A TWO-SIDED CELL IN AN AFFINE WEYL GROUP

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1. *Introduction*

Let $G = (G, S)$ be a Coxeter group with S its distinguished generator set. We define, following Lusztig [4], the corresponding Hecke algebra H over $A = \mathbb{Z}[u, u^{-1}]$, where u is an indeterminate, as follows. The set H has basis elements T_w, $w \in G$ as a free A-module and multiplication is defined by the rules

$$T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w'),$$
$$(T_s + u)(T_s - u^{-1}) = 0 \quad \text{if } s \in S.$$

As an A-module, H also has the basis $(C_w)_{w \in G}$:

$$C_w = \sum_{y \leqslant w} u^{l(w)-l(y)} P_{y,w}(u^{-2}) T_y,$$

where l is the length function on G. The $P_{y,w}(x)$ are known as the Kazhdan–Lusztig polynomials [2].

We define for any $x, y, z \in G$, elements $h_{x,y,z} \in A$ such that

$$C_x C_y = \sum_z h_{x,y,z} C_z.$$

Let $A^+ = \mathbb{Z}[u]$. Given $z \in G$, if there exists an integer $N \geqslant 0$ such that $u^N h_{x,y,z} \in A^+$ for all $x, y \in G$ then we define $a(z)$ to be the smallest integer satisfying $u^{a(z)} h_{x,y,z} \in A^+$ for all $x, y \in G$; if there is no such integer then we define $a(z) = \infty$.

From now on, we assume G to be an indecomposable affine Weyl group, denoted by W_a. Let Φ be the root system whose type is determined by W_a. Let Φ^+ be a positive root system of Φ with Δ its simple root system. Set $v = |\Phi^+|$. Then in [3], Lusztig proved that $a(z) \leqslant v$ for all $z \in W_a$. Let $W_{(v)} = \{w \in W_a \mid a(w) = v\}$.

For any subset $J \subset S$, let W_J be the subgroup of W_a generated by J, which is isomorphic to some Weyl group. Let w_J be the longest element in W_J.

Let **S** be the set of all subsets J of S such that W_J is isomorphic to the Weyl group on Φ. Let

$$W(\mathbf{S}) = \{w \in W_a \mid w = x \cdot w_J \cdot y \text{ with } J \in \mathbf{S}\},$$

where the notation $z = x \cdot y$ means that $z = xy$ and $l(z) = l(x) + l(y)$ for any $x, y, z \in W_a$.

Lusztig proved that $W(\mathbf{S}) \subseteq W_{(v)}$ [3]. In the present paper, we shall prove the following theorem.

THEOREM 1.1. *For any indecomposable affine Weyl group $W_a = (W_a, S)$, we have $W(\mathbf{S}) = W_{(v)}$.*

This result will be used in §5 to deduce a new result on cells of W_a.

We shall prove Theorem 1.1 in §§2 to 4. In the case when the rank of W_a is less than

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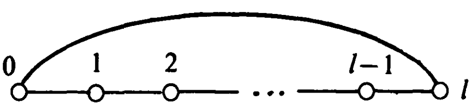
or equal to 2, Theorem 1.1 can be deduced easily from Lusztig's results on cells of W_a [3]. Thus, in §§2 to 4, we always assume that the rank of W_a is greater than 2.

2. *The coordinate form of* $w \in W_a$

Let E be the euclidean space spanned by Φ with inner product $\langle , \rangle$ such that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 1$ for any short root $\alpha \in \Phi$. Then the affine Weyl group W_a can be regarded as a group of right isometric transformations on E. More precisely, let W be the Weyl group of Φ generated by the reflections s_α on E for $\alpha \in \Phi$; s_α sends $x \in E$ to $x - \langle x, \alpha^\vee \rangle \alpha$, where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. Let Q denote the root lattice $\mathbb{Z}\Phi$. Let N denote the group consisting of all translations T_λ, $\lambda \in Q$, on E; T_λ sends x to $x + \lambda$. Then W_a can be regarded as the semi-direct product $N \rtimes W$. There is a canonical homomorphism from W_a to $W: w \mapsto \bar{w}$.

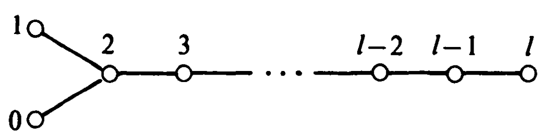
Let $-\alpha_0$ be the highest short root of Φ. We define $s_0 = s_{\alpha_0} T_{-\alpha_0}$ and $s_i = s_{\alpha_i}$, $1 \leqslant i \leqslant l$, where $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Then the generator set S of W_a can be taken as $S = \{s_0, s_1, \ldots, s_l\}$. The Dynkin diagram of W_a has one of the following figures. (Recall that we assume that the rank of W_a is greater than 2.)

$\tilde{A}_l$



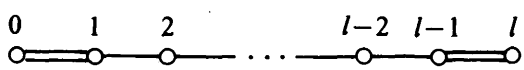
$l \geqslant 3$

$\tilde{B}_l$



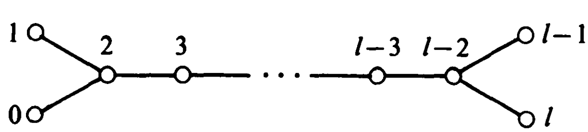
$l \geqslant 3$

$\tilde{C}_l$



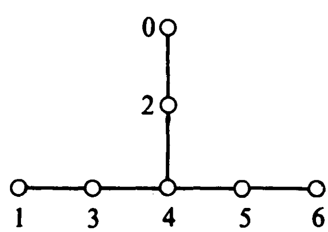
$l \geqslant 3$

$\tilde{D}_l$

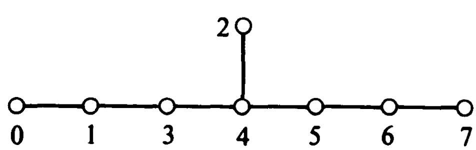


$l \geqslant 4$

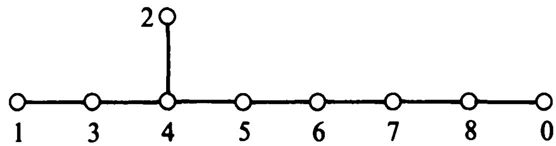
$\tilde{E}_6$



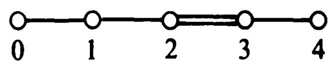
$\tilde{E}_7$



$\tilde{E}_8$



$\tilde{F}_4$



In [7], the author defined a set $E(\Phi)$ of Φ-tuples $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi}$ which satisfy

(1) $k_\alpha = -k_{-\alpha} \in \mathbb{Z}$ for $\alpha \in \Phi$.
(2) For any $\alpha, \beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$, the inequality

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta < |\alpha+\beta|^2(k_{\alpha+\beta}+1) < |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 \quad (2.0.1)$$

holds. It was proved that there exists a bijection between W_a and $E(\Phi)$ such that if $w \in W_a$ corresponds to $\mathbf{k}^w = (k(w, \alpha))_{\alpha\in\Phi}$ then

$$k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(\bar{w}, \alpha) \quad \text{for } \alpha \in \Phi, \quad (2.0.2)$$

where $w = \bar{w}T_\lambda$ for $\bar{w} \in W$ and $\lambda \in Q$, and for any $\alpha \in \Phi^+$ and $x \in W$, the integer $k(x, \alpha)$ is defined as follows:

$$k(x, \alpha) = \begin{cases} 0 & \text{if } (\alpha)x^{-1} \in \Phi^+, \\ -1 & \text{if } (\alpha)x^{-1} \in \Phi^-. \end{cases} \quad (2.0.3)$$

We call $\mathbf{k}^w$ the coordinate form of w. We shall identify $\mathbf{k}^w$ with w.

For $w \in W_a$, we define

$$\mathscr{L}(w) = \{s \in S \mid sw < w\}, \qquad \mathscr{R}(w) = \{s \in S \mid ws < w\},$$

where the relation $\leqslant$ is the Bruhat order on W_a [8].

PROPOSITION 2.1. *The coordinate form* $\mathbf{k}^w$ *of* w *has the following properties.*

(1) $l(w) = \sum_{\alpha\in\Phi^+} |k(w, \alpha)|$.
(2) *Let* $x = ws_t$ *and* $y = s_t w$, $0 \leqslant t \leqslant l$. *Then for* $\alpha \in \Phi$,

$$k(x, \alpha) = k(w, (\alpha)\bar{s}_t) + k(s_t, \alpha) \quad \textit{and} \quad k(y, \alpha) = k(w, \alpha) + k(s_t, (\alpha)\bar{w}^{-1}),$$

where

$$k(s_t, \alpha) = \begin{cases} 0 & \textit{if } \alpha \neq \pm\alpha_t, \\ 1 & \textit{if } \alpha = -\alpha_t, \\ -1 & \textit{if } \alpha = \alpha_t. \end{cases} \quad (2.1.1)$$

(3) $\mathscr{L}(w) = \{s_i \mid k(w, (\alpha_i)\bar{w}) > 0,\ 0 \leqslant i \leqslant l\}$.
(4) $\mathscr{R}(w) = \{s_i \mid k(w, \alpha_i) < 0,\ 0 \leqslant i \leqslant l\}$.

The proof of the above results can be found in [6].

Given S_1, $S_2 \in \mathbf{S}$, there always exists an automorphism φ of W_a which preserves S and sends S_1 to S_2. Conversely, let φ be an automorphism of W_a which preserves S and sends s_i to $s_{i'}$, $0 \leqslant i \leqslant l$, where $i \mapsto i'$ is a permutation on the set $\{0, 1, 2, \ldots, l\}$. Then φ induces a permutation on the set of subsets of S which preserves the set $\mathbf{S}$. The automorphism φ also gives rise to an automorphism of the root system Φ, which we shall also denote by ϕ, so that $\phi(\alpha_i) = \alpha'_i$, $0 \leqslant i \leqslant l$. If $w = (k_\alpha)_{\alpha\in\Phi} \in W_a$ and

$w' = \varphi(w) = (k'_\alpha)_{\alpha\in\Phi}$ then $k'_{\varphi(\alpha)} = k_\alpha$ for all $\alpha\in\Phi$. Thus by the formulae (2.0.2) and (2.0.3) and Proposition 2.1, it is easily seen that we have the following.

LEMMA 2.2. *Let $w = w_J$, $J \subset S$. Then $|k(w,\alpha)| = 1$ for all $\alpha\in\Phi$. Thus when $J\in\mathbf{S}$, we have $|k(w,\alpha)| = 1$ for all $\alpha\in\Phi$. In particular, when $J = \{s_1, s_2, \ldots, s_l\}$, we have $k(w,\alpha) = -1$ for all $\alpha\in\Phi^+$.*

We define $\mathbf{W} = \{w\in W_a \mid k(w,\alpha) \neq 0, \text{ for } \alpha\in\Phi\}$. We state the following two results whose proof will be given in subsequent sections.

THEOREM 2.3. $\mathbf{W} = W(\mathbf{S})$.

THEOREM 2.4. $\mathbf{W} = W_{(v)}$.

Clearly, Theorem 1.1 is a direct consequence of these two theorems. From these two theorems, we get three equivalent descriptions of the set $W_{(v)}$.

Let us conclude this section by recording an empirical result on the set $\mathbf{S}$ which will be used in §3.

Put $-\alpha_0 = \sum_{i=1}^{l} a_i\alpha_i$.

PROPOSITION 2.5. *$J\in\mathbf{S}$ if and only if $J = S-\{s_i\}$, where either $i = 0$ or $\alpha_i\in\Delta$ is a short root with $a_i = 1$.*

3. *Proof of Theorem* 2.3

In this section we prove Theorem 2.3.

For any $\alpha\in Q$, set $\Delta_\alpha = \{\gamma\in\Delta \mid \gamma < \alpha\}$.

LEMMA 3.1. *Let $\alpha,\beta\in\Phi^+$ with $\beta < \alpha$. Then there exists a sequence*

$$\beta_0 = \beta, \beta_1, \ldots, \beta_r = \alpha$$

in Φ^+ such that $\gamma_i = \beta_i - \beta_{i-1}\in\Delta_\alpha$ for every i, $1 \leqslant i \leqslant r$.

Proof. The proof is by induction on $m = \mathrm{ht}(\alpha) - \mathrm{ht}(\beta) > 0$, where $\mathrm{ht}(\alpha)$ denotes the height of the root $\alpha\in\Phi$. It is obvious when $m = 1$. Now assume that $m > 1$. Put $\eta = \alpha - \beta$ (note that η is not necessarily in Φ). We claim that there exists some $\gamma\in\Delta_\eta$ with $\langle\gamma,\beta^\vee\rangle < 0$, for otherwise

$$\langle\gamma,\beta^\vee\rangle \geqslant 0 \quad \text{for } \gamma\in\Delta_\eta. \tag{3.1.1}$$

Hence $\langle\eta,\beta^\vee\rangle \geqslant 0$ which implies that $\langle\alpha,\beta^\vee\rangle = \langle\eta,\beta^\vee\rangle + \langle\beta,\beta^\vee\rangle \geqslant 2$. Thus $\langle\alpha,\beta^\vee\rangle = 2$ since we have assumed that the rank of Φ is greater than 2. Then $\beta' = \alpha - 2\beta\in\Phi^+$ and $\langle\beta',\beta^\vee\rangle = -2$. Clearly, $\Delta_{\beta'} \subseteq \Delta_\eta$ and there exists some $\delta\in\Delta_{\beta'}$ such that $\langle\delta,\beta^\vee\rangle < 0$, which contradicts (3.1.1).

By the above claim, we have $\zeta = \beta + \gamma\in\Phi^+$ and $\zeta < \alpha$. By the inductive hypothesis, there exists a sequence $\gamma_0 = \zeta, \gamma_1, \ldots, \gamma_t = \alpha$ in Φ^+ with $\gamma_i - \gamma_{i-1}\in\Delta_\alpha$ for $i = 1, \ldots, t$. Thus β, $\gamma_0, \gamma_1, \ldots, \gamma_t = \alpha$ is the required sequence.

Let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$.

LEMMA 3.2. *Let $w \in \mathbf{W}$ be such that either $k(w,\alpha) \geqslant 1$ or $k(w,\alpha) = -1$ for any $\alpha \in \tilde{\Delta}$. If $k(w,\gamma) \geqslant 1$ for some $\gamma \in \Delta$ then $k(w,\beta) \geqslant 1$ for all $\beta \in \Phi^+$ with $\gamma \leqslant \beta$.*

Proof. We know from Lemma 3.1 that for any $\beta \in \Phi^+$ with $\gamma \leqslant \beta$ there exists a sequence of roots $\beta_0 = \gamma, \beta_1, \ldots, \beta_t = \beta$ in Φ^+ such that $\gamma_i = \beta_i - \beta_{i-1} \in \Delta$, $i = 1, \ldots, t$. We shall show that $k(w,\beta_i) \geqslant 1$ for all $i \geqslant 0$. Suppose not, then there exists some $j \geqslant 1$ such that $k(w,\beta_i) \geqslant 1$ for all i, $0 \leqslant i < j$, and $k(w,\beta_j) < 0$. The inequality

$$-1 \leqslant |\beta_{j-1}|^2 - |\gamma_j|^2 \leqslant |\beta_{j-1}|^2 k(w,\beta_{j-1}) + |\gamma_j|^2 k(w,\gamma_j) < |\beta_j|^2 (k(w,\beta_j)+1) \leqslant 0 \quad (3.2.1)$$

implies that γ_j is a long root and β_{j-1} is a short root. This implies that β_j is a short root. That is, $|\gamma_j|^2 = 2|\beta_{j-1}|^2 = 2|\beta_j|^2 = 2$. Again by the inequality (3.2.1) we have $k(w,\gamma_j) = k(w,\beta_j) = -1$ and $k(w,\beta_{j-1}) = 1$. In this case, $\zeta_j = \beta_j + \beta_{j-1} \in \Phi^+$ and $0 < 2(k(w,\zeta_j)+1) < 4$ by (2.0.1). Hence $k(w,\zeta_j) = 0$ which contradicts our assumption that $w \in \mathbf{W}$. Now we have shown that $k(w,\beta_i) \geqslant 1$ for all $i \geqslant 0$. In particular, $k(w,\beta) \geqslant 1$.

LEMMA 3.3. *Put $-\alpha_0 = \sum_{\alpha \in \Delta} a_\alpha \alpha$.*

(1) *If $a_\alpha > 1$ for some $\alpha \in \Phi$ then there exist γ, $\delta \in \Phi^+$ such that $-\alpha_0 = \gamma + \delta$ and $\alpha \leqslant \gamma, \delta$.*
(2) *If $\alpha \neq \beta$ in Δ then there exist $\gamma, \delta \in \Phi^+$ such that $-\alpha_0 = \gamma + \delta$, $\alpha < \gamma$ and $\beta < \delta$.*

Proof. This can be verified case by case.

LEMMA 3.4. *Let $w \in \mathbf{W}$ be such that either $k(w,\alpha) \geqslant 1$ or $k(w,\alpha) = -1$ for any $\alpha \in \tilde{\Delta}$. Suppose that there exists some $\beta \in \Delta$ such that $k(w,\beta) \geqslant 1$. If $\gamma \in \Delta$ and $-\alpha_0 + \gamma \in \Phi^+$ then γ is a short root and $k(w,\gamma) \geqslant 1$.*

Proof. By Lemma 3.2 and our assumption, we have $k(w, -\alpha_0) = 1$. Suppose that $k(w,\gamma) \ngeqslant 1$. Then $k(w,\gamma) = -1$. Since $\delta = -\alpha_0 + \gamma \in \Phi^+$, and since $-\alpha_0$ is the highest short root of Φ, we see that γ is a short root and that δ is a long one. Thus we have $0 < 2(k(w,\delta)+1) < 4$. This forces $k(w,\delta) = 0$ which contradicts our assumption. Hence the result follows.

LEMMA 3.5. *Let $w \in \mathbf{W}$. Suppose that either $k(w,\alpha) \geqslant 1$ or $k(w,\alpha) = -1$ for any $\alpha \in \tilde{\Delta}$. Then the following cases cannot occur.*

(1) *$k(w,\beta) \geqslant 1$ for some $\beta \in \Delta$ with $a_\beta > 1$.*
(2) *$k(w,\alpha)$, $k(w,\beta) \geqslant 1$ for some $\alpha \neq \beta$ in Δ.*
(3) *$k(w,\beta) \geqslant 1$ for some long root $\beta \in \Delta$ when Φ contains two roots of different lengths.*

Proof. By Lemma 3.3, there exist $\gamma, \delta \in \Phi^+$ with $-\alpha_0 = \gamma + \delta$ such that $\beta < \gamma, \delta$ in case (1), or $\alpha < \gamma$ and $\beta < \delta$ in (2). Then by Lemma 3.2, we have $k(w,\gamma)$, $k(w,\delta) \geqslant 1$. By the inequality

$$k(w, -\alpha_0) + 1 > |\gamma|^2 k(w,\gamma) + |\delta|^2 k(w,\delta) \geqslant 2,$$

we obtain $k(w, -\alpha_0) \geqslant 2$, which contradicts our assumption.

In the case when Φ contains roots of two different lengths, $-\alpha_0$ is not the highest root of Φ. Thus there must exist some $\gamma \in \Delta$ such that $-\alpha_0 + \gamma \in \Phi^+$. By Lemma 3.4 and its proof, we see that γ is a short root and $k(w, \gamma) \geqslant 1$. Hence we are back in case (2).

PROPOSITION 3.6. *Let $w \in \mathbf{W}$ be such that either $k(w, \alpha) \geqslant 1$ or $k(w, \alpha) = -1$ for any $\alpha \in \tilde{\Delta}$. Then $\mathscr{R}(w) \in \mathbf{S}$.*

Proof. If $k(w, \alpha) = -1$ for all $\alpha \in \Delta$ then by Proposition 2.1(4),

$$\mathscr{R}(w) = \{s_1, s_2, \ldots, s_l\} \in \mathbf{S}.$$

If there exists some $\alpha \in \Delta$ with $k(w, \alpha) \geqslant 1$ then, by Lemmas 3.2 and 3.5, we have that $k(w, \beta) = -1$ for all $\beta \in \tilde{\Delta}$ with $\beta \neq \alpha$ and that α is a short root with $a_\alpha = 1$. Thus by Propositions 2.1(4) and 2.5, we also have $\mathscr{R}(w) \in \mathbf{S}$.

Recall that for $x, y \in W_a$, the notation $w = x \cdot y$ means $w = xy$ and $l(w) = l(x) + l(y)$. Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. By Proposition 2.1, we see that if $w = x \cdot y \in W_a$ and if either x or y is in $\mathbf{W}$, then w is in $\mathbf{W}$. By Lemma 2.2, we see that the elements w_J, $J \in \mathbf{S}$, are all in $\mathbf{W}$. This implies that $W(\mathbf{S}) \subseteq \mathbf{W}$.

Conversely, let $w \in \mathbf{W}$. Then there must exist $x \in \mathbf{W}$ and $y \in W_a$ with $w = x \cdot y$ such that for any $s \in \mathscr{R}(x)$, $xs \notin \mathbf{W}$. By Proposition 2.1(2), (4), we see that either $k(x, \alpha) \geqslant 1$ or $k(x, \alpha) = -1$ for any $\alpha \in \tilde{\Delta}$. Hence by Proposition 3.6, we have $\mathscr{R}(x) \in \mathbf{S}$, that is, $x = x' \cdot w_J$ for some $J \in \mathbf{S}$. This implies that $w = x' \cdot w_J \cdot y$. Thus $w \in W(\mathbf{S})$ and hence $\mathbf{W} \subseteq W(\mathbf{S})$ and Theorem 2.3 is proved.

4. *Proof of Theorem* 2.4

In this section, we shall prove Theorem 2.4. To do this, we need some results.

LEMMA 4.1. *If $x, y \in W_a$ and $s \in S$ are such that $x = s \cdot y$ and $\mathscr{R}(x) \neq \mathscr{R}(y)$ then $a(x) > a(y)$.*

Proof. We have $\mathscr{L}(x) \not\subset \mathscr{L}(y)$ since $s \in \mathscr{L}(x) - \mathscr{L}(y)$. On the other hand, we have $\mathscr{R}(x) \supseteq \mathscr{R}(y)$ in general. Thus $\mathscr{R}(x) \neq \mathscr{R}(y)$ implies that $\mathscr{R}(x) \not\subset \mathscr{R}(y)$. Hence our result is a special case of a result of Lusztig [3].

In [7], the author defined sign types of type Φ.

A Φ-tuple $X = (X_\alpha)_{\alpha \in \Phi}$ is called a sign type of type Φ (or briefly, a sign type), if the set $\{X_\alpha, X_{-\alpha}\}$ is either $\{\bigcirc, \bigcirc\}$ or $\{+, -\}$ for any $\alpha \in \Phi$. Since X is determined uniquely by the Φ^+-tuple $(X_\alpha)_{\alpha \in \Phi^+}$, we shall identify $(X_\alpha)_{\alpha \in \Phi^+}$ with X. Let $\bar{\mathscr{S}} = \bar{\mathscr{S}}(\Phi)$ be the set of all sign types of type Φ. Let

$$G_1 = \left\{ \begin{matrix} + \\ ++ \end{matrix}, \begin{matrix} + \\ +\bigcirc \end{matrix}, \begin{matrix} + \\ +- \end{matrix}, \begin{matrix} \bigcirc \\ +- \end{matrix}, \begin{matrix} - \\ +- \end{matrix}, \begin{matrix} + \\ \bigcirc+ \end{matrix}, \begin{matrix} + \\ \bigcirc\bigcirc \end{matrix}, \begin{matrix} \bigcirc \\ \bigcirc\bigcirc \end{matrix}, \begin{matrix} \bigcirc \\ \bigcirc- \end{matrix}, \begin{matrix} - \\ \bigcirc- \end{matrix}, \right.$$

$$\left. \begin{matrix} + \\ -+ \end{matrix}, \begin{matrix} \bigcirc \\ -+ \end{matrix}, \begin{matrix} - \\ -+ \end{matrix}, \begin{matrix} \bigcirc \\ -\bigcirc \end{matrix}, \begin{matrix} - \\ -\bigcirc \end{matrix}, \begin{matrix} - \\ -- \end{matrix} \right\}$$

and

$$G_2=\left\{\begin{array}{cccccccccc} \bigcirc & \bigcirc & \bigcirc & - & - & - & + & \bigcirc & \bigcirc & \bigcirc \\ \bigcirc\bigcirc & +\bigcirc & +\bigcirc & \bigcirc\bigcirc & -\bigcirc & -\bigcirc & +\bigcirc & \bigcirc- & \bigcirc- & +- \\ \bigcirc, & \bigcirc, & +, & \bigcirc, & \bigcirc, & -, & +, & \bigcirc, & -, & \bigcirc, \end{array}\right.$$

$$\begin{array}{ccccccccc} \bigcirc & + & + & - & + & + & + & - & - \\ -- & \bigcirc- & +- & -- & -- & +- & +- & \bigcirc+ & \bigcirc+ \\ -, & -, & \bigcirc, & -, & -, & -, & +, & \bigcirc, & +, \end{array}$$

$$\left.\begin{array}{cccccc} - & \bigcirc & - & - & - & + \\ -+ & ++ & ++ & -+ & -+ & ++ \\ \bigcirc, & +, & +, & +, & -, & + \end{array}\right\}.$$

For any subsystem Φ' of Φ, $\Phi'^+ = \Phi^+ \cap \Phi'$ is a positive subsystem of Φ.

Given an indecomposable positive subsystem Φ'^+ of Φ of rank 2, we say that a sign type $(X_\alpha)_{\alpha\in\Phi'^+}$ is admissible if we have one of the following cases.

(1) Φ'^+ has type A_2, say $\Phi'^+ = \{\alpha,\beta,\alpha+\beta\}$. Then

$$\begin{array}{ccc} & X_{\alpha+\beta} & \\ X_\alpha & & X_\beta \end{array}$$

belongs to G_1.

(2) Φ'^+ has type B_2, say $\Phi'^+ = \{\alpha,\beta,\alpha+\beta,2\alpha+\beta\}$. Then

$$\begin{array}{ccc} & X_\beta & \\ X_{\alpha+\beta} & & X_\alpha \\ & X_{2\alpha+\beta} & \end{array}$$

belongs to G_2.

We say that a sign type $(X_\alpha)_{\alpha\in\Phi}$ is admissible if for any indecomposable positive subsystem Φ'^+ of Φ of rank 2, the sign type $(X_\alpha)_{\alpha\in\Phi'^+}$ is admissible. Let $\mathscr{S} = \mathscr{S}(\Phi)$ be the set of all admissible sign types of $\bar{\mathscr{S}}$.

We know from [7] that there exists a surjective map $\zeta\colon W_a \to \mathscr{S}$ which maps $x = (k(x,\alpha))_{\alpha\in\Phi}$ to $X = (X_\alpha)_{\alpha\in\Phi}$ such that, for any $\alpha\in\Phi$,

$$k(x,\alpha) > 0 \text{ if and only if } X_\alpha = +\,,$$

$$k(x,\alpha) = 0 \text{ if and only if } X_\alpha = \bigcirc\,,$$

$$k(x,\alpha) < 0 \text{ if and only if } X_\alpha = -\,.$$

We call X the sign type of x. We usually denote the sign types of elements $x, y, \ldots$ of W_a by the corresponding capital letters X, $Y, \ldots$.

The following results can be deduced easily from Proposition 2.1.

LEMMA 4.1. *Let X be the sign type of $x\in W_a$. Then $\mathscr{R}(x) = \{s_i \mid 0 \leqslant i \leqslant l, X_{\alpha_i} = -\}$.*

In a Coxeter group G, we say that two elements $x, y\in G$ have the same right extension property (r.e.p.) if, for any $w\in G$, $xw = x\cdot w$ if and only if $yw = y\cdot w$. Clearly, if x, y have the same r.e.p. and $xw = x\cdot w$ then $\mathscr{R}(xw) = \mathscr{R}(yw)$. In particular, $\mathscr{R}(x) = \mathscr{R}(y)$.

If G is a finite Coxeter group then it is easily shown that $x, y \in G$ have the same r.e.p. if and only if $x = y$.

Now we return to the case where $G = W_a$.

PROPOSITION 4.2. *If $x, y, \in W_a$ have the same sign type then x, y have the same r.e.p.*

Proof. This follows from Proposition 2.1 (2), (4).

The converse of the above proposition is not true in general. For example, let W_a have type $\tilde{B}_2$. Then the elements $s_0 s_2$ and $s_1 s_0 s_2$ have the same r.e.p. but different sign types. However, in certain circumstances, such a converse is true. This is just what we shall consider next.

LEMMA 4.3. *If $x, y \in W_a$ have the sign types $X = (X_\alpha)_{\alpha\in\Phi}$ and $Y = (Y_\alpha)_{\alpha\in\Phi}$ such that for some $\beta \in \Phi^+$, $X_\alpha = Y_\alpha$ for $\alpha \in \Phi^+ - \{\beta\}$ and $X_\beta \neq Y_\beta$, $\{X_\beta, Y_\beta\} \cap \{+, \bigcirc\} \neq \varnothing$.*

(1) *There exists $w \in W_a$ satisfying*
   (a) *$xw = x \cdot w$ and $yw = y \cdot w$.*
   (b) *Let $\tilde{X} = O(\tilde{X}_\alpha)_{\alpha\in\Phi}$ and $\tilde{Y} = (\tilde{Y}_\alpha)_{\alpha\in\Phi}$ be the sign types of xw and yw, respectively. Then there exists some $\gamma \in \Delta$ such that $\tilde{X}_\alpha = \tilde{Y}_\alpha$ for $\alpha \in \Phi^+ - \Delta$, $\tilde{X}_{\alpha'} = \tilde{Y}_{\alpha'} = -$ for $\alpha' \in \Delta - \{\gamma\}$ and $\tilde{X}_\gamma = X_\beta$, $\tilde{Y}_\gamma = Y_\beta$.*

(2) *If $- \in \{X_\beta, Y_\beta\}$ then x and y have different r.e.p.*

Proof. (1) Let $m(X) = {}^\#\{X_\alpha \mid X_\alpha \in \{+, \bigcirc\}, \alpha \in \Delta - \{\beta\}\}$ and $m(Y) = {}^\#\{Y_\alpha \mid Y_\alpha \in \{+, \bigcirc\}, \alpha \in \Delta - \{\beta\}\}$. Then define $m = m(X) = m(Y)$. If $m = 0$ we claim that $\beta \in \Delta$. For otherwise, $\beta \notin \Delta$. Then $X_\alpha = Y_\alpha = -$ for all $\alpha \in \Delta$. Hence $X_\alpha = Y_\alpha = -$ for all $\alpha \in \Phi^+$ and in particular, $X_\beta = Y_\beta = -$ which contradicts $\{X_\beta, Y_\beta\} \cap \{+, \bigcirc\} \neq \varnothing$. Therefore $w = 1$ satisfies the required conditions. Now assume that $m > 0$. Say $\gamma \in \Delta - \{\beta\}$ satisfying $X_\gamma = Y_\gamma \in \{+, \bigcirc\}$. Let $x^1 = xs_\gamma$ and $y^1 = ys_\gamma$. Let $X^1 = (X^1_\alpha)_{\alpha\in\Phi}$ and $Y^1 = (Y^1_\alpha)_{\alpha\in\Phi}$ be the sign types of x^1, y^1, respectively. Since $\beta^1 = (\beta)s_\gamma \in \Phi^+$, we have $X^1_\alpha = Y^1_\alpha$ for $\alpha \in \Phi^+ - \{\beta^1\}$ and $X^1_{\beta^1} = X_\beta$, $Y^1_{\beta^1} = Y_\beta$. If $m^1 := m(X^1) = m(Y^1) = 0$ then by the above argument, the element $w = s_\gamma$ satisfies the required conditions. If $m^1 > 0$ then there exists $\delta \in \Delta - \{\beta^1\}$ satisfying $X^1_\delta = Y^1_\delta \in \{+, \bigcirc\}$. Let $x^2 = x^1 s_\delta$ and $y^2 = y^1 s_\delta$. In this way, we get two sequences of elements: $x^0 = x, x^1, x^2, \ldots$ and $y^0 = y, y^1, y^2, \ldots$ in W_a. We also get the corresponding two sequences of sign types $X^0 = X, X^1, X^2, \ldots$ and $Y^0 = Y, Y^1, Y^2, \ldots$. Here for every $i \geqslant 1, x^i = x^{i-1}s_{\gamma_i}$ with $\gamma_i \in \Delta - (\beta)s_{\gamma_1}s_{\gamma_2}\ldots s_{\gamma_{i-1}}$ and $m(X^{i-1}) = m(Y^{i-1}) > 0$. Let

$$m(X^i) = {}^\#\{X^i_\alpha \mid X^i_\alpha \in \{+, \bigcirc\},\ \alpha \in \Phi^+ - (\beta)s_{\gamma_1}\ldots s_{\gamma_i}\}$$

and

$$M(Y^i) = {}^\#\{Y^i_\alpha \mid Y^i_\alpha \in \{+, \bigcirc\}, \alpha \in \Phi^+ - (\beta)s_{\gamma_1}\ldots s_{\gamma_i}\}.$$

Then $m(X^i) = M(Y^i)$ for all i and $M(X^0) > M(X^1) > \ldots$. Since $M(X^0) < \infty$, there must exist some $j \geqslant 0$ such that $M(X^j) = M(Y^j) = 0$. Thus $w = s_{\gamma_1}s_{\gamma_2}\ldots s_{\gamma_j}$ satisfies the required conditions and we have proved (1).

(2) If $- \in \{X_\beta, Y_\beta\}$ then $\mathscr{R}(xw) \neq \mathscr{R}(yw)$ and so x, y have different r.e.p. and we obtain (2).

LEMMA 4.4. *Assume that* $x, y, w \in W_a$, $\gamma \in \Delta$, $\beta \in \Phi^+$ *and* X, Y, $\tilde{X}$, $\tilde{Y} \in \mathscr{S}$ *are as in Lemma* 4.3 (1). *Assume that* $\bigcirc \in \{X_\beta, Y_\beta\}$ *and that one of the following conditions is satisfied:*

(1) Φ *has type* A_l, $l \geqslant 1$,
(2) $\{x, y\} \cap \mathbf{W} \neq \varnothing$.

Then $\tilde{X}_\alpha = \tilde{Y}_\alpha \in \{-, \bigcirc\}$ *for* $\alpha \in \Phi^+ - \Delta$, $\tilde{X}_{\alpha'} = \tilde{Y}_{\alpha'} = -$ *for* $\alpha' \in \Delta - \{\gamma\}$ *and* $\tilde{X}_\gamma = X_\beta$, $\tilde{Y}_\gamma = Y_\beta$.

Proof. By Lemma 4.3 (1), it is enough to show that $\tilde{X}_\alpha = \tilde{Y}_\alpha \in \{-, \bigcirc\}$ for $\alpha \in \Phi^+ - \Delta$. We may assume without loss of generality that $X_\beta = \bigcirc$. If condition (1) holds then by the admissibility of the sign type X, our result follows. Now assume that condition (2) holds. Then $\tilde{X}_\alpha = \tilde{Y}_\alpha \in \{+, -\}$ for $\alpha \in \Phi^+ - \Delta$. Suppose that there is some $\delta \in \Phi^+ - \Delta$ with $\tilde{X}_\delta = \tilde{Y}_\delta = +$. We can choose δ such that $\mathrm{ht}(\delta)$ is as small as possible. Thus by the admissibility of $\tilde{X}$, there must exist some $\alpha \in \Phi^+ - \{\gamma\}$ such that $\delta = \gamma + \alpha$ and $\alpha' = \alpha + \delta \in \Phi^+$. Clearly, $\{\alpha, \gamma, \delta, \alpha'\}$ forms a positive subsystem of Φ^+ of type B_2 with γ, α' long roots and α, δ short roots. Let $\tilde{x} = xw$ and $\tilde{y} = yw$. Then by the fact that $k(\tilde{x}, \gamma) = 0$, $k(\tilde{x}, \alpha) < 0$ and $k(\tilde{x}, \delta) > 0$, we get $k(\tilde{x}, \alpha) = -1$ and $k(\tilde{x}, \delta) = 1$ from the inequality (2.0.1). Again by (2.0.1) we get $k(\tilde{x}, \alpha') = 0$ and hence $k(\tilde{y}, \alpha') = 0$. Thus $\{\tilde{x}, \tilde{y}\} \cap \mathbf{W} = \varnothing$. Since $\tilde{x} = x \cdot w$ and $\tilde{y} = y \cdot w$, we have $\{x, y\} \cap \mathbf{W} = \varnothing$ by Proposition 2.1, which contradicts our condition. Hence the result follows.

LEMMA 4.5. *Assume that* $x, y \in W_a$ *have the sign types* $X = (X_\alpha)_{\alpha \in \Phi}$ *and* $Y = (Y_\alpha)_{\alpha \in \Phi}$ *such that for some* $\alpha_t \in \Delta$, *we have* $X_\alpha = Y_\alpha$ *for* $\alpha \in \Phi^+ - \{\alpha_t\}$ *and* $\{X_{\alpha_t}, Y_{\alpha_t}\} = \{+, \bigcirc\}$, *where* $S - \{s_t\} \in \mathbf{S}$. *Then* x, y *have different r.e.p.*

Proof. There exists an automorphism φ of W_a which preserves S and sends s_t to s_0. Let $\tilde{x} = \varphi(x)$ and $\tilde{y} = \varphi(y)$. Clearly, $s_t \notin \mathscr{R}(x) \cup \mathscr{R}(y)$ implies that $s_0 \notin \mathscr{R}(\tilde{x}) \cup \mathscr{R}(\tilde{y})$. Then x, y have different r.e.p. if and only if $\tilde{x}, \tilde{y}$ have different r.e.p. Let $\tilde{X} = (\tilde{X}_\alpha)_{\alpha \in \Phi}$ and $\tilde{Y} = (\tilde{Y}_\alpha)_{\alpha \in \Phi}$ be the sign types of $\tilde{x}, \tilde{y}$, respectively. Then we have $\tilde{X}_\alpha = \tilde{Y}_\alpha$ for $\alpha \in \Phi^+ - \{-\alpha_0\}$ and $\{\tilde{X}_{-\alpha_0}, \tilde{Y}_{-\alpha_0}\} = \{-, \bigcirc\}$ by Lemma 4.1 and the fact that $s_0 \notin \mathscr{R}(\tilde{x}) \cup \mathscr{R}(\tilde{y})$. Thus by Lemma 4.3(2), $\tilde{x}, \tilde{y}$ have different r.e.p. and hence so do x, y.

LEMMA 4.6. *Assume that* $x, y \in W_a$ *have the sign types* $X = (X_\alpha)_{\alpha \in \Phi}$ *and* $Y = (Y_\alpha)_{\alpha \in \Phi}$ *such that, for some* $\alpha_t \in \Delta$ *with* $\langle \alpha_t, (-\alpha_0)^\vee \rangle \neq 0$, *we have* $X_\alpha = Y_\alpha$ *for* $\alpha \in \Phi^+ - \{\alpha_t, -\alpha_0\}$, $X_{-\alpha_0} = Y_{-\alpha_0} \in \{-, \bigcirc\}$ *and* $\{X_{\alpha_t}, Y_{\alpha_t}\} = \{+, \bigcirc\}$. *Then* x, y *have different r.e.p.*

Proof. Let $x' = xs_0$ and $y' = ys_0$. Let $X' = (X'_\alpha)_{\alpha \in \Phi}$ and $Y' = (Y'_\alpha)_{\alpha \in \Phi}$ be the sign types of x', y', respectively. Note that $(\alpha_t)\bar{s}_0 \in \Phi^-$ by our condition. Let $\gamma = -(\alpha_t)\bar{s}_0$. Then by Proposition 2.1(2), $X'_\alpha = Y'_\alpha$ for $\alpha \in \Phi^+ - \{\gamma\}$ and $\{X'_\gamma, Y'_\gamma\} = \{-, \bigcirc\}$. Then by Lemma 4.3(2). x', y' have different r.e.p. But $x' = x \cdot s_0$ and $y' = y \cdot s_0$. Hence x, y have different r.e.p.

By Lemmas 4.5 and 4.6, we obtain the following.

COROLLARY 4.7. *Let $x, y \in W_a$, $X, Y \in \mathscr{S}$ and $\beta \in \Phi^+$ be as in Lemma 4.4. In addition, suppose that $\beta = \alpha_i \in \Delta$ and that we have one of the following cases:*

(1) *Φ has type A_l, $l \geqslant 1$;*
(2) *Φ has type B_l, $l \geqslant 3$, $i = 1, 2$;*
(3) *Φ has type C_l, $l \geqslant 2$, $i = 1, l$;*
(4) *Φ has type D_l, $l \geqslant 4$, $i = 1, 2, l-1, l$;*
(5) *Φ has type E_6, $i = 1, 2, 6$;*
(6) *Φ has type E_7, $i = 1, 7$;*
(7) *Φ has type E_8, $i = 8$;*
(8) *Φ has type F_4, $i = 1$;*
(9) *Φ has type G_2, $i = 1$.*

Then x, y have different r.e.p.

Proposition 4.8. *Assume that $x, y \in W_a$ have the sign types $X = (X_\alpha)_{\alpha\in\Phi}$ and $Y = (Y_\alpha)_{\alpha\in\Phi}$ such that for some $\alpha_i \in \Delta$, $X_\alpha = Y_\alpha = -$, for $\alpha \in \Phi^+ - \{\alpha_i\}$ and $X_{\alpha_i} \neq Y_{\alpha_i}$. Then x, y have different r.e.p.*

Proof. If $- \in \{X_{\alpha_i}, Y_{\alpha_i}\}$ then by $X_{\alpha_i} \neq Y_{\alpha_i}$, we have $\mathscr{R}(x) \neq \mathscr{R}(y)$ and hence x, y have different r.e.p. Now assume that $- \notin \{X_{\alpha_i}, Y_{\alpha_i}\}$. Then $\{X_{\alpha_i}, Y_{\alpha_i}\} = \{+, \bigcirc\}$. We may assume without loss of generality that $X_{\alpha_i} = \bigcirc$ and $Y_{\alpha_i} = +$. We shall prove our result by a case-by-case argument.

(1) Φ has type A_l. The result follows from Corollary 4.7(1).

In each of the remaining cases, except for cases (6a, c), we shall find an element $w \in W_a$ with $xw = x \cdot w$, $yw = y \cdot w$ such that for some $\gamma \in \Phi^+$, $\tilde{X}_\alpha = \tilde{Y}_\alpha$, for $\alpha \in \Phi^+ - \{\gamma\}$ and $\{\tilde{X}_\gamma, \tilde{Y}_\gamma\} = \{-, \bigcirc\}$, where $\tilde{X} = (\tilde{X}_\alpha)_{\alpha\in\Phi}$ and $\tilde{Y} = (\tilde{Y}_\alpha)_{\alpha\in\Phi}$ are sign types of xw and yw, respectively. In either of cases (6a) and (6c), we shall find $w \in W_a$ which satisfies the following conditions:

(i) $xw = x \cdot w$ and $yw = y \cdot w$;
(ii) if $X' = (X'_\alpha)_{\alpha\in\Phi}$ and $Y' = (Y'_\alpha)_{\alpha\in\Phi}$ are the sign types of xw and yw, respectively, then $X'_\alpha = Y'_\alpha$ for all $\alpha \in \Phi - \{\alpha_i\}$, and $\{X'_{\alpha_i}, Y'_{\alpha_i}\} = \{+, \bigcirc\}$, where $S - \{s_i\} \in \mathbf{S}$.

Once we have done this, our result follows immediately from Lemmas 4.3(2) and 4.5.

(2) Φ has type B_l or D_l. By Corollary 4.7(2), (4) we may assume that $i > 2$. Moreover, in the case when Φ has type D_l, we may further assume that $i < l-1$. Let $a_i = s_0 s_2 s_3 \dots s_i$ and $b_j = s_1 s_2 \dots s_j$, $i \geqslant 2, j \geqslant 1$.
   (a) If $i < l$ is even, let $w = a_i b_{i-1} a_{i-2} b_{i-3} \cdots a_4 b_3 s_0$.
   (b) If $i < l$ is odd, let $w = a_i b_{i-1} a_{i-2} b_{i-3} \cdots a_3 b_2 a_l^{-1}$.

   The following two cases only occur for Φ of type B_l.
   (c) If $i = l$ is even, let $w = a_{l-1} b_{l-2} a_{l-3} b_{l-4} \cdots a_3 b_2 s_0$.
   (d) If $i = l$ is odd, let $w = a_{l-1} b_{l-2} a_{l-3} b_{l-4} \cdots a_2 b_1 a_l^{-1}$.

(3) Φ has type C_l. By Corollary 4.7, we may assume that $1 < i < l$. Let $a_i = s_0 s_1 s_2 \dots s_i$, $i \geqslant 0$, and let $w = a_i a_{i-1} \dots a_2 a_0$.

(4) Φ has type F_4. By Corollary 4.7, we may assume that $i \neq 1$. Let $a_1 = s_1 s_2$, $a_2 = s_2 s_3$, $b_1 = s_0 s_1 s_2$, $b_2 = s_1 s_2 s_3$, $c = s_0 s_1 s_2 s_3$ and $d = s_0 s_1 s_2 s_3 s_4$.
   (a) If $i = 2$, let $w = d b_1^{-1} c^{-1} b_2^{-1} a_2^{-1} d^{-1}$.
   (b) If $i = 3$, let $w = d a_1^{-1} c^{-1}$.
   (c) If $i = 4$, let $w = c b_1^{-1}$.

(5) Φ has type E_6. By Corollary 4.7, we may assume that $i = 3, 4, 5$. Let $a = s_3 s_4$, $b_1 = s_3 s_4 s_2$, $b_2 = s_3 s_4 s_5$, $c_1 = s_0 s_2 s_4 s_5$, $c_2 = s_1 s_3 s_4 s_5$, $c_3 = s_3 s_4 s_2 s_0$, $d_1 = s_0 s_2 s_4 s_5 s_6$, $d_2 = s_1 s_3 s_4 s_2 s_0$, $d_3 = s_1 s_3 s_4 s_5 s_6$.
   (a) If $i = 3$, let $w = d_1 b_2 b_1^{-1} c_2 d_2^{-1} c_1^{-1}$.
   (b) If $i = 4$, let $w = d_1 b_2 d_2$.
   (c) If $i = 5$, let $w = d_1 b_1 d_3 a^{-1} d_1 c_3$.

(6) Φ has type E_7. By Corollary 4.7, we may assume that $1 < i < 7$. Let $a = s_2$, $b_1 = s_2 s_4 s_5$, $b_2 = s_3 s_4 s_2$, $b_3 = s_3 s_4 s_5$, $b_4 = s_4 s_5 s_6$, $c_1 = s_1 s_3 s_4 s_2$, $c_2 = s_1 s_3 s_4 s_5$, $c_3 = s_2 s_4 s_5 s_6$, $c_4 = s_3 s_4 s_5 s_6$, $d_1 = s_0 s_1 s_3 s_4 s_2$, $d_2 = s_0 s_1 s_3 s_4 s_5$, $d_3 = s_1 s_3 s_4 s_5 s_6$, $e_1 = s_0 s_1 s_3 s_4 s_5 s_6$, $e_2 = s_1 s_3 s_4 s_5 s_6 s_7$ and $f = s_0 s_1 s_3 s_4 s_5 s_6 s_7$.
   (a) If $i = 2$, let $w = f c_3$.
   (b) If $i = 3$, let $w = f c_3 b_3 d_1^{-1}$.
   (c) If $i = 4$, let $w = f c_3 b_2 e_2 e_1 b_1 b_2 e_2 b_4 b_1 f c_3 d_3 d_1 f b_3 a d_3$.
   (d) If $i = 5$, let $w = f c_3 b_3 c_1 f c_4 c_2 d_1 b_4 b_3 c_1 f b_2 d_2^{-1}$.
   (e) If $i = 6$, let $w = d_2 d_1^{-1}$.

(7) Φ has type E_8. By Corollary 4.7, we may assume that $i < 8$. Let $a = s_2$, $b = s_2 s_4$, $c_1 = s_1 s_3 s_4$, $c_2 = s_2 s_4 s_5$, $c_3 = s_3 s_4 s_2$, $c_4 = s_3 s_4 s_5$, $c_5 = s_4 s_5 s_6$, $c_6 = s_5 s_6 s_7$, $d_1 = s_1 s_3 s_4 s_2$, $d_2 = s_1 s_3 s_4 s_5$, $d_3 = s_2 s_4 s_5 s_6$, $d_4 = s_3 s_4 s_5 s_6$, $d_5 = s_4 s_5 s_6 s_7$, $e_1 = s_1 s_3 s_4 s_5 s_6$, $e_2 = s_2 s_4 s_5 s_6 s_7$, $e_3 = s_3 s_4 s_5 s_6 s_7$, $e_4 = s_5 s_6 s_7 s_8 s_0$, $f_1 = s_1 s_3 s_4 s_5 s_6 s_7$, $f_2 = s_2 s_4 s_5 s_6 s_7 s_8$, $f_3 = s_3 s_4 s_5 s_6 s_7 s_8$, $g_1 = s_1 s_3 s_4 s_5 s_6 s_7 s_8$, $g_2 = s_2 s_4 s_5 s_6 s_7 s_8 s_0$, $g_3 = s_3 s_4 s_5 s_6 s_7 s_8 s_0$ and $h = s_1 s_3 s_4 s_5 s_6 s_7 s_8 s_0$.
   (a) If $i = 1$, let $w = h^{-1} g_2$.
   (b) If $i = 2$, let $w = h^{-1} g_2 c_3 d_2^{-1} d_4^{-1} a f_1^{-1} f_3^{-1} g_2$.
   (c) If $i = 3$, let $w = h^{-1} c_2 c_3 c_1^{-1} c_4^{-1} d_5^{-1} e_4^{-1} g_2 e_3 d_3 h f_2 e_3 b g_1 d_1^{-1} c_4^{-1} g_2$.
   (d) If $i = 4$, let $w = h^{-1} f_2 e_3 d_1^{-1} c_2^{-1} e_1^{-1} b^{-1} c_4^{-1} g_2^{-1} g_3 g_1 d_2^{-1} d_4^{-1} g_2 c_3 d_2 c_1^{-1} c_3^{-1} d_3^{-1} g_3 g_1 e_2 d_4 d_2 e_2 d_4 c_2 d_1 d_2^{-1} f_3^{-1} g_2^{-1} g_1^{-1} g_3^{-1} g_2$.
   (e) If $i = 5$, let $w = h^{-1} g_2 f_3 e_2 d_4 c_2 h d_3 c_4 h f_2 e_3 d_2 a f_3 e_1 g_2 g_1 d_3 d_1 c_4^{-1} e_2^{-1} h$.
   (f) If $i = 6$, let $w = h^{-1} g_2 f_3 d_3 e_3 d_2 g_2 f_3 e_2 f_1 c_2 c_3 h$.
   (g) If $i = 7$, let $w = h^{-1} e_2 d_4 c_2 c_3 h$.

COROLLARY 4.9. *Assume that $x, y \in W_a$ have sign types $X = (X_\alpha)_{\alpha \in \Phi}$ and $Y = (Y_\alpha)_{\alpha \in \Phi}$ respectively such that $X_\alpha = Y_\alpha \in \{-, \bigcirc\}$ for all $\alpha \in \Phi^+ - \Delta$, $X_\beta = Y_\beta = -$ for $\beta \in \Delta - \{\gamma\}$ and $X_\gamma = \bigcirc$, $Y_\gamma = +$, where $\gamma \in \Delta$. Then x, y have different r.e.p.*

Proof. Define $\tilde{X} = (\tilde{X}_\alpha)_{\alpha \in \Phi}$ and $\tilde{Y} = (\tilde{Y}_\alpha)_{\alpha \in \Phi}$ by $\tilde{X}_\alpha = \tilde{Y}_\alpha = -$ for $\alpha \in \Phi^+ - \{\gamma\}$, $\tilde{X}_\gamma = \bigcirc$ and $\tilde{Y}_\gamma = +$. Then $\tilde{X}$ and $\tilde{Y}$ are two admissible sign types. Thus there exist $\tilde{x} \in \zeta^{-1}(\tilde{X})$ and $\tilde{y} \in \zeta^{-1}(\tilde{Y})$. By Proposition 4.8, $\tilde{x}$ and $\tilde{y}$ have different r.e.p. Hence we can find $w \in W_a$ such that $\tilde{x}w = \tilde{x} \cdot w$, $\tilde{y}w = \tilde{y} \cdot w$ and $\mathscr{R}(\tilde{x}w) \neq \mathscr{R}(\tilde{y}w)$. But this implies from Proposition 2.1 (2), (4) that $xw = x \cdot w$, $yw = y \cdot w$ and $\mathscr{R}(xw) \neq \mathscr{R}(yw)$. That is, x, y have different r.e.p.

The following result gives a necessary and sufficient condition for two elements of W_a to have the same r.e.p. in certain circumstance. This result will be crucial in the proof of Theorem 2.4.

THEOREM 4.10. *Assume that* $x, y \in W_a$ *have sign types* $X = (X_\alpha)_{\alpha\in\Phi}$, $Y = (Y_\alpha)_{\alpha\in\Phi}$, *respectively. Assume that one of the following conditions is satisfied.*

(1) Φ has type A_l, $l \geqslant 1$.
(2) $\{x, y\} \cap \mathbf{W} \neq \varnothing$.

Then $X = Y$ *if and only if* x, y *have the same r.e.p.*

Proof. ($\Rightarrow$) This follows from Proposition 4.2.

($\Leftarrow$) We must show that if $X \neq Y$ then x, y have different r.e.p. Suppose to the contrary that $X \neq Y$ and x, y have the same r.e.p. We can find a sequence of elements $x(0) = x, x(1), \ldots, x(r)$ such that $x(i-1)^{-1}x(i) \in S - \{s_0\}$, $x(i-1) < x(i)$, for every i, $1 \leqslant i \leqslant r$, and $x(r)$ has the sign type whose entries are all $-$. Since y has the same r.e.p. as x, there also exists a sequence of elements $y(0) = y, y(1), \ldots, y(r)$ such that for every i, $1 \leqslant i \leqslant r$, $y(i-1)^{-1}y(i) = x(i-1)^{-1}x(i)$, and $y(r)$ has the same sign type as $x(r)$. We see that for each j, $0 \leqslant j \leqslant r$, $x(j)$ and $y(j)$ have the same r.e.p. Let $X(j)$, $Y(j)$ be the sign types of $x(j)$, $y(j)$, respectively. Since $X(0) \neq Y(0)$ and $X(r) = Y(r)$, there must exist some l, $0 \leqslant l < r$, such that $X(l)_\alpha = Y(l)_\alpha$ for all $\alpha \in \Phi^+ - \Delta$, $X(l)_\beta = Y(l)_\beta = -$ for $\beta \in \Delta - \{\gamma\}$ and $X(l)_\gamma \neq Y(l)_\gamma$, where $\gamma \in \Delta$. Since $\mathscr{R}(x(l)) = \mathscr{R}(y(l))$, we have $\{X(l)_\gamma, Y(l)_\gamma\} = \{+, \bigcirc\}$. We may assume without loss of generality that $X(l)_\gamma = \bigcirc$ and $Y(l)_\gamma = +$. Then by Lemmas 4.3 and 4.4, we have $X(l)_\alpha = Y(l)_\alpha \in \{-, \bigcirc\}$ for all $\alpha \in \Phi^+ - \Delta$. Hence it follows from Corollary 4.9 that $x(l)$ and $y(l)$ have different r.e.p., which is a contradiction. This implies that x and y have different r.e.p. and our result follows.

We need the following result which is a special case of Lusztig's result [3].

LEMMA 4.11. For $x \in W_a$, *let* $s \in S - \mathscr{L}(w)$, $t \in S - \mathscr{R}(w)$. *Then* $a(sx)$, $a(xt) \geqslant a(x)$.

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Since $\mathbf{W} = W(\mathbf{S}) \subset W_{(v)}$, it suffices to show that if $x \notin \mathbf{W}$ then $a(x) < v$.

There exists a sequence of elements $x_0 = x, x_1, \ldots, x_r$ in W_a such that for every i, $1 \leqslant i \leqslant r$, $x_i x_{i-1}^{-1} \in S$, $x_{i-1} < x_i$, $x_{r-1} \notin \mathbf{W}$ and $x_r \in \mathbf{W}$. By Lemma 4.11, we have $a(x_r) \geqslant a(x_{r-1}) \geqslant \ldots \geqslant a(x_0)$. To prove that $a(x) < v$, it suffices to show that $a(x_{r-1}) < v$. Let $s_t = x_r x_{r-1}^{-1}$, $z = x_{r-1}$ and $z' = x_r$. Let Z and Z' be the sign types of z and z', respectively. Then $Z \neq Z'$. By Theorem 4.10, z and z' have different r.e.p. Thus there is $z_0 \in W_a$ such that $zz_0 = z \cdot z_0$, $z'z_0 = z' \cdot z_0$ and $\mathscr{R}(zz_0) \neq \mathscr{R}(z'z_0)$. This implies that $\mathscr{R}(z'z_0) \supsetneqq \mathscr{R}(zz_0)$ since $s_t \cdot zz_0 = z'z_0$. On the other hand, $s_t \in \mathscr{L}(z'z_0) - \mathscr{L}(zz_0)$ implies that $\mathscr{L}(z'z_0) \not\subset \mathscr{L}(zz_0)$. By Lemma 4.1, we have $a(zz_0) < a(z'z_0) \leqslant v$. But $a(x) \leqslant a(z) \leqslant a(zz_0)$. This implies that $a(x) < v$.

5. *The two-sided cell $W_{(v)}$ of W_a*

For the time being, we consider an arbitrary Coxeter group (G, S).

It is known that the Kazhdan–Lusztig polynomial $P_{y,w}(x)$, $y, w \in G$, has degree no greater than $\frac{1}{2}(l(w)-l(y)-1)$ if $y < w$. We write $y \prec w$ if $y < w$ and $\deg P_{y,w}(x)$ is exactly $\frac{1}{2}(l(w)-l(y)-1)$. We write $y - w$ if either $y \prec w$ or $w \prec y$.

We write $y \leqslant_L w$ if there exists a sequence $y_0 = y, y_1, \ldots, y_t$ in W_a such that for every i, $1 \leqslant i \leqslant t$, $y_{i-1} - y_i$ and $\mathscr{L}(y_{i-1}) \not\subset \mathscr{L}(y_i)$. We define $y \sim_L w$ if $y \leqslant_L w \leqslant_L y$. We write $y \leqslant_R w$ if $y^{-1} \leqslant_L w^{-1}$, and $y \sim_R w$ if $y^{-1} \sim_L w^{-1}$. Finally, we write $y \leqslant_\Gamma w$ if there exists a sequence $y_0 = y, y_1, \ldots, y_r$ in W_a such that either $y_{i-1} \leqslant_L y_i$ or $y_{i-1} \leqslant_R y_i$ for every i, $1 \leqslant i \leqslant r$; we define $y \sim_\Gamma w$ if $y \leqslant_\Gamma w \leqslant_\Gamma y$.

The relation $\sim_L$ (respectively $\sim_R$, $\sim_\Gamma$) is an equivalence relation on W_a. We call the corresponding equivalence classes left (respectively right, two-sided) cells. Clearly, any two-sided cell of G is a union of left (respectively right) cells of G. These cells play an important role in the representation theory of Coxeter groups and Hecke algebras.

The following results are well known [3].

THEOREM 5.1. *Let $x, y \in G$.*

(1) $x \sim_\Gamma y \Rightarrow a(x) = a(y)$.
(2) $x \leqslant_L y$ *and* $a(x) = a(y) \Rightarrow x \sim_L y$.
(3) $x \leqslant_R y$ *and* $a(x) = a(y) \Rightarrow x \sim_R y$.
(4) $w = x \cdot y \Rightarrow w \leqslant_R x$ *and* $w \leqslant_L y$.

Now we can prove the main result of this paper.

THEOREM 5.2. *The set $W_{(v)}$ is a two-sided cell of W_a.*

Proof. Let $D = \{w_J \mid J \in \mathbf{S}\}$. Then by Lemma 2.5, $D \subset W_{(v)}$. By Theorems 1.1 and 5.1 we have that for any $x \in W_{(v)}$ there exists some $y \in D$ satisfying $x \sim_\Gamma y$. To prove our result, it suffices to show that for any $I, J \in \mathbf{S}$, we have $w_I \sim_\Gamma w_J$. There exists a unique expression $w_I = xy$ such that $y \in W_J$ and x is the shortest element in the coset $w_I W_J$. Thus $w_I = x \cdot y$. Let $z = y^{-1} w_J$ and $w = xyz$. Then $w = x \cdot y \cdot z = w_I \cdot z = x \cdot w_J$ and, in particular, $w \in W_{(v)}$. By Theorem 5.1, we get $w_I \sim_\Gamma w \sim_\Gamma w_J$ and hence the result follows.

Let $\tilde{\mathscr{S}}$ be the set of all sign types $X = (X_\alpha)_{\alpha \in \Phi}$ of $\mathscr{S}$ such that $X_\alpha \neq \bigcirc$ for all $\alpha \in \Phi$. Then we know from [7] that $\tilde{\mathscr{S}}$ is in one–one correspondence with the Weyl chambers of the euclidean space E. Hence the cardinal of $\tilde{\mathscr{S}}$ is equal to the order of the Weyl group W on Φ. On the other hand, for any $X \in \mathscr{S}$, the fibre $\zeta^{-1}(X)$ is a left connected set of W_a in the sense that, for any $x, y \in \zeta^{-1}(X)$, there exists a sequence $x_0 = x, x_1, \ldots, x_r = y$ in $\zeta^{-1}(X)$ such that $x_{i-1} x_i^{-1} \in S$. Since $\zeta^{-1}(X) \subset W_{(v)}$, from Theorem 5.1 we have the following theorem.

THEOREM 5.3. *For any $X \in \tilde{\mathscr{S}}$, the fibre $\zeta^{-1}(X)$ is contained in some left cell of W_a in $W_{(v)}$. Thus $W_{(v)}$ is a union of m left cells of W_a with $m \leqslant |W|$.*

Finally, we conclude our paper with a conjecture.

CONJECTURE 5.4. For any $X \in \tilde{\mathscr{S}}$, the fibre $\zeta^{-1}(X)$ is a left cell of W_a. Thus $W_{(v)}$ is a union of $|w|$ left cells of W_a.

The above conjecture is supported by computation in the following cases:

(1) W_a has type $\tilde{A}_l$, $l \geqslant 1$ [5];
(2) W_a has type $\tilde{B}_3$ [1];
(3) W_a has rank at most 2 [3].

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