A TWO-SIDED CELL IN AN AFFINE WEYL GROUP, II

JIAN-YI SHI

1. *Introduction*

This paper is a continuation of [6]. One of the main themes in [6] was to find a two-sided cell, written $W_{(\nu)}$, of an affine Weyl group and to give an upper bound for the number of left cells in $W_{(\nu)}$. We shall describe all left cells of W_a in $W_{(\nu)}$ and give the exact number of left cells of $W_{(\nu)}$.

Let $G = (G, S)$ be a Coxeter group with S the set of simple reflections. Let $l: G \to N$ be the usual length function of G.

Let $\leqslant$ be the Bruhat order on G (see [7]). We associate to each $w \in G$ two subsets of S:

$$\mathscr{L}(w) = \{s \in S \mid sw < w\} \quad \text{and} \quad \mathscr{R}(w) = \{s \in S \mid ws < w\}.$$

Let u be an indeterminate and let $\mathbf{A} = \mathbb{Z}[u, u^{-1}]$. We define, following Lusztig (see [3]), the Hecke algebra $\mathbf{H}$ of G over $\mathbf{A}$, that is the free $\mathbf{A}$-module with basis T_w, $w \in G$, and multiplication defined by

$$\left.\begin{aligned} &(T_s + u)(T_s - u^{-1}) = 0 && \text{if } s \in S, \\ &T_w T_{w'} = T_{ww'} && \text{if } l(ww') = l(w) + l(w'). \end{aligned}\right\} \tag{1}$$

Regarded as an $\mathbf{A}$-module, $\mathbf{H}$ has also the basis $(C_w)_{w \in G}$:

$$C_w = \sum_{y \leqslant w} u^{l(w)-l(y)} P_{y,w}(u^{-2})\, T_y, \tag{2}$$

where the $P_{y,w}(u) \in \mathbb{Z}[u]$ are known as the Kazhdan–Lusztig polynomials (see [1]), which satisfy

$$\deg P_{y,w}(u) \leqslant \tfrac{1}{2}l(w) - l(y) - 1)$$

if $y < w$ and $P_{w,w}(u) = 1$.

Let $\mathbf{A}^+ = \mathbb{Z}[u]$ and let $\mathbf{A}^{++}$ be the set of all polynomials of $\mathbb{Z}[u]$ which have non-negative coefficients. It is known (see [2, (1.2.1.), (1.2.2.)]) that

$$C_w \in T_w + u \sum_{y < w} \mathbf{A}^{++} T_y \tag{3}$$

and

$$T_w \in C_w + u \sum_{y < w} \mathbf{A}^{+} C_y. \tag{4}$$

We define for any $x, y, z \in G$ some elements $f_{x,y,z}$, $h_{x,y,z} \in \mathbf{A}$ such that

$$T_x T_y = \sum_z f_{x,y,z} T_z \tag{5}$$

and

$$C_x C_y = \sum_z h_{x,y,z} C_z. \tag{6}$$

Received 3 March 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*) 20H15.

Supported by the Science Fund of the Chinese Academy of Sciences (842106).

Throughout this paper, we shall assume that (G, S) is crystallographic. We shall also assume that (G, S) satisfies the following property.

(A) There exists an integer $n \geqslant 0$ such that $u^n f_{x,y,z} \in \mathbf{A}^+$ for all $x, y, z \in G$, or equivalently, $u^n h_{x,y,z} \in \mathbf{A}^+$ for all $x, y, z \in G$ (see [**2**, 3.1]).

This includes Weyl groups and affine Weyl groups (see [**2**, Theorem 7.2]).

It is well known that for any $x, y, z \in G$, $h_{x,y,z}$ has non-negative coefficients as a Laurent polynomial in u (see [**2**, (3.1.1)]). Let $\xi = u^{-1} - u$ and let $\mathbb{Z}^+[\xi]$ be the set of all polynomials of $\mathbb{Z}[\xi]$ which have non-negative coefficients. Then, by (1), we see that $f_{x,y,z} \in \mathbb{Z}^+[\xi]$. The degree of $f_{x,y,z}$ satisfies the inequality

$$\deg_\xi f_{x,y,z} \leqslant \min\{l(x), l(y), l(z)\} \tag{7}$$

(see [**2**, Lemma 7.4]). It is also known that for each $z \in G$, there is a well defined integer $a(z) \geqslant 0$ such that

$$\left.\begin{aligned} &u^{a(z)} h_{x,y,z} \in \mathbf{A}^+ \quad \text{for all } x, y \in G, \\ &u^{a(z)-1} h_{x,y,z} \notin \mathbf{A}^+ \quad \text{for some } x, y \in G. \end{aligned}\right\} \tag{8}$$

For any $x, y, z \in G$, we define $\gamma_{x,y,z} \in \mathbb{Z}$ by

$$u^{a(z)} h_{x,y,z^{-1}} - \gamma_{x,y,z} \in u\mathbf{A}^+. \tag{9}$$

Let $W_a = (W_a, S)$ be an irreducible affine Weyl group regarded as a Coxeter group: S is the set of simple reflections. Let Φ be the root system whose type is determined by W_a (see [**5**, §1]). Let Φ^+ be a positive root system of Φ. Set $v = |\Phi^+|$. Then Lusztig proved that

$$\deg_\xi f_{x,y,z} \leqslant v \quad \text{for any } x, y, z \in W_a \tag{10}$$

(see [**2**, Theorem 7.2]). He also proved that

$$a(z) \leqslant v \quad \text{for any } z \in W_a \tag{11}$$

(see [**2**, Corollary 7.3]). We define

$$W_{(v)} = \{w \in W_a \mid a(w) = v\}. \tag{12}$$

Kazhdan and Lusztig defined the preorders $\leqslant_L$, $\leqslant_\Gamma$ on W_a (see [**2**] or [**1**]), and the associated equivalence relations $\sim_L$, $\sim_\Gamma$ on W_a; the equivalence classes for $\sim_L$ (respectively $\sim_\Gamma$) are called left (respectively two-sided) cells.

In [**6**], the author proved that the set $W_{(v)}$ is a two-sided cell of W_a, which is a union of m left cells of W_a for some $m \leqslant |W|$, where W is the Weyl group on Φ (see [**6**, Theorems 5.2 and 5.3]). It was conjectured that the equality $m = |W|$ should always hold (see [**6**, Conjecture 5.4]).

In the present paper, we shall verify this conjecture.

THEOREM 1.1. *The two-sided cell $W_{(v)}$ consists of exactly $|W|$ left cells of W_a.*

The proof of Theorem 1.1 will be given in §6.

For any subset $J \subset S$, let W_J be the standard parabolic subgroup of (W_a, S) generated by J. Let w_J be the longest element of W_J. Let $\mathbf{S}$ be the set of all subsets J of S such that W_J is isomorphic to the Weyl group W on Φ. Let

$$W(\mathbf{S}) = \{w \in W_a \mid w = x \cdot w_J \cdot y \text{ with } J \in \mathbf{S}\}, \tag{13}$$

where, for any $x, y, z \in W_a$, the notation $z = x \cdot y$ means that $z = xy$ and $l(z) = l(x) + l(y)$.

In [6, Theorem 1.1], the author proved that $W(\mathbf{S}) = W_{(\nu)}$. Now we define a subset M of $W_{(\nu)}$ as follows:

$$M = \{w_J \cdot y \mid J \in \mathbf{S}, sw_J y \notin W_{(\nu)} \text{ for any } s \in J (= \mathscr{L}(w_J y))\}. \tag{14}$$

Recall that in [5, 6], the author introduced the sign types for the elements of W_a. The elements of W_a having the same sign type form an equivalence class of W_a. Let us call it an ST-class of W_a. Then it was proved that $W_{(\nu)}$ is a union of $|W|$ ST-classes of W_a (see [6, §5]). It was proved that each ST-class of W_a in $W_{(\nu)}$ is contained in some left cell of W_a (see [6, §5]). It was also proved that each ST-class of W_a contains a unique shortest element and that M is the set of the shortest elements for the ST-classes of W_a in $W_{(\nu)}$ (see [5, Proposition 7.3]). Thus we have $|M| = |W|$. It is known that the ST-class $Y(y, J)$ containing the element $w_J \cdot y \in M$ with $J \in \mathbf{S}$ has the form

$$Y(y, J) = \{z = xw_J y \mid x \in W_a, z = x \cdot w_J \cdot y\}. \tag{15}$$

Thus there is an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *Every left cell of W_a in $W_{(\nu)}$ is an* ST-*class of W_a which has the form* (15).

2. *The elements $\lambda(x, y)$*

For any $x, y \in G$, we define two subsets $F(x, y)$ and $H(x, y)$ of G as follows:

$$F(x, y) = \{z \in G \mid f_{x,y,z} \neq 0\}, \tag{1}$$

$$H(x, y) = \{z \in G \mid h_{x,y,z} \neq 0\}, \tag{2}$$

where the $f_{x,y,z}$, $h_{x,y,z} \in \mathbf{A}$ are defined as in §1 (5), (6). We shall prove that there exists a unique maximal element in the set $F(x, y)$ or $H(x, y)$. The results of this section will be used in the proof of Theorem 6.1.

LEMMA 2.1. *Let $x, y \in G$.*
(a) *If $s \in \mathscr{R}(x) \cap \mathscr{L}(y)$, then*

$$F(x, y) = F(xs, sy) \cup F(xs, y). \tag{3}$$

(b) *If $s \in \mathscr{R}(x) \cup \mathscr{L}(y) - \mathscr{R}(x) \cap \mathscr{L}(y)$, then*

$$F(x, y) = F(xs, sy). \tag{4}$$

Proof. We have $T_x T_y = T_{xs}(T_s T_y) = T_{xs} T_{sy} + \xi T_{xs} T_y$ in case (a) and $T_x T_y = T_{xs} T_{sy}$ in case (b). Thus our results follow by the fact that the coefficients of the polynomials $f_{\alpha,\beta,\gamma}$ in ξ are positive.

The following result is well known (see [4, Proposition 3.2. and Corollary 3.3]).

LEMMA 2.2. *Assume that $x \leqslant y$ in G and $s \in S$.*
(a) *If $sy \leqslant y$, then $sx \leqslant y$.*
(b) *If $ys \leqslant y$, then $xs \leqslant y$.*
(c) *If $x \leqslant sx$, then $x \leqslant sy$.*
(d) *If $x \leqslant xs$, then $x \leqslant ys$.*

For any $x, y \in G$, we define

$$K(x, y) = \{(x', y') \mid x' \leqslant x, y' \leqslant y\}. \tag{5}$$

PROPOSITION 2.3. *Let $x, y \in G$.*

(a) *There exists a unique element $\lambda(x, y) \in F(x, y)$ such that for any $(x', y') \in K(x, y)$ and $z \in F(x', y')$, we have $\lambda(x, y) \geqslant z$.*

(b) *For any reduced form of x, say $x = s_1 s_2 \ldots s_r$, $s_i \in S$, there exists a unique subsequence $i_1, i_2, \ldots, i_p$ of $1, 2, \ldots, r$ such that*

$$s_{i_l} \ldots \hat{s}_{i_{l+1}} \ldots \hat{s}_{i_p} \ldots s_r y < \hat{s}_{i_l} \ldots \hat{s}_{i_{l+1}} \ldots \hat{s}_{i_p} \ldots s_r y$$

for l, $1 \leqslant l \leqslant p$, and

$$s_j s_{j+1} \ldots \hat{s}_{i_{m+1}} \ldots \hat{s}_{i_p} \ldots s_r y > s_{j+1} \ldots \hat{s}_{i_{m+1}} \ldots \hat{s}_{i_p} \ldots s_r y$$

for j, $i_m < j < i_{m+1}$, $1 \leqslant m \leqslant p$, with the convention that $i_{p+1} = r+1$. We have

$$\lambda(x, y) = s_1 \ldots \hat{s}_{i_1} \ldots \hat{s}_{i_2} \ldots \hat{s}_{i_p} \ldots s_r y. \tag{6}$$

(c) *For any reduced form of y, say $y = t_1 t_2 \ldots t_k$, $t_j \in S$, there exists a unique subsequence $j_1, j_2, \ldots, j_q$ of $1, 2, \ldots, t$ such that* ·

$$x t_1 \ldots \hat{t}_{j_1} \ldots \hat{t}_{j_{l-1}} \ldots t_{j_l} < x t_1 \ldots \hat{t}_{j_1} \ldots \hat{t}_{j_l}$$

for l, $1 \leqslant l \leqslant q$, and

$$x t_1 \ldots \hat{t}_{j_1} \ldots \hat{t}_{j_{m-1}} \ldots t_i > x t_1 \ldots \hat{t}_{j_1} \ldots \hat{t}_{j_{m-1}} \ldots t_{i-1}$$

for i, $j_{m-1} < i < j_m$, $1 \leqslant m \leqslant q$, with the convention that $j_0 = 0$. We have

$$\lambda(x, y) = x t_1 \ldots \hat{t}_{j_1} \ldots \hat{t}_{j_q} \ldots t_k. \tag{7}$$

(d) *The integer p in* (b) *is equal to the integer q in* (c). *Let us denote this integer by $\pi(x, y)$. Then*

$$\pi(x, y) = l(x) + l(y) - l(\lambda(x, y)) \tag{8}$$

and

$$f_{x, y, \lambda(x, y)} = (u^{-1} - u)^{\pi(x, y)}. \tag{9}$$

Proof. First we shall prove (a) and (b) by induction on $n = l(x) \geqslant 0$. The result is obviously true in the case when $n = 0$. In that case, we have $p = 0$ and $\lambda(x, y) = y$. Now assume that $n > 0$. Let (x', y') be any pair in $K(x, y)$. If there exists some $s \in \mathscr{R}(x) \cap \mathscr{R}(x')$, then we have

$$T_x T_y = T_{xs}(T_s T_y) \quad \text{and} \quad T_{x'} T_{y'} = T_{x's}(T_s T_{y'}).$$

In this case, there are the following two possibilities.

(i) If $s \notin \mathscr{L}(y)$, then we have $F(s, y) = \{sy\}$ and $F(s, y') \subseteq \{sy', y'\}$. By Lemma 2.2, we see that $sy', y' \leqslant sy$ and $x's \leqslant xs$. Thus it follows from Lemma 2.1 that

$$F(x, y) = F(xs, sy) \quad \text{and} \quad F(x', y') \subseteq F(x' s, sy') \cup F(x's, y').$$

(ii) If $s \in \mathscr{L}(y)$, then we have $F(s, y) = \{sy, y\}$ and $F(s, y') \subseteq \{sy', y'\}$. We have $sy', y' \leqslant y$ and $x's \leqslant xs$ by Lemma 2.2. Hence we obtain

$$F(x, y) = F(xs, sy) \cup F(xs, y) \quad \text{and} \quad F(x', y') \subseteq F(x's, sy') \cup F(x's, y')$$

by Lemma 2.1.

We have $l(xs) < l(x)$. So by the inductive hypothesis, in the case (i), there exists a unique element $\lambda(xs, sy) \in F(xs, sy)$ such that $\lambda(xs, sy) \geqslant z$ for any

$$z \in F(x's, sy') \cup F(x's, y')$$

and such that (b) is satisfied by xs and sy instead of x and y; in (ii), there exists a unique element $\lambda(xs, y) \in F(xs, y)$ such that $\lambda(xs, y) \geqslant z$ for any

$$z \in F(x's, sy') \cup F(x's, y')$$

and such that (b) is satisfied by replacing x by xs. Let us take $\lambda(x, y)$ to be $\lambda(xs, sy)$ in the case (i) or to be $\lambda(xs, y)$ in the case (ii). Then $\lambda(x, y)$ satisfies (b) and also $\lambda(x, y) \geqslant z$ for any $z \in F(x, y) \cup F(x', y')$.

If $\mathscr{R}(x) \cap \mathscr{R}(x') = \varnothing$, then there exists some $s \in \mathscr{R}(x) - \mathscr{R}(x')$. We have

$$T_x T_y = T_{xs}(T_s T_y).$$

We see that $x' \leqslant xs$ by Lemma 2.2. Hence our result follows by Lemmas 2.1, 2.2 and the inductive hypothesis in the same way as above. Thus (a) and (b) are proved. Part (c) can be proved by induction on $n = l(y) \geqslant 0$ in the similar way to (b). Finally, (d) is an immediate consequence of (a) to (c) and §1 (1).

The following are some simple properties of the element $\lambda(x, y)$ for any $x, y \in G$. We leave the proofs to the reader.

COROLLARY 2.4. *Let* $x, y, z \in G$.

(a) $\lambda(1, x) = \lambda(x, 1) = x$.

(b) *For any* $s \in S$, *we have*

$$\lambda(s, x) = \begin{cases} x & \text{if } s \in \mathscr{L}(x), \\ sx & \text{if } s \notin \mathscr{L}(x) \end{cases}$$

and

$$\lambda(x, s) = \begin{cases} x & \text{if } s \in \mathscr{R}(x), \\ xs & \text{if } s \notin \mathscr{R}(x). \end{cases}$$

(c) *If* $xy = x \cdot y$, *then* $\lambda(x, y) = xy$.

(d) $\lambda(\lambda(x, y), z) = \lambda(x, \lambda(y, z))$. *In particular, if* $x = x' \cdot x''$, *then*

$$\lambda(x, y) = \lambda(x', \lambda(x'', y)) \quad \text{and} \quad \lambda(y, x) = \lambda(\lambda(y, x'), x'').$$

(e) $\lambda(y^{-1}, x^{-1}) = \lambda(x, y)^{-1}$. *In particular,* $\lambda(x, x^{-1})$ *is an involution of* G.

(f) *For any* $s \in \mathscr{R}(x)$, *we have*

$$\lambda(x, y) = \begin{cases} \lambda(xs, sy) & \text{if } s \notin \mathscr{L}(y), \\ \lambda(xs, y) & \text{if } s \in \mathscr{L}(y). \end{cases}$$

REMARK. Proposition 2.3 asserts that for any pair $x, y \in G$, there is a unique maximal element $\lambda(x, y)$ in the set $F(x, y)$. We can also show that the element xy is the unique minimal element in the set $F(x, y)$ in the sense that $xy \leqslant z$ for any $z \in F(x, y)$.

PROPOSITION 2.5. *Keep the notation of Proposition* 2.3. *Let* x, y *be two elements of* G.

(a) *For any* $(x', y') \in K(x, y)$ *and* $z \in H(x', y')$, *we have* $\lambda(x, y) \geqslant z$.

(b) $u^{\pi(x,y)} h_{x,y,\lambda(x,y)}$ *is in* $\mathbf{A}^+$, *and its constant term is* 1.

Proof. We can write

$$C_x = T_x + \sum_{z<x} a_z T_z \quad \text{and} \quad C_y = T_y + \sum_{w<y} b_w T_w,$$

with $0 \neq a_z, b_w \in u\mathbf{A}^{++}$ for all $z < x$, $w < y$. Hence we have

$$C_x C_y = T_x T_y + \sum_{(z,w) \in K'(x,y)} a_z b_w T_z T_w = \sum_v d_{x,y,v} T_v, \tag{10}$$

where $K'(x, y) = K(x, y) - \{(x, y)\}$, $a_x = b_y = 1$, and the $d_{x,y,v}$ are Laurent polynomials in u, the coefficients of whose lowest degree terms are positive integers whenever $d_{x,y,v} \neq 0$.

Let $F'(x, y) = \{v \in G \mid d_{x,y,v} \neq 0\}$. Then by Proposition 2.3 and Corollary 2.4, we see that

$$F'(x, y) = \bigcup_{(z,w)\in K(x,y)} F(z, w) = \{v \in G \mid \lambda(x, y) \geqslant v\}. \tag{11}$$

By (10), (11) and §1 (4), we have that

$$\lambda(x, y) \in H(x, y) \subseteq F'(x, y). \tag{12}$$

Thus (a) follows by (11) and (12).

By Proposition 2.3, we have

$$\pi(x, y) = \deg_\xi f_{x,y,\lambda(x,y)} \geqslant \deg_\xi f_{z,w,\lambda(x,y)} \tag{13}$$

for any $(z, w) \in K'(x, y)$. We also have $a_z b_w \in u\mathbf{A}^{++}$ for such a pair (z, w). Thus by (9) and §1 (4), we obtain (b).

3. *The subset N of W_a*

Recall the definition of the subset M of W_a (see §1 (14)). Now we define

$$N = \{y^{-1} w_J y \mid J \in \mathbf{S},\ w_J \cdot y \in M\}. \tag{1}$$

We shall prove that $y^{-1} w_J y = y^{-1} \cdot w_J \cdot y$ for any $J \in \mathbf{S}$ and $w_J \cdot y \in M$, which will imply that N is a set of representatives for the ST-classes of W_a in $W_{(\nu)}$. The set N will play a crucial role in the proof of Theorem 1.1.

LEMMA 3.1. *In the group (W_a, S), let $J \subset S$. Then for any $z \in W_J$, we have $\deg_\xi f_{w_J, w_J, z} = l(z)$.*

Proof. We see that for any $z \in W_J$, the equalities

$$l(w_J z^{-1} w_J) = l(z) \quad \text{and} \quad l(w_J) = l(z w_J) + l(w_J z^{-1} w_J)$$

hold. So by §1 (1), we have

$$\begin{aligned} T_{w_J} T_{w_J} &= T_{zw_J} T_{w_J z^{-1} w_J} T_{w_J} = T_{zw_J}\Big(\xi^{l(z)} T_{w_J} + \sum_{x \leqslant w_J} a_x T_x\Big) \\ &= \xi^{l(z)} T_z + \sum_{y \leqslant w_J} b_y T_y, \end{aligned} \tag{2}$$

where all the a_x, b_y are in $\mathbb{Z}^+[\xi]$. Thus we have $\deg_\xi f_{w_J, w_J, z} \geqslant l(z)$. Hence our result follows by §1 (7).

LEMMA 3.2. *Let $x, y \in W_a$ and $J \in \mathbf{S}$. If $xw_J = x \cdot w_J$ and $w_J y = w_J \cdot y$, then we have $xw_J y = x \cdot w_J \cdot y$.*

Proof. Suppose not. Then there exists a counterexample such that $l(x)$ is the minimum possible. Clearly, $l(x) > 0$. There must exist some $s \in \mathscr{L}(x)$ such that

$$sxw_J y = sx \cdot w_J \cdot y \quad \text{and} \quad l(xw_J y) < l(x) + (w_J) + l(y).$$

Hence we have

$$\begin{aligned} T_{xw_J} T_{w_J y} &= T_x (T_{w_J} T_{w_J}) T_y \\ &= \xi^{\nu} T_x T_{w_J} T_y + \sum_{w \leqslant w_J} b_w T_x T_w T_y \qquad \text{by (1)} \\ &= \xi^{\nu} T_s T_{sxw_J y} + \sum_{w \leqslant w_J} b_w T_x T_{wy} \\ &= \xi^{\nu+1} T_{sxw_J y} + \xi^{\nu} T_{xw_J y} + \sum_z \Big(\sum_{w \leqslant w_J} b_w f_{x, wy, z} \Big) T_z, \end{aligned}$$

where all the $b_w, f_{x,wy,z}$ are in $\mathbb{Z}^+[\xi]$. This implies that

$$\deg_\xi f_{xw_J, w_J y, sxw_J y} \geqslant \nu + 1$$

which contradicts § 1 (10). Hence the result follows.

By Lemma 3.2 and § 1 (15), we obtain the following result.

PROPOSITION 3.3. *In the group* (W_a, S), *we have*

$$y^{-1} w_J y = y^{-1} \cdot w_J \cdot y \quad \textit{for any } J \in \mathrm{S} \textit{ and } w_J \cdot y \in M.$$

Hence N is a set of representatives for the ST-*classes of W_a in $W_{(\nu)}$.*

4. *A geometric description of the affine Weyl groups*

We shall give, following Lusztig (see [**2**, 7.5; **4**]), a geometric description of the affine Weyl group $W_a = (W_a, S)$.

Let E be an affine euclidean space with a given set of hyperplanes $\mathscr{F}$. Let Ω be the set of right affine motions in E generated by the orthogonal reflections in the various hyperplanes P in $\mathscr{F}$. We assume that Ω is an infinite discrete group acting irreducibly on the space of translations of E and leaving stable the set $\mathscr{F}$. The connected components of the set $E - \bigcup_{P \in \mathscr{F}} P$ are called alcoves of E. Let X be the set of all alcoves of E. Then Ω acts simply transitively on X. Let S_1 be the set of Ω-orbits in the set of codimension 1 facets of alcoves. Each $s \in S_1$ defines an involution $A \mapsto sA$ of X, where, for an alcove A, sA is the alcove differing from A and having a common facet of type s with A. The maps $A \mapsto sA$ generate a group of permutations of X. This group, together with its subset S_1, is a Coxeter group. We call it an affine Weyl group. We shall assume that (W_a, S) is this particular Coxeter group with $S = S_1$.

We assume that W_a acts on the left on X. It acts simply transitively and commutes with the action of Ω on X. A special point in E is a 0-dimensional facet v of an alcove such that the number n of hyperplanes $P \in \mathscr{F}$ passing through v is the maximum possible. Let Φ be the root system determined by W_a. Then it is well known that $n = |\Phi^+| = \nu$. For such v, we denote by W_v the subgroup of W_a which is the stabilizer of the set of alcoves containing v in their closure. Then W_v is a standard parabolic subgroup of W_a isomorphic to the Weyl group of Φ. We denote by w_v the longest element of W_v. Then w_v has the form $w_v = w_J$ for some $J \in \mathrm{S}$. Hence $l(w_v) = \nu$. Conversely, for any w_J with $J \in \mathrm{S}$, there exists some special points v in E such that $w_J = w_v$. We choose, for each special point v, a connected component C_v^+ of the set

$$\hat{C} = E - \bigcup_{\substack{P \in \mathscr{F} \\ v \in P}} P$$

in such a way that for any two special points v, v' in E, $C_{v'}^{+}$ is a translate of C_v^{+}. Let A_v^{+} be the unique alcove contained in C_v^{+} and having v in its closure, and let $A_v^{-} = w_v A_v^{+}$. We denote by C_v^{-} the connected component of the set $\hat{C}$ which contains A_v^{-}.

REMARK. Fix a special point v in E and let $A_w = wA_v^{+}$. Then the above description of (W_a, S) coincides with that in [5].

Let $\mathscr{F}^*$ be the set of hyperplanes $P \in \mathscr{F}$ such that P is a wall of C_v^{+} for some special point v. The connected components of $E - \bigcup_{P \in \mathscr{F}^*} P$ will be called boxes. Clearly, any alcove is contained in a unique box. If v is a special point, we denote by Π_v the box containing A_v^{+}.

To any alcove A, we associate a subset $\mathscr{L}'(A) \subset S$ as follows. Let $s \in S$ and let P be the hyperplane in $\mathscr{F}$ supporting the common facet of type s of A and sA. We say that $s \in \mathscr{L}'(A)$ if A is in that half-space determined by P which meets C_v^{+} for any special point v.

Following [2, 7.6], we consider the free **A**-module $\mathscr{M}$ with basis corresponding to the alcoves in X. It can be regarded as a left **H**-module. For $s \in S$ and $A \in X$,

$$T_s A = \begin{cases} sA & \text{if } s \in S - \mathscr{L}'(A), \\ sA + \xi A & \text{if } s \in \mathscr{L}'(A). \end{cases} \tag{1}$$

Given a special point v in E, the following facts are well known:

$$T_w(A_v^{-}) = w(A_v^{-}) \quad \text{for any } w \leqslant w_v, \tag{2}$$

$$T_y(A_v^{+}) = y(A_v^{+}) \quad \text{and} \quad T_{y^{-1}}(y(A_v^{-})) = A_v^{-} \quad \text{for any } y \in W_a \text{ with } yw_v = y \cdot w_v. \tag{3}$$

The following result is due to Lusztig (see [4, the proof of Proposition 4.2 (b)]).

PROPOSITION 4.1. *Let v be a special point, let $A \in X$ have v in its closure and let $y, w \in W_a$ be such that $y(A_v^{+}) \subset C_v^{+}$ and $wA = A_v^{+}$. Then in the expression*

$$T_y A = \sum_{B \in X} M_{y,A,B} B \tag{4}$$

we have $M_{y,A,B} \in \mathbb{Z}^{+}[\xi]$ for any $B \in X$. In particular, if $l(w) > 0$ and $y(A_v^{+}) \subset \Pi_v$, then $\deg_\xi M_{y,A,B} < l(w)$ for any $B \in X$.

LEMMA 4.2. *Let v, v' be two special points of E and let $x \in W_a$. Then we have the following results:*

(a) *$A_{v'}^{+} \subset C_v^{+}$ if and only if $A_v^{-} \subset C_{v'}^{-}$;*
(b) *$x(A_v^{+}) \subset C_v^{+}$ if and only if $xw_v = x \cdot w_v$;*
(c) *$x(A_v^{-}) \subset C_v^{-}$ if and only if $xw_v = x \cdot w_v$.*

Proof. Part (a) follows directly by the definitions of $A_{v'}^{+}$, A_v^{-}, C_v^{+} and $C_{v'}^{-}$; (b) was proved by Lusztig (see [4, Lemma 3.6]). Finally, (c) is a simple consequence of (a) and (b).

Recall that in §1 we defined a subset M of W_a. The following result gives a geometric description of the set M.

PROPOSITION 4.3. *Let v be a special point in E and let $z \in W_a$ be such that $w_v z = w_v \cdot z$. Then*

$$w_v z \in M \quad \text{if and only if} \quad z^{-1}(A_v^+) \subset \Pi_v \tag{5}$$

Proof. ($\Rightarrow$) By the condition that $w_v z = w_v \cdot z$ and by Lemma 4.2 (b), we have

$$A = z^{-1}(A_v^+) \subset C_v^+. \tag{6}$$

Let $\bar{C}_v^+$ be the closure of C_v^+ in E. Then there exists a unique special point $v' \in \bar{C}_v^+$ such that $A \subset \Pi_{v'}$. Let $x \in W_a$ be such that

$$A = x(A_{v'}^+). \tag{7}$$

Then

$$x w_{v'} = x \cdot w_{v'}. \tag{8}$$

Since $A_{v'}^+ \subset C_v^+$, by Lemma 4.2 (a), we have $A_v^- \subset C_{v'}^-$. Let $y \in W_a$ be such that

$$A_v^- = y(A_{v'}^-) = y w_{v'}(A_{v'}^+). \tag{9}$$

Then, by Lemma 4.2 (c), we have

$$y w_{v'} = y \cdot w_{v'}. \tag{10}$$

Thus by (7) and (9), we obtain

$$A = x(A_{v'}^+) = x w_{v'} y^{-1}(A_v^-). \tag{11}$$

Comparing (6) with (11), we see that

$$z^{-1} w_v = x w_{v'} y^{-1} = x \cdot w_{v'} \cdot y^{-1}, \tag{12}$$

where the last equality holds by (8), (10) and Lemma 3.2. Thus $y \cdot w_{v'} \cdot x^{-1} \in M$. This forces $y = 1$ and hence $v' = v$ by (9), that is, $A \subset \Pi_v$.

($\Leftarrow$) Let V be a set of representatives for the Ω-orbits of the special points in E. We define

$$\mathscr{B} = \{A \in X \mid A \subset \Pi_v \text{ for some } v \in V\}. \tag{13}$$

This is the fundamental region for the actions of all translations of Ω on X in the strict sense that for every $A' \in X$, there exists a unique $A \in \mathscr{B}$ such that $A = (A')\tau$ for some translation τ of Ω on E. Hence we have

$$|\mathscr{B}| = |W| = |M|. \tag{14}$$

Now we define a map $\theta: M \to \mathscr{B}$ as follows. For any $x \in M$, we may write $x = w_v \cdot z$ for some special point v in E and $z \in W_a$. Then by the above proof, we see that $z^{-1}(A_v^+) \subset \Pi_v$. Let τ be the translation of Ω such that $A = (z^{-1}(A_v^+))\tau \in \mathscr{B}$. Then A is uniquely determined by x. We define $\theta(x) = A$. Clearly, the map θ is well defined. To reach our goal, it is enough to show that the map θ is bijective.

Suppose that there exists some $y \in M$ such that $\theta(y) = \theta(x)$. We write $y = w_{v'} \cdot z'$ for some special point v' and $z' \in W_a$. Then $z'^{-1}(A_{v'}^+) \subset \Pi_{v'}$. By our assumption, there exists some translation $\iota \in \Omega$ such that

$$z^{-1}(A_v^+) = (z'^{-1}(A_{v'}^+))\iota. \tag{15}$$

Thus

$$z^{-1}(A_v^+) = z'^{-1}((A_{v'}^+)\iota) = z'^{-1}(A_{(v')\iota}^+). \tag{16}$$

This implies that $v = (v')\iota$ and hence $w_v = w_{v'}$. This also implies that $z = z'$, that is, $x = y$. Thus the map θ is injective and hence by (14), θ must be bijective. The result follows.

PROPOSITION 4.4. *Let* $x, y \in W_a$ *and* $J \in S$ *be such that* $w_J x^{-1} \in M$ *and* $xw_J y = x \cdot w_J \cdot y$. *Let* $w < w_J$. *Then for any* $z \in W_a$, *the inequality*

$$\deg_\xi f_{x,wy,z} < \nu - l(w) \tag{17}$$

holds.

Proof. There exists some special point v in E such that $w_J = w_v$. Let $A = y^{-1}(A_v^-)$. Then by the assumption that $w_v y = w_v \cdot y$ and by Lemma 4.2 (c), it follows that $A \in C_v^-$. Thus by (2), (3) and the fact that $wy = w \cdot y$, we have

$$A' = T_{wy} A = wyA, \tag{18}$$

where A' contains v in its closure. Let $w' = w_v w^{-1}$. Then

$$l(w') = \nu - l(w) > 0$$

and $w'(A') = A_v^+$. Since $w_v \cdot x^{-1} \in M$, we have $x(A_v^+) \subset \Pi_v$ by Proposition 4.3. Hence, by Proposition 4.1, this implies that

$$\deg_\xi M_{x,wyA,B} < \nu - l(w) \tag{19}$$

for any $B \in X$. Thus we have

$$T_x T_{wy} A = T_x(wyA) = \sum_B M_{x,wyA,B} B = \sum_z f_{x,wy,z} T_z A$$

$$= \sum_B \left(\sum_z f_{x,wy,z} M_{z,A,B}\right) B.$$

Hence $\sum_{z \in W_a} f_{x,wy,z} M_{z,A,B} = M_{x,wyA,B}$ for any $B \in X$. Since $f_{x,wy,z} M_{z,A,B} \in \mathbb{Z}^+[\xi]$, it follows from (19) that

$$\deg_\xi f_{x,wy,z} M_{z,A,B} < \nu - l(w) \quad \text{for any } z \in W_a \text{ and } B \in X. \tag{20}$$

We take $B = zA$. Then $M_{z,A,B} \neq 0$ (its value for $\xi = 0$ is equal to 1). Thus, by (20), we reach our goal.

5. *The distinguished involutions*

Recall that the group (G, S) is assumed to be crystallographic with property (A) (p. 254).

Let $\delta(z) = \deg P_{1,z}(u)$ for any $z \in G$. In [3], Lusztig proved that

$$a(z) \leqslant l(z) - 2\delta(z) \quad \text{for all } z \in G \tag{1}$$

[3, 1.3 (a)]. He defined the following subset $\mathcal{D}$ of G:

$$\mathcal{D} = \{z \in G \mid a(z) = l(z) - 2\delta(z)\} \tag{2}$$

[3, 1.3 (b)]. Then he proved the following results.

PROPOSITION 5.1. (a) *All elements of* $\mathcal{D}$ *are involutions* [3, Proposition 1.4 (a)].

(b) *Any left cell* L *of* G *contains a unique element* d *of* $\mathcal{D}$ *which satisfies* $\gamma_{x^{-1},x,d} = 1$ *for all* $x \in L$ *(see* §1 (9) *and* [3, Theorem 1.10]).

(c) *If* $z \in G$ *belongs to a standard parabolic subgroup* G_0 *of* G, *then* $a(z)$ *(respectively* $\delta(z)$*) computed with respect to* G_0 *is equal to* $a(z)$ *(respectively* $\delta(z)$*) computed with respect to* G [3, Corollary 1.9 (d)].

The elements of $\mathscr{D}$ are called the distinguished involutions of G [3, 1.4]. Some results on the cells of the group (G, S) may be deduced by making use of the properties of $\mathscr{D}$.

For any $J \subset S$, we write

$$\mathscr{D}_J = W_J \cap \mathscr{D}. \tag{3}$$

Then by Proposition 5.1 (c), we see that $\mathscr{D}_J$ is the set of all distinguished involutions of W_J. Suppose that L is a left cell of G with $L \cap W_J \neq \varnothing$. Then $L \cap W_J$ is a union of m left cells of W_J for some $m \geqslant 1$. Thus by Proposition 5.1 (b), we have

$$|\mathscr{D}_J \cap (L \cap W_J)| = m.$$

But on the other hand, we have $|\mathscr{D} \cap L| = 1$ and hence

$$1 \leqslant m = |\mathscr{D}_J \cap (L \cap W_J)| = |(\mathscr{D} \cap W_J) \cap (L \cap W_J)| = |(\mathscr{D} \cap L) \cap W_J| \leqslant 1.$$

This forces $m = 1$. We obtain the following result.

PROPOSITION 5.2. *Let L be a left cell of (G, S) and let $J \subset S$. Then the intersection $L \cap W_J$ is either empty or a left cell of W_J.*

6. *The proof of Theorem* 1.1

Now we consider the affine Weyl group $W_a = (W_a, S)$. The following result is crucial for the proof of Theorem 1.1.

Note that any ST-class Y of W_a in $W_{(\nu)}$ is a maximal set in $W_{(\nu)}$ with the property that, for any $x, y \in Y$, there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in Y such that for every i, $1 \leqslant i \leqslant r$, $x_{i-1} x_i^{-1} \in S$ (see [5, §8]).

THEOREM 6.1. $\mathscr{D} \cap W_{(\nu)} = N$.

Proof. For any $x \in \mathscr{D} \cap W_{(\nu)}$, we may write

$$x = y \cdot w_J \cdot z \in Y(z, J) \tag{1}$$

(see §1 (15)) for some $J \in \mathbf{S}$ and $y, z \in W_a$. By §1 (15), we have $x \sim_L w_J \cdot z$. So by Proposition 5.1 (b), we have $h_{z^{-1}w_J, w_J z, x} \neq 0$. We see from Propositions 2.3 and 2.5 that

$$x \leqslant \lambda(z^{-1} w_J, w_J z) = z^{-1} \cdot w_J \cdot z. \tag{2}$$

By (1) and (2), this implies that $y \leqslant z^{-1}$. By Proposition 5.1 (a), we have

$$x = x^{-1} = z^{-1} \cdot w_J \cdot y^{-1}.$$

From (1) and the preceding remark, it follows that $w_J \cdot y^{-1} \in Y(z, J)$ and hence $l(y) = l(y^{-1}) \geqslant l(z) = l(z^{-1})$. Thus $y = z^{-1}$, that is, $x \in N$. Thus the inclusion $\mathscr{D} \cap W_{(\nu)} \subseteq N$ is proved. Now let $z^{-1} \cdot w_J \cdot z \in N$ with $J \in \mathbf{S}$. Then

$$\begin{aligned} T_{z^{-1}w_J} T_{w_J z} &= T_{z^{-1}}(T_{w_J} T_{w_J}) T_z = T_{z^{-1}}\Big(\sum_{w \leqslant w_J} f_{w_J, w_J, w} T_w\Big) T_z \\ &= f_{w_J, w_J, w_J} T_{z^{-1} w_J z} + \sum_{w < w_J} f_{w_J, w_J, w} T_{z^{-1}} T_{wz} \\ &= f_{w_J, w_J, w_J} T_{z^{-1} w_J z} + \sum_{y \in W_a} \Big(\sum_{w < w_J} f_{w_J, w_J, w} f_{z^{-1}, wz, y}\Big) T_y. \end{aligned}$$

This implies that for any $y \in W_a$ with $y \neq z^{-1}w_J z$, we have

$$f_{z^{-1}w_J, w_J z, y} = \sum_{w < w_J} f_{w_J, w_J, w} f_{z^{-1}, wz, y}.$$

By Lemma 3.1 and Proposition 4.4, we see that

$$\deg_\xi f_{w_J, w_J, w} f_{z^{-1}, wz, y} < l(w) + (v - l(w)) = v \quad \text{for any } w < w_J.$$

Hence $\deg_\xi f_{z^{-1}w_J, w_J z, y} < v$ for any $y \in W_a$ with $y \neq z^{-1}w_J z$. We know from §1 (2) that

$$C_{z^{-1}w_J} C_{w_J z} = T_{z^{-1}w_J} T_{w_J z} + \sum_{\alpha, \beta} a_{\alpha\beta} T_\alpha T_\beta, \tag{3}$$

where the pair (α, β) in the above sum ranges over $K'(z^{-1}w_J, w_J z)$ (see §2 (10)). The $a_{\alpha\beta}$ are all in $u\mathbf{A}^{++}$. Thus by the fact that $f_{\alpha,\beta,\gamma} \in \mathbb{Z}^+[\xi]$, the inequality $\deg_\xi f_{\alpha,\beta,\gamma} \leqslant v$, (3) and §1 (4), we have

$$u^v h_{z^{-1}w_J, w_J z, y} \in u\mathbf{A}^+ \quad \text{for all } y \in W_a \text{ with } y \neq z^{-1}w_J z.$$

That is, $\gamma_{z^{-1}w_J, w_J z, y} = 0$ for all such y. Thus by Proposition 5.1 (b), we have $z^{-1} \cdot w_J \cdot z \in \mathscr{D}$. This implies that $N \subseteq \mathscr{D} \cap W_{(v)}$. Hence our result follows.

Proof of Theorem 1.1. We know from §1 that the set $W_{(v)}$ consists of $|W|$ ST-classes of W_a. We also know from Proposition 3.3 that N is a set of representatives for the ST-classes of W_a in $W_{(v)}$. By Theorem 6.1, N consists of all distinguished involutions of W_a in $W_{(v)}$ and so our result follows from Proposition 5.1 (b).

References

1. D. KAZHDAN and G. LUSZTIG, 'Representations of Coxeter groups and Hecke algebras', *Invent. Math.* 53 (1979) 165–184.
2. G. LUSZTIG, 'Cells in affine Weyl groups', *Algebraic groups and related topics*, Advanced Studies in Pure Mathematics 6 (1985) 255–287.
3. G. LUSZTIG, 'Cells in affine Weyl groups, II', *J. Algebra* 109 (1987) 536–548.
4. G. LUSZTIG, 'Hecke algebras and Jantzen's generic decomposition patterns', *Adv. in Math.* 37 (1980) 121–164.
5. J.-Y. SHI, 'Sign types corresponding to an affine Weyl group', *J. London Math. Soc.* (2) 35 (1987) 56–74.
6. J.-Y. SHI, 'A two-sided cell in an affine Weyl group', *J. London Math. Soc.* (2) 36 (1987) 407–420.
7. D. N. VERMA, 'The role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras', *Lie groups and their representaions* (ed. I. M. Gelfand, Halsted, New York, 1975) 653–722.

Department of Mathematics
East China Normal University
Shanghai
China