SIGN TYPES CORRESPONDING TO AN AFFINE WEYL GROUP

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ABSTRACT

The sign types corresponding to an affine Weyl group W_a were first studied in [3]. In the present paper, I generalize all the results of [3] on sign types to the case when W_a is an indecomposable affine Weyl group of an arbitrary type. As a result, I verify Carter's conjecture on the cardinality of sign types of type Φ, where Φ is the root system determined by W_a.

In [3], we defined sign types of the Euclidean space E spanned by the root system Φ of type A_{n-1}. These sign types are the connected components of the complement of a certain set of hyperplanes in E and can be regarded as certain equivalence classes of W_a, where W_a is the affine Weyl group of type $\tilde{A}_{n-1}$ identified with the set of alcoves of E via its action on E. The sign types play an important role in the study of the affine Weyl groups [3]. We described sign types of E and showed that the number of sign types of E is $(n+1)^{n-1}$ in [3].

I am very grateful to Professor R. W. Carter who told me that the formula $(n+1)^{n-1}$ can be rewritten $(h+1)^l$, where $l = n-1$ is the rank of Φ and h is the Coxeter number of Φ. He then conjectured that this result can be generalized to the case when Φ is an indecomposable root system of any other type.

In the present paper, I shall generalize all the results of [3] on sign types to the case when Φ is an indecomposable root system of an arbitrary type. The main results are Theorems 2.1 and 8.1. We start with the definition of an admissible sign type in terms of a Φ-tuple over $\mathbb{Z}$. Then §§3–5 are reserved for the proof of Theorem 2.1. Theorem 2.1 asserts that the set $\mathscr{S}(\Phi)$ of admissible sign types can be identified with the set of certain equivalence classes of W_a. We also deduce in §6 that $\mathscr{S}(\Phi)$ can be identified with the set of connected components of the complement of a certain set of hyperplanes in E. Finally, we prove Theorem 8.1 in §§7–8 and thus verify the above conjecture of Carter.

1. *Preliminary*

Let Φ be an indecomposable reduced root system. Choose a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of Φ. Let Φ^+, Φ^- be the corresponding positive and negative root systems of Φ. Let E be the Euclidean space spanned by Φ with positive definite inner product $\langle\ ,\ \rangle$ such that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 1$ for any short root α of Φ. For any $\alpha \in \Phi$, $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is called the coroot of α. The set $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ of coroots is again a root system such that the set $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ affords a choice of simple root system in it. Let $-\alpha_0$ be the highest short root of Φ. Then $(-\alpha_0)^\vee$ is the highest (co)root of $\Phi^\vee$. Let h be the Coxeter number of Φ. Then h is also the Coxeter number of $\Phi^\vee$.

Let W be the Weyl group of Φ generated by the reflections s_α on E for $\alpha \in \Phi$, where

Received 14 February 1986.

1980 *Mathematics Subject Classification* 20H15.

s_α sends x to $x-\langle x,\alpha^\vee\rangle\alpha$. Let Q denote the root lattice $\mathbb{Z}\Phi$. Let N denote the group consisting of all translations T_λ operating on E for $\lambda\in Q$, where T_λ sends x to $x+\lambda$. We denote by W_a the group of affine transformations of E generated by N and W. It is well known that W_a is the semidirect extension of W by the normal subgroup N on which the action of W is known. Any $w\in W_a$ has a unique decomposition $w=\bar{w}T_\lambda$ with $\bar{w}\in W$ and $\lambda\in Q$.

For linear and affine transformations, we shall denote the operation on the right and compose them accordingly. With this convention, we define $s_0=s_{\alpha_0}T_{-\alpha_0}$, $s_i=s_{\alpha_i}$, $1\leqslant i\leqslant l$. It is known that W_a (respectively W) is a Coxeter group on generators $s_0,s_1,\ldots,s_l$ (respectively $s_1,\ldots,s_l$). We denote $\Delta=\{s_0,s_1,\ldots,s_l\}$. The group W_a will be called an affine Weyl group.

We define the length $l(w)$ of an element $w\in W_a$ to be the smallest number r such that there exists an expression $w=s(1)s(2)\ldots s(r)$ with $s(i)\in\Delta$. An expression of w is called a reduced form if it is a product of $l(w)$ generators.

The symbol $\leqslant$ denotes the Bruhat order on W_a (defined, for example, in [5]). For any $w\in W_a$, we associate two subsets of Δ as follows:

$$\mathscr{L}(w)=\{s\in\Delta\mid sw<w\},$$

$$\mathscr{R}(w)=\{s\in\Delta\mid ws<w\}.$$

Given any two sets S, R, we call $\mathbf{x}=(x_i)_{i\in R}$ an R-tuple over S if $x_i\in S$ for all $i\in R$. Sometimes we simply call $\mathbf{x}$ an R-tuple when there is no danger of confusion. Two R-tuples $\mathbf{x}=(x_i)_{i\in R}$ and $\mathbf{y}=(y_j)_{j\in R}$ are said to be equal if $x_i=y_i$ for all $i\in R$.

For any $\alpha\in\Phi^+$, $k\in\mathbb{Z}$ and a positive real number m, we define a hyperplane

$$H_{\alpha;k}=\{v\in E\mid\langle v,\alpha^\vee\rangle=k\}$$

and a stripe

$$H^m_{\alpha;k}=H^m_{-\alpha;-k}=\{v\in E\mid k<\langle v,\alpha^\vee\rangle<k+m\}.$$

We call any non-empty connected simplex of

$$E-\bigcup_{\substack{\alpha\in\Phi\\ k\in\mathbb{Z}}}H_{\alpha;k}$$

an alcove of E. Each alcove of E has the form $\bigcap_{\alpha\in\Phi^+}H^1_{\alpha;k_\alpha}$ for a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ over $\mathbb{Z}$. The following results are well known.

Lemma 1.1 [4, Lemma 1.1]. *Let $A_1=\bigcap_{\alpha\in\Phi^+}H^1_{\alpha;0}$. Then A_1 is an alcove of E which can also be expressed in the form $(\bigcap_{\alpha\in\Pi}H^{1/c_\alpha}_{\alpha;0})\cap H^1_{-\alpha_0;0}$, where the c_α satisfy the equation*

$$(-\alpha_0)^\vee=\sum_{\alpha\in\Pi}c_\alpha\alpha^\vee.$$

Theorem 1.2 [4, Theorem 5.2]. *Let $A_k=\bigcap_{\alpha\in\Phi^+}H^1_{\alpha;k_\alpha}$ with $k_\alpha\in\mathbb{Z}$. Then A_k is an alcove of E if and only if for any $\alpha,\beta\in\Phi^+$ with $\alpha+\beta\in\Phi^+$, the inequality*

$$|\alpha|^2k_\alpha+|\beta|^2k_\beta+1\leqslant|\alpha+\beta|^2(k_{\alpha+\beta}+1)\leqslant|\alpha|^2k_\alpha+|\beta|^2k_\beta+|\alpha|^2+|\beta|^2+|\alpha+\beta|^2-1$$

holds.

It is well known that the right action of W_a on E induces a bijective map $w\mapsto(A_1)w=A_w$ from the set of elements of W_a to the set $\mathfrak{A}$ of alcoves of E. Thus any alcove of E has the form A_w, $A_w=\bigcap_{\alpha\in\Phi^+}H^1_{\alpha;k(w,\alpha)}$ or $A_w=\bigcap_{\alpha\in\Phi}H^1_{\alpha;k(w,\alpha)}$

with the convention that $k(w,-\alpha)=-k(w,\alpha)$ for any $\alpha\in\Phi^+$. We shall identify W_a with $\mathfrak{A}$ as a set under the correspondence $w\mapsto A_w$. Later the integers $k(w,\alpha)$ indexed by $w\in W_a$ and $\alpha\in\Phi$ always stand for the coordinates of the alcove A_w. The following result is known.

PROPOSITION 1.3 [4, Proposition 4.2]. *Let $w'=ws_j$ with $w\in W_a$ and $s_j\in\Delta$. Then for any $\alpha\in\Phi$, we have*

$$k(w',\alpha)=k(w,(\alpha)s_j)+k(s_j,\alpha).$$

2. *Admissible sign types*

A Φ-tuple $X=(X_\alpha)_{\alpha\in\Phi}$ over the set $\{+,-,\bigcirc\}$ is called a sign type of type Φ if the set $\{X_\alpha,X_{-\alpha}\}$ is either $\{\bigcirc,\bigcirc\}$ or $\{+,-\}$ for any $\alpha\in\Phi$. We see that a sign type $(X_\alpha)_{\alpha\in\Phi}$ is entirely determined by the Φ^+-tuple $(X_\alpha)_{\alpha\in\Phi^+}$. So sometimes we can identify $(X_\alpha)_{\alpha\in\Phi^+}$ with $(X_\alpha)_{\alpha\in\Phi}$ and call $(X_\alpha)_{\alpha\in\Phi^+}$ a sign type.

Let $\overline{\mathscr{S}}=\overline{\mathscr{S}}(\Phi)$ be the set of all sign types of type Φ. Let

$$G_1=\Big\{\substack{+\\++},\ \substack{+\\+\bigcirc},\ \substack{+\\+-},\ \substack{\bigcirc\\+-},\ \substack{-\\+-},\ \substack{+\\\bigcirc+},\ \substack{+\\\bigcirc\bigcirc},\ \substack{\bigcirc\\\bigcirc\bigcirc},$$
$$\substack{\bigcirc\\\bigcirc-},\ \substack{-\\\bigcirc-},\ \substack{+\\-+},\ \substack{\bigcirc\\-+},\ \substack{-\\-+},\ \substack{\bigcirc\\-\bigcirc},\ \substack{-\\-\bigcirc},\ \substack{-\\--}\Big\}.$$

Let

$$G_2=\Big\{\substack{\bigcirc\\\bigcirc\bigcirc\\\bigcirc},\ \substack{\bigcirc\\+\bigcirc\\\bigcirc},\ \substack{\bigcirc\\+\bigcirc\\+},\ \substack{-\\\bigcirc\bigcirc\\\bigcirc},\ \substack{-\\-\bigcirc\\\bigcirc},\ \substack{-\\-\bigcirc\\-},\ \substack{+\\+\bigcirc\\+},\ \substack{\bigcirc\\\bigcirc-\\\bigcirc},$$
$$\substack{\bigcirc\\\bigcirc-\\-},\ \substack{\bigcirc\\+-\\\bigcirc},\ \substack{\bigcirc\\--\\-},\ \substack{+\\\bigcirc-\\-},\ \substack{+\\+-\\\bigcirc},\ \substack{-\\--\\-},\ \substack{+\\--\\-},\ \substack{+\\+-\\-},\ \substack{+\\+-\\+},$$
$$\substack{-\\\bigcirc+\\\bigcirc},\ \substack{-\\\bigcirc+\\+},\ \substack{-\\-+\\\bigcirc},\ \substack{\bigcirc\\++\\+},\ \substack{-\\++\\+},\ \substack{-\\-+\\+},\ \substack{-\\-+\\-},\ \substack{+\\++\\+}\Big\}.$$

Let

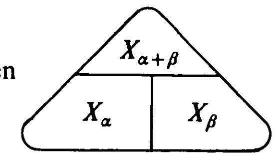
$$G_3=\Big\{\substack{\bigcirc\bigcirc\\\bigcirc\ \ \bigcirc\\\bigcirc\bigcirc},\ \substack{\bigcirc\bigcirc\\\bigcirc\ \ -\\\bigcirc\bigcirc},\ \substack{\bigcirc-\\\bigcirc\ \ -\\\bigcirc\bigcirc},\ \substack{--\\\bigcirc\ \ -\\\bigcirc\bigcirc},\ \substack{--\\-\ \ -\\\bigcirc\bigcirc},\ \substack{--\\-\ \ -\\-\bigcirc},\ \substack{++\\+\ \ +\\+\bigcirc},$$
$$\substack{++\\+\ \ \bigcirc\\+\bigcirc},\ \substack{++\\+\ \ \bigcirc\\\bigcirc\bigcirc},\ \substack{\bigcirc+\\+\ \ \bigcirc\\\bigcirc\bigcirc},\ \substack{\bigcirc\bigcirc\\+\ \ \bigcirc\\\bigcirc\bigcirc},\ \substack{\bigcirc\bigcirc\\+\ \ -\\\bigcirc\bigcirc},\ \substack{\bigcirc\bigcirc\\\bigcirc\ \ \bigcirc\\\bigcirc-},\ \substack{\bigcirc\bigcirc\\\bigcirc\ \ \bigcirc\\--},$$
$$\substack{\bigcirc\bigcirc\\-\ \ \bigcirc\\--},\ \substack{-\bigcirc\\-\ \ \bigcirc\\--},\ \substack{--\\-\ \ \bigcirc\\--},\ \substack{--\\-\ \ -\\--},\ \substack{\bigcirc+\\\bigcirc\ \ \bigcirc\\\bigcirc-},\ \substack{\bigcirc+\\\bigcirc\ \ \bigcirc\\--},\ \substack{\bigcirc+\\-\ \ \bigcirc\\\bigcirc\bigcirc},$$
$$\substack{-\bigcirc\\-\ \ +\\--},\ \substack{--\\-\ \ +\\--},\ \substack{\bigcirc+\\+\ \ \bigcirc\\\bigcirc-},\ \substack{\bigcirc+\\\bigcirc\ \ +\\--},\ \substack{\bigcirc+\\-\ \ +\\--},\ \substack{-+\\-\ \ +\\--},\ \substack{++\\+\ \ \bigcirc\\\bigcirc-},$$
$$\substack{++\\\bigcirc\ \ +\\--},\ \substack{++\\-\ \ +\\--},\ \substack{++\\+\ \ +\\\bigcirc-},\ \substack{++\\+\ \ +\\--},\ \substack{++\\+\ \ +\\+-},\ \substack{\bigcirc-\\\bigcirc\ \ -\\\bigcirc+},\ \substack{--\\\bigcirc\ \ -\\\bigcirc+},$$

$$\begin{matrix} - & - \\ - & - \\ \bigcirc & + \end{matrix},\ \begin{matrix} - & - \\ - & - \\ - & + \end{matrix},\ \begin{matrix} \bigcirc & \bigcirc \\ + & - \\ \bigcirc & + \end{matrix},\ \begin{matrix} \bigcirc & - \\ + & - \\ \bigcirc & + \end{matrix},\ \begin{matrix} - & - \\ \bigcirc & - \\ + & + \end{matrix},\ \begin{matrix} - & - \\ - & - \\ + & + \end{matrix},\ \begin{matrix} \bigcirc & \bigcirc \\ + & - \\ + & + \end{matrix},$$

$$\left.\begin{matrix} \bigcirc & - \\ + & - \\ + & + \end{matrix},\ \begin{matrix} - & - \\ + & - \\ + & + \end{matrix},\ \begin{matrix} + & \bigcirc \\ + & - \\ + & + \end{matrix},\ \begin{matrix} + & - \\ + & - \\ + & + \end{matrix},\ \begin{matrix} + & + \\ + & \bigcirc \\ + & + \end{matrix},\ \begin{matrix} + & + \\ + & - \\ + & + \end{matrix},\ \begin{matrix} + & + \\ + & + \\ + & + \end{matrix}\right\}.$$

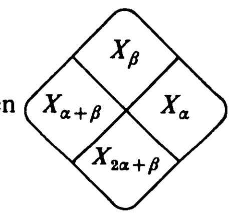
Given an indecomposable positive subsystem Φ'^{+} of Φ of rank 2, we say that a sign type $(X_\alpha)_{\alpha\in\Phi'^{+}}$ is admissible if one of the following conditions is satisfied:

(1) Φ'^{+} has type A_2, say $\Phi'^{+} = \{\alpha, \beta, \alpha+\beta\}$. Then



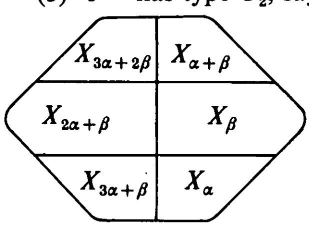
$\in G_1$.

(2) Φ'^{+} has type B_2, say $\Phi'^{+} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$. Then



$\in G_2$.

(3) Φ'^{+} has type G_2, say $\Phi'^{+} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$. Then



$\in G_3$.

We say that a sign type $(X_\alpha)_{\alpha\in\Phi}$ is admissible if for any indecomposable positive subsystem Φ'^{+} of Φ of rank 2, the sign type $(X_\alpha)_{\alpha\in\Phi'^{+}}$ is admissible.

Let $\mathscr{S} = \mathscr{S}(\Phi)$ be the set of all admissible sign types of $\overline{\mathscr{S}}$.

Define a map
$$\zeta\colon W_a \longrightarrow \overline{\mathscr{S}}$$
by sending $A_w = \bigcap_{\alpha\in\Phi} H^1_{\alpha;\,k(w,\alpha)}$ to $X_w = (X(w,\alpha))_{\alpha\in\Phi}$ such that for any $\alpha\in\Phi$,
$$k(w,\alpha) > 0 \Leftrightarrow X(w,\alpha) = +,$$
$$k(w,\alpha) = 0 \Leftrightarrow X(w,\alpha) = \bigcirc,$$
$$k(w,\alpha) < 0 \Leftrightarrow X(w,\alpha) = -.$$
By Theorem 1.2, one can check that $\zeta(W_a) \subseteq \mathscr{S}$. Thus ζ induces a map from W_a to $\mathscr{S}$ which we still denote by ζ. In particular, one can check directly that $\zeta(W_a) = \mathscr{S}$ when Φ has rank 2. In the following sections we shall go further and show the following.

THEOREM 2.1. *$\zeta(W_a) = \mathscr{S}(\Phi)$ for any indecomposable root system Φ.*

We denote $\Pi \cup \{-\alpha_0\}$ by $\tilde{\Pi}$.

3. *Some results on* X_β, $\beta\in\tilde{\Pi}$

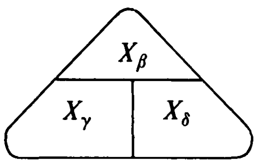
Sections 3 and 4 will be reserved mainly for the proof of Theorem 2.1. We assume that rank $\Phi > 2$ in these two sections.

For any $X=(X_\alpha)_{\alpha\in\Phi}\in\overline{\mathscr{S}}$, we define $m_X = {}^{\#}\{\alpha\in\Phi^+ \mid X_\alpha = -\}$.

LEMMA 3.1. *Assume that* $X=(X_\alpha)_{\alpha\in\Phi}\in\mathscr{S}$ *and* $m_X>0$. *Then there exists some* $\beta\in\Pi$ *satisfying* $X_\beta = -$.

Proof. It is enough to show that if $X_\alpha\in\{+,\bigcirc\}$ for all $\alpha\in\Pi$ then $m_X=0$, that is, for any $\beta\in\Phi^+$, we have $X_\beta\in\{+,\bigcirc\}$. Now we assume that $X_\alpha\in\{+,\bigcirc\}$ for all $\alpha\in\Pi$. We apply induction on $\mathrm{ht}(\beta)\geqslant 1$, the height of $\beta\in\Phi^+$. The result is obviously true when $\mathrm{ht}(\beta)=1$, by our assumption. Now assume that $\mathrm{ht}(\beta)>1$. Then we can write $\beta=\gamma+\delta$ for some $\gamma,\delta\in\Phi^+$. By the inductive hypothesis, $X_\gamma, X_\delta\in\{+,\bigcirc\}$. By symmetry, we need only to consider the following cases.

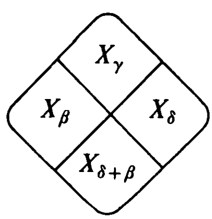
(i) $\{\gamma,\delta,\beta\}$ forms a positive subsystem of Φ of type A_2. By the hypothesis that $X\in\mathscr{S}$ and $X_\gamma, X_\delta\in\{+,\bigcirc\}$, we have



$$\in\left\{\begin{matrix}+\\++\end{matrix},\ \begin{matrix}+\\+\bigcirc\end{matrix},\ \begin{matrix}+\\\bigcirc+\end{matrix},\ \begin{matrix}+\\\bigcirc\bigcirc\end{matrix},\ \begin{matrix}\bigcirc\\\bigcirc\bigcirc\end{matrix}\right\},$$

that is, $X_\beta\in\{+,\bigcirc\}$.

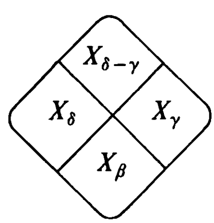
(ii) $\{\gamma,\delta,\beta,\delta+\beta\}$ forms a positive subsystem of Φ of type B_2. Then by the same reasoning as in (i), we have



$$\in\left\{\begin{matrix}\bigcirc\\+\bigcirc\\\bigcirc\end{matrix},\ \begin{matrix}\bigcirc\\+\bigcirc\\+\end{matrix},\ \begin{matrix}+\\+\bigcirc\\+\end{matrix},\ \begin{matrix}\bigcirc\\++\\+\end{matrix},\ \begin{matrix}+\\++\\+\end{matrix},\ \begin{matrix}\bigcirc\\\bigcirc\bigcirc\\\bigcirc\end{matrix}\right\},$$

that is, $X_\beta\in\{+,\bigcirc\}$.

(iii) $\{\delta-\gamma,\gamma,\delta,\beta\}$ forms a positive subsystem of Φ of type B_2. Then by the same reasoning as above, we have



$$\in\left\{\begin{matrix}\bigcirc\\\bigcirc\bigcirc\\\bigcirc\end{matrix},\ \begin{matrix}\bigcirc\\+\bigcirc\\\bigcirc\end{matrix},\ \begin{matrix}\bigcirc\\+\bigcirc\\+\end{matrix},\ \begin{matrix}+\\+\bigcirc\\+\end{matrix},\ \begin{matrix}\bigcirc\\++\\+\end{matrix},\ \begin{matrix}+\\++\\+\end{matrix}\right\},$$

that is, $X_\beta\in\{+,\bigcirc\}$.

Therefore, our result follows by induction.

Call a sign type $X=(X_\alpha)_{\alpha\in\Phi}\in\overline{\mathscr{S}}$ dominant if $m_X=0$.

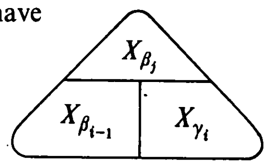
LEMMA 3.2. *Assume that $X=(X_\alpha)_{\alpha\in\Phi}\in\mathscr{S}$ is dominant. Assume that not all X_α, $\alpha\in\Phi$, are equal to* $\bigcirc$. *Then $X_{-\alpha_0}=+$.*

Proof. One can check the result directly when rank $\Phi=2$. Now assume that rank $\Phi>2$. By our condition, there exists some $\alpha\in\Phi^+$ with $X_\alpha=+$.

(i) First assume that α is a short root of Φ (including the case when the roots in Φ all have the same length). Then $\alpha\leqslant-\alpha_0$. By a well-known result, there exists a sequence of roots

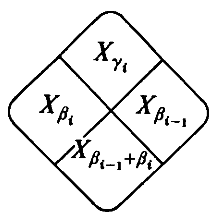
$$\beta_0=\alpha,\beta_1,\ldots,\beta_r=-\alpha_0$$

in Φ^+ such that for every i, $1\leqslant i\leqslant r$, $\beta_{i-1}<\beta_i$ and $\beta_i=(\beta_{i-1})\bar{s}_{\gamma_i}$ with some $\gamma_i\in\Pi$. Clearly, all β_j, $0\leqslant j\leqslant r$, are short roots. Now it is enough to show that if $X_{\beta_{i-1}}=+$ for some i, $1\leqslant i\leqslant r$, then $X_{\beta_i}=+$. Our conditions on β_{i-1}, β_i clearly imply that $\langle\beta_{i-1},\gamma_i^\vee\rangle=-1$ and hence $\beta_i=\beta_{i-1}+\gamma_i$. If γ_i is short then $\{\beta_{i-1},\gamma_i,\beta_i\}$ forms a positive subsystem of Φ of type A_2. Then by the assumption that $X\in\mathscr{S}$, $X_{\beta_{i-1}}=+$ and $X_{\gamma_i}\in\{+,\bigcirc\}$, we have



$$\in\left\{\begin{array}{c}+\\ +\bigcirc\end{array},\ \begin{array}{c}+\\ ++\end{array}\right\}$$

which implies that $X_{\beta_i}=+$. If γ_i is long then $\{\beta_{i-1},\gamma_i,\beta_i,\beta_{i-1}+\beta_i\}$ forms a positive subsystem of Φ of type B_2. Then by the assumption that $\beta_{i-1}=+$, $\gamma_i\in\{+,\bigcirc\}$ and $X\in\mathscr{S}$, we have



$$\in\left\{\begin{array}{c}\bigcirc\\ ++\\ +\end{array},\ \begin{array}{c}+\\ ++\\ +\end{array}\right\}$$

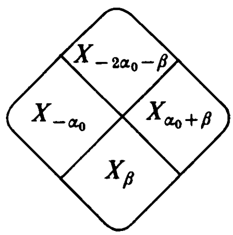
which also implies that $X_{\beta_i}=+$. As i runs over $1,2,\ldots,r$ in turn, we can show that $X_{-\alpha_0}=+$ from X_α by repeatedly using the above argument.

(ii) Now assume that the roots in Φ have two different lengths and that α is a long root. Let β be the highest (long) root of Φ. Then there exists a sequence of long roots

$$\beta_0=\alpha,\beta_1,\ldots,\beta_r=\beta$$

in Φ^+ such that for every i, $1\leqslant i\leqslant r$, $\beta_{i-1}<\beta_i$ and $\beta_i=(\beta_{i-1})\bar{s}_{\gamma_i}$ with some $\gamma_i\in\Pi$. By a similar argument to that in (i), we can show that $X_\beta=+$ from $X_\alpha=+$.

We see that Φ has type B_l, C_l or F_4 according to our assumption. In any of these cases, $\{-2\alpha_0-\beta,\alpha_0+\beta,-\alpha_0,\beta\}$ forms a positive subsystem of Φ of type B_2. So by the hypothesis that $X\in\mathscr{S}$, $X_\beta=+$ and $X_{-2\alpha_0-\beta},X_{\alpha_0+\beta},X_{-\alpha_0}\in\{+,\bigcirc\}$, we have



$$\in\left\{\begin{array}{c}\bigcirc\\ +\bigcirc\\ +\end{array},\ \begin{array}{c}+\\ +\bigcirc\\ +\end{array},\ \begin{array}{c}\bigcirc\\ ++\\ +\end{array},\ \begin{array}{c}+\\ ++\\ +\end{array}\right\}$$

which implies that $X_{-\alpha_0}=+$.

Putting (i) and (ii) together we conclude that $X_{-\alpha_0}=+$.

4. *The sign types X' and X'' for $X \in \mathscr{S}$*

Here we shall give three key lemmas for the proof of Theorem 2.1. We assume that rank $\Phi > 2$ in this section.

LEMMA 4.1. *Assume that $X = (X_\alpha)_{\alpha \in \Phi} \in \mathscr{S}$ is dominant and assume that not all X_α, $\alpha \in \Phi$, are zero. Then we have $X_{-\alpha_0} = +$ by Lemma* 3.2. Let $X' = (X'_\alpha)_{\alpha \in \Phi}$ *be in* $\overline{\mathscr{S}}$ *satisfying $X'_\alpha = X_{(\alpha)s_0}$ for any $\alpha \in \Phi^+$. Let $X'' = (X''_\alpha)_{\alpha \in \Phi}$ be obtained from X' by replacing $X'_{\varepsilon\alpha_0} = \varepsilon$ by $X''_{\varepsilon\alpha_0} = \bigcirc$, $\varepsilon = \pm$. Then $\{X', X''\} \cap \mathscr{S} \neq \emptyset$.*

Sketch of the proof. We must show either that $(X'_\alpha)_{\alpha \in \Phi'^+}$ are admissible for all indecomposable positive subsystems Φ'^+ of Φ of rank 2, or that $(X''_\alpha)_{\alpha \in \Phi'^+}$ are admissible for all these subsystems Φ'^+ of Φ. To do this, we need only show that if Φ'^+, Φ''^+ are two indecomposable positive subsystems of Φ of rank 2 then either $(X'_\alpha)_{\alpha \in \Phi'^+}$ or $(X''_\alpha)_{\alpha \in \Phi''^+}$ must be admissible. If $-\alpha_0 \notin \Phi'^+ \cap \Phi''^+$ then either

$$(X'_\alpha)_{\alpha \in \Phi'^+} = (X_\alpha)_{\alpha \in (\Phi'^+)s_0}$$

with $-\alpha_0 \notin \Phi'^+$ or

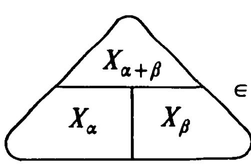
$$(X''_\alpha)_{\alpha \in \Phi''^+} = (X_\alpha)_{\alpha \in (\Phi''^+)s_0}$$

with $-\alpha_0 \notin \Phi''^+$. So our result follows by the assumption that $X \in \mathscr{S}$. If $-\alpha_0 \in \Phi'^+ \cap \Phi''^+$ then we have one of the following cases.

(a) $\Phi'^+ = \Phi''^+$ and both have type A_2 or B_2.

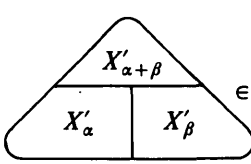
(b) $\Phi'^+ \neq \Phi''^+$ and they are both in some indecomposable positive subsystem of Φ of type A_3, B_3 or C_3.

We can verify our result case by case. For example, in case (a) with Φ'^+ of type A_2, we assume that $(X'_\alpha)_{\alpha \in \Phi'} \notin \mathscr{S}(\Phi')$. Say $\Phi'^+ = \{\alpha, \beta, \alpha+\beta\}$. Then $-\alpha_0 = \alpha+\beta$. Since $X \in \mathscr{S}$ is dominant and $X_{-\alpha_0} = +$, we have



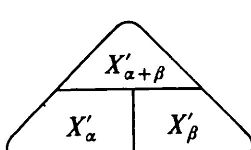
$$\in \left\{ \begin{smallmatrix} + \\ ++ \end{smallmatrix}, \begin{smallmatrix} + \\ +\bigcirc \end{smallmatrix}, \begin{smallmatrix} + \\ \bigcirc+ \end{smallmatrix}, \begin{smallmatrix} + \\ \bigcirc\bigcirc \end{smallmatrix} \right\}.$$

Thus



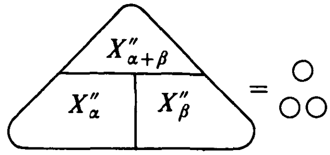
$$\in \left\{ \begin{smallmatrix} - \\ -- \end{smallmatrix}, \begin{smallmatrix} - \\ \bigcirc- \end{smallmatrix}, \begin{smallmatrix} - \\ -\bigcirc \end{smallmatrix}, \begin{smallmatrix} - \\ \bigcirc\bigcirc \end{smallmatrix} \right\}.$$

By the assumption of $(X'_\alpha)_{\alpha \in \Phi'} \notin \mathscr{S}(\Phi')$, we get



$$= \begin{smallmatrix} - \\ \bigcirc\bigcirc \end{smallmatrix}.$$

But then



is admissible.

LEMMA 4.2. *Assume that* $X=(X_\alpha)_{\alpha\in\Phi}\in\mathscr{S}$ *and* $X_\beta=-$ *for some* $\beta\in\Pi$. *Let* $X'=(X'_\alpha)_{\alpha\in\Phi}$ *be such that* $X'_\alpha=X_{(\alpha)s_\beta}$ *for all* $\alpha\in\Phi$. *Let* $X''=(X''_\alpha)_{\alpha\in\Phi}$ *be obtained from* X' *by replacing* $X'_{\varepsilon\beta}=\varepsilon$ by $X''_{\varepsilon\beta}=\bigcirc$, $\varepsilon=\pm$. *Then either* X' *or* X'' (*or both*) *must be in* $\mathscr{S}$.

The strategy of the proof for this lemma is similar to that for Lemma 4.1 but is more complicated. We omit the detail.

LEMMA 4.3. *Assume that* $X=(X_\alpha)_{\alpha\in\Phi}\in\mathscr{S}$ *and* $m_X=1$. *Then by Lemma* 3.1, *we have* $X_\beta=-$ *for some* $\beta\in\Pi$. *Let* $X''=(X''_\alpha)_{\alpha\in\Phi}$ *be a sign type of* $\overline{\mathscr{S}}$ *satisfying*

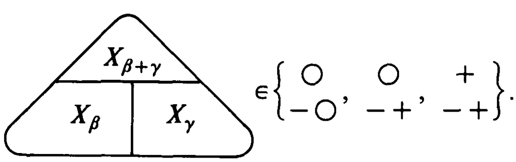
$$X''_\alpha=\begin{cases}X_{(\alpha)s_\beta} & \text{if } \alpha\neq\beta,\\ \bigcirc & \text{if } \alpha=\beta,\end{cases}$$

for any $\alpha\in\Phi^+$. *Then* $X''\in\mathscr{S}$.

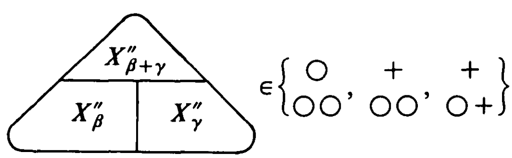
Proof. Let Φ'^+ be any positive subsystem of Φ of rank 2. We must show that $(X''_\alpha)_{\alpha\in\Phi'^+}$ is admissible.

If $\beta\notin\Phi'^+$ then $(X''_\alpha)_{\alpha\in\Phi'^+}=(X_\alpha)_{\alpha\in(\Phi'^+)s_\beta}$. Since $(\Phi'^+)s_\beta$ is also a positive subsystem of Φ of rank 2, the admissibility of $(X''_\alpha)_{\alpha\in\Phi'^+}$ follows from $X\in\mathscr{S}$.

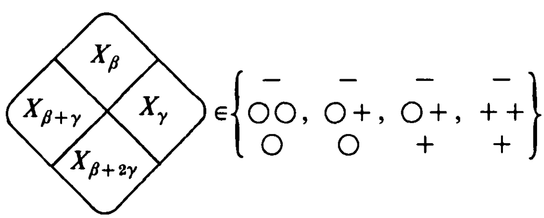
Now assume that $\beta\in\Phi'^+$. We know that Φ'^+ either has type A_2 or B_2. First suppose that Φ'^+ has type A_2 with $\Phi'^+=\{\beta,\gamma,\beta+\gamma\}$. By the assumption that $X\in\mathscr{S}$, $m_X=1$ and $X_\beta=-$, we have



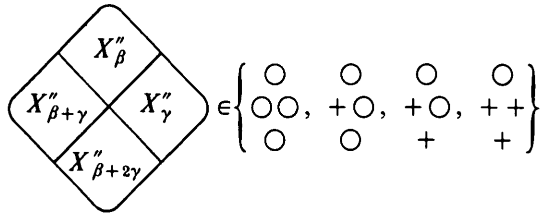
Thus



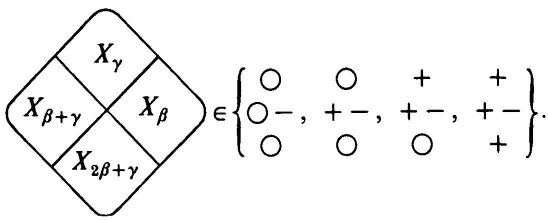
which is admissible. Next suppose that Φ'^+ has type B_2 with $\Phi'^+=\{\beta,\gamma,\beta+\gamma,\beta+2\gamma\}$. By the same reasoning as above, we have



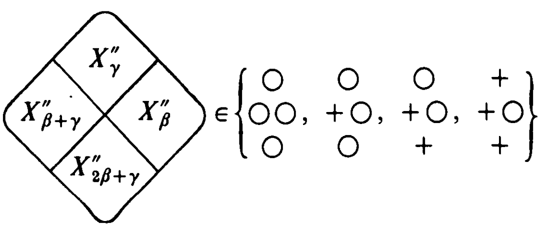
So



which is admissible also. Finally, suppose that Φ'^{+} has type B_2 with $\Phi'^{+} = \{\beta, \gamma, \beta+\gamma, 2\beta+\gamma\}$. We have



So



which is again admissible.

Therefore we have $X'' \in \mathscr{S}$.

5. *The proof of Theorem* 2.1

It is known that the result is true when Φ has rank 2. So we may assume that $\operatorname{rank}\Phi > 2$. It is also known that the inclusion $\zeta(W_a) \subseteq \mathscr{S}$ holds in general. Now we define $n_X = \#\{\alpha \in \Phi^+ \mid X_\alpha \neq \bigcirc\}$ and $m_X = \#\{\alpha \in \Phi^+ \mid X_\alpha = -\}$ for any $X \in \overline{\mathscr{S}}$.

Assume that we are given a sign type $X = (X_\alpha)_{\alpha \in \Phi}$ in $\mathscr{S}$. We must find an element w of W_a such that $\zeta(w) = X$. We apply induction on $n_X \geqslant 0$. It is clear that $\zeta(1) = X$ in the case when $n_X = 0$. Now assume that $n_X > 0$.

(a) If $m_X = 1$ then by Lemma 3.1 there exists some $\beta \in \Pi$ such that $X_\beta = -$. Let $X'' = (X''_\alpha)_{\alpha \in \Phi} \in \overline{\mathscr{S}}$ be defined by

$$X''_\alpha = \begin{cases} X_{(\alpha)s_\beta} & \text{if } \alpha \neq \beta, \\ \bigcirc & \text{if } \alpha = \beta, \end{cases}$$

for any $\alpha \in \Phi^+$. Then by Lemma 4.3 we have $X'' \in \mathscr{S}$ with $n_{X'} < n_X$. By the inductive hypothesis, there exists some $w' \in \zeta^{-1}(X'')$. Let $w = w's_\beta$. Then $\zeta(w) = X$ by Proposition 1.3.

(b) If $m_X > 1$ then there exists some $\beta(1) \in \Pi$ with $X_{\beta(1)} = -$ by Lemma 3.1. Let $X' = (X'_\alpha)_{\alpha \in \Phi} \in \overline{\mathscr{S}}$ be defined by $X'_\alpha = X_{(\alpha)s_{\beta(1)}}$ for any $\alpha \in \Phi$. Let $X'' = (X''_\alpha)_{\alpha \in \Phi} \in \overline{\mathscr{S}}$ be obtained from X' by replacing $X'_{\varepsilon\beta(1)} = \varepsilon$ by $X''_{\varepsilon\beta(1)} = \bigcirc$, $\varepsilon = \pm$. Then by Lemma 4.2, one of X' and X'' must be in $\mathscr{S}$. We denote this sign type by $X(1)$ (note that when both X' and X'' are in $\mathscr{S}$ we can freely choose one of them and call it $X(1)$). Clearly $m_{X(1)} = m_X - 1$. If $m_{X(1)}$ is still greater than 1 then the same process can be carried on and we get a sequence of sign types $X(0) = X, X(1), \ldots, X(m)$ in $\mathcal{S}$ with $m = m_X$ such that for every i, $1 \leqslant i \leqslant m$, $X(i-1)_{\beta(i)} = -$ with some $\beta(i) \in \Pi$, and either $X(i)_\alpha = X(i-1)_{(\alpha)s_{\beta(i)}}$ for all $\alpha \in \Phi$ or $X(i)_\alpha = X(i-1)_{(\alpha)s_{\beta(i)}}$ for all $\alpha \in \Phi - \{\varepsilon\beta(i)\}$ and $X(i)_{\varepsilon\beta(i)} = \bigcirc$. In particular, $X(m)_{\varepsilon\beta(m)} = \bigcirc$. Since $m_{X(i)} = m_{X(i-1)} - 1$, we have $m_{X(i)} = m_X - i$ and in particular $m_{X(m-1)} = 1$. Hence such a sequence $X(0), X(1), \ldots, X(m)$ does exist in $\mathcal{S}$. Clearly $n_{X(m)} < n_X$. Thus by the inductive hypothesis, there exists some $x \in \zeta^{-1}(X(m))$. Let $w = xs_{\beta(m)}s_{\beta(m-1)} \cdots s_{\beta(1)}$. Then by Proposition 1.3, we have $\zeta(w) = X$.

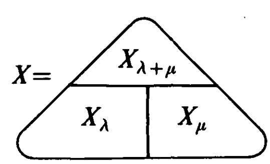
(c) If $m_X = 0$ then X is dominant. Since $n_X > 0$, we see by Lemma 3.2 that $X_{-\alpha_0} = +$. Let $X' = (X'_\alpha)_{\alpha \in \Phi}$ be in $\overline{\mathcal{S}}$ satisfying $X'_\alpha = X_{(\alpha)s_0}$ for all $\alpha \in \Phi$ and let $X'' = (X''_\alpha)_{\alpha \in \Phi} \in \overline{\mathcal{S}}$ be obtained from X' by replacing $X'_{\varepsilon\alpha_0} = \varepsilon$ by $X''_{\varepsilon\alpha_0} = \bigcirc$. Then Lemma 4.1 asserts that $\{X', X''\} \cap \mathcal{S} \neq \varnothing$. Denote one of $\{X', X''\} \cap \mathcal{S}$ by $X(1)$. Then either $n_{X(1)} < n_X$, or $n_{x(1)} = n_X$ and $m_{X(1)} > 0$. So either by the inductive hypothesis or by (b) we can find some $x \in \zeta^{-1}(X(1))$. Let $w = xs_0$. Then by Proposition 1.3, we get $\zeta(w) = X$, again.

Therefore our result follows by induction.

6. *The geometrical interpretation of admissible sign types*

Let $\mathcal{T} = \{H_{\alpha;\tau} \mid \alpha \in \Phi^+, \tau = 0, 1\}$. Then the connected components of $E - \bigcup_{H \in \mathcal{T}} H$ are open simplices. We see that any alcove of E lies in some connected component of $E - \bigcup_{H \in \mathcal{T}} H$ and that two alcoves correspond to the same sign type if and only if they are in the same connected component of $E - \bigcup_{H \in \mathcal{T}} H$. So by Theorem 2.1, the map ζ induces a bijection between the set of connected components of $E - \bigcup_{H \in \mathcal{T}} H$ and the set $\mathcal{S}$. Then we can identify these two sets.

EXAMPLES 6.1. (1) When Φ has type A_2, say $\Phi^+ = \{\lambda, \mu, \lambda+\mu\}$, the number of connected components of $E - \bigcup_{H \in \mathcal{T}} H$ is 16. Each of these components determines a sign type



as in Figure 1.

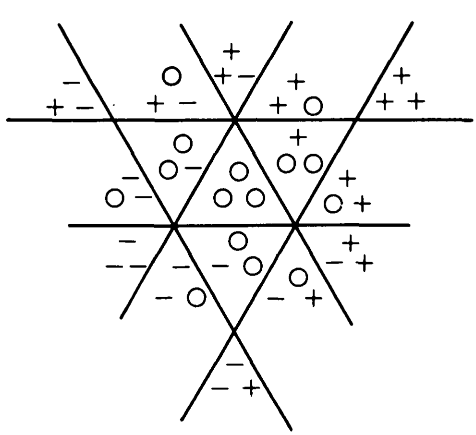
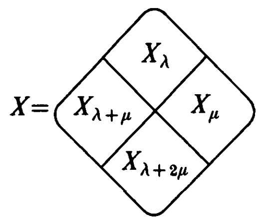


FIG. 1

(2) When Φ has type B_2, say $\Phi^+ = \{\lambda, \mu, \lambda+\mu, \lambda+2\mu\}$, then $E - \bigcup_{H \in \mathscr{F}} H$ has 25 connected components each of which determines a sign type



as in Figure 2.

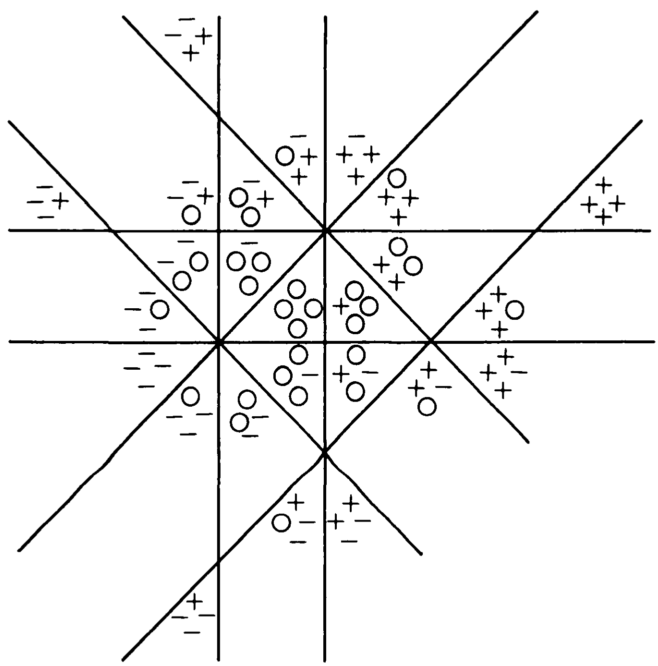
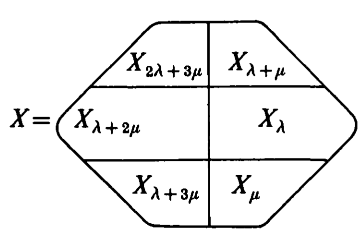


FIG. 2

(3) Let $\Phi^+ = \{\lambda, \mu, \lambda+\mu, \lambda+2\mu, \lambda+3\mu, 2\lambda+3\mu\}$ be the positive system of Φ of type G_2. Then $E - \bigcup_{H \in \mathscr{F}} H$ has 49 connected components each of which determines a sign type



as in Figure 3.

It is well known that any alcove of E has $l+1$ facets. In [4], we labelled any facet of an alcove by an element $s \in \Delta$ such that the following result holds.

LEMMA 6.2 [4, Lemma 6.1]. *If $w, w' \in W_a$ have the relation $w' = s_t w$ for some $s_t \in \Delta$ then the alcoves A_w and $A_{w'}$ share the common s_t-facet. Conversely, if A_w and $A_{w'}$ are*

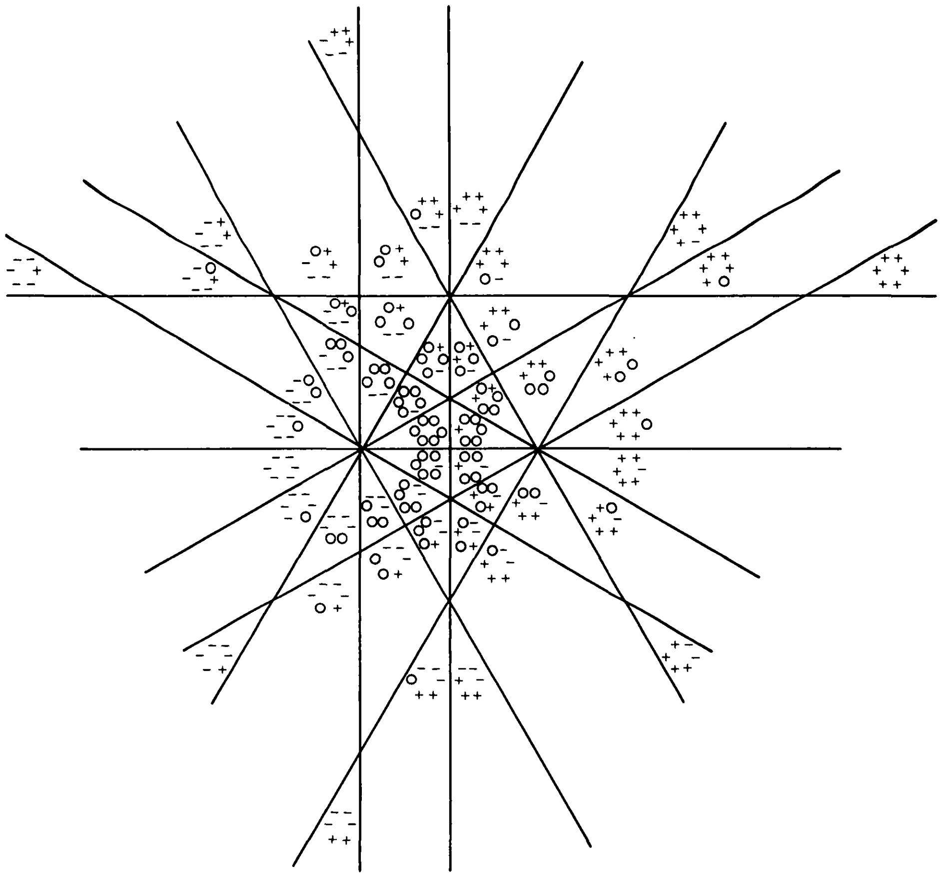


FIG. 3

two alcoves of E which share a common facet then the labelling of this facet for A_w is the same as for $A_{w'}$, say s_t-facet. We have $w' = s_t w$.

The following result is due to the convexity of an admissible sign type.

PROPOSITION 6.3. *For any $X \in \mathscr{S}$ and $w, y \in \zeta^{-1}(X)$, there exists a sequence of elements $w_0 = w, w_1, \ldots, w_r = y$ in W_a such that for every h, j with $0 \leqslant h \leqslant r$ and $1 \leqslant j \leqslant r$, $w_h \in \zeta^{-1}(X)$ and $w_j = s_j w_{j-1}$ for some $s_j \in \Delta$.*

Proof. We see that each connected component of $E - \bigcup_{H \in \mathscr{T}} H$ is convex. Our condition means that w, y are in the same connected component X of $E - \bigcup_{H \in \mathscr{T}} H$. So there exists a sequence of alcoves $A_0 = A_w, A_1, \ldots, A_r = A_y$ in X such that for every j, $1 \leqslant j \leqslant r$, A_j and A_{j-1} share a common wall. Hence our result follows by Lemma 6.2.

Recall that $\mathscr{R}(w) = \{s \in \Delta \mid ws < w\}$ for any $w \in W_a$. Now we need the following.

LEMMA 6.4 [**4**, Proposition 4.3(ii)].

$$\mathscr{R}(w) = \{s_j \in \Delta \mid k(w, \alpha_j) < 0\}.$$

By the definition of the map ζ and Lemma 6.4, the function $\mathscr{R}(w)$ on $\zeta^{-1}(X)$ for any $X\in\mathscr{S}$ is constant. So we can define $\mathscr{R}(X)=\mathscr{R}(w)$ for any $w\in\zeta^{-1}(X)$. We see from Lemma 6.4 that $\mathscr{R}(X)=\{s_i\in\Delta \mid X_{\alpha_i}=-\}$.

7. *The shortest elements of* $\zeta^{-1}(X)$, $X\in\mathscr{S}$

The shortest elements of $\zeta^{-1}(X)$, $X\in\mathscr{S}$, have very nice properties. They will play a crucial role in the calculation of the cardinality of $\mathscr{S}$.

PROPOSITION 7.1. *Let U be a set of sign types of $\mathscr{S}$ such that there exists some $Y\in U$ which can be obtained from any $Z\in U$ by substituting some non-zero signs by zero signs. Then there exists an element $y\in\zeta^{-1}(Y)$ such that $|k(y,\alpha)|=\min\{|k(x,\alpha)| \mid x\in\zeta^{-1}(U)\}$ for all $\alpha\in\Phi^+$.*

Proof. We apply induction on $l=n_Y\geqslant 0$. The result is trivial in the case when $l=0$.

(1) First assume that $m_Y>0$. Then by Lemma 3.1 there must exist some $\lambda=\eta(1)\in\Pi$ with $Y_\lambda=-$. Then we also have $Z_\lambda=-$ for any $Z\in U$. Let $Y'=(Y'_\alpha)_{\alpha\in\Phi}=\overline{\mathscr{S}}$ be defined by $Y'_\alpha=Y_{(\alpha)s_\lambda}$ for all $\alpha\in\Phi$. Let $Y''=(Y''_\alpha)_{\alpha\in\Phi}\in\overline{\mathscr{S}}$ be obtained from Y' by replacing $Y'_{\varepsilon\lambda}$ with $Y''_{\varepsilon\lambda}=\bigcirc$, $\varepsilon=\pm$. We define $Z',Z''\in\overline{\mathscr{S}}$ from any $Z\in U$ in the same way as Y', Y'' from Y. Then by our assumption on U we see that Y'' can be obtained from Z'' (respectively Z') with $Z\in U$ by replacing some non-zero signs by zero signs. We also see that Y' can be obtained from Z' with $Z\in U$ by replacing some non-zero signs by zero signs. We can show that if $Z''\in\mathscr{S}$ for some $Z\in U$ then $Y''\in\mathscr{S}$. Thus by Lemma 4.2, we have either $Y''\in\mathscr{S}$ or

$$\{Z',Z''\mid Z\in U\}\cap\mathscr{S}=\{Z'\mid Z\in U\}.$$

Let $U^1=\{Z',Z''\mid Z\in U\}\cap\mathscr{S}$ and let

$$Y(1)=\begin{cases} Y'' & \text{if } Y''\in\mathscr{S},\\ Y' & \text{if } Y''\notin\mathscr{S}.\end{cases}$$

Then $Y(1)$ can be obtained from any sign type of U^1 by replacing some non-zero signs by zero signs.

If $m_{Y(1)}=m_Y-1>0$ then the same process can be carried on by substituting U^1 and $Y(1)$ for U and Y. We get a sequence of subsets $U^0=U$, U^1, $\ldots$, U^m in $\mathscr{S}$, a sequence of sign types $Y(0)=Y, Y(1),\ldots,Y(m)$ in $\mathscr{S}$ and a sequence of simple roots $\eta(1),\eta(2),\ldots,\eta(m)$ in Π with $m=m_Y$. These roots are such that for every i, $1\leqslant i\leqslant m$, and for any $X=(X_\alpha)_{\alpha\in\Phi}\in U^{i-1}$, the following conditions hold.

(a) $X_{\eta(i)}=-$.

(b) Let $X'=(X'_\alpha)_{\alpha\in\Phi}\in\overline{\mathscr{S}}$ be defined by $X'_\alpha=X_{(\alpha)s_{\eta(i)}}$ for all $\alpha\in\Phi$ and let $X''=(X''_\alpha)_{\alpha\in\Phi}\in\overline{\mathscr{S}}$ be obtained from X' by replacing $X'_{\varepsilon\eta(i)}$ by $X''_{\varepsilon\eta(i)}=\bigcirc$. Then $U^i=\{X', X''\mid X\in U^{i-1}\}\cap\mathscr{S}$ and $Y(i)\in U^i$ is defined to be $Y(i-1)''$ if $Y(i-1)''\in\mathscr{S}$ or to be $Y(i-1)'$ otherwise. Then by Lemma 4.2 and the above result we see that for every i, $0\leqslant i\leqslant m$, $Y(i)$ can be obtained from any $X\in U^i$ by substituting zero signs for some non-zero signs. In particular, we see from Lemma 4.3 that $Y(m)=Y(m-1)''$ and so $n_{Y(m)}<n_Y$.

Let $\mathscr{M}=\{w\in W_a\mid\zeta(w)\in U^m\}$ and $\mathscr{Y}=\zeta^{-1}(U)$. Then by Proposition 1.3, the map

$\phi\colon w \mapsto ws_{\eta(m)} s_{\eta(m-1)} \cdots s_{\eta(1)}$ gives a bijection from $\mathscr{M}$ to $\mathscr{Y}$ which satisfies $l(w)+m = l(\phi(w))$ for any $w \in \mathscr{M}$.

By the inductive hypothesis, there exists an element y of $\zeta^{-1}(Y^{(m)})$ satisfying $|k(y,\alpha)| = \min\{|k(x,\alpha)| \mid x \in \mathscr{M}\}$ for all $\alpha \in \Phi^+$. By the rule of the right action of W_a on $\mathfrak{A}$, we see that for any $\alpha \in \Phi$, the difference $|k(\phi(w),\alpha)| - |k(w,\alpha)|$ is a non-negative constant on $w \in \mathscr{M}$. Thus we have

$$|k(\phi(y),\alpha)| = \min\{|k(\phi(x),\alpha)| \mid x \in \mathscr{M}\} = \min\{|k(x,\alpha)| \mid x \in \mathscr{Y}\},$$

for all $\alpha \in \Phi^+$. Clearly, $\phi(y) \in \zeta^{-1}(Y)$. So our result follows in this case.

(2) Next assume that $m_Y = 0$. Then by Lemma 3.2, we have $Y_{-\alpha_0} = +$ and also $Z_{\alpha_0} = +$ for all $Z = (Z_\alpha)_{\alpha \in \Phi} \in U$. Let $Y' = (Y'_\alpha)_{\alpha \in \Phi} \in \overline{\mathscr{S}}$ be defined by $Y'_\alpha = Y_{(\alpha)s_0}$ for all $\alpha \in \Phi$. Let $Y'' = (Y''_\alpha)_{\alpha \in \Phi} \in \mathscr{S}$ be obtained from Y' by replacing $Y'_{\varepsilon\alpha_0}$ by $Y''_{\varepsilon\alpha_0} = \bigcirc$. We define $Z', Z'' \in \overline{\mathscr{S}}$ from any $Z \in U$ in the same way as Y', Y'' were defined from Y. Then Y'' can be obtained from Z'' (respectively Z') with $Z \in U$ by replacing some non-zero signs by zero signs. Also, Y' can be obtained from Z' with $Z \in U$ by replacing some non-zero signs by zero signs.

We claim that if $Z'' \in \mathscr{S}$ for some $Z \in U$ then $Y'' \in \mathscr{S}$, since otherwise, this would imply $Y' \notin \mathscr{S}$ which contradicts the fact that $\{Y', Y''\} \cap \mathscr{S} \neq \varnothing$. Thus we have either $Y'' \in \mathscr{S}$ or

$$\{Z', Z'' \mid Z \in U\} \cap \mathscr{S} = \{Z' \mid Z \in U\}.$$

We define

$$\tilde{Y} = \begin{cases} Y'' & \text{if } Y'' \in \mathscr{S}, \\ Y' & \text{otherwise.} \end{cases}$$

Let $U' = \{Z', Z'' \mid Z \in U\} \cap \mathscr{S}$. Then $\tilde{Y}$ can be obtained from any $X \in U'$ by substituting zero signs for some non-zero signs. We also have $n_{\tilde{Y}} \leqslant n_Y$.

If $\tilde{Y} = Y''$ then $n_{\tilde{Y}} < n_Y$. By the inductive hypothesis, we have

$$|k(y,\alpha)| = \min\{|k(x,\alpha)| \mid x \in \zeta^{-1}(U)\}$$

for any $\alpha \in \Phi$ and some $y \in \zeta^{-1}(\tilde{Y})$. If $\tilde{Y} = Y'$ then $m_{\tilde{Y}} > 0$. By the case which we have discussed with Y' and U' instead of Y and U, we also have

$$|k(y,\alpha)| = \min\{|k(x,\alpha)| \mid x \in \zeta^{-1}(U')\}$$

for any $\alpha \in \Phi$ and some $y \in \zeta^{-1}(\tilde{Y})$. Then in either case, $\phi\colon w \mapsto ws_0$ gives a bijection from $\zeta^{-1}(U')$ to $\zeta^{-1}(U)$ which satisfies

$$|k(w,\alpha)| = \begin{cases} |k(\phi(w),\alpha)| & \text{if } \alpha \in \Phi^+ - \{-\alpha_0\}, \\ |k(\phi(w),\alpha)| - 1 & \text{if } \alpha = -\alpha_0, \end{cases}$$

for all $w \in \zeta^{-1}(U')$. So

$$\begin{aligned} |k(\phi(y),\alpha)| &= \min\{|k(\phi(x),\alpha)| \mid x \in \zeta^{-1}(U')\} \\ &= \min\{|k(x,\alpha)| \mid x \in \zeta^{-1}(U)\}. \end{aligned}$$

Clearly, $\phi(y)$ is in $\zeta^{-1}(Y)$. So our result is also true in this case. By induction, we reach our goal.

The element $y \in \zeta^{-1}(Y)$ in the above proposition is clearly the shortest element of $\zeta^{-1}(U)$ which is unique. In particular, when U consists of a single sign type, we get the following.

PROPOSITION 7.2. *For any $X \in \mathscr{S}$, there exists a unique shortest element, say y, of $\zeta^{-1}(X)$ which is characterized by the requirement that*

$$|k(y,\alpha)| = \min\{|k(x,\alpha)| \mid x \in \zeta^{-1}(X)\}$$

for all $\alpha \in \Phi$, where the $k(x,\alpha)$ are as in Proposition 7.1.

Now we shall give another criterion for an element to be the shortest element of $\zeta^{-1}(X)$ for any $X \in \mathscr{S}$.

Let w', $w \in W_a$ and $s_\lambda \in \Delta$ satisfy $w' = s_\lambda w$ and $l(w') = l(w) - 1$. Then by the definition of the left action of W_a on the alcove set $\mathfrak{A}$ we have $k(w',\alpha) = k(w,\alpha)$ for all $\alpha \in \Phi^+$ but one. Let the exceptional one be $\beta \in \Phi^+$. Then we have $|k(w',\beta)| = |k(w,\beta)| - 1$ and $k(w',\beta) = k(w,\beta) \pm 1$. Now assume that w is the shortest element of $\zeta^{-1}(\zeta(w))$. Then by Proposition 7.2, we must have $k(w,\beta) = \pm 1$ and $k(w',\beta) = 0$. In particular, $w' \notin \zeta^{-1}(\zeta(w))$.

PROPOSITION 7.3. *Let $X \in \mathscr{S}$ and $w \in \zeta^{-1}(X)$. Then w is the shortest element of $\zeta^{-1}(X)$ if and only if, for any $s \in \mathscr{L}(w)$, we have $sw \notin \zeta^{-1}(X)$.*

Proof. Let w be the shortest element of $\zeta^{-1}(X)$. By the above discussion, it is sufficient to show that if $y \in \zeta^{-1}(X)$ with $y \neq w$ then there must exist some $s \in \mathscr{L}(y)$ such that $sy \in \zeta^{-1}(X)$. Now assume that $y \in \zeta^{-1}(X)$ with $y \neq w$. Then by Proposition 7.2, we have, for any $\alpha \in \Phi^+$,

$$k(y,\alpha)\begin{cases} \geqslant k(w,\alpha) > 0 & \text{if } X_\alpha = +, \\ \leqslant k(w,\alpha) < 0 & \text{if } X_\alpha = -, \\ = k(w,\alpha) = 0 & \text{if } X_\alpha = \bigcirc, \end{cases}$$

and the set $D = \{\alpha \in \Phi^+ \mid k(y,\alpha) \neq k(w,\alpha)\}$ is non-empty. Let

$$D^+ = \{\alpha \in D \mid X_\alpha = +\} \quad \text{and} \quad D^- = \{\alpha \in D \mid X_\alpha = -\}.$$

Set $a = k(y,\alpha) + 1 - k(w,\alpha)$ and $b = 2 - a$. Let

$$K_1 = \bigcap_{\alpha \in \Phi^+ - D} H^1_{\alpha;k(w,\alpha)}, \qquad K_2 = \bigcap_{\alpha \in D^+} H^a_{\alpha;k(w,\alpha)}, \qquad K_3 = \bigcap_{\alpha \in D^-} H^b_{\alpha;k(y,\alpha)}$$

with the convention that $K_i = E$ if the set of indices for the corresponding intersection is empty. Let $K = K_1 \cap K_2 \cap K_3$. We have $A_w, A_y \subset K \subset X$ regarded as sets of vectors of E. On the other hand, we see that for any alcove $A \in \mathfrak{A}$, either $A \subset K$ or $A \cap K = \varnothing$. So K can be regarded as a set of all elements x of W_a with $A_x \subset K$. Thus w (respectively y) is the shortest (respectively longest) element in K. Since K is a convex set of E, which contains more than one alcove of E, there must exist some alcove A_x in K with $x \neq y$ such that A_x and A_y share a common facet. That is, $x = sy$ for some $s \in \Delta$ by Lemma 6.2. Clearly, we have $s \in \mathscr{L}(y)$ and $x \in \zeta^{-1}(X)$. Thus our result follows.

8. *The cardinality of $\mathscr{S}(\Phi)$*

In this section, our aim is to prove Carter's conjecture.

THEOREM 8.1 (Carter's conjecture). $|\mathscr{S}(\Phi)| = (h+1)^l$, *where* $l = \operatorname{rank}\Phi$ *and h is the Coxeter number of* Φ.

To prove our result, we need an earlier result.

PROPOSITION 8.2 [4, Proposition 3.4]. *Let* $w \in W_a$. *Then for any* $\alpha \in \Phi$, $k(w^{-1}, \alpha) = k(w, -(\alpha)\bar{w})$.

Let
$$E(S) = \{w \in W_a \mid w \text{ is the shortest element of } \zeta^{-1}(\zeta(w))\}.$$

Let $E(S)^{-1} = \{w \mid w^{-1} \in E(S)\}$. For any $w \in W_a$, we see from Propositions 8.2, 7.2 and 7.3 that $w \in E(S)^{-1}$ if and only if $k(w, \lambda) = -1$ for any $s_\lambda \in \mathscr{R}(w)$ and this is the case if and only if $k(w, \lambda) \geqslant -1$ for any $s_\lambda \in \Delta$, where we assume that $A_w = \bigcap_{\alpha \in \Phi} H^1_{\alpha; k(w, \alpha)}$.

We define

$$H^+_{\alpha; k} = \{v \in E \mid \langle v, \alpha^\vee \rangle > k\}, \quad H^-_{\alpha; k} = \{v \in E \mid \langle v, \alpha^\vee \rangle < k\}$$

for any $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. Then regarded as a set of alcoves of E, $E(S)^{-1}$ is the set of all alcoves of E contained in $H = (\bigcap_{\alpha \in \Pi} H^+_{\alpha; -1}) \cap H^-_{-\alpha_0; 2}$.

Define $Z \subset W_a$ to be a left connected set of W_a if for any $x, y \in Z$, there exists a sequence of elements $x_0 = x, x_1, \ldots, x_r = y$ in Z such that for every i, $1 \leqslant i \leqslant r$, $x_{j-1} x_j^{-1} \in \Delta$. Then by the convexity of H we see that $E(S)^{-1}$ is a left connected set of W_a.

By the way, any admissible sign type, regarded as a set of elements of W_a, is a left connected set of W_a by the convexity of the sign type in E.

EXAMPLES 8.3. When the rank of Φ is 2, $E(S)^{-1}$ is the set of all alcoves in the fully shaded area of each of Figures 4, 5 and 6. From these figures, we see that

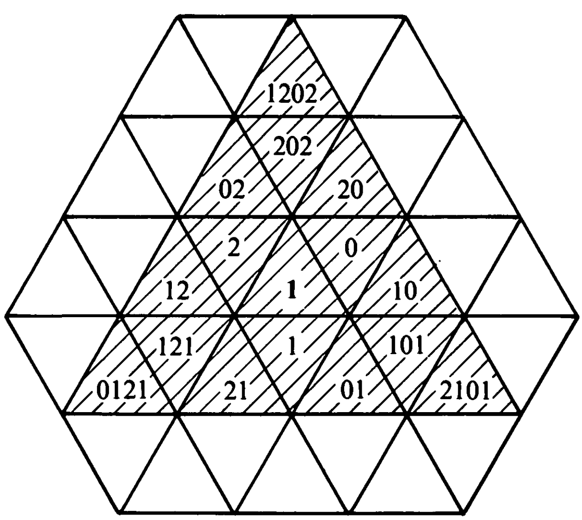


FIG. 4. Type A_2

$H = H^+_{\alpha_1; -1} \cap H^+_{\alpha_2; -1} H^-_{-\alpha_0; 2}$ are all triangles (the fully shaded areas) similar to the corresponding alcoves A_1 (the alcoves labelled by 1). The scale of A_1 to H is $1 : h+1$ and so the area of H is $(h+1)^{\dim E}$ times that of A_1, where h is the Coxeter number of Φ. Then H contains $(h+1)^{\dim E}$ alcoves altogether.

Recall from §1 that $H^m_{\alpha; k} = \{v \in E \mid k < \langle v, \alpha^\vee \rangle < k+m\}$ for any $\alpha \in \Phi^+$, $k \in \mathbb{Z}$ and $m > 0$ in $\mathbb{R}$.

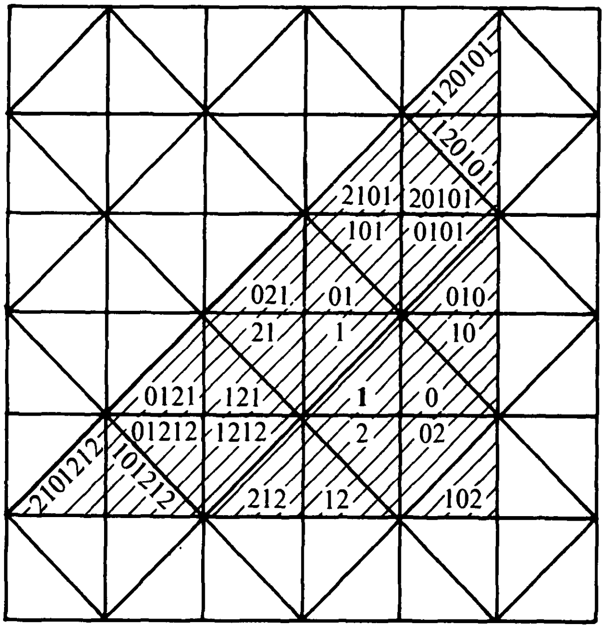


FIG. 5. Type B_2

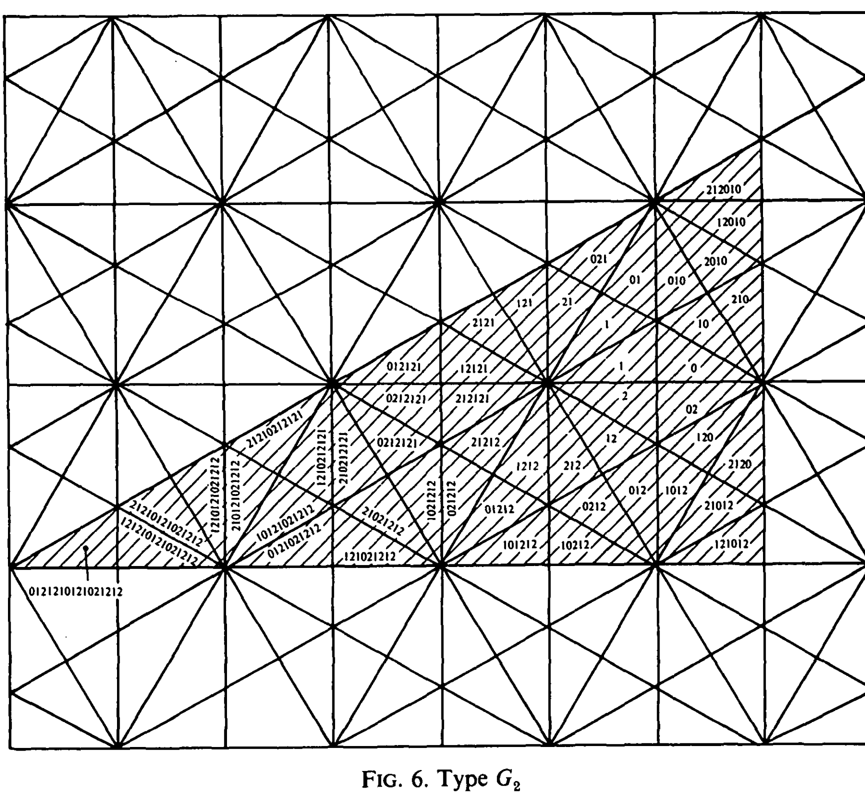


FIG. 6. Type G_2

LEMMA 8.4. *Let* $H^{h+1} = (\bigcap_{\alpha\in\Pi} H_{\alpha;-1}^{((h+1)/c_\alpha)}) \cap H_{-\alpha_0;1-h}^{h+1}$. *Then* $H = H^{h+1}$, *where* $(-\alpha_0)^\vee = \sum_{\alpha\in\Pi} c_\alpha \alpha^\vee$.

Proof. Since $H_{\alpha;-1}^{(h+1)/c_\alpha} \subset H_{\alpha;-1}^{+}$ for all $\alpha\in\Pi$, and $H_{-\alpha_0;1-h}^{h+1} \subset H_{-\alpha_0;2}^{-}$, the inclusion $H \supseteq H^{h+1}$ is obvious.

Now let $v\in H$. To prove that $v\in H^{h+1}$, it is enough to show that

$$\langle v,\alpha^\vee\rangle < (h+1)/c_\alpha - 1$$

for $\alpha\in\Pi$ and that $\langle v,(-\alpha_0)^\vee\rangle > 1-h$.

For $\alpha\in\Pi$, we have $\alpha^\vee = ((-\alpha_0)^\vee - \sum_{\alpha\neq\beta\in\Pi} c_\beta \beta^\vee)/c_\alpha$. So

$$\begin{aligned}\langle v,\alpha^\vee\rangle &= (\langle v,(-\alpha_0)^\vee\rangle - \sum_{\alpha\neq\beta\in\Pi} c_\beta\langle v,\beta^\vee\rangle)/c_\alpha < (2+h-1-c_\alpha)/c_\alpha \\ &= (h+1)/c_\alpha - 1.\end{aligned}$$

We also have

$$\langle v,(-\alpha_0)^\vee\rangle = \langle v, \sum_{\alpha\in\Pi} c_\alpha\alpha^\vee\rangle = \sum_{\alpha\in\Pi} c_\alpha\langle v,\alpha^\vee\rangle > -\sum_{\alpha\in\Pi} c_\alpha = 1-h.$$

So $H \subseteq H^{h+1}$ and hence $H = H^{h+1}$.

Recall that $\tilde{\Pi} = \Pi \cup \{-\alpha_0\}$.

LEMMA 8.5. *Let* $m > 0$ *be an integer and let the* $\tilde{\Pi}$-*tuple* $\mathbf{k} = (k_\alpha)_{\alpha\in\tilde{\Pi}}$ *over* $\mathbb{Z}$ *satisfy the condition that* $k_{-\alpha_0} = \sum_{\alpha\in\Pi} c_\alpha k_\alpha$. *Let*

$$H_{\mathbf{k}}^m = (\bigcap_{\alpha\in\Pi} H_{\alpha;k_\alpha}^{m/c_\alpha}) \cap H_{-\alpha_0;k_{-\alpha_0}}^m.$$

Then $H_{\mathbf{k}}^m$ *contains exactly* $m^{\dim E}$ *alcoves of* E.

Proof. We have $A_1 = H_{\mathbf{k}_0}^1$ by Lemma 1.1, where $\mathbf{k}_0 = (k_\alpha)_{\alpha\in\tilde{\Pi}}$, with $k_\alpha = 0$ for all α, satisfies the condition $k_{-\alpha_0} = \sum_{\alpha\in\Pi} c_\alpha k_\alpha$. Then for any integer $m > 0$, $H_{\mathbf{k}_0}^m$ is similar to A_1 in geometrical shape and the scale of A_1 is one mth part of that of $H_{\mathbf{k}_0}^m$. So the volume of $H_{\mathbf{k}_0}^m$ is $m^{\dim E}$ times that of A_1. This implies that $H_{\mathbf{k}_0}^m$ contains exactly $m^{\dim E}$ alcoves of E. Now we take any other $\tilde{\Pi}$-tuple $\mathbf{k} = (k_\alpha)_{\alpha\in\tilde{\Pi}}$ with $k_{-\alpha_0} = \sum_{\alpha\in\Pi} c_\alpha k_\alpha$. Then there exists a unique vector $v\in E$ satisfying $\langle v,\alpha^\vee\rangle = k_\alpha$ for all $\alpha\in\Pi$ and $\langle v,(-\alpha_0)^\vee\rangle = k_{-\alpha_0}$. Let T_v be the translation on E which sends the origin to v. Then T_v also sends $H_{\mathbf{k}_0}^m$ to $H_{\mathbf{k}}^m$. Hence $H_{\mathbf{k}}^m$ also contains $m^{\dim E}$ alcoves of E by the condition that the k_α, $\alpha\in\tilde{\Pi}$, are all integers.

COROLLARY 8.6. *Let* H^{h+1} *be defined as in Lemma* 8.5. *Then* H^{h+1} *contains exactly* $(h+1)^l$ *alcoves of* E, *where* h *is the Coxeter number of* Φ *and* l *is the rank of* Φ.

Proof. Let $\mathbf{k} = (k_\alpha)_{\alpha\in\tilde{\Pi}}$ with $k_\alpha = -1$ for all $\alpha\in\Pi$ and $k_{-\alpha_0} = 1-h$. Then $H^{h+1} = H_{\mathbf{k}}^{h+1}$. Since $\dim E = l$, the result follows by Lemma 8.5.

Proof of Theorem 8.1. By Proposition 7.2, it is enough to show that $|E(S)| = (h+1)^l$ or, equivalently, to show that $|E(S)^{-1}| = (h+1)^l$. But this follows by Lemma 8.5 and Corollary 8.6.

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