 1 - k_m$ for $m \in [b]$ is an order-preserving bijection.

(3) $(p)w' = (i_p)w - 2l'n$ and $(a + k_q)w' = (j_q)w$ for any $p \in [a]$ and $q \in [b]$, where $l' \in \mathbb{Z}$ and $l' \geqslant l$;

(4) $(\overline{\langle c\rangle})w' < (\langle c\rangle)w'$ for any $c \in [a] \cup \{a + k_m \mid m \in [b]\}$;

(5) If $b > 0$ then $0 < \min\{(c)w' - (a + k_m)w' \mid c \in [a], m \in [b]\} < 2n$; if $b = 0$ then $n < \min\{(c)w' \mid c \in [n]\} \leqslant 3n$.

We see by Lemma 3.2 that $\psi(w') = \psi(w)$ (denoted by λ) and by Corollary 2.17 that w, w' are in the same lcc of E_λ.

Example 3.4. (a) Let $w = [8, 30, 4, -11, 27, 2] \in \widetilde{C}_6$. Then $E_1 = \{2, 5, 9\}$ and $E_2 = \{1, 7, 10\}$ satisfy 3.3 (i)-(iv) with $n = 6$ and $(a, b, l) = (3, 3, 1)$. Let $w' = t_{4,9}t_{5,8}t_{9,4}w$. Then $w' = [18, 15, 12, 8, 4, 2] \in \widetilde{C}_6$. Hence w' satisfies 3.3 (1)-(5) with $b > 0$ and $\psi(w') = \psi(w) = \mathbf{93}$.

(b) Let $w = [20, 30, -8, -11, 27, -10] \in \widetilde{C}_6$. Then $E_1 = \{1, 2, 5, 7, 9, 10\}$ and $E_2 = \emptyset$ satisfy 3.3 (i)-(iv) with $n = 6$ and $(a, b, l) = (6, 0, 1)$. Let $w' = t_{6,7}t_{6,7}t_{6,7}t_{7,6}t_{9,4}t_{10,3}w$. Then $w' = [8, 18, 15, 11, 12, 9] \in \widetilde{C}_6$. Hence w' satisfies 3.3 (1)-(5) with $b = 0$ and $\psi(w') = \psi(w) = \mathbf{82^2}$.

3.5. By a *composition* of $2n$, we mean an r-tuple $(a_1, a_2, ..., a_r)$ of positive integers $a_1, ..., a_r$ with some $r \geqslant 1$ such that $\sum_{i=1}^r a_i = 2n$. Let $\widetilde{\Lambda}_{2n}$ be the set of all compositions of $2n$. Clearly, $\Lambda_{2n} \subseteq \widetilde{\Lambda}_{2n}$.

A generalized tabloid of rank $2n$ is, by definition, an r-tuple $\mathbf{T} = (T_1, T_2, ..., T_r)$ with some $r \in \mathbb{N}$ such that $[2n]$ is a disjoint union of some non-empty subsets T_j, $j \in [r]$. We have $\xi(\mathbf{T}) := (|T_1|, |T_2|, ..., |T_r|) \in \widetilde{\Lambda}_{2n}$, where $|T_i|$ denotes the cardinal of the set T_i. Let $i_1, i_2, ..., i_r$ be a permutation of $1, 2, ..., r$ such that $|T_{i_1}| \geqslant |T_{i_2}| \geqslant \cdots \geqslant |T_{i_r}|$. Then $\zeta(\mathbf{T}) := (|T_{i_1}|, |T_{i_2}|, ..., |T_{i_r}|) \in \Lambda_{2n}$. Let $\mathcal{C}_{2n}$ be the set of all generalized tabloids of rank $2n$. Then both $\xi : \mathcal{C}_{2n} \longrightarrow \widetilde{\Lambda}_{2n}$ and $\zeta : \mathcal{C}_{2n} \longrightarrow \Lambda_{2n}$ are surjective maps.

Let Ω be the set of all $w \in \widetilde{A}_{2n-1}$ such that there is some $\mathbf{T} = (T_1, T_2, ..., T_r) \in \mathcal{C}_{2n}$ satisfying:

(i) If $i < j$ in $[r]$ then $\langle (a)w^{-1} \rangle \prec_w \langle (b)w^{-1} \rangle$ for any $a \in T_i$ and $b \in T_j$;

(ii) $\langle (a)w^{-1} \rangle$, $\langle (b)w^{-1} \rangle$ are w-uncomparable if $a \neq b$ in T_i, $i \in [r]$.

Clearly, $\mathbf{T}$ is determined entirely by $w \in \Omega$, denote $\mathbf{T}$ by $T(w)$. The map $T : \Omega \longrightarrow \mathcal{C}_{2n}$ is surjective by [8, Proposition 19.1.2]. By a result of Greene in [2], $\zeta(T(w))$ is the dual partition of $\psi(w)$.

The following known result will be crucial in subsequent discussion.

Lemma 3.6. (see [8, Lemma 19.4.6]) *Let* $y, w \in \widetilde{A}_{2n-1}$ *be in* Ω *with* $\xi(T(y)) = \xi(T(w))$. *Then* $y \underset{L}{\sim} w$ *if and only if* $T(y) = T(w)$.

§4. The set $E_{\mathbf{k1^{2n-k}}}$.

Fix $\lambda \in \Lambda_{2n}$. Recall the set E_λ defined in 2.15. We have $E_\lambda^{-1} = E_\lambda$. The group automorphism η of $\widetilde{C}_n$ (see 2.7) stabilizes each E_λ.

In the present section, we shall describe all the cells of $\widetilde{C}_n$ in the set $E_{\mathbf{k1^{2n-k}}}$ for all $k \in [2n]$. $E_{\mathbf{1^{2n}}}$ consists of the identity element of $\widetilde{C}_n$. In the subsequent discussion, we shall always assume $k > 1$.

4.1. First assume $k = 2m+1 \in [2n]$ odd. Let $l = n - m$. By Lemma 3.2, $w \in \widetilde{C}_n$ is in $E_{\mathbf{k1^{2n-k}}}$ if and only if w satisfies the condition (4.1.1) below.

(4.1.1) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m$ in $[2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$ and with either $\bar{i}_1, i_1 \prec_w j_1$ or $\bar{i}_l, i_l \prec_w j_1$.

Let F_1^o (respectively, F_2^o) be the set of all $w \in \widetilde{C}_n$ satisfying the condition (4.1.2) below.

(4.1.2) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m$ in $[2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $0 < (j_{a+1})w - (j_a)w < 2n$ for any $a \in [m-1]$; (iii) $(i_1)w < (j_1)w < (\bar{i}_l)w + 2n$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$ (respectively, $(\bar{i}_l)w + 2n < (j_1)w < (i_1)w + 2n$ and $(i_1, i_2, ..., i_{l-1}, j_m, j_{m-1}, ..., j_1, i_l) = (n+1, n+2, ..., 2n)$).

4.2. Next assume $k = 2m \in [2n]$ even. Let $l = n - m$. By Lemma 3.2, $w \in \widetilde{C}_n$ is in $E_{\mathbf{k1^{2n-k}}}$ if and only if w satisfies the condition (4.2.1) below.

(4.2.1) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m$ in $[2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$; (iii) j_1 is w-uncomparable with $i_a, \bar{i}_a$ for all $a \in [l]$.

If $m = n$ then (4.2.1) (iii) is an empty condition. Now assume $m < n$.

Under the assumption of (4.2.1) (i)-(ii), the condition (4.2.1) (iii) is equivalent to that either $\bar{i}_1 < j_1 < i_1$ and $(\bar{i}_1)w < (j_1)w < (i_1)w$, or $i_l < j_1 < \bar{i}_l + 2n$ and $(i_l)w < (j_1)w < (\bar{i}_l)w + 2n$. Since j_1 is a w-wild head, this is also equivalent to that either $\bar{i}_1 < j_1 \leqslant n$ and $n < (j_1)w < (i_1)w$, or $i_l < j_1 \leqslant 2n$ and $2n < (j_1)w < (\bar{i}_l)w + 2n$. Let $E'_{\mathbf{k1^{2n-k}}}$ (respectively, $E''_{\mathbf{k1^{2n-k}}}$) be the set of all $w \in E_{\mathbf{k1^{2n-k}}}$ such that $\bar{i}_1 < j_1 \leqslant n$ and $n < (j_1)w < (i_1)w$ (respectively, $i_l < j_1 \leqslant 2n$ and $2n < (j_1)w < (\bar{i}_l)w + 2n$). Then $E_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}} \dot{\cup} E''_{\mathbf{k1^{2n-k}}}$ (disjoint union).

Let F_1^e (respectively, F_2^e) be the set of all $w \in \widetilde{C}_n$ satisfying the condition (4.2.2) below.

(4.2.2) There exist some pairwise not $2n$-dual $i_1, i_2, ..., i_l, j_1, j_2, ..., j_m$ in $[2n]$ such that (i) $i_1, i_2, ..., i_l$ are all w-tame heads with $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (ii) $j_1, j_2, ..., j_m$ are all w-wild heads with $0 < (j_{a+1})w - (j_a)w < 2n$ for any $a \in [m-1]$; (iii) $n < (j_1)w < (i_1)w$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ (respectively, $2n < (j_1)w < (\bar{i}_l)w + 2n$ and $(i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1) = (n+1, n+2, ..., 2n)$).

When $m = n$, the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ (respectively, F_1^e and F_2^e) can also be defined by the condition (4.2.1) (respectively, (4.2.2)) if we stipulate $i_1 = (i_1)w = 2n+1$, $\bar{i}_1 = (\bar{i}_1)w = 0$, $i_l = (i_l)w = n$ and $\bar{i}_l = (\bar{i}_l)w = n+1$.

Clearly, $F_1^e \subset E'_{\mathbf{k1^{2n-k}}}$ and $F_2^e \subset E''_{\mathbf{k1^{2n-k}}}$.

Lemma 4.3. *$F_1^\epsilon \cup F_2^\epsilon \subset E_{\mathbf{k1^{2n-k}}}$, where ϵ is o if k is odd and e if k is even. For any $w \in E_{\mathbf{k1^{2n-k}}}$, there exists some $w' \in F_1^\epsilon \cup F_2^\epsilon$ such that w', w are in the same lcc of $E_{\mathbf{k1^{2n-k}}}$.*

Proof. It is a direct consequence of 3.3 and 4.1-4.2. □

Lemma 4.4. *Let ϵ be given as in Lemma 4.3.*

(1) The map η (see 2.7) interchanges the sets F_1^ϵ and F_2^ϵ.

(2) If $k \in [2n-2]$ is even, then $E'^{-1}_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}}$ and $E''^{-1}_{\mathbf{k1^{2n-k}}} = E''_{\mathbf{k1^{2n-k}}}$. The map η interchanges the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$.

(3) Each of F_1^ϵ and F_2^ϵ is contained in an rcc of $E_{\mathbf{k1^{2n-k}}}$.

Proof. (1)-(2) follow by 4.1-4.2 and Proposition 2.8 (3). For (3), we need only to show that F_1^ϵ is contained in an rcc of $E_{\mathbf{k1^{2n-k}}}$.

(I) First assume ϵ being o. For $J = \{t_l, t_{l+1}, ..., t_n\}$, $w^{(o)} := t_n w_J = [1, 2, ..., l-1, 2n+1-l, 2n-l, ..., n+2, n]$ is the unique shortest element in F_1^o. Take any $w \in F_1^o$. Keep the notation in (4.1.2).

(a) First assume $(j_a)w - (j_{a-1})w = 1$ for any $a \in [2, m]$. Since $(i_1)w < (j_1)w < (\bar{i}_l)w + 2n$, there exists the largest $b \in [l]$ with $(i_b)w < (j_1)w$. If $b = l$ then $w \in \{w_1, w_2\}$ with $w_1 = [n+1-l, n+2-l, ..., n-1, 2n, 2n-1, ..., n+l+1, n]$ and $w_2 = [n+1-l, n+2-l, ..., n-1, 3n-l, 3n-l-1, ..., 2n+1, n]$. Let $J = \{t_1, t_2, ..., t_{n-2}\}$, $J_1 = J \setminus \{t_{n-l}\}$, $I_1 = \{t_1, t_2, ..., t_{n-l-1}\}$ and $I = I_1 \cup \{t_0\}$. Then $w_2 = w_1 w_{I_1} w_I$, $w_1 = w^{(o)} w_J w_{J_1}$ satisfy $\ell(w_2) = \ell(w_1) + \ell(w_{I_1} w_I)$, $\ell(w_1) = \ell(w^{(o)}) + \ell(w_J w_{J_1})$ by Proposition 2.8 (2). Hence w_1, w_2, $w^{(o)}$ are in the same rcc of $E_{\mathbf{k1^{2n-k}}}$ by Corollary 2.17. Now assume $b < l$. Since $(i_1)w < (j_1)w < (\bar{i}_l)w + 2n$, we have $w = [1, 2, ..., l-b, n+1-b, n+2-b, ..., n-1, 2n+b-l, 2n+b-l-1, ..., n+b+1, n]$. Let $J = \{t_{l+1-b}, t_{l+2-b}, ..., t_{n-2}\}$, $J_1 = J \setminus \{t_{n-b}\}$. Then $w = w^{(o)} w_J w_{J_1}$ and $\ell(w) = \ell(w^{(o)}) + \ell(w_J w_{J_1})$ by Proposition 2.8 (2). So w, $w^{(o)}$ are in the same rcc of $E_{\mathbf{k1^{2n-k}}}$ by Corollary 2.17.

(b) Next assume $(j_a)w - (j_{a-1})w > 1$ for some $a \in [2, m]$. Take a the largest with such a property. Then $d := (j_a)w - 1 \equiv (k)w \pmod{2n}$ for some $k \in \{j_b, \bar{j}_b, i_c, \bar{i}_c, \bar{j}_a \mid b \in [a-1], c \in [l]\}$. When $d \not\equiv (\bar{j}_a)w \pmod{2n}$, let $y_1 = w t_d t_{d+1} \cdots t_{m+d-a}$. Then for any $t \in \mathbb{Z}$, we have

$$(t)y_1 = \begin{cases} (t)w - 1, & \text{if } t \equiv (j_h)w \pmod{2n} \quad \text{for some } h \in [a, m], \\ (t)w + 1, & \text{if } t \equiv (\bar{j}_h)w \pmod{2n} \quad \text{for some } h \in [a, m], \\ (t)w + (m+1-a), & \text{if } t \equiv (k)w \pmod{2n}, \\ (t)w - (m+1-a), & \text{if } t \equiv (\bar{k})w \pmod{2n}, \\ (t)w, & \text{if otherwise.} \end{cases}$$

We see that either $(j_h)w - (k)w > 2n$ for all $h \in [a, m]$, or $k = i_f$ with $(j_a)w = (i_f)w + 1$ for some $f \in [l]$ (hence $j_h < k$ for any $h \in [a, m]$ in the latter case) by the condition (4.1.2) on $w \in F_1^o$ and by the choice of a. We see by Corollary 2.6 and Proposition 2.8 (2) that $\ell(y_1) = \ell(w) - (m+1-a)$ and $y_1 \in F_1^o$. When $d \equiv (\bar{j}_a)w \pmod{2n}$, we have $d \equiv n, 0 \pmod{2n}$ and $(j_a)w - (\bar{j}_a)w > 2n$. In this case, let $y_1 = ww_{J_1}w_J$ with $J = \{t_n, t_{n-1}, ..., t_{n+a-m}\}$ and $J_1 = J \setminus \{t_n\}$ if $d \equiv n \pmod{2n}$ and $J = \{t_0, t_1, ..., t_{m-a}\}$ and $J_1 = J \setminus \{t_0\}$ if $d \equiv 0 \pmod{2n}$. By Corollary 2.6 and Proposition 2.8 (2), we have $\ell(y_1) = \ell(w) - (w_{J_1}w_J)$ and $y_1 \in F_1^o$. By induction on $p := \ell(w) \geqslant \ell(w^{(o)})$, we see that there exists a sequence $y_0 = w, y_1, ..., y_r$ in F_1^o with some $r \geqslant 0$ such that $\ell(y_h) = \ell(y_{h-1}) - \ell(y_{h-1}^{-1}y_h)$ and $(j_a)y_r - (j_{a-1})y_r = 1$ for any $h \in [r]$ and $a \in [2, m]$. This implies by Corollary 2.17 that $y_0, y_1, ..., y_r$ are in the same rcc of $E_{\mathbf{k1^{2n-k}}}$. Since y_r and $w^{(o)}$ are in the same rcc of $E_{\mathbf{k1^{2n-k}}}$ by (a), F_1^o is contained in an rcc of $E_{\mathbf{k1^{2n-k}}}$.

(II) Next assume ϵ being e. Then $w^{(e)} := w_J = [1, 2, ..., l, 2n-l, 2n-l-1, ..., n+1]$ is the unique shortest element in F_1^e with $J = \{t_{l+1}, t_{l+2}, ..., t_n\}$. Take any $w \in F_1^e \setminus \{w^{(e)}\}$. There exists some $a \in [2, m]$ with $(j_a)w - (j_{a-1})w > 1$. Take a the largest with such a property. By the same argument as that in (I) (b), we can find $y_0 = w, y_1, ..., y_r = w^{(e)}$ in F_1^e with some $r \geqslant 0$ such that $\ell(y_h) = \ell(y_{h-1}) - \ell(y_{h-1}^{-1}y_h)$ for every $h \in [r]$. By Corollary 2.17, we see that F_1^e is contained in an rcc of $E_{\mathbf{k1^{2n-k}}}$. □

Lemma 4.5. *For $k \in [2, 2n]$, $\epsilon \in \{o, e\}$ and $i = 1, 2$, let $F_i^\epsilon \subset E_{\mathbf{k1^{2n-k}}}$ be defined as in 4.1-4.2. Then $|F_1^\epsilon| = |F_2^\epsilon| = 2^{\lfloor \frac{k}{2} \rfloor - 1} n!/(n - \lfloor \frac{k-1}{2} \rfloor)!$.*

Proof. We have $|F_1^\epsilon| = |F_2^\epsilon|$ by Lemma 4.4 (1).

First we enumerate the set $F^o := F_1^o \cup F_2^o$. Let G^o be the set of all $w \in \widetilde{C}_n$ satisfying (4.1.2) but with (iii) replaced by (iii)′ below:

(iii)′ $(i_1)w < (j_1)w < (i_1)w + 2n$ and $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$.

Then $F_1^o \subset G^o$. There exists a bijection $\lambda_x : G^o \setminus F_1^o \longrightarrow F_2^o$ given by $\lambda_x(w) = xw$, where, when $l > 1$, let $x = w_I w_{I_1} w_J w_{J_1} t_n$ with $J = \{t_l, t_{l+1}, ..., t_n\}$, $J_1 = J \setminus \{t_{n-1}, t_n\}$, $I = \{t_2, t_3, ..., t_{n-2}\}$ and $I_1 = I \setminus \{t_{l-1}\}$;

when $l=1$, let $x=w_{J_1}w_Jd_{n-1,n-1}$ (see (3.3.1)) with $J=\{t_2,t_3,...,t_n\}$ and $J_1=J\setminus\{t_n\}$.

Note that in either case, we have $\ell(xw)=\ell(w)-\ell(x)$ and $\psi(w)=\psi(xw)$, hence $xw \underset{L}{\sim} w$ by Corollary 2.17 and Lemma 2.16.

Now we enumerate the set G^o. Any $w\in G^o$ is determined entirely by the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$, $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ under the conditions (4.1.2) (i)-(ii) and (iii)$'$. There are $\binom{n}{l}=\dfrac{n!}{l!(n-l)!}$ different choices for the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$ by the condition $n<(i_1)w<(i_2)w<\cdots<(i_l)w\leqslant 2n$. Once they are fixed, the numbers of different choices for $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ are $2m, 2(m-1),...,2$ in turn by the conditions (4.1.2) (i)-(ii), (iii)$'$ and the facts that $m+l=n$ and $b\not\equiv c,\bar{c}\,(\text{mod } 2n)$ for any $b\neq c$ in $\{(i_1)w,...,(i_l)w,(j_1)w,...,(j_m)w\}$. So $|G^o|=\binom{n}{l}2^m m!$. The assertion is proved for ϵ being o by the facts $|F_1^o|=|F_2^o|=\frac{1}{2}|G^o|$ and $m+l=n$.

Next we compute $|F_1^e|$. Any $w\in F_1^e$ is determined entirely by the integers $(i_1)w$, $(i_2)w$, ..., $(i_l)w$, $(j_1)w$, $(j_2)w$, ..., $(j_m)w$ under the condition (4.2.2). There are $\binom{n}{l+1}$ different choices for the integers $(j_1)w$, $(i_1)w$, $(i_2)w$, ..., $(i_l)w$ by the condition $n<(j_1)w<(i_1)w<(i_2)w<\cdots<(i_l)w\leqslant 2n$. Once they are fixed, the numbers of different choices for $(j_2)w$, $(j_3)w$, ..., $(j_m)w$ are $2(m-1),2(m-2),...,2$ in turn by the condition (4.2.2) and the facts that $m+l=n$ and $b\not\equiv c,\bar{c}\,(\text{mod } 2n)$ for any $b\neq c$ in $\{(i_1)w,...,(i_l)w,(j_1)w,...,(j_m)w\}$. So $|F_1^e|=\binom{n}{l+1}2^{m-1}(m-1)!$. The assertion is proved for ϵ being e by the fact $m+l=n$. $\square$

Lemma 4.6. *No two elements of* $F_1^\epsilon\cup F_2^\epsilon$ *are in the same left cell of* $\widetilde{C}_n$.

Proof. Let $w\in F_1^o$ be as in (4.1.2). If $l=1$ then let $w'=w$; if $l>1$ then let $w'=w_{J_6}w_{J_7}t_nw_{J_1}w_{J_5}w_{J_3}w_{J_4}w_{J_1}w_{J_2}w$, where $J_1=\{t_1,t_2,...,t_{n-2}\}$, $J_2=J_1\setminus\{t_{l-1}\}$, $J_3=\{t_0,t_1,...,t_{m-1}\}$, $J_4=J_3\setminus\{t_0\}$, $J_5=J_1\setminus\{t_m\}$, $J_6=\{t_l,t_{l+1},...,t_n\}$ and $J_7=J_6\setminus\{t_n,t_{n-1}\}$ (see Figure 1). Regarding w' as an element of $\widetilde{A}_{2n-1}$, we have $w'\in\Omega$ (see 3.5), which satisfies $\psi(w)=\psi(w')$ and $\ell(w')=\ell(w)+\ell(w'w^{-1})$, hence $w \underset{L}{\sim} w'$ by Corollary 2.17.

Let $\alpha = (1, ..., 1, 2l, 1, ..., 1) \in \widetilde{\Lambda}_{2n}$ with $2l$ the $(m+1)$-th component. We see that $T(w') = (T_1, T_2, ..., T_{2m+1}) \in \xi^{-1}(\alpha)$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$ (see 2.7 for the notation $\langle q\rangle$).

Similarly, for $w \in F_2^o$ as in (4.1.2), we can find some $w' \in \widetilde{C}_n$ satisfying $w \underset{L}{\sim} w'$ and $w' \in \Omega$ as an element of $\widetilde{A}_{2n-1}$. We again get $T(w') = (T_1, T_2, ..., T_{2m+1}) \in \xi^{-1}(\alpha)$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$.

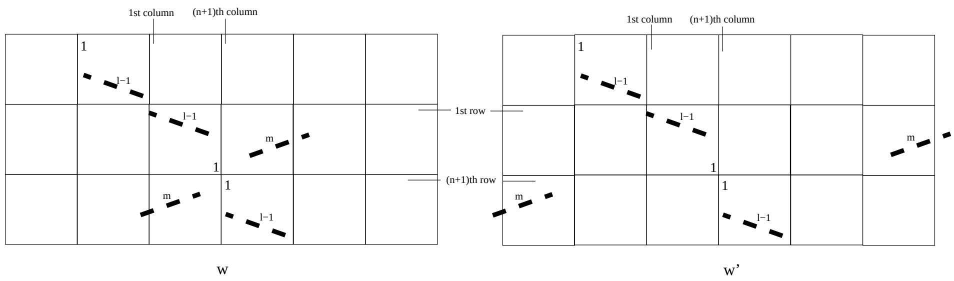
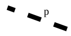
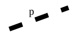


Figure 1

Figure 1 displays the corresponding parts for the matrix forms of w and w' if $l > 1$, where the symbol  (respectively, ) stands for a rectangular submatrix A with p rows each row has a unique non-zero entry 1, the entries 1 of A are going down to the right (respectively, to the left).

We see that the above $T(w')$ with $w' \underset{L}{\sim} w$ and $w' \in \Omega$ depends only on $w \in F_1^o \cup F_2^o$ and α but not on the choice of w' in Ω. So we can denote $T(w')$ by $T_\alpha(w)$. We claim that $T_\alpha(w)$ should be pairwise different in $\xi^{-1}(\alpha)$ as w ranges over $F_1^o \cup F_2^o$. For, recall that in the proof of Lemma 4.5, there is a bijective map τ from G^o to $F_1^o \cup F_2^o$ which satisfies $w \underset{L}{\sim} \tau(w)$ for any $w \in G^o$. We see that $T_\alpha(w) = (T_1, T_2, ..., T_{2m+1})$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle(\bar{i}_a)w\rangle, \langle(i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle(j_{d-m-1})w\rangle\}$ for $d \in [m+2, 2m+1]$ should be pairwise different as w ranges over G^o. This proves our assertion by Lemma 3.6 when ϵ is o.

If $m = n$, then $F_1^e \cup F_2^e \subseteq \Omega$. The set $\{T(w) \mid w \in F_1^e \cup F_2^e\}$ is equal to $\{(\{a_1\}, ..., \{a_{2n}\}) \mid \{a_1, ..., a_{2n}\} = [2n]; \overline{a_i} = a_{2n+1-i}, \forall\ i \in [n]\}$. So our result in this case follows by Lemmas 3.6 and 2.13. Now assume $m < n$. Let $\beta = (1, ..., 1, 2l+1, 1, ..., 1) \in \widetilde{\Lambda}_{2n}$ with $2l+1$ the $(m+1)$th component. Let $w \in F_1^e$ be as in (4.2.2). When $m = 1$, let $w' = w_J w_{J_1} w$ with $J = \widetilde{S} \setminus \{s_n\}$ and $J_1 = J \setminus \{s_{n+1}\}$; when $m > 1$, let $w' = w_{I_3} w_{I_4} w_{I_1} w_{I_2} w$ with $I_1 = \widetilde{S} \setminus \{s_{n-1}, s_n\}$, $I_2 = I_1 \setminus \{s_l\}$, $I_3 = \widetilde{S} \setminus \{s_{n+m-1}\}$ and $I_4 = I_3 \setminus \{s_{n+2m-1}\}$ (see Figure 2 if $m > 1$). Then w' is in $\widetilde{A}_{2n-1}$, but not in $\widetilde{C}_n$. We have $w' \in \Omega$, which satisfies $\psi(w) = \psi(w')$ and $\widetilde{\ell}(w') = \widetilde{\ell}(w) + \widetilde{\ell}(w'w^{-1})$, hence $w \underset{L}{\sim} w'$ by Corollary 2.17.

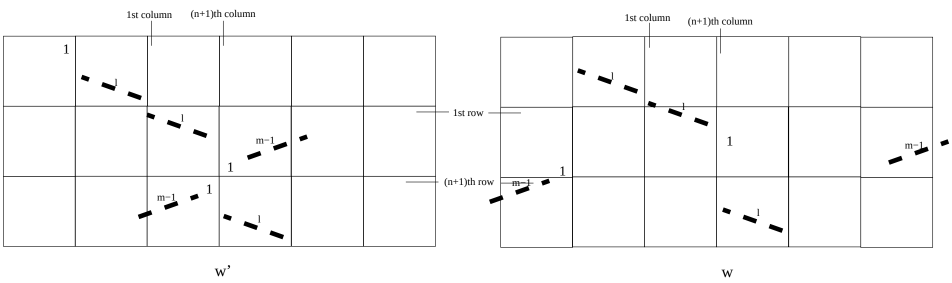


Figure 2

We have $T(w') = (T_1, T_2, ..., T_{2m}) \in \xi^{-1}(\beta)$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle (j_1)w\rangle, \langle (\bar{i}_a)w\rangle, \langle (i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle (j_{d-m})w\rangle\}$ for $d \in [m+2, 2m]$.

Similarly, for $w \in F_2^e$ as in (4.2.2), we can find some $w' \in \widetilde{A}_{2n-1}$ satisfying $w \underset{L}{\sim} w'$ and $w' \in \Omega$. We again get $T(w') = (T_1, T_2, ..., T_{2m})$ with $T_c = (\langle \bar{j}_{m+1-c})w\rangle$ for $c \in [m]$, $T_{m+1} = \{\langle (j_1)w\rangle, \langle (\bar{i}_a)w\rangle, \langle (i_a)w\rangle \mid a \in [l]\}$ and $T_d = \{\langle (j_{d-m})w\rangle\}$ for $d \in [m+2, 2m]$.

Again, the above $T(w')$ with $w' \underset{L}{\sim} w$ and $w' \in \Omega$ depends only on $w \in F_1^e \cup F_2^e$ and β but not on the choice of w' in Ω. So we can denote $T(w')$ by $T_\beta(w)$. Then $T_\beta(w)$ are pairwise different in $\xi^{-1}(\beta)$ as w ranges over F_1^e (respectively, F_2^e) by the proof of Lemma 4.5. We claim that $T_\beta(w) = (T_1, ..., T_{2m})$ for $w \in F_1^e$ is different from that for $w \in F_2^e$. For, $T_m = \{\langle (\bar{j}_1)w\rangle\}$ satisfies $\langle (\bar{j}_1)w\rangle \leqslant n$ if $w \in F_1^e$ and $\langle (\bar{j}_1)w\rangle > \langle (i_l)w\rangle > n$ if $w \in F_2^e$. The claim is proved. So our assertion follows by Lemmas 3.6 and 2.13 when ϵ is e. $\square$

Lemma 4.7. *The set $E_{\mathbf{k1^{2n-k}}}$ forms a single two-sided cell of $\widetilde{C}_n$ if either $k \in [2n]$ is odd or $k = 2n$. In particular, $E_{\mathbf{2n}}$ is the lowest two-sided cell of $\widetilde{C}_n$ under the relation $\underset{LR}{\leqslant}$.*

Proof. First assume $k = 2m+1 \in [2n]$ odd. Let $w^{(o)} = t_n w_J$ and $y^{(o)} = t_0 w_I$ with $J = \{t_l, t_{l+1}, ..., t_n\}$ and $I = \{t_0, t_1, ..., t_{n-l}\}$, where $l = n - m$. Then $y^{(o)} = \eta(w^{(o)})$. By (4.1.1)-(4.1.2) and the proof of Lemma 4.4, we see that any $w \in E_{\mathbf{k1^{2n-k}}}$ is in a tcc of $E_{\mathbf{k1^{2n-k}}}$ containing either $w^{(o)}$ or $y^{(o)}$. Thus by Lemma 2.16, in order to show our result, we need only to show that $w^{(o)}$ and $y^{(o)}$ are contained in the same tcc of $E_{\mathbf{k1^{2n-k}}}$.

When $l = 1$, let $I_1 = S \setminus \{t_{n-1}, t_n\}$, $I_2 = I_1 \setminus \{t_0\}$, $I_3 = S \setminus \{t_0, t_1\}$ and $I_4 = I_3 \setminus \{t_n\}$ and let $y_0 = w^{(o)}$, $y_1 = w_{I_2} w_{I_1} y_0$, $y_2 = d_{n-1,n-1} y_1$, $y_3 = y_2 t_{1,n-1}$ and $y_4 = y_3 w_{I_4} w_{I_3}$; when $l > 1$, let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \setminus \{t_{l-1}\}$, $J_3 = \{t_0, t_1, ..., t_m\}$, $J_4 = J_3 \setminus \{t_0\}$, $J_5 = \{t_l, t_{l+1}, ..., t_n\}$, $J_6 = J_5 \setminus \{t_{n-1}, t_n\}$, $J_7 = \{t_2, t_3, ..., t_{n-2}\}$ and $J_8 = J_7 \setminus \{t_{l-1}\}$. Let $y_0 = w^{(o)}$, $y_1 = w_{J_1} w_{J_2} y_0$, $y_2 = t_0 w_{J_3} w_{J_4} y_1$, $y_3 = y_2 t_n w_{J_6} w_{J_5}$ and $y_4 = y_3 w_{J_8} w_{J_7}$. In either case, we have $y_4 = y^{(o)}$.

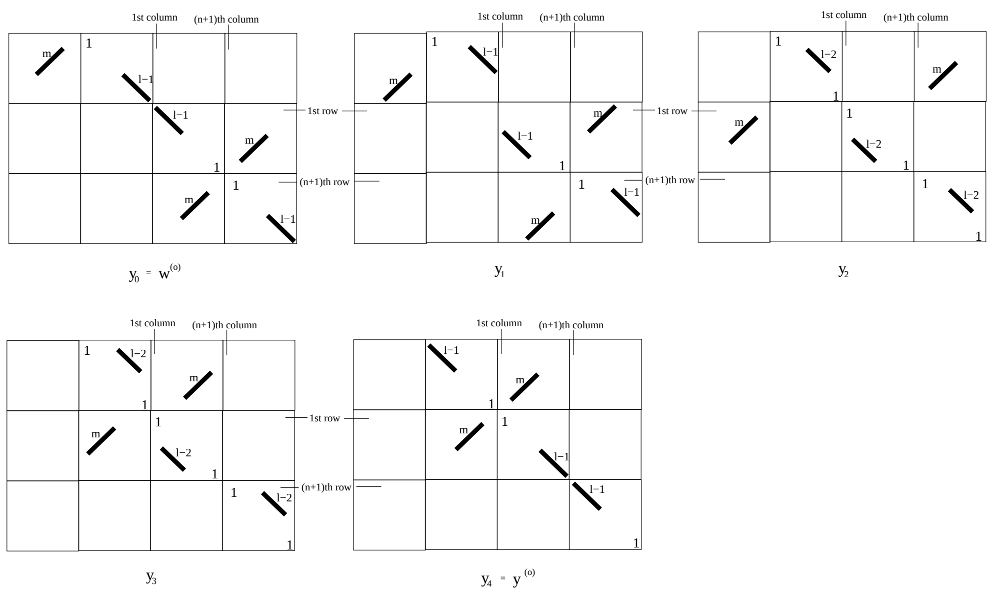


Figure 3

In Figure 3, we display the corresponding parts of the matrix forms of $y_0, ..., y_4$ for $l > 1$, the notation $\diagdown^{l}$ stands for the $l \times l$ identity submatrix, while $\diagup^{m}$ stands for the $m \times m$ anti-diagonal submatrix with all the anti-diagonal entries being 1.

We have $y_i \in E_{\mathbf{k1^{2n-k}}}$ for any $i \in [0, 4]$. Also, $\ell(y_1) = \ell(y_0) + \ell(y_1 y_0^{-1})$, $\ell(y_2) = \ell(y_1) + \ell(y_2 y_1^{-1})$, $\ell(y_3) = \ell(y_2) - \ell(y_2^{-1} y_3)$ and $\ell(y_4) = \ell(y_3) - \ell(y_3^{-1} y_4)$ (see Figure 3 for $l > 1$). This implies by Corollary 2.17 that $w^{(o)}$ and $y^{(o)}$ are contained in the same tcc of $E_{\mathbf{k1^{2n-k}}}$.

Next assume $k = 2n$. By the part (II) in the proof of Lemma 4.4, we see that any $w \in E_{\mathbf{2n}}$ is in the tcc of $E_{\mathbf{2n}}$ containing either w_J or w_I with $J = S \backslash \{t_0\}$ and $I = S \backslash \{t_n\}$. Let $K = S \backslash \{t_0, t_n\}$ and let $y = w_K w_I w_J$. Then $y = w_I w_J w_K \in E_{\mathbf{2n}}$ satisfies $\ell(y) = \ell(w_J) + \ell(w_K w_I) = \ell(w_I) + \ell(w_J w_K)$. So w_J, w_I are contained in the same tcc of $E_{\mathbf{2n}}$ by Corollary 2.17. Hence $E_{\mathbf{2n}}$ is two-sided-connected and forms a two-sided cell of $\widetilde{C}_n$ by Lemma 2.16, which is the lowest one under the relation $\underset{LR}{\leqslant}$ by Lemmas 2.13-2.14. □

In the proof of Lemma 4.7, we actually show that if $k \in [2n]$ is either odd or 2n then the set $E_{\mathbf{k1^{2n-k}}}$ is two-sided-connected. By 3.3, 4.2 and Lemmas 4.3-4.4, we see that if $k = 2m < 2n$ is even then each of the sets $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is contained in some tcc of $E_{\mathbf{k1^{2n-k}}}$. Now we have

Lemma 4.8. *If $k = 2m \in [2n-2]$ is even, then the set $E_{\mathbf{k1^{2n-k}}}$ has two tccs $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$.*

Proof. Keep the notation in (4.2.1) for $w \in E_{\mathbf{k1^{2n-k}}}$. Denote the integers $j_a, \bar{j}_a, i_b, \bar{i}_b$ by $j'_a, \bar{j}'_a, i'_b, \bar{i}'_b$, resp,. $j''_a, \bar{j}''_a, i''_b, \bar{i}''_b$ for $a \in [m]$ and $b \in [l]$, according to w being $w' \in E'_{\mathbf{k1^{2n-k}}}$, resp., $w'' \in E''_{\mathbf{k1^{2n-k}}}$. Observe the following facts: If w'' is obtained from w' by left multiplying some $t \in S$, then $j''_1 = \langle (j'_1)t \rangle$ (see 2.7) and $(j''_1)w'' = (j'_1)w'$. If w'' is obtained from w' by right multiplying some $t \in S$, then $j''_1 = j'_1$ and $(j''_1)w'' = (j'_1)w't$.

We see that $w' \in E'_{\mathbf{k1^{2n-k}}}$ satisfies $\bar{i}'_1 < j'_1 \leqslant n$ and $n < (j'_1)w' < (i'_1)w'$, and that $w'' \in E''_{\mathbf{k1^{2n-k}}}$ satisfies $i''_l < j''_1 \leqslant 2n$ and $2n < (j''_1)w'' < (\bar{i}''_l)w'' + 2n$.

Since $i''_l \geqslant n+1$ and $\bar{i}'_1 \geqslant 1$, we have $j'_1 \in [2, n]$ and $j''_1 \in [n+2, 2n]$, hence $j''_1 \neq \langle (j'_1)t\rangle$ for any $t \in S$. So no element of $E''_{\mathbf{k1^{2n-k}}}$ could be obtained from an element of $E'_{\mathbf{k1^{2n-k}}}$ by left multiplying some $t \in S$. Since $(i'_1)w' \leqslant 2n$, we have $(j'_1)w' \leqslant 2n-1$ and $2n+1 \leqslant (j''_1)w''$, hence $(j''_1)w'' \neq (j'_1)w't$ for any $t \in S$. So no element of $E''_{\mathbf{k1^{2n-k}}}$ could be obtained from an element of $E'_{\mathbf{k1^{2n-k}}}$ by right multiplying some $t \in S$. So $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$ form two different tccs of $E_{\mathbf{k1^{2n-k}}}$ by the fact $E_{\mathbf{k1^{2n-k}}} = E'_{\mathbf{k1^{2n-k}}} \dot{\cup} E''_{\mathbf{k1^{2n-k}}}$. □

Theorem 4.9. *(1) If $k = 2m+1 \in [2n]$ is odd, then $E_{\mathbf{k1^{2n-k}}}$ is a two-sided cell of $\widetilde{C}_n$ containing $2^m n!/(n-m)!$ left cells.*

(2) $E_{\mathbf{2n}}$ is the lowest two-sided cell of $\widetilde{C}_n$ consists of $2^n n!$ left cells.

(3) If $k = 2m \in [2n-2]$ is even, then $E_{\mathbf{k1^{2n-k}}}$ is a union of two two-sided cells $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$ of $\widetilde{C}_n$, each of $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$ contains $2^{m-1}n!/(n-m+1)!$ left cells. The group automorphism η interchanges $E'_{\mathbf{k1^{2n-k}}}$, $E''_{\mathbf{k1^{2n-k}}}$.

(4) Each left (respectively, two-sided) cell of $\widetilde{C}_n$ in $E_{\mathbf{k1^{2n-k}}}$ is left- (respectively, two-sided-) connected.

(5) The set $E_{\mathbf{k1^{2n-k}}}$ is infinite unless $k = 1, 2$.

Proof. By Lemma 2.16, we see that E_λ is either empty or a union of some two-sided cells of $\widetilde{C}_n$ for any $\lambda \in \Lambda_{2n}$. Hence (1)-(2) follow by Lemmas 4.3 and 4.5-4.7. For (3), we see by Lemmas 4.3-4.4 and 4.6 that each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ contains $2^{m-1}n!/(n-m+1)!$ left cells. By Lemmas 1.4, 4.4 and 2.12, we see that each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is a union of some two-sided cells of $\widetilde{C}_n$. On the other hand, each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ is a tcc of $E_{\mathbf{k1^{2n-k}}}$ by Lemma 4.8, which should be contained in some two-sided cell of $\widetilde{C}_n$ by Lemma 2.16. So each of $E'_{\mathbf{k1^{2n-k}}}$ and $E''_{\mathbf{k1^{2n-k}}}$ forms a single two-sided cell of $\widetilde{C}_n$. The last assertion of (3) follows by Lemma 4.4. This proves (3). (4) follows by (1)-(3) and Lemmas 4.3, 4.6. Finally, $E_{\mathbf{1^{2n}}} = \{1\}$ and $E_{\mathbf{21^{2n-2}}} = \{t_0, t_n\}$. When $k = 2m \geqslant 4$ or $k = 2m+1 \geqslant 3$, the number of the choices for $(j_m)w$ in (4.1.1) or (4.2.1) is infinite. This proves (5). □

§5. The set $E_{(k,2,1,...,1)}$ with $(k,2,1,...,1)\in\Lambda_{2n}$.

In this section, we describe cells of $\widetilde{C}_n$ in the set $E_{(k,2,1,...,1)}$ with $(k,2,1,...,1)\in\Lambda_{2n}$. The main result is as follows.

Theorem 5.1. *Let $\lambda=(2m,2,1,...,)$ and $\mu=(2m+1,2,1,...,1)$ be in Λ_{2n}.*

(1) The set E_λ forms a single two-sided cell of $\widetilde{C}_n$ if $m=n-1$ and is a union of two two-sided cells (say E'_λ and E''_λ) of $\widetilde{C}_n$ if $m<n-1$. The set E_μ is a union of two two-sided cells (say E'_μ and E''_μ) of $\widetilde{C}_n$.

(2) Let $n(\nu)$ be the number of left cells of $\widetilde{C}_n$ in E_ν for $\nu=\lambda,\mu$. Then $n(\lambda)=\frac{2^{m-1}n!(n+3-m)}{(n+1-m)!}$ and $n(\mu)=\frac{2^m\cdot n!}{(n-m)!}$.

Let $n'(\nu)$ and $n''(\nu)$ be the numbers of left cells in E'_ν and E''_ν respectively for $\nu=\lambda,\mu$. Then $\{n'(\lambda),n''(\lambda)\}=\left\{\frac{2^{m-1}n!}{(n+1-m)!},\frac{2^{m-1}n!(n+2-m)}{(n+1-m)!}\right\}$ and $n'(\mu)=n''(\mu)=\frac{2^{m-1}\cdot n!}{(n-m)!}$.

(3) Any left (respectively, two-sided) cell of $\widetilde{C}_n$ in $E_\lambda\cup E_\mu$ is left- (respectively, two-sided-) connected.

(4) $|E_{\mathbf{k21^{2n-k-2}}}|=\infty$ unless $k=2,3$.

We shall prove Theorem 5.1 in the remaining part of the section.

5.2. Let $l=n-m-1$. Then $w\in\widetilde{C}_n$ is in E_λ if and only if one of the conditions (a)-(c) on w holds:

(a) There are some pairwise not $2n$-dual $j_1,j_2,...,j_m,\ k,\ i_1,i_2,...,i_l$ in $[2n]$ with $j_1,j_2,...,j_m,\ k$ w-wild heads and $i_1,i_2,...,i_l$ w-tame heads such that

(a1) $j_1\prec_w j_2\prec_w\cdots\prec_w j_m$;

(a2) $i_1<i_2<\cdots<i_l$ and $(i_1)w<(i_2)w<\cdots<(i_l)w$;

(a3) j_1 (respectively, k) is w-comparable with none of $i_h,\bar{i}_h$ for $h\in[l]$;

(a4) k is w-uncomparable with j_p for some $p\in[m]$.

Both (a2) and (a3) become empty condition if $m=n-1$.

(b) There are some pairwise not $2n$-dual $j_1,j_2,...,j_m,\ i_1,i_2,...,i_l,i_{l+1}$ in $[2n]$ with $j_1,j_2,...,j_m$ w-wild heads and $i_1,i_2,...,i_l,i_{l+1}$ w-tame heads such that

(b1) $j_1\prec_w j_2\prec_w\cdots\prec_w j_m$;

(b2) $i_1<i_2<\cdots<i_l<i_{l+1}$ and $(i_1)w<(i_2)w<\cdots<(i_l)w<(i_{l+1})w$;

(b3) j_1 is w-comparable with at least one of $i_1, \bar{i}_1, i_{l+1}, \bar{i}_{l+1}$, but not with $i_h, \bar{i}_h$ simultaneously for any $h \in \{1, l+1\}$.

(c) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_{m-1}$, $i_1, i_2, ..., i_{l+2}$ in $[2n]$ with $j_1, j_2, ..., j_{m-1}$ w-wild heads and $i_1, i_2, ..., i_{l+2}$ w-tame heads such that

(c1) $\bar{j}_1 \prec_w i_q \prec_w i_p \prec_w j_1 \prec_w j_2 \prec_w \cdots \prec_w j_{m-1}$ for some $p, q \in [l+2]$ with $l+2 \in \{p, q\}$;

(c2) $i_1 < i_2 < \cdots < i_l < i_{l+1}$ and $(i_1)w < (i_2)w < \cdots < (i_l)w < (i_{l+1})w$.

For any $w_1 \in E_\lambda$ satisfying (c), there exists some $w_2 \in E_\lambda$ satisfying (b) such that w_1 and w_2 are in the same lcc of E_λ.

5.3. Let E'_λ be the set of all $w \in E_\lambda$ satisfying 5.2 (a) with one additional requirement that $\bar{k}$ and j_1 are w-uncomparable, that is, at least one of the following two cases occurs:

(a5) $\bar{i}_1 < j_1 < \bar{j}_1 < i_1$ and $(\bar{i}_1)w < (\bar{j}_1)w < (j_1)w < (i_1)w$ and $i_l - 2n < k - 2n < \bar{k} < \bar{i}_l$ and $(i_l)w - 2n < (\bar{k})w < (k)w - 2n < (\bar{i}_l)w$;

(a6) $\bar{i}_1 < k < \bar{k} < i_1$ and $(\bar{i}_1)w < (\bar{k})w < (k)w < (i_1)w$ and $i_l - 2n < j_1 - 2n < \bar{j}_1 < \bar{i}_l$ and $(i_l)w - 2n < (\bar{j}_1)w < (j_1)w - 2n < (\bar{i}_l)w$.

Let $E''_\lambda = E_\lambda \setminus E'_\lambda$.

Lemma 5.4. *$E'^{-1}_\lambda = E'_\lambda$ and $E''^{-1}_\lambda = E''_\lambda$ for $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$ with $m < n-1$.*

Proof. From the matrix forms of elements, we see that if w is in E_λ and satisfies (a1)-(a5) (respectively, (a1)-(a4) and (a6)), then so does w^{-1}. Hence $E'^{-1}_\lambda = E'_\lambda$. We also have $E''^{-1}_\lambda = E''_\lambda$ by the fact $E^{-1}_\lambda = E_\lambda$. $\square$

5.5. Let F'_λ be the set of all $w' \in \widetilde{C}_n$ satisfying the condition (a$'$) below.

(a$'$) Let $j_1, ..., j_m$, k be w'-wild heads and $i_1, ..., i_l$ w'-tame heads such that

(a$'$1) either

(a$'$11) $(\bar{k}, \bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ with $0 < (k)w' - 2n < (\bar{i}_l)w' < (\bar{i}_{l-1})w' < \cdots < (\bar{i}_1)w' < (j_1)w' < (i_1)w'$ if $m < n-1$ and with $0 < (k)w' - 2n \leqslant n < (j_1)w' \leqslant 2n$ if $m = n-1$,

or

(a$'$12) $(\bar{k}, i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1) = (n+1, n+2, ..., 2n)$ with $n < (k)w' < (i_1)w' < (i_2)w' < \cdots < (i_l)w' < (j_1)w' < (\bar{i}_l)w' + 2n$ if $m < n-1$ and with $0 < (j_1)w' - 2n \leqslant n < (k)w' \leqslant 2n$ if $m = n-1$;

(a$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$.

Let F''_λ be the set of all $w' \in \widetilde{C}_n$ satisfying one of (b$'$), (c$'$) below.

(b$'$) Let $j_1, ..., j_m$, k be w'-wild heads and $i_1, ..., i_l$ w'-tame heads such that

(b$'$1) either that $(\bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1, k) = (1, 2, ..., n)$ with $n < (j_1)w' < (k)w' < (i_1)w'$ if $m < n-1$ and with $n < (j_1)w' < (k)w' \leqslant 2n$ if $m = n-1$,

or that $(i_1, i_2, ..., i_l, j_m, j_{m-1}, ..., j_1, k) = (n+1, n+2, ..., 2n)$ with $2n < (j_1)w' < (k)w' < (\bar{i}_l)w' + 2n$ if $m < n-1$ and $2n < (j_1)w' < (k)w' \leqslant 3n$ if $m = n-1$;

(b$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(b$'$3) $(i_1)w' < (i_2)w' < \cdots < (i_l)w'$.

(c$'$) Let $j_1, ..., j_m$ be w'-wild heads and $i_1, ..., i_{l+1}$ w'-tame heads with $m < n-1$ such that

(c$'$1) either

(c$'$11) $(\bar{i}_{l+1}, \bar{i}_l, ..., \bar{i}_1, j_m, j_{m-1}, ..., j_1) = (1, 2, ..., n)$ and $(i_1)w' < (j_1)w' \leqslant 2n$,

or

(c$'$12) $(i_1, i_2, ..., i_{l+1}, j_m, j_{m-1}, ..., j_1) = (n+1, n+2, ..., 2n)$ and $(\bar{i}_{l+1})w' + 2n < (j_1)w' \leqslant 3n$;

(c$'$2) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(c$'$3) $(i_1)w' < (i_2)w' < \cdots < (i_l)w' < (i_{l+1})w'$.

5.6. $F'_\lambda \subseteq E'_\lambda$ and $F''_\lambda \subseteq E''_\lambda$ by 5.2-5.3 and 5.5. Any lcc of E'_λ (respectively, E''_λ) contains some element of F'_λ (respectively, F''_λ) by 3.3 and Lemma 3.2.

Let $\alpha = (1, ..., 1, 2, 2(n-m), 1, ..., 1) \in \widetilde{\Lambda}_{2n}$ with 2 its m-th component. Let $F_\lambda := F'_\lambda \cup F''_\lambda$. By the argument for Lemma 4.6, there is some $y \in \Omega$ with $y \underset{L}{\sim} w'$ and $T(y) \in \xi^{-1}(\alpha)$ for any $w' \in F_\lambda$. Now we describe $T(y)$.

(1) If w' satisfies (a') or (b$'$) in 5.5, then $T(y)$ is equal to

$$\big(\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(\bar{k})w'\rangle\},$$
$$\{\langle(j_1)w'\rangle,\langle(k)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle \mid h\in[l]\},\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}\big),$$

where (i) $0<(k)w'-2n<(\bar{i}_l)w'$ and $n<(j_1)w'<(i_1)w'$
if $(\bar{k},\bar{i}_l,\bar{i}_{l-1},...,\bar{i}_1,j_m,j_{m-1},...,j_1)=(1,2,...,n)$;
(ii) $n<(k)w'<(i_1)w'$ and $2n<(j_1)w'<(\bar{i}_l)w'+2n$
if $(\bar{k},i_1,i_2,...,i_l,j_m,j_{m-1},...,j_1)=(n+1,n+2,...,2n)$;
(iii) $n<(j_1)w'<(k)w'<(i_1)w'$
if $(\bar{i}_l,\bar{i}_{l-1},...,\bar{i}_1,j_m,j_{m-1},...,j_1,k)=(1,2,...,n)$;
(iv) $2n<(j_1)w'<(k)w'<(\bar{i}_l)w'+2n$
if $(i_1,i_2,...,i_l,j_m,j_{m-1},...,j_1,k)=(n+1,n+2,...,2n)$.
Here we stipulate $(\bar{i}_l)w'=n+1$ and $(i_1)w'=2n+1$ if $l=0$.

(2) If w' satisfies 5.5 (c′) with $(\bar{i}_{l+1},...,\bar{i}_1,j_m,...,j_1)=(1,2,...,n)$ and $(i_p)w'<(j_1)w'<(i_{p+1})w'$ for some $p\in[l+1]$ with the convention that $(i_{l+2})w'=2n+1$, then $T(y)$ is equal to

$$(\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(i_p)w'\rangle\},$$
$$\{\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle \mid h\in[l+1]\}\setminus\{\langle(i_p)w'\rangle\},\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}).$$

(3) If w' satisfies 5.5 (c′) with $(i_1,...,i_{l+1},j_m,...,j_1)=(n+1,n+2,...,2n)$ and $(\bar{i}_p)w'+2n<(j_1)w'<(\bar{i}_{p-1})w'+2n$ for some $p\in[l+1]$ with the convention that $(\bar{i}_0)w'=n+1$, then $T(y)$ is equal to

$$(\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_2)w'\rangle\},\{\langle(\bar{j}_1)w'\rangle,\langle(\bar{i}_p)w'\rangle\},$$
$$\{\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle \mid h\in[l+1]\}\setminus\{\langle(\bar{i}_p)w'\rangle\},\{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}).$$

5.7. By Lemma 3.6, we see that $T(y)\in\xi^{-1}(\alpha)$ given in 5.6 only depends on $w'\in F_\lambda$ and α but not on the choice of $y\in\Omega$. We can denote $T(y)$ by $T_\alpha(w')$. This defines a map $T_\alpha : F_\lambda \longrightarrow \xi^{-1}(\alpha)$. By 5.5-5.6, $\mathbf{T}=(T_1,T_2,...,T_{2m})\in\xi^{-1}(\alpha)$ is in the image of the map T_α if and only if $\mathbf{T}$ satisfies the following conditions:

(1) $\overline{T_i} = T_{2m+1-i}$ for any $i \in [2m] \setminus \{m, m+1\}$ (see 3.1).

(2) $T_m \cup T_{m+1} = \{\bar{q}_{n+1-m}, \bar{q}_{n-m}, ..., \bar{q}_1, q_1, ..., q_{n-m}, q_{n+1-m}\}$ for some $\bar{q}_1 < q_1 < q_2 < \cdots < q_{n+1-m}$ in $[2n]$, and

(5.7.1) $\quad T_m \in \{\{\bar{q}_1, q_{n+1-m}\}, \{\bar{q}_1, \bar{q}_2\}, \{q_{n-m}, q_{n+1-m}\}, \{\bar{q}_{i+1}, q_i\} \mid i \in [n-m]\}.$

When the equivalent conditions hold, we have $|T_\alpha^{-1}(\mathbf{T})| = 1$ if $T_m \in \{\{\bar{q}_1, \bar{q}_2\}, \{q_{n-m}, q_{n+1-m}\}\}$ (i.e., 5.5 (b′)) and $|T_\alpha^{-1}(\mathbf{T})| = 2$ if otherwise.

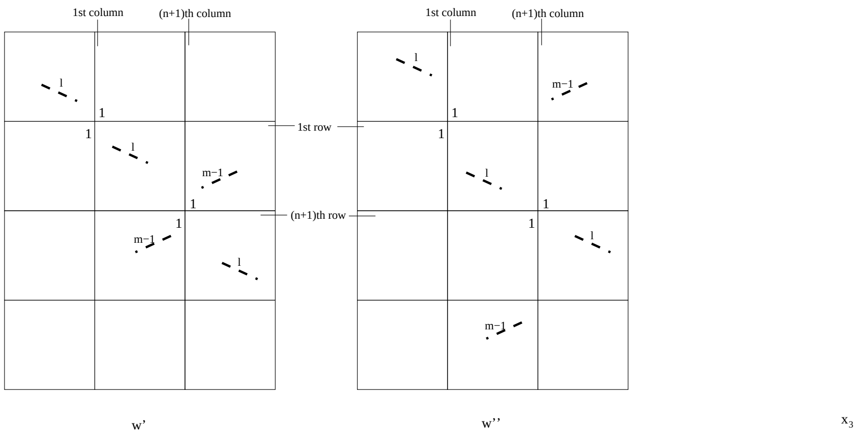


Figure 4

Suppose $m < n-1$. By 5.6, we see that $T_\alpha(w') \neq T_\alpha(w'')$ for any $w' \in F'_\lambda$ and any $w'' \in F''_\lambda$. This implies by 5.6 and Lemmas 3.6, 2.13-2.14 that each of E'_λ and E''_λ is a union of some left cells of $\widetilde{C}_n$.

First consider the case 5.5 (a′). Let F'_i be the set of all $w' \in F'_\lambda$ satisfying (a′1i) and (a′2) for $i = 1, 2$. Then $F'_\lambda = F'_1 \dot{\cup} F'_2$. Use the notation in 5.6 and in (2) above, $w' \in F'_1$ (respectively, $w' \in F'_2$) means that $\langle(\bar{k})w'\rangle$, $\langle(\bar{j}_1)w'\rangle$ (respectively, $\langle(\bar{j}_1)w'\rangle$, $\langle(\bar{k})w'\rangle$) in 5.6 are q_{n+1-m}, $\bar{q}_1$ in (2), respectively. Take any $w' \in F'_1$ with the notation as in 5.5 (a′). Let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \backslash \{t_{l+1}\}$, $J_3 = \{t_0, t_1, ..., t_{m-2}\}$, $J_4 = \{t_1, t_2, ..., t_{m-1}\}$. Let $J'_j = \eta(J_j)$ for $j \in [4]$. If $(j_2)w' < (k)w'$, let $w'' = w_{J_4} w_{J_3} w_{J_1} w_{J_2} w'$, then $w'' \in F'_2$ with $\ell(w'') = \ell(w') + \ell(w_{J_4} w_{J_3} w_{J_1} w_{J_2})$ (see Figure 4). If $(j_2)w' > (k)w'$, let $w'' = w_{J'_2} w_{J'_1} w_{J'_3} w_{J'_4} w'$, then $w'' \in F'_2$ with $\ell(w'') = \ell(w') - \ell(w_{J'_2} w_{J'_1} w_{J'_3} w_{J'_4})$. In either case, w'', w' are in the same lcc of E'_λ by Corollary 2.17. So $w'' \underset{L}{\sim} w'$ and $T_\alpha(w') = T_\alpha(w'')$ by Lemma 3.6. $w' \mapsto w''$ is a bijection from F'_1 to F'_2.

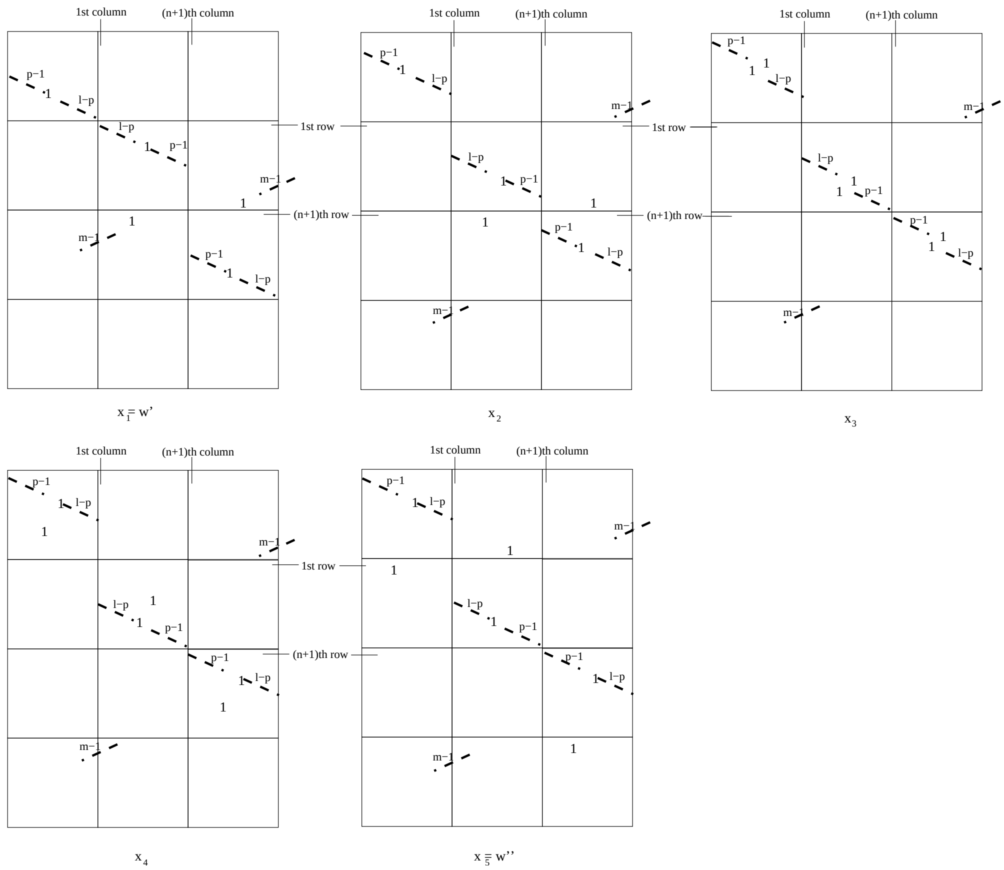


Figure 5

Next consider the case 5.5 (c$'$). Let F''_i be the set of all $w' \in F''_\lambda$ satisfying (c$'1i$) and (c$'$2)-(c$'$3) for $i = 1, 2$. Then $F''_1 \cap F''_2 = \emptyset$. Use the notation in 5.6 and in (2) above, $w' \in F''_1$ (respectively, $w' \in F''_2$) means that $\langle(\bar{j}_1)w'\rangle$, $\langle(i_p)w'\rangle$ (respectively, $\langle(\bar{i}_p)w'\rangle$, $\langle(\bar{j}_1)w'\rangle$) in 5.6 are $\bar{q}_{p+1}$, q_p in (2), respectively. Take any $w' \in F''_1$. Let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \setminus \{t_l\}$, $J_3 = \{t_0, t_1, ..., t_{m-2}\}$, $J_4 = J_3 \setminus \{t_0\}$, $J_5 = \{t_{n-1}, t_{n-2}, ..., t_{n+1-p}\}$, $J_6 = J_5 \setminus \{t_{n-1}\}$, $J_7 = \{t_m, t_{m+1}, ..., t_{n-2-p}\}$, $J_8 = J_7 \setminus \{t_{n-2-p}\}$ and $J_9 = \{t_1, t_2, ..., t_{m-1}\}$. Let $x_1 = w'$, $x_2 = w_{J_3} w_{J_4} w_{J_1} w_{J_2} x_1$, $x_3 = w_{J_5} w_{J_6} t_n x_2$, $x_4 = t_0 w_{J_9} w_{J_4} w_{J_7} w_{J_8} x_3$. Let $w'' = x_4$. Then $x_i \in E''_\lambda$ for $i \in [4]$ and

$w'' \in F_2''$ with $\ell(x_2) = \ell(x_1) + \ell(w_{J_3}w_{J_4}w_{J_1}w_{J_2})$, $\ell(x_3) = \ell(x_2) - \ell(w_{J_5}w_{J_6}t_n)$ and $\ell(x_4) = \ell(x_3) + \ell(t_0w_{J_9}w_{J_4}w_{J_7}w_{J_8})$ (see Figure 5). w', w'' are in the same lcc of E''_λ by Corollary 2.17. So $w' \underset{L}{\sim} w''$ and hence $T_\alpha(w') = T_\alpha(w'')$ by Lemma 3.6. $w' \mapsto w''$ is a bijection from F_1'' to F_2''.

From 5.6 and the above discussion, we conclude that

Lemma 5.8. *Let* $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$.

(1) Each of E'_λ *and* E''_λ *is a union of some left cells of* $\widetilde{C}_n$ *if* $m < n-1$.

(2) Any left cell of $\widetilde{C}_n$ *in* E_λ *is left-connected.*

Now consider the two-sided cells of $\widetilde{C}_n$ in E_λ.

Lemma 5.9. *Let* $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$.

(1) If $m < n-1$, *then each of* E'_λ *and* E''_λ *is two-sided-connected and is a two-sided cell of* $\widetilde{C}_n$.

(2) $E_{(2n-2,2)}$ *is two-sided-connected and is a single two-sided cell of* $\widetilde{C}_n$.

Proof. By 1.3 (1)-(2), Lemmas 1.4, 5.4 and 5.8, to show our result, we need only to prove that each of E'_λ and E''_λ is two-sided-connected if $m < n-1$ and that $E_{(2n-2,2)}$ is two-sided-connected.

(I) First assume $m < n-1$.

(Ia) E'_λ **is two-sided-connected.**

Let $w_1 = [0, 2, 3, ..., n-m, n+m, n+m-1, ..., n+2, n+1]$ and $w_2 = [0, -1, -2, ..., -m+1, m+1, m+2, ..., n-1, n+1]$ be in $\widetilde{C}_n$ (see Figure 6). Then $w_1, w_2 \in F'_\lambda$. Let F_1', F_2' be defined as in 5.7. Then

$$(5.9.1) \quad \eta(E'_\lambda) = E'_\lambda; \quad w_i \in F_i', \ \eta(w_i) = w_{3-i}, \ \eta(F_i') = F_{3-i}' \ \text{ for } i = 1, 2.$$

by Proposition 2.8 (3). By 5.6, to show (Ia), we need only to prove that

(a) Any $x \in F_i'$ is in the rcc of E'_λ containing w_i for $i = 1, 2$;

(b) w_1 and w_2 are in the same tcc of E'_λ.

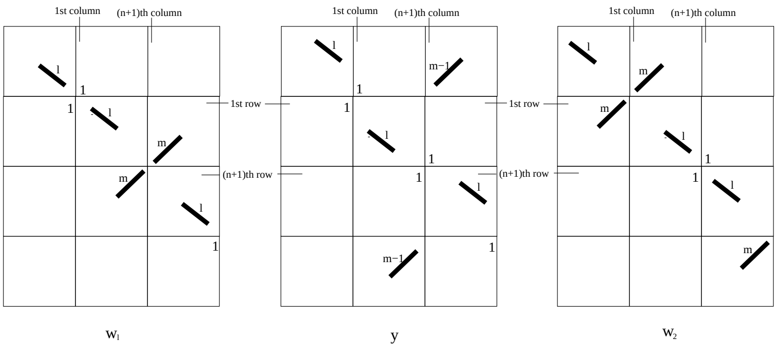


Figure 6

For (a), we need only to deal with the case of $i = 1$ by the fact (5.9.1), while the argument for this part is similar to that for Lemma 4.4 (3) (hence leaving it to the readers). Now consider (b). Let $J_1 = \{t_2, t_3, ..., t_{n-2}\}$, $J_2 = J_1 \backslash \{t_{n-m}\}$, $J_3 = \{t_0, t_1, ..., t_{m-1}\}$, $J_4 = J_3 \backslash \{t_1\}$, $J_5 = \{t_n, t_{n-1}, ..., t_{n+1-m}\}$, $J_6 = J_5 \setminus \{t_{n-1}\}$ and $y = w_{J_3} w_{J_4} w_{J_1} w_{J_2} w_1$. Then $y = w_2 w_{J_1} w_{J_2} w_{J_6} w_{J_5}$, which is in E'_λ and satisfies $\ell(y) = \ell(w_1) + \ell(w_{J_3} w_{J_4} w_{J_1} w_{J_2}) = \ell(w_2) + \ell(w_{J_1} w_{J_2} w_{J_6} w_{J_5})$ (see Figure 6). This proves (b) by Corollary 2.17.

(Ib) E''_λ **is two-sided-connected.**

Let $w_1 = [1, 2, ..., n-m-1, n+m, n+m-1, ..., n+1, n+m+1]$ and $w_2 = [-m, 0, -1, -2, ..., -m+1, m+2, m+3, ..., n]$ be in $\widetilde{C}_n$ (see Figure 7). Let F''_1, F''_2 be defined as in 5.7. Then

$$w_i \in F''_i, \quad \eta(w_i) = w_{3-i}, \quad \eta(F''_i) = F''_{3-i} \text{ for } i = 1, 2. \tag{5.9.2}$$

by Proposition 2.8 (3). By 5.6, to show (Ib), we need only to prove that

(a) Any $x \in F''_i$ is in the rcc of E''_λ containing w_i for $i = 1, 2$;

(b) w_1 and w_2 are in the same tcc of E''_λ.

By (5.9.2), to prove (a), we need only to deal with the case of $i = 1$, the latter can be proved by the argument similar to that for Lemma 4.4 (3) (hence leaving it to the readers). Next consider (b).

Let $J_1 = \{t_1, t_2, ..., t_{n-2}\}$, $J_2 = J_1 \setminus \{t_{n-m-1}\}$, $J_3 = \{t_0, t_1, ..., t_{m-1}\}$, $J_4 = J_3 \setminus \{t_0\}$, $J_5 = \{t_n, t_{n-1}, ..., t_{n-m+1}\}$, $J_6 = J_5 \setminus \{t_n\}$, $J_7 = \{t_2, t_3, ..., t_{n-1}\}$, $J_8 = J_7 \setminus \{t_{n-m}\}$ and $J_9 = J_4 \setminus \{t_1\}$. Let $x_0 = w_1$, $x_1 = t_n x_0$, $x_2 = w_{J_3} w_{J_4} w_{J_1} w_{J_2} x_1$, $x_3 = x_2 w_{J_6} w_{J_5} w_{J_8} w_{J_7}$, $x_4 = x_3 t_0 w_{J_9} w_{J_4}$ and $x_5 = w_{J_9} w_{J_4} x_4$. Then $x_5 = w_2$. We have $x_i \in E''_\lambda$ for any $i \in [0, 5]$ and $\ell(x_1) = \ell(x_0) - 1$, $\ell(x_2) = \ell(x_1) + \ell(w_{J_3} w_{J_4} w_{J_1} w_{J_2})$, $\ell(x_3) = \ell(x_2) - \ell(w_{J_6} w_{J_5} w_{J_8} w_{J_7})$, $\ell(x_4) = \ell(x_3) + \ell(t_0 w_{J_9} w_{J_4})$, $\ell(x_5) = \ell(x_4) - \ell(w_{J_9} w_{J_4})$ (see Figure 7).

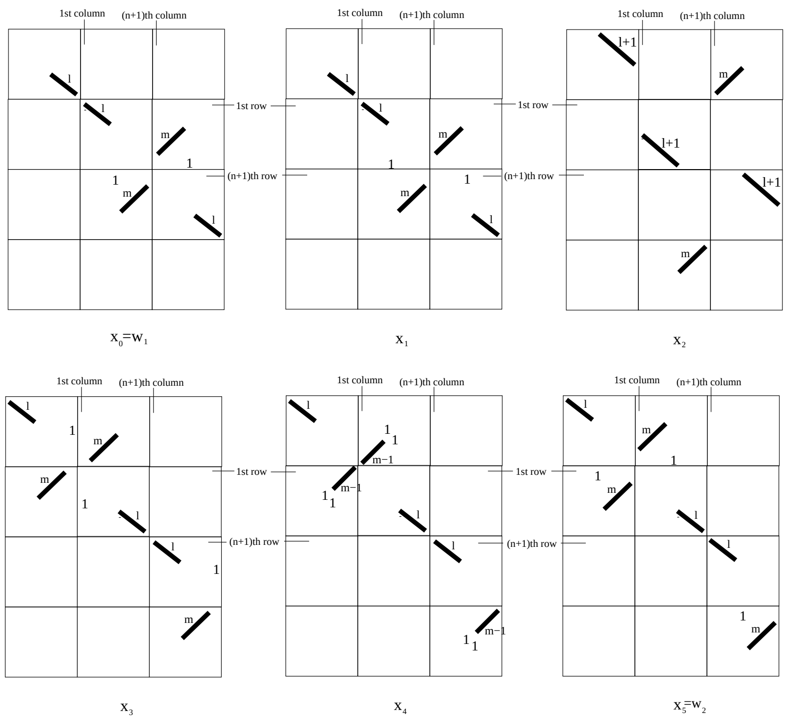


Figure 7

By Corollary 2.17, we see that x_{i-1}, x_i are contained in the same tcc of E''_λ for any $i \in [5]$. This proves (b) and hence E''_λ is two-sided connected.

(II) $E_{(2n-2,2)}$ **is two-sided-connected**.

Denote $\lambda = (2n-2, 2)$. Let $w_1 = [2n-1, 2n-2, ..., n+1, 2n]$, $w_2 = [0, -1, ..., -n+2, n+1]$, $w_3 = [0, 2n-1, 2n-2, ..., n+1]$ and $w_4 = [-n+1, 0, -1, ..., -n+2]$ be in $\widetilde{C}_n$ (see Figure 8). Then $w_i \in F_\lambda := F'_\lambda \cup F''_\lambda$ for $i \in [4]$. By the argument similar to that for Lemma 4.4 (3), we can show that any element of F_λ is in the rcc of E_λ containing w_i for some $i \in [4]$ (hence leaving it to the readers). Note that $\eta(w_i) = w_{5-i}$ for $i \in [4]$ by Proposition 2.8 (3). Thus to show that E_λ is two-sided-connected, we need only to prove that w_1, w_2 (respectively, w_2, w_3) are in the same tcc of E_λ.

Let $x_1 = w_1$, $x_2 = w_{J_4}w_{J_2}x_1$, $x_3 = x_2w_{J_5}w_{J_3}$, $x_1' = w_3$, $x_2' = t_nw_{J_4}w_{J_2}w_{J_3}w_{J_1}x_1'$, $x_3' = x_2'w_{J_1}w_{J_5}t_0w_{J_1}w_{J_3}$, where $J_1 = \{t_2, t_3, ..., t_{n-1}\}$, $J_2 = \{t_1, t_2, ..., t_{n-2}\}$, $J_3 = J_2 \cup \{t_{n-1}\}$, $J_4 = J_2 \cup \{t_0\}$, $J_5 = J_1 \cup \{t_n\}$. Then $x_3 = x_3' = w_2$ and $x_i, x_i' \in E_\lambda$ for $i \in [3]$ and $\ell(x_2) = \ell(x_1) + \ell(w_{J_4}w_{J_2}) = \ell(x_3) + \ell(w_{J_3}w_{J_5})$ and $\ell(x_2') = \ell(x_1') + \ell(t_nw_{J_4}w_{J_2}w_{J_3}w_{J_1}) = \ell(x_3') + \ell(w_{J_3}w_{J_1}t_0w_{J_5}w_{J_1})$ (see Figure 8). We see by Corollary 2.17 that w_1, w_2, (respectively, w_2, w_3) are in the same tcc of E_λ. So E_λ is two-sided-connected. $\square$

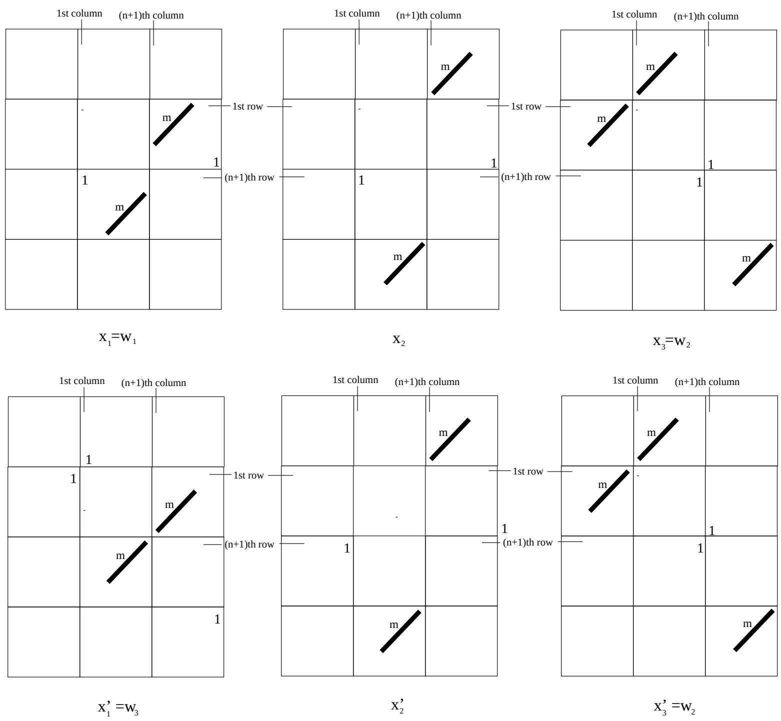


Figure 8

Recall the notation $n(\lambda)$, $n'(\lambda)$, $n''(\lambda)$ in Theorem 5.1 (2).

Lemma 5.10. $n(\lambda) = \frac{2^{m-1}n!(n+3-m)}{(n+1-m)!}$ *for* $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$. *In this case, if* $m < n-1$, *then* $n'(\lambda) = \frac{2^{m-1}n!}{(n+1-m)!}$ *and* $n''(\lambda) = \frac{2^{m-1}n!(n+2-m)}{(n+1-m)!}$.

Proof. Consider $\mathbf{T} = (T_1, ..., T_{2m}) \in \xi^{-1}(\alpha)$ satisfying 5.7 (1)-(2). The number of the choices is $\binom{n}{n+1-m}$ for $E := T_m \cup T_{m+1} = \{\bar{q}_{n+1-m}, \bar{q}_{n-m}, ..., \bar{q}_1, q_1, ..., q_{n-m}, q_{n+1-m}\}$ with $\bar{q}_1 < q_1 < q_2 < \cdots < q_{n+1-m}$ in $[2n]$. Once E is fixed, the number of the choices is $n+3-m$ for T_m satisfying (5.7.1), while that is $2^{m-1}(m-1)!$ for $(T_1, ..., T_{m-1})$. We know that w' is in E'_λ (respectively, E''_λ) if and only if the m-th component of $T(w')$ is $\{\bar{q}_1, q_{n+1-m}\}$ (respectively, satisfies (5.7.1) but is not $\{\bar{q}_1, q_{n+1-m}\}$). This proves our result. $\square$

If $m > 1$, then the number of the choices is infinite for the integer $(j_m)w$ in the case 5.2 (a), hence $|E_{(2m,2,1,...,1)}| = \infty$. On the other hand, the set $E_{\mathbf{2^21^{2n-4}}} = \{t_0t_n, t_it_{i+1}\cdots t_j, t_jt_{j-1}\cdots t_i \mid 0 \leqslant i \leqslant j \leqslant n\} \setminus \{t_0, t_n\}$ is finite.

So far we have proved all the assertions of Theorem 5.1 involving the partition $\lambda = (2m, 2, 1, ..., 1) \in \Lambda_{2n}$.

5.11. Let $\mu = (2m+1, 2, 1, ..., 1) \in \Lambda_{2n}$ and $l = n-m-1$. Then $w \in \widetilde{C}_n$ is in E_μ if and only if w satisfies the condition (5.11.1) below.

(5.11.1) There are some pairwise not $2n$-dual $j_1, j_2, ..., j_m$, k, $i_1, i_2, ..., i_l$ in $[2n]$ with $j_1, j_2, ..., j_m$, k w-wild heads and $i_1, i_2, ..., i_l$ w-tame heads such that (i) $j_1 \prec_w j_2 \prec_w \cdots \prec_w j_m$; (ii) either $\bar{i}_1, i_1 \prec_w j_1$, or $\bar{i}_l, i_l \prec_w j_1$; (iii) $i_1 < i_2 < \cdots < i_l$ and $(i_1)w < (i_2)w < \cdots < (i_l)w$; (iv) k is w-comparable with none of $\bar{i}_h, i_h, j_q$ for any $h \in [l]$ and some $q \in [m]$.

According to (i)-(ii) and (iv), if k is w-comparable with j_p for some $p \in [m]$, then $k \prec_w j_p$ and $p > 1$ by Lemma 3.2. Thus under the assumption of (i)-(ii), the condition (iv) is equivalent to that k is w-comparable with none of $\bar{i}_h, i_h, j_1$ for any $h \in [l]$.

5.12. Under the condition (5.11.1) on $w \in E_\mu$, there are two possible cases:

(a) $\bar{i}_1, i_1 \prec_w j_1$. Then $j_1 < \bar{i}_1 < i_1 < \bar{j}_1$ and $(\bar{j}_1)w < (\bar{i}_1)w < (i_1)w < (j_1)w$ and $i_l - 2n < k - 2n < \bar{k} < \bar{i}_l$ and $(i_l)w - 2n < (\bar{k})w < (k)w - 2n < (\bar{i}_l)w$;

(b) $\bar{i}_l, i_l \prec_w j_1$. Then $j_1 - 2n < i_l - 2n < \bar{i}_l < \bar{j}_1$ and $(\bar{j}_1)w < (i_l)w - 2n < (\bar{i}_l)w < (j_1)w - 2n$ and $\bar{i}_1 < k < \bar{k} < i_1$ and $(\bar{i}_1)w < (\bar{k})w < (k)w < (i_1)w$.

Let E'_μ (respectively, E''_μ) be the set of all $w \in E_\mu$ in the case (a) (respectively, (b)).

From the matrix forms of elements, we see that $w \in E_\mu$ is in the case (a) (respectively, (b)) if and only if so does w^{-1}. So by 5.11, we get

Lemma 5.13. *Let $\mu = (2m+1, 2, 1, ..., 1) \in \Lambda_{2n}$. Then*

(1) ${E'_\mu}^{-1} = E'_\mu$ and ${E''_\mu}^{-1} = E''_\mu$.

(2) The group automorphism η of $\widetilde{C}_n$ interchanges the sets E'_μ and E''_μ.

(3) $E_\mu = E'_\mu \dot{\cup} E''_\mu$.

5.14. Let F'_μ (respectively, F''_μ) be the set of all $w' \in \widetilde{C}_n$ satisfying (a$'$) (respectively, (b$'$)) below.

(a$'$) There exist w'-wild heads $j_1, j_2, ..., j_m$, k and w'-tame heads $i_1, i_2, ..., i_l$ such that

(i) $(\bar{k}, \bar{i}_l, \bar{i}_{l-1}, ..., \bar{i}_2, j_m, j_{m-1}, ..., j_1, \bar{i}_1) = (1, 2, ..., n)$;

(ii) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(iii) $(i_1)w' < (i_2)w' < \cdots < (i_l)w' < (\bar{k})w' + 2n \leqslant 2n$;

(iv) $(i_p)w' < (j_1)w' < (i_{p+1})w'$ for some $p \in [l]$ with the convention that $(i_{l+1})w' = (k)w'$.

(b$'$) There exist w'-wild heads $j_1, j_2, ..., j_m$, k and w'-tame heads $i_1, i_2, ..., i_l$ such that

(i) $(\bar{k}, i_1, i_2, ..., i_{l-1}, j_m, j_{m-1}, ..., j_1, i_l) = (n+1, n+2, ..., 2n)$;

(ii) $0 < (j_{h+1})w' - (j_h)w' < 2n$ for any $h \in [m-1]$;

(iii) $n < (k)w' < (i_1)w' < (i_2)w' < \cdots < (i_l)w'$;

(iv) $(\bar{i}_p)w' + 2n < (j_1)w' < (\bar{i}_{p-1})w' + 2n$ for some $p \in [l]$ with the convention that $(\bar{i}_0)w' = (k)w'$.

5.15. By 5.11-5.12 and 5.14, we have $F'_\mu \subseteq E'_\mu$ and $F''_\mu \subseteq E''_\mu$. Also, by 3.3 and Lemma 3.2, any lcc of E'_μ (respectively, E''_μ) contains some element of F'_μ (respectively,F''_μ).

Let $\beta = (1, ..., 1, 2, 2l+1, 1, ..., 1) \in \widetilde{\Lambda}_{2n}$ with 2 the $(m+1)$-th component. Let $F_\mu := F'_\mu \cup F''_\mu$. By the argument for Lemma 4.6, there is some $z \in \Omega$ with $z \underset{L}{\sim} w'$ and $T(z) \in \xi^{-1}(\beta)$ for any $w' \in F_\mu$. Now we describe $T(z)$. If $w' \in F'_\mu$ is as in 5.14 (a$'$), then

$$(5.15.1)\quad T(z)=\big(\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_1)w'\rangle\},\{\langle(\bar{k})w'\rangle,\langle(i_p)w'\rangle\},\\ \{\langle(k)w'\rangle,\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l]\}\setminus\{\langle(i_p)w'\rangle\},\\ \{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}\big),$$

where (i) $\langle(\bar{k})w'\rangle\in[(i_l)w'+1,2n]$; (ii) $p\in[l]$. If $p<l$, then $(\bar{i}_{p+1})w'<(\bar{j}_1)w'<(\bar{i}_p)w'$; if $p=l$ then $(\bar{j}_1)w'$ is in one of the three cases: $(k)w'-2n<(\bar{j}_1)w'<(\bar{i}_l)w'$, $1\leqslant(\bar{j}_1)w'<(k)w'-2n$, $(\bar{k})w'<(\bar{j}_1)w'\leqslant 0$.

If $w'\in F''_\mu$ is as in 5.14 (b′), then

$$(5.15.2)\quad T(z)=\big(\{\langle(\bar{j}_m)w'\rangle\},...,\{\langle(\bar{j}_1)w'\rangle\},\{\langle(\bar{k})w'\rangle,\langle(\bar{i}_p)w'\rangle\},\\ \{\langle(k)w'\rangle,\langle(j_1)w'\rangle,\langle(\bar{i}_h)w'\rangle,\langle(i_h)w'\rangle\mid h\in[l]\}\setminus\{\langle(\bar{i}_p)w'\rangle\},\\ \{\langle(j_2)w'\rangle\},...,\{\langle(j_m)w'\rangle\}\big),$$

where (i) $\langle(\bar{k})w'\rangle\in[(\bar{i}_1)w'+1,n]$; (ii) $p\in[l]$. If $p>1$ then $(i_{p-1})w'<(\bar{j}_1)w'+2n<(i_p)w'$; if $p=1$ then $(\bar{j}_1)w'$ is in one of the three cases: $(k)w'<(\bar{j}_1)w'+2n<(i_1)w'$, $n<(\bar{j}_1)w'+2n<(k)w'$, $(\bar{k})w'<(\bar{j}_1)w'+2n\leqslant n$.

5.16. We see that $T(z)$ only depends on $w'\in F_\mu$ and β, but not on the choice of $z\in\Omega$. We can denote $T(z)$ by $T_\beta(w')$. This defines a map $T_\beta: F_\mu\longrightarrow\xi^{-1}(\beta)$. By 5.14-5.15, we see that $\mathbf{T}=(T_1,T_2,...,T_{2m+1})\in\xi^{-1}(\beta)$ is in the image of T_β if and only if $\mathbf{T}$ satisfies the following conditions:

(1) $\overline{T_i}=T_{2m+2-i}$ for $i\in[2m+1]\setminus\{m,m+1,m+2\}$;

(2) $\bigcup_{i=m}^{m+2}T_i=\{\bar{q}_{n-m+1},\bar{q}_{n-m},...,\bar{q}_1,q_1,q_2,...,q_{n-m+1}\}$ with $\bar{q}_1<q_1<q_2<\cdots<q_{n-m+1}$ in $[2n]$ and $(T_m,T_{m+1})\in E_1\cup E_2\cup E_3\cup E_4$, where $E_1=\{(\{\bar{q}_{i+1}\},\{q_{n-m+1},q_i\})\mid i\in[n-m-1]\}$, $E_2=\{(\{q_j\},\{\bar{q}_1,\bar{q}_{j+1}\})\mid j\in[2,n-m]\}$, $E_3=\{(\{\bar{q}_{n-m+1}\},\{q_{n-m},q_{n-m-1}\}),(\{q_{n-m+1}\},\{q_{n-m},q_{n-m-1}\})\}$ and $E_4=\{(\{q_1\},\{\bar{q}_2,\bar{q}_3\}),(\{\bar{q}_1\},\{\bar{q}_2,\bar{q}_3\})\}$.

5.17. Keep the notation in 5.14-5.15. For $w\in F_\mu$, let $T_i(w)$ be the i-th component of $T_\beta(w)$ for $i\in[2m+1]$. Then $w'\in F'_\mu$ if and only if $(T_m(w'),T_{m+1}(w'))\in E_1\cup E_3$; $w''\in F''_\mu$ if and only if $(T_m(w''),T_{m+1}(w''))\in$

$E_2 \cup E_4$. So $(T_m(w'), T_{m+1}(w')) \neq (T_m(w''), T_{m+1}(w''))$ (hence $T_\beta(w') \neq T_\beta(w'')$) for any $w' \in F'_\mu$ and any $w'' \in F''_\mu$. Each of E'_μ and E''_μ is a union of some left cells of $\widetilde{C}_n$ by Lemmas 3.6 and 2.13. This further implies by Lemmas 2.12, 2.14, 5.13 and 1.4 that each of E'_μ and E''_μ is a union of some two-sided cells of $\widetilde{C}_n$. Let $w_1 = [0, 2, 3, ..., n-m-1, n+m+1, n+m, ..., n+2, n]$ and $w_2 = [1, -1, -2, ..., -m, m+2, m+3, ..., n-1, n+1]$ be in $\widetilde{C}_n$ (see Figure 9). Then $w_1 \in F'_\mu$ and $w_2 \in F''_\mu$. By the argument similar to that for Lemma 4.4 (3), we can prove that F'_μ (respectively, F''_μ) is in the rcc of E'_μ (respectively, E''_μ) containing w_1 (respectively, w_2) (the proof is left to the readers). So by 5.15 and Lemma 2.16, we conclude that

(5.17.1) Each of E'_μ and E''_μ is two-sided-connected and is a two-sided cell of $\widetilde{C}_n$.

Let $\mathbf{T} = (T_1, T_2, ..., T_{2m+1}) \in \xi^{-1}(\beta)$ satisfy 5.16 (1)-(2). If $(T_m, T_{m+1}) \in E_1 \cup E_3$, then $w' \in F'_\mu$ with $T_\beta(w') = \mathbf{T}$ is uniquely determined by 5.14 (a$'$) and (5.15.1). This implies by 5.15 that any left cell of $\widetilde{C}_n$ in E'_μ is left-connected. Since $E''_\mu = \eta(E'_\mu)$ by Lemma 5.13, any left cell of $\widetilde{C}_n$ in E''_μ is also left-connected. So we conclude that

(5.17.2) All left cells of $\widetilde{C}_n$ in E_μ are left-connected.

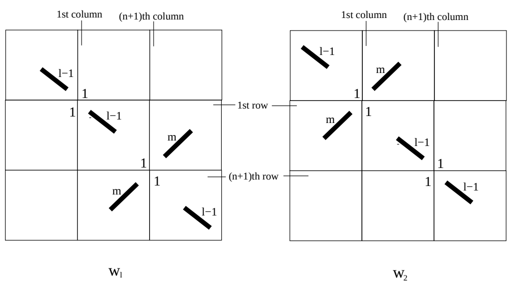


Figure 9

5.18. Now we want to enumerate the left cells in E'_μ and E''_μ. Since $\eta(E'_\mu) = E''_\mu$, we need only deal with the set E'_μ. By 5.16 and Lemma 3.6, we need only to enumerate $\mathbf{T} = (T_1, T_2, ..., T_{2m+1})$ in $\xi^{-1}(\beta)$ satisfying 5.16 (1)-(2) but with the condition $(T_m, T_{m+1}) \in E_1 \cup E_2 \cup E_3 \cup E_4$ replaced by $(T_m, T_{m+1}) \in$

$E_1\cup E_3$.

The number of the choices for $E:=\bigcup_{i=m}^{m+2}T_i=\{\bar{q}_{n-m+1},\bar{q}_{n-m},...,\bar{q}_1,q_1,q_2,...,q_{n-m+1}\}$ is $\binom{n}{n+1-m}$. Once E is fixed, the number of the choices for (T_m,T_{m+1},T_{m+2}) is $|E_1\cup E_3|=n+1-m$, while that for $(T_1,T_2,...,T_{m-1})$ is $2^{m-1}(m-1)!$. Recall the notation $n(\mu)$, $n'(\mu)$, $n''(\mu)$ in Theorem 5.1 (3).

(5.18.1) $\quad n'(\mu)=n''(\mu)=\frac{1}{2}n(\mu)=\frac{2^{m-1}\cdot n!}{(n-m)!}$ for $\mu=(2m+1,2,1,...,1)\in\Lambda_{2n}$.

When $m>1$, the number of the choices for $(j_m)w$ in (5.11.1) is infinite.

(5.18.2) $\quad |E_{(2m+1,2,1,\ldots,1)}|=\infty$.

Denote $p_{i,j}:=t_it_{i-1}\cdots t_1t_0t_nt_1\cdots t_{j-1}t_j$ for $i,j\in[n]$ and $q_{i,j}=\eta(p_{i,j})$. Then $E'_{\mathbf{321^{2n-5}}}=\{q_{i,j}\mid i,j\in[n]\}$ and $E''_{\mathbf{321^{2n-5}}}=\{p_{i,j}\mid i,j\in[n]\}$.

(5.18.3) $\quad$ The set $E_{\mathbf{321^{2n-5}}}=E'_{\mathbf{321^{2n-5}}}\cup E''_{\mathbf{321^{2n-5}}}$ is finite.

By (5.17.1)-(5.17.2) and (5.18.1)-(5.18.3), it is proved for all the assertions of Theorem 5.1 involving $\mu=(2m+1,2,1,...,1)\in\Lambda_{2n}$.

§6. The cells in the weighted Coxeter group $(\widetilde{C}_3,\widetilde{\ell})$.

As an application of Theorems 4.9 and 5.1, we shall describe all the cells of the weighted Coxeter group $(\widetilde{C}_3,\widetilde{\ell})$ in this section.

Recall the notation E_λ for $\lambda\in\Lambda_{2n}$ and $\eta:\widetilde{C}_n\longrightarrow\widetilde{C}_n$ defined before (see 2.15 and 2.7). Let $n(\lambda)$ be the number of left cells of $\widetilde{C}_n$ in E_λ. When E_λ is a union of two two-sided cells (say E'_λ, E''_λ) of $\widetilde{C}_n$, denote by $n'(\lambda)$, $n''(\lambda)$ the numbers of left cells of $\widetilde{C}_n$ in E'_λ, E''_λ, respectively.

The main result of the section is as follows.

Theorem 6.1. *In the weighted Coxeter group $(\widetilde{C}_3,\widetilde{\ell})$, we have*

(1) E_λ is a single two-sided cell of $\widetilde{C}_3$ if $\lambda\in\{\mathbf{6},\mathbf{51},\mathbf{42},\mathbf{3^2},\mathbf{31^3},\mathbf{2^3},\mathbf{1^6}\}$ and is a union of two two-sided cells of $\widetilde{C}_3$ if $\lambda\in\{\mathbf{41^2},\mathbf{321},\mathbf{2^21^2},\mathbf{21^4}\}$. E_λ is finite if $\lambda\in\{\mathbf{1^6},\mathbf{21^4},\mathbf{2^21^2},\mathbf{321}\}$, and infinite if otherwise.

(2) η stabilizes the two-sided cells $E'_{\mathbf{2^21^2}}$ and $E''_{\mathbf{2^21^2}}$, and interchanges the following pairs of two-sided cells: $E'_{\mathbf{41^2}}$, $E''_{\mathbf{41^2}}$; $E'_{\mathbf{321}}$, $E''_{\mathbf{321}}$; $E'_{\mathbf{21^4}}$, $E''_{\mathbf{21^4}}$.

(3) The numbers $n(\lambda)$ for any $\lambda\in\Lambda_6$ are listed as follows.

λ	$\mathbf{6}$	$\mathbf{51}$	$\mathbf{42}$	$\mathbf{41^2}$	$\mathbf{3^2}$	$\mathbf{321}$	$\mathbf{31^3}$	$\mathbf{2^3}$	$\mathbf{2^21^2}$	$\mathbf{21^4}$	$\mathbf{1^6}$
$n(\lambda)$	*48*	*24*	*24*	*12*	*12*	*6*	*6*	*8*	*5*	*2*	*1*

we have $n'(\mathbf{41^2}) = n''(\mathbf{41^2}) = 6$, $n'(\mathbf{321}) = n''(\mathbf{321}) = 3$, $n'(\mathbf{2^21^2}) = 4$, $n''(\mathbf{2^21^2}) = n'(\mathbf{21^4}) = n''(\mathbf{21^4}) = 1$.

(4) Each left (respectively, two-sided) cell of $\widetilde{C}_3$ *is left- (respectively, two-sided-) connected.*

6.2. All the results in Theorem 6.1 follow by Theorems 4.9 and 5.1 except for those involving the partitions $\mathbf{3^2}$ and $\mathbf{2^3}$.

The following equivalent conditions on $w \in \widetilde{C}_3$ hold by Lemma 3.2:

(1) $\psi(w) = \mathbf{3^2}$ if and only if one of the conditions (1a)-(1c) holds for some pairwise not 6-dual i, j, k in $[6]$:

(1a) i is w-tame and j, k are w-wild heads such that $i \prec_w k$, that $\bar{i} \prec_w j$ and that j, k are w-uncomparable;

(1b) k is a w-wild head and i, j are w-tame such that $j \prec_w i \prec_w k$ and that $\bar{k}$ is w-uncomparable with j;

(1c) i, j, k are all w-tame with $i \prec_w j \prec_w k$.

(2) $\psi(w) = \mathbf{2^3}$ if and only if one of the conditions (2a)-(2c) holds for some pairwise not 6-dual i, j, k in $[6]$:

(2a) i, j, k are all w-wild heads and pairwise w-uncomparable;

(2b) i is w-tame; j, k are w-wild heads and w-uncomparable; i is w-comparable with some element in either $\{j, k\}$ or $\{\bar{j}, \bar{k}\}$ but not both;

(2c) k is a w-wild head and i, j are w-tame heads such that $j \prec_w i$ and that k is w-uncomparable with $i, \bar{j}$.

Since $\{[6i-1, 6i, 3] \mid i \in \mathbb{Z} \setminus \{0\}\} \subset E_{\mathbf{3^2}}$ and $\{[3i+1, 3i+2, 3i+3] \mid i \in \mathbb{Z} \setminus \{0\}\} \subset E_{\mathbf{2^3}}$, we have

(6.2.1) $|E_{\mathbf{3^2}}| = |E_{\mathbf{2^3}}| = \infty$.

6.3. Let $F'_{\mathbf{3^2}}$ be the set of all $w' \in \widetilde{C}_3$ satisfying (6.3.1) below.

(6.3.1) There exists some pairwise not 6-dual i, j, k in $[6]$ such that one of the following conditions holds:

(a) i is a w'-tame head and j,k are w'-wild heads such that (i) $j<k$ and $(j)w'<(k)w'<(j)w'+6$; (ii) $j<\bar{i}$ and $k<i$; (iii) $(\bar{i})w'<(j)w'$ and $(i)w'<(k)w'$; (iv) either $(k)w'<(\bar{i})w'+6$ or $(j)w'<(i)w'$;

(b) k is a w-wild head and i,j are w-tame heads such that (i) $i<j$ and $(i)w'>(j)w'$; (ii) either $(k,i,j)=(4,5,6)$ and $6<(4)w'<(1)w'+6$, or $(k,i,j)=(1,4,5)$ and $3<(1)w'<(4)w'$;

(c) $w'=[3,2,1]$.

We see by 3.3 and 6.2 (1) that for any $w\in E_{\mathbf{3^2}}$, there exists some $w'\in F'_{\mathbf{3^2}}$ such that w,w' are in the same lcc of $E_{\mathbf{3^2}}$.

6.4. Let $F_1=\{[4,2,0],[4,1,-1],[5,3,0],[5,1,-2],[5,1,-3],[6,3,-1],[7,3,-1],[4,2,6],[5,3,6]\}$, $F_2=\{[3,2,0],[3,1,-1],[4,2,1],[5,3,1]\}$, $F_3=\{[3,2,1]\}$.

Then $w'\in F'_{\mathbf{3^2}}$ satisfies (6.3.1) (a) if and only if w' is in the lcc of $E_{\mathbf{3^2}}$ containing some $w\in F_1$. $w'\in F'_{\mathbf{3^2}}$ satisfies (6.3.1) (b) (respectively, (6.3.1) (c)) if and only if $w'\in F_2$ (respectively, $w'\in F_3$).

6.5. Let $x_1=[4,2,6]$, $x_2=[4,2,1]$, $y_1=[5,3,6]$ and $y_2=[5,3,1]$. Then $x_1,y_1\in F_1$, $x_2,y_2\in F_2$, $x_2=t_3x_1$ and $y_2=t_3y_1$. So x_1,x_2 (respectively, y_1,y_2) are in the same lcc of $E_{\mathbf{3^2}}$. Let $F_{\mathbf{3^2}}=(F_1\cup F_2\cup F_3)\setminus\{[4,2,6],[5,3,6]\}$. We see from Figure 10 that all the elements of $F_{\mathbf{3^2}}$ are in the same rcc of $E_{\mathbf{3^2}}$ and have pairwise different generalized τ-invariants (see 2.9 and 2.10).

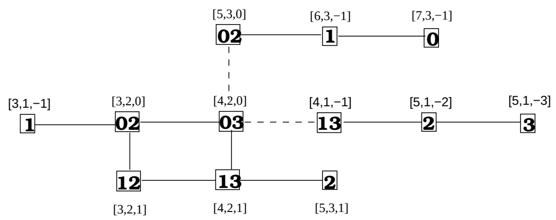


Figure 10

So by Lemmas 2.13-2.14 and 2.16, we see that

(6.5.1) $E_{\mathbf{3^2}}$ is two-sided-connected and forms a two-sided cell of $\widetilde{C}_3$ with $n(\mathbf{3^2})=|F_{\mathbf{3^2}}|=12$, each left cell of $\widetilde{C}_3$ in $E_{\mathbf{3^2}}$ is left-connected.

6.6. Next consider $E_{\mathbf{2^3}}$. Let $F'_{\mathbf{2^3}}$ be the set of all $w'\in\widetilde{C}_3$ satisfying (6.6.1) below.

(6.6.1) There exists some pairwise not 6-dual i, j, k in [6] satisfying one of the conditions (a)-(c) below:

(a) i, j, k are all w'-wild heads satisfying: (i) $i < j < k$ and $4 \leqslant (i)w' < (j)w' < (k)w' \leqslant 9$; (ii) $i \in [3]$ unless $(i)w' > 6$; (iii) $k = 6$ unless $(k)w' \leqslant 6$;

(b) i is a w'-tame head and j, k are w'-wild heads such that (i) $j < k$ and $(j)w' < (k)w' < (j)w' + 6$; (ii) Assume $j < \bar{i}$. If $k < i$ then $(k)w' < (i)w'$; if $k > i$ then $(j)w' < (i)w' < (k)w'$; (iii) Assume $\bar{i} < j < i$. If $k < i$ then $(i)w' < (k)w' < (\bar{i})w' + 6$; if $k > i$ then either $(i)w' < (j)w' < (k)w' < (\bar{i})w' + 6$, or $(\bar{i})w' < (j)w' < (i)w'$ and $(\bar{i})w' < (k)w' - 6 < (i)w'$; (iv) If $i < j$ then $(\bar{i})w' < (k)w' - 6 < (i)w'$;

(c) k is a w'-wild head and i, j are w'-tame heads with $i < j$ and $(j)w' < (i)w'$ such that (i) $\bar{j} < k$ and $3 < (k)w' < (\bar{j})w' + 6$; (ii) Either $i < k$, or $k < i$ and $3 < (k)w' < (i)w'$.

By 6.2 (2) and 3.3, we see that for any $w \in E_{\mathbf{2^3}}$, there exists some $w' \in F'_{\mathbf{2^3}}$ such that w, w' are in the same lcc of $E_{\mathbf{2^3}}$.

6.7. Let $F'_1 = \{[4,5,6], [0,4,5], [-1,4,6], [-2,5,6], [-2,-1,0], [-1,0,4], [-2,0,5], [-2,-1,6]\}$, $F'_2 = \{[4,1,5], [0,4,2], [2,4,6], [3,5,6], [2,0,4], [0,3,5], [-1,3,6], [-1,1,4], [-2,1,5], [-1,3,0], [-2,-1,1], [-2,0,2]\}$, $F'_3 = \{[2,1,4], [3,1,5], [0,3,2], [-1,3,1]\}$.

We see by 3.3 that any $x \in F'_{\mathbf{2^3}}$ satisfying (a) (respectively, (b), (c)) in (6.6.1) is in a lcc of $E_{\mathbf{2^3}}$ containing some element of F'_1 (respectively, F'_2, F'_3).

Let $F_{\mathbf{2^3}} = F_1 \cup F_2$, where $F_1 = \{[0,4,2], [0,3,2], [-1,3,1], [-1,3,0]\}$ and $F_2 = \{[2,0,4], [2,1,4], [3,1,5], [4,1,5]\}$.

Then any $x \in \bigcup_{k=1}^{3} F'_k$ is in a lcc of $E_{\mathbf{2^3}}$ containing some element of $F_{\mathbf{2^3}}$.

6.8. We see from Figure 11 that no two elements of $F_{\mathbf{2^3}}$ have the same generalized τ-invariants (see 2.9-2.10).

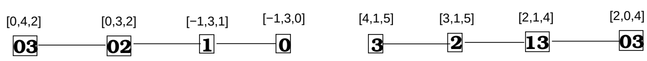


Figure 11

Since $[0,4,2]=t_2t_0t_3 \underset{R}{\sim} t_0t_2t_3t_1 \underset{L}{\sim} t_1t_3 \underset{R}{\sim} t_1t_0t_3=[2,0,4]$, the set $F_{\mathbf{2^3}}$ is contained in a tcc of $E_{\mathbf{2^3}}$. By Lemmas 2.13-2.14 and 2.16, we see that

(6.8.1) $E_{\mathbf{2^3}}$ is two-sided-connected and is a two-sided cell of $\widetilde{C}_3$ with $n(\mathbf{2^3})=|F_{\mathbf{2^3}}|=8$, each left cell of $\widetilde{C}_3$ in $E_{\mathbf{2^3}}$ is left-connected.

So we complete the proof of Theorem 6.1 by Theorems 4.9, 5.1 and the results (6,2,1), (6.5.1), (6.8.1).

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