ALCOVES CORRESPONDING TO AN AFFINE WEYL GROUP

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ABSTRACT

In this paper, I study the alcoves of a Euclidean space E corresponding to an affine Weyl group W_a. I give the coordinate form of an alcove of E and establish an explicit correspondence between the elements of W_a and the alcoves of E. In particular, I characterize an alcove by a Φ-tuple over $\mathbb{Z}$ subject to certain conditions, where Φ is the root system determined by W_a.

In [3], I gave the coordinate form of alcoves in the Euclidean space E spanned by a root system Φ of type A_{n-1}; these alcoves are in 1–1 correspondence with the elements of the affine Weyl group W_a of type $\tilde{A}_{n-1}$. The coordinate form of an alcove of E is a Φ-tuple over $\mathbb{Z}$ subject to certain conditions. I gave necessary and sufficient conditions for a Φ-tuple over $\mathbb{Z}$ to be the coordinate form of some alcove of E.

In the present paper, I shall generalize the above results on Φ from type $\tilde{A}$ to an arbitrary type, provided that Φ is indecomposable. Our main results are Theorems 3.3 and 5.2.

1. *The alcoves of E*

Let Φ be an indecomposable reduced root system. Let E be the Euclidean space spanned by Φ with positive definite inner product $\langle\ ,\ \rangle$ such that $|\alpha|^2 = \langle \alpha, \alpha \rangle = 1$ for any short root α of Φ. Choose a simple root system $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of Φ. Then Φ^+, Φ^- are the corresponding positive and negative root systems of Φ. Define the fundamental weights $\lambda_1, \ldots, \lambda_l$ by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ (the Kronecker delta), where for any $\alpha \in \Phi$, $a^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is called the coroot of α. Let $-\alpha_0$ be the highest short root of Φ. Then the set $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ has the property that $\langle \alpha_i, \alpha_j^\vee \rangle$ consists of non-positive integers for all pairs of distinct i, j in $\{0, 1, \ldots, l\}$. The set $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ of coroots is again a root system such that the set $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ affords a choice of a simple root system in it. The root $(-\alpha_0)^\vee$ is the highest (co)root of $\Phi^\vee$. Let h be the Coxeter number of Φ. Then h is also the Coxeter number of $\Phi^\vee$.

Let W be the Weyl group of Φ generated by the reflections s_α on E for $\alpha \in \Phi$, where s_α sends x to $x - \langle x, \alpha^\vee \rangle \alpha$. Let Q denote the root lattice $\mathbb{Z}\Phi$. Let N denote the group consisting of all translations T_λ operating on E for $\lambda \in Q$, where T_λ sends x to $x + \lambda$. We denote by W_a the group NW of affine transformations of E generated by N and W. It is well known that W_a is the semidirect extension of W by the normal subgroup N on which the action of W is known.

For linear and affine transformations, we shall denote the operation on the right and compose them accordingly. With this convention, we define $s_0 = s_{\alpha_0} T_{-\alpha_0}$, $s_i = s_{\alpha_i}$, $1 \leqslant i \leqslant l$. It is known that W_a (respectively W) is a Coxeter group on generators $s_0, s_1, \ldots, s_l$ (respectively $s_1, \ldots, s_l$). We write $\Delta = \{s_0, s_1, \ldots, s_l\}$. The group W_a will be called an affine Weyl group [1, 4].

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Any $w \in W_a$ can be written (being not necessarily unique) as a product of these generators. We define the length $l(w)$ of w to be the smallest number r such that there exists an expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$ with $s_{i_j} \in \Delta$. An expression of w is called a reduced form if it is a product of $l(w)$ generators.

The Bruhat order $\leqslant$ on W_a is a partial order of W_a which is defined as follows. Say $y \leqslant w$ in W_a if there are two reduced forms $w = s_{i_1} s_{i_2} \dots s_{i_r}$ and $y = s_{j_1} s_{j_2} \dots s_{j_t}$ such that $j_1, j_2, \dots, j_t$ is a subsequence of $i_1, i_2, \dots, i_r$ [4].

Given any two sets S, R, we call $\mathbf{x} = (x_i)_{i \in R}$ an R-tuple over S if $x_i \in S$ for all $i \in R$. Sometimes we simply call $\mathbf{x}$ an R-tuple when there is no danger of confusion. Two R-tuples $\mathbf{x} = (x_i)_{i \in R}$ and $\mathbf{y} = (y_j)_{j \in R}$ are said to be equal if $x_i = y_i$ for all $i \in R$.

For any $\alpha \in \Phi^+$, $k \in \mathbb{Z}$ and a positive real number m, we define a hyperplane

$$H_{\alpha;k} = \{v \in E \mid \langle v, \alpha^\vee \rangle = k\}$$

and a stripe

$$H^m_{\alpha;k} = H^m_{-\alpha;-k} = \{v \in E \mid k < \langle v, \alpha^\vee \rangle < k+m\}.$$

We call any non-empty connected simplex of

$$E - \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha;k}$$

an alcove of E. Each alcove of E has the form $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ for a Φ^+-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ over $\mathbb{Z}$. Since $H^1_{-\alpha;-k_\alpha} = H^1_{\alpha;-k_\alpha}$, sometimes it is more convenient to denote the alcove $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ by $\bigcap_{\alpha \in \Phi} H^1_{\alpha;k_\alpha}$ with the convention that $k_{-\alpha} = -k_\alpha$ for $\alpha \in \Phi^+$.

Let $(-\alpha_0)^\vee = \sum_{i=1}^l c_i \alpha_i^\vee$. Then c_i, $1 \leqslant i \leqslant l$, are all positive integers satisfying $h = \sum_{i=1}^l c_i + 1$. The following lemma gives an example of an alcove of E which can be shown directly by definition.

LEMMA 1.1. *Let* $A_1 = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha;0}$. *Then*

(i) A_1 *is an alcove of* E,

(ii) A_1 *can also be expressed as the form* $(\bigcap_{i=1}^l H^{1/c_i}_{\alpha_i;0}) \cap H^1_{-\alpha_0;0}$,

(iii) $\{(1/c_i)\lambda_i : 1 \leqslant i \leqslant l; 0\}$ *is the set of vertices of the closure of* A_1 *in* E,

(iv) $\{H_{\alpha_i;0} : 1 \leqslant i \leqslant l; H_{-\alpha_0;1}\}$ *is the set of facets of* A_1 *of codimension* 1 *in* E.

One should note that not every Φ^+-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ gives rise to an alcove of E as above. The following lemma gives a necessary condition on a Φ^+-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ over $\mathbb{Z}$ such that $\bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ is an alcove. Later, we shall show that this is also a sufficient condition.

LEMMA 1.2. *Suppose that* $A_{\mathbf{k}} = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha;k_\alpha}$ *is an alcove of* E. *Then for any* $\alpha, \beta \in \Phi^+$ *with* $\alpha+\beta \in \Phi^+$, *we have*

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leqslant |\alpha+\beta|^2 (k_{\alpha+\beta}+1) \leqslant |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1.$$

Proof. Let $v \in A_{\mathbf{k}}$. Then $k_\alpha < \langle v, \alpha^\vee \rangle < k_\alpha + 1$, $k_\beta < \langle v, \beta^\vee \rangle < k_\beta + 1$ and $k_{\alpha+\beta} < \langle v, (\alpha+\beta)^\vee \rangle < k_{\alpha+\beta} + 1$. Hence

$$|\alpha|^2 k_\alpha < 2\langle v, \alpha \rangle < |\alpha|^2 k_\alpha + |\alpha|^2, \; |\beta|^2 k_\beta < 2\langle v, \beta \rangle < |\beta|^2 k_\beta + |\beta|^2$$

and

$$|\alpha+\beta|^2 k_{\alpha+\beta} < 2\langle v, \alpha+\beta \rangle < |\alpha+\beta|^2 k_{\alpha+\beta} + |\alpha+\beta|^2.$$

This imples that

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta < 2\langle v, \alpha\rangle + 2\langle v, \beta\rangle = 2\langle v, \alpha+\beta\rangle < |\alpha+\beta|^2(k_{\alpha+\beta}+1)$$

and

$$|\alpha+\beta|^2 k_{\alpha+\beta} < 2\langle v, \alpha+\beta\rangle = 2\langle v, \alpha\rangle + 2\langle v, \beta\rangle < |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2.$$

So our conclusion follows immediately.

LEMMA 1.3. *Let* $A_{\mathbf{k}} = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ *be an alcove of* E *satisfying* $k_\alpha \geqslant 0$ *for all* $\alpha\in\Pi$. *Then*

(i) $k_\alpha \geqslant 0$ *for all* $\alpha\in\Phi^+$.

(ii) *If there exists some* $\gamma\in\Phi^+$ *with* $k_\gamma > 0$ *then* $k_{-\alpha_0} > 0$.

Proof. Take any $v\in A_{\mathbf{k}}$. For any $\beta\in\Phi^+$, $k_\beta \geqslant 0$ if and only if $\langle v, \beta^\vee\rangle > 0$. Given any $\alpha\in\Phi^+$, we can write $\alpha^\vee = \sum_{i=1}^{l} a_i\alpha_i^\vee$ with each a_i a non-negative integer, not all zero. By our condition, we have $\langle v, \alpha^\vee\rangle = \sum_{i=1}^{l} a_i\langle v, \alpha_i^\vee\rangle > 0$. So $k_\alpha \geqslant 0$ and (i) follows. For any $\beta\in\Phi^+$, $k_\beta > 0$ if and only if $\langle v, \beta^\vee\rangle > 1$. Thus we have $\langle v, \gamma^\vee\rangle > 1$. Since $(-\alpha_0)^\vee$ is the highest coroot of $\Phi^\vee$, we can write $(-\alpha_0)^\vee = \gamma^\vee + \sum_{i=1}^{l} a_i\alpha_i^\vee$ with the a_i non-negative integers. By our condition, we get $\langle v, (-\alpha_0)^\vee\rangle \geqslant \langle v, \gamma^\vee\rangle > 1$. This implies that $k_{-\alpha_0} > 0$.

COROLLARY 1.4. *Let* $A_{\mathbf{k}} = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ *be an alcove of* E *with* $k_\beta < 0$ *for some* $\beta\in\Phi^+$. *Then there exists some* $\gamma\in\Pi$ *satisfying* $k_\gamma < 0$.

Proof. This follows immediately from Lemma 1.3 (i).

EXAMPLES 1.5. The alcoves corresponding to the root systems Φ of types A_2, B_2 and G_2 are as in the following diagrams, where each small triangle in these diagrams represents an alcove. We label each alcove by its coordinate form. That is, let $\triangle$ be an alcove $\bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$. Then when Φ has type A_2, say $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$, we put

$$k_{\alpha+\beta}$$

$$k_\alpha \qquad k_\beta$$

into this triangle. When Φ has type B_2, say $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$, we put

$$k_\beta$$

$$k_{\alpha+\beta} \qquad\qquad k_\alpha$$

$$k_{2\alpha+\beta}$$

into this triangle. When Φ has type G_2, say $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$, we put

$$k_{3\alpha+2\beta} \qquad k_{\alpha+\beta}$$

$$k_{2\alpha+\beta} \qquad\qquad k_\beta$$

$$k_{3\alpha+\beta} \qquad k_\alpha$$

into this triangle.

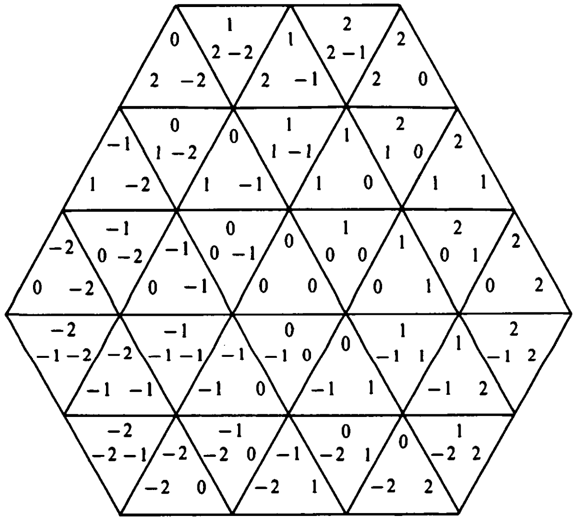

FIG. 1. Type A_2

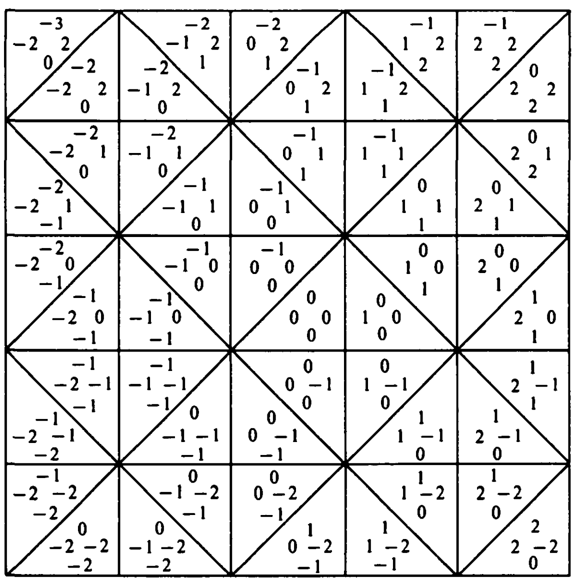

FIG. 2. Type B_2

2. *Some properties of root systems*

We shall study some properties of root systems which will be used later.

We say that a subset of Φ^+ is a positive subsystem of Φ if it has the form $\Phi^+ \cap \Phi'$ for some subsystem Φ' of Φ. We denote such a subset by Φ'^+. We say that Φ'^+ is indecomposable if Φ' is, and has type X if Φ' does so.

LEMMA 2.2. *Assume that $\alpha, \beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$. Then one of the following cases must occur:*

(i) *α, β, $\alpha+\beta$ have the same length and they span a subsystem of Φ of type A_2;*

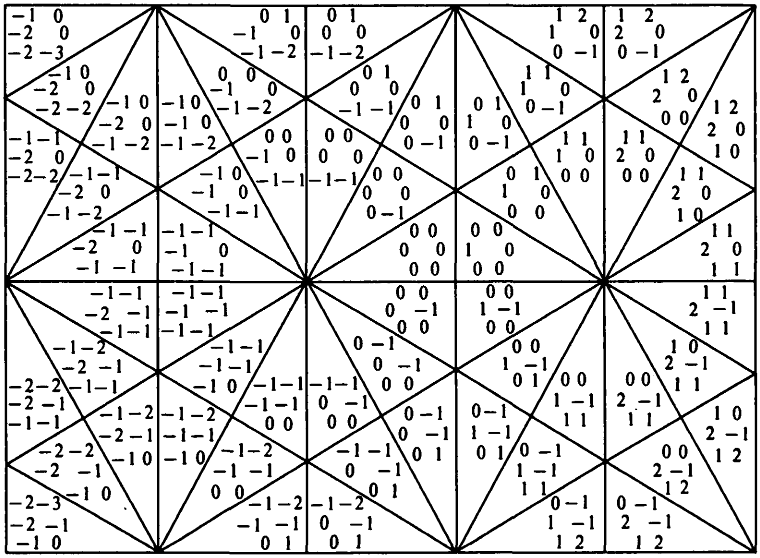

FIG. 3. Type G_2.

(ii) *α and β are short roots but $\alpha+\beta$ is a long one. They span a subsystem of Φ of type B_2 or G_2;*

(iii) *α and β have different lengths and they form a simple root system of the subsystem of Φ spanned by α, β. In that case, $\alpha+\beta$ is always a short one.*

Proof. This can be reduced to the case when Φ has rank 2, and the results checked directly.

LEMMA 2.3. *Suppose that both α and $(\alpha)s_{\alpha_0}$ are in Φ^+. Then $\alpha = (\alpha)s_{\alpha_0}$.*

Proof. Without loss of generality we may assume that Φ is spanned by α, $-\alpha_0$ over $\mathbb{Z}$. We can then check the result case by case.

LEMMA 2.4. *Assume that $\alpha, \beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$ and $-\alpha_0 \notin \{\alpha, \beta, \alpha+\beta\}$. Assume that $(\alpha)s_{\alpha_0}, (\beta)s_{\alpha_0} \in \Phi^-$. Then $(\alpha+\beta)s_{\alpha_0} \in \Phi^-$ and $|\alpha+\beta|^2 = 2|\alpha|^2 = 2|\beta|^2$.*

Proof. Obviously, $(\alpha+\beta)s_{\alpha_0} = (\alpha)s_{\alpha_0} + (\beta)s_{\alpha_0} \in \Phi^-$. To show the rest, we may assume without loss of generality that Φ is spanned by α, β, $-\alpha_0$ over $\mathbb{Z}$. The condition that $(\alpha)s_{\alpha_0}, (\beta)s_{\alpha_0} \in \Phi^-$ implies that

$$(\alpha+\beta)s_{\alpha_0} = \alpha+\beta-(\langle\alpha, (-\alpha_0)^\vee\rangle + \langle\beta, (-\alpha_0)^\vee\rangle)(-\alpha_0) \leqslant \gamma - 2(-\alpha_0)$$

with γ the highest root of Φ. Then $-\alpha_0 \neq \alpha+\beta$ implies that Φ has two different lengths of roots and that $\alpha+\beta$ must be a long root. On the other hand, by Lemma 2.2, α and β are either both short or both long roots. If they are both long, $\langle\alpha, (-\alpha_0)^\vee\rangle \geqslant 2$, $\langle\beta, (-\alpha_0)^\vee\rangle \geqslant 2$ and so $(\alpha+\beta)s_{\alpha_0} \leqslant \gamma - 4(-\alpha_0)$. But there is no root δ of Φ satisfying $\delta \leqslant \gamma - 4(-\alpha_0)$. Thus both α and β must be short roots. Since $-\alpha_0 \notin \{\alpha, \beta, \alpha+\beta\}$, it follows that Φ cannot have type G_2. Thus $|\alpha+\beta|^2 = 2|\alpha|^2 = 2|\beta|^2$.

LEMMA 2.5. *Assume that* $\alpha, \beta \in \Phi^+$ *with* $\alpha+\beta \in \Phi^+$. *Assume that* $(\alpha) s_{\alpha_0} \in \Phi^+$ *and* $(\beta) s_{\alpha_0} \in \Phi^-$. *Then* $(\alpha+\beta) s_{\alpha_0} \in \Phi^-$ *and* β, $\alpha+\beta$ *have the same length.*

Proof. Suppose that $(\alpha+\beta) s_{\alpha_0} \in \Phi^+$. Then by Lemma 2.3,

$$\alpha+\beta = (\alpha+\beta) s_{\alpha_0} = (\alpha) s_{\alpha_0} + (\beta) s_{\alpha_0} = \alpha + (\beta) s_{\alpha_0}.$$

That is, $(\beta) s_{\alpha_0} = \beta \in \Phi^+$ which contradicts our condition. Thus $(\alpha+\beta) s_{\alpha_0} \in \Phi^-$.

Since $(\alpha) s_{\alpha_0} = \alpha$, we have $\langle \alpha, (-\alpha_0)^\vee \rangle = 0$. Thus

$$\langle \beta, (-\alpha_0)^\vee \rangle = \langle \alpha+\beta, (-\alpha_0)^\vee \rangle > 0$$

since $(\beta) s_{\alpha_0} \in \Phi^-$. This implies that β, $\alpha+\beta$ have the same length.

3. *The correspondence between the alcoves of E and the affine Weyl group W_a*

In this section, we shall establish the correspondence between the alcoves of E and the elements of the affine Weyl group W_a. The main results of this section are Theorem 3.3 and Proposition 3.4.

It is well known that the right action of W_a on E gives rise to the permutations of the set

$$\{H_{\alpha;k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}\}.$$

So it induces the permutations of the set $\mathfrak{A}$ of alcoves of E. It is well known that $\mathfrak{A}$ is simply transitive under W_a[2]. Denote $A_w = (A_1) w$ for any $w \in W_a$. Thus any alcove of $\mathfrak{A}$ has the form A_w, written

$$A_w = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha; k(w,\alpha)} \quad \text{or} \quad A_w = \bigcap_{\alpha \in \Phi} H^1_{\alpha; k(w,\alpha)}$$

with the convention that $k(w, -\alpha) = -k(w, \alpha)$ for any $\alpha \in \Phi^+$. We shall identify W_a with $\mathfrak{A}$ as a set under the correspondence $w \mapsto A_w$. The integers $k(w, \alpha)$ labelled by $w \in W_a$ and $\alpha \in \Phi$ always stand for the coordinates of the alcove $A_w = \bigcap_{\alpha \in \Phi} H^1_{\alpha; k(w,\alpha)}$.

As $W_a = W \ltimes N$, any $w \in W_a$ has a unique decomposition $w = \bar{w} T_\lambda$ with $\bar{w} \in W$ and $\lambda \in Q$. We shall describe the integers $k(w, \alpha)$, $\alpha \in \Phi^+$ in terms of $\bar{w}$ and λ.

LEMMA 3.1. *For any* $w \in W$ *and any* $\alpha \in \Phi^+$, *we have*

$$k(w, \alpha) = \begin{cases} 0 & \text{if } (\alpha) w^{-1} \in \Phi^+, \\ -1 & \text{if } (\alpha) w^{-1} \in \Phi^-. \end{cases}$$

Proof. Let $v \in A_1$ and $\alpha \in \Phi^+$. Then $(v) w \in A_w$. It is well known that

$$\langle (v) w, \alpha^\vee \rangle = \langle v, ((\alpha) w^{-1})^\vee \rangle.$$

If $(\alpha) w^{-1} \in \Phi^+$ then $0 < \langle v, ((\alpha) w^{-1})^\vee \rangle < 1$ and hence $0 < \langle (v) w, \alpha^\vee \rangle < 1$. Thus $k(w, \alpha) = 0$. If $(\alpha) w^{-1} \in \Phi^-$ then $-(\alpha) w^{-1} \in \Phi^+$. So $0 < \langle v, (-(\alpha) w^{-1})^\vee \rangle < 1$ and hence $-1 < \langle v, ((\alpha) w^{-1})^\vee \rangle < 0$. That is, $-1 < \langle (v) w, \alpha^\vee \rangle < 0$. Then $k(w, \alpha) = -1$.

LEMMA 3.2. *Assume that* $A_{\mathbf{k}} = \bigcap_{\alpha \in \Phi} H^1_{\alpha; k_\alpha}$ *is an alcove of E. Let* $\lambda \in Q$. *Then* $A_{\mathbf{k}'} = (A_{\mathbf{k}}) T_\lambda$ *is also an alcove of E, say* $A_{\mathbf{k}'} = \bigcap_{\alpha \in \Phi} H^1_{\alpha; k'_\alpha}$. *Hence for any* $\alpha \in \Phi$, $k'_\alpha = k_\alpha + \langle \lambda, \alpha^\vee \rangle$.

Proof. We know that there exists some $w \in W_a$ with $A_w = A_{\mathbf{k}}$. Since $T_\lambda \in W_a$ $A_{\mathbf{k}'} = A_{wT_\lambda}$ is clearly an alcove of E. For any $\alpha \in \Phi^+$ and $v \in A_{\mathbf{k}}$, we have $k_\alpha < \langle v, \alpha^\vee \rangle < k_\alpha + 1$. Since

$$\langle v, \alpha^\vee \rangle + \langle \lambda, \alpha^\vee \rangle = \langle (v) T_\lambda, \alpha^\vee \rangle,$$

we get

$$k_\alpha + \langle \lambda, \alpha^\vee \rangle < \langle (v) T_\lambda, \alpha^\vee \rangle < k_\alpha + \langle \lambda, \alpha^\vee \rangle + 1.$$

But $(v) T_\lambda \in A_{\mathbf{k}'}$. This implies that $k'_\alpha = k_\alpha + \langle \lambda, \alpha^\vee \rangle$. For $\alpha \in \Phi^-$,

$$k'_\alpha = -k'_{-\alpha} = -(k_{-\alpha} + \langle \lambda, (-\alpha)^\vee \rangle) = k_\alpha + \langle \lambda, \alpha^\vee \rangle$$

and the result is proved.

From Lemmas 3.1 and 3.2, we get the following result immediately.

THEOREM 3.3. *For any $\bar{w} \in W$ and $\lambda \in Q$, let $w = \bar{w} T_\lambda$. Then the equation*

$$k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(\bar{w}, \alpha)$$

holds for any $\alpha \in \Phi$.

By Lemma 3.1 and Theorem 3.3, one can easily show that for any j, $0 \leqslant j \leqslant l$, and $\alpha \in \Phi$,

$$k(s_j, \alpha) = \begin{cases} 0 & \text{if } \alpha \neq \pm \alpha_j, \\ 1 & \text{if } \alpha = -\alpha_j, \\ -1 & \text{if } \alpha = \alpha_j. \end{cases} \tag{3.3.1}$$

We can deduce the following result from Theorem 3.3.

PROPOSITION 3.4 *Let $w \in W_a$. Then for any $\alpha \in \Phi$,*

$$k(w^{-1}, \alpha) = k(w, -(\alpha)\bar{w}).$$

Proof. Write $w = \bar{w} T_\lambda$ with $\bar{w} \in W$ and $\lambda \in Q$. Then $w^{-1} = \bar{w}^{-1} T_{(-\lambda)\bar{w}^{-1}}$. By Theorem 3.3, we have

$$k(w, -(\alpha)\bar{w}) = \langle \lambda, (-(\alpha)\bar{w})^\vee \rangle + k(\bar{w}, -(\alpha)\bar{w}),$$
$$k(w^{-1}, \alpha) = \langle (-\lambda)\bar{w}^{-1}, \alpha^\vee \rangle + k(\bar{w}^{-1}, \alpha)$$

for any $\alpha \in \Phi$. To show that $k(w^{-1}, \alpha) = k(w, -(\alpha)\bar{w})$ is equivalent to showing that $k(\bar{w}^{-1}, \alpha) = k(\bar{w}, -(\alpha)\bar{w})$. It is enough to show that $k(\bar{w}^{-1}, \alpha) = k(\bar{w}, -(\alpha)\bar{w})$ for $\alpha \in \Phi^+$. If $-(\alpha)\bar{w} \in \Phi^+$ then $(\alpha)(\bar{w}^{-1})^{-1} = (\alpha)\bar{w} \in \Phi^-$ and so $k(\bar{w}^{-1}, \alpha) = -1$. Also $(-(\alpha)\bar{w})\bar{w}^{-1} = -\alpha \in \Phi^-$ implies that $k(\bar{w}, -(\alpha)\bar{w}) = -1$. Thus $k(\bar{w}^{-1}, \alpha) = k(\bar{w}, -(\alpha)\bar{w})$ in this case. If $-(\alpha)\bar{w} \in \Phi^-$ then $(\alpha)(\bar{w}^{-1})^{-1} = (\alpha)\bar{w} \in \Phi^+$, which implies that $k(\bar{w}^{-1}, \alpha) = 0$. Also, $k(\bar{w}, -(\alpha)\bar{w}) = -k(\bar{w}, (\alpha)\bar{w})$. But $((\alpha)\bar{w})\bar{w}^{-1} = \alpha \in \Phi^+$ implies that $k(\bar{w}, (\alpha)\bar{w}) = 0$ and hence that $k(\bar{w}, -(\alpha)\bar{w}) = 0$. Also we have $k(\bar{w}^{-1}, \alpha) = k(\bar{w}, -(\alpha)\bar{w})$. This implies that we always have $k(\bar{w}^{-1}, \alpha) = k(\bar{w}, -(\alpha)\bar{w})$ and the result follows.

EXAMPLES 3.5. Recall that in Examples 1.5 we drew the diagrams for the alcoves of E when Φ has type A_2, B_2 or G_2. We labelled each alcove by the corresponding Φ^+-tuple there. Now we shall label them by the corresponding elements of W_a instead of Φ^+-tuples. We assume that $s_1 = s_\alpha$ and $s_2 = s_\beta$ and denote s_i by i for short.

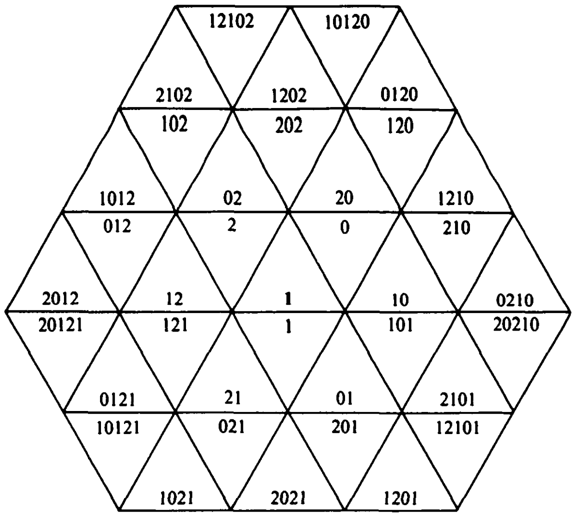

FIG. 4. Type A_2

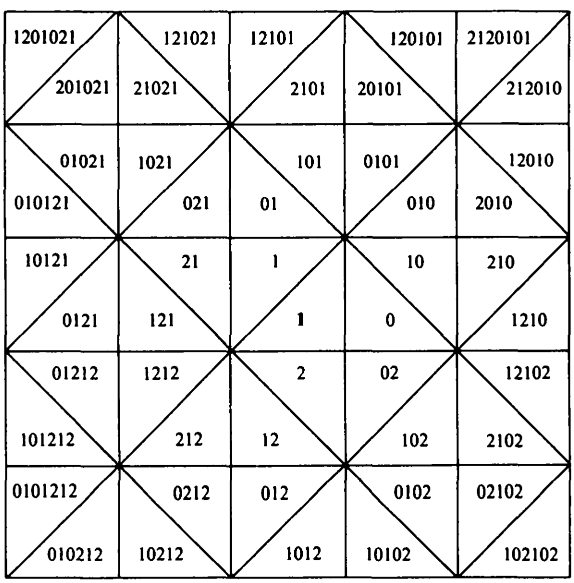

FIG. 5. Type B_2

4. *The actions of W_a on the alcoves of E*

Let $w' = s_j w$ with $w \in W_a$ and $0 \leqslant j \leqslant l$. We wish to express the $k(w', \alpha)$ in terms of the $k(w, \beta)$.

Write $w = \bar{w} T_\lambda$ with $\bar{w} \in W$ and $\lambda \in Q$. First assume that $1 \leqslant j \leqslant l$. Then $w' = s_j \bar{w} T_\lambda$ with $s_j \bar{w} \in W$. By Theorem 3.3,

$$k(w, \alpha) = \langle \lambda, \alpha^\vee \rangle + k(\bar{w}, \alpha)$$

and

$$k(w', \alpha) = k(w, \alpha) + k(s_j \bar{w}, \alpha) - k(\bar{w}, \alpha)$$

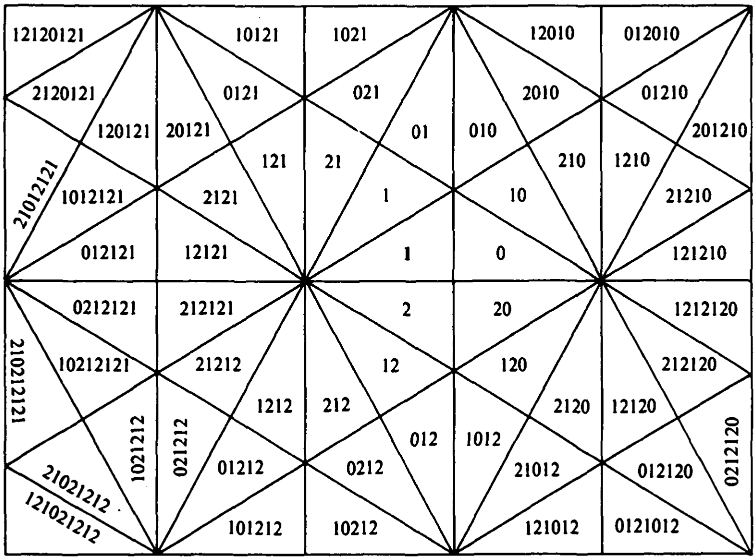

FIG. 6. Type G_2

for any $\alpha\in\Phi^+$. When $(\alpha)\bar{w}^{-1}\neq\pm\alpha_j$, we have $(\alpha)\bar{w}^{-1}\in\Phi^+$ if and only if $(\alpha)\bar{w}^{-1}s_j\in\Phi^+$. So $k(\bar{w},\alpha)=k(s_j\bar{w},\alpha)$ and thus $k(w,\alpha)=k(w',\alpha)$ in this case. When $(\alpha)\bar{w}^{-1}=\alpha_j$ we have $(\alpha)\bar{w}^{-1}\in\Phi^+$ and $(\alpha)\bar{w}^{-1}s_j\in\Phi^-$. Thus $k(s_j\bar{w},\alpha)=-1$ and $k(\bar{w},\alpha)=0$ by Lemma 3.1. This implies that $k(w',\alpha)=k(w,\alpha)-1$.

Next assume that $j=0$. Then $w'=s_{\alpha_0}\bar{w}T_{\lambda+(-\alpha_0)\bar{w}}$. By Theorem 3.3, we have, for any $\alpha\in\Phi^+$,

$$k(w',\alpha)=k(w,\alpha)+\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle+k(s_{\alpha_0}\bar{w},\alpha)-k(\bar{w},\alpha).$$

When $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle=0$, we have $(\alpha)\bar{w}^{-1}\in\Phi^+$ if and only if $(\alpha)\bar{w}^{-1}s_{\alpha_0}\in\Phi^+$. Thus $k(s_{\alpha_0}\bar{w},\alpha)=k(\bar{w},\alpha)$. So in this case, $k(w',\alpha)=k(w,\alpha)$. When $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle>0$ and $(\alpha)\bar{w}^{-1}\neq\pm\alpha_0$ we have $(\alpha)\bar{w}^{-1}\in\Phi^+$ and $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle=1$ since $-\alpha_0$ is the highest short root of Φ. We also have $(\alpha)\bar{w}^{-1}s_{\alpha_0}\in\Phi^-$. Thus $k(w',\alpha)=k(w,\alpha)$. When $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle<0$ and $(\alpha)\bar{w}^{-1}\neq\pm\alpha_0$, we have $(\alpha)\bar{w}^{-1}\in\Phi^-$ and $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle=-1$. We also have $(\alpha)\bar{w}^{-1}s_{\alpha_0}\in\Phi^+$. Thus $k(w',\alpha)=k(w,\alpha)$. When $(\alpha)\bar{w}^{-1}=-\alpha_0$, we get $(\alpha)\bar{w}^{-1}\in\Phi^+$, $(\alpha)\bar{w}^{-1}s_{\alpha_0}\in\Phi^-$ and $\langle-\alpha_0,((\alpha)\bar{w}^{-1})^\vee\rangle=2$. Thus $k(w',\alpha)=k(w,\alpha)+1$.

To sum up, we get the following result, by using (3.3.1).

PROPOSITION 4.1. *Let $w'=s_jw$ with $w\in W_a$ and $0\leqslant j\leqslant l$. Then for any $\alpha\in\Phi^+$, we have $k(w',\alpha)=k(w,\alpha)+k(s_j,(\alpha)\bar{w}^{-1})$.*

Now assume that, $w'=ws_j$ instead of $w'=s_jw$ in the above. We shall find the relations between the $k(w,\alpha)$ and the $k(w',\beta)$.

First assume that $1\leqslant j\leqslant l$. Then $w'=ws_j=\bar{w}T_\lambda s_j=\bar{w}s_jT_{(\lambda)s_j}$. By Theorem 3.3, $k(w,\alpha)=\langle\lambda,\alpha^\vee\rangle+k(\bar{w},\alpha)$ and $k(w',\alpha)=\langle\lambda,((\alpha)s_j)^\vee\rangle+k(\bar{w}s_j,\alpha)$ for any $\alpha\in\Phi^+$. This implies that

$$k(w',(\alpha)s_j)=k(w,\alpha)+k(\bar{w}s_j,(\alpha)s_j)-k(\bar{w},\alpha).$$

When $\alpha \neq \alpha_j$, we have $(\alpha) s_j \in \Phi^+$. We also have $(\alpha) \overline{w}^{-1} \in \Phi^+$ if and only if $(\alpha) s_j s_j \overline{w}^{-1} \in \Phi^+$. So $k(\overline{w} s_j, (\alpha) s_j) = k(\overline{w}, \alpha)$ and hence $k(w', (\alpha) s_j) = k(w, \alpha)$ in that case. When $\alpha = \alpha_j$, we have $(\alpha_j) s_j = -\alpha_j \in \Phi^-$. Thus

$$k(w', \alpha_j) = -k(w, \alpha_j) + k(\overline{w}, \alpha_j) + k(\overline{w} s_j, \alpha_j).$$

Since $(\alpha_j) \overline{w}^{-1} \in \Phi^+$ if and only if $(\alpha_j) s_j \overline{w}^{-1} \in \Phi^-$, we get $k(\overline{w}, \alpha_j) + k(\overline{w} s_j, \alpha_j) = -1$ and so $k(w', \alpha_j) = -k(w, \alpha_j) - 1$.

Next assume that $j = 0$. Then $w' = w s_0 = \overline{w} T_\lambda s_{\alpha_0} T_{-\alpha_0} = \overline{w} s_{\alpha_0} T_a$, $a = (\lambda) s_{\alpha_0} - \alpha_0$. Hence

$$k(w', \alpha) = \langle \lambda, ((\alpha) s_{\alpha_0})^\vee \rangle + \langle -\alpha_0, \alpha^\vee \rangle + k(\overline{w} s_{\alpha_0}, \alpha)$$

for any $\alpha \in \Phi^+$. When $\langle -\alpha_0, \alpha^\vee \rangle = 0$, we have $(\alpha) s_{\alpha_0} = \alpha$, and $(\alpha) \overline{w}^{-1} \in \Phi^+$ if and only if $(\alpha) s_{\alpha_0} \overline{w}^{-1} \in \Phi^+$. In that case, $k(w', \alpha) = k(w, \alpha)$. When $\langle -\alpha_0, \alpha^\vee \rangle \neq 0$ and $\alpha \neq -\alpha_0$, we have $\langle -\alpha_0, \alpha^\vee \rangle = 1$ and $(\alpha) s_{\alpha_0} \in \Phi^-$ by Lemma 2.3. Thus

$$k(w', -(\alpha) s_{\alpha_0}) = \langle \lambda, (-\alpha)^\vee \rangle + 1 + k(\overline{w} s_{\alpha_0}, -(\alpha) s_{\alpha_0}).$$

Since $(\alpha) \overline{w}^{-1} \in \Phi^+$ if and only if $(-(\alpha) s_{\alpha_0}) s_{\alpha_0} \overline{w}^{-1} \in \Phi^-$, we get

$$k(\overline{w}, \alpha) + k(\overline{w} s_{\alpha_0}, -(\alpha) s_{\alpha_0}) = -1.$$

Thus $k(w', -(\alpha) s_{\alpha_0}) = -k(w, \alpha)$. When $\alpha = -\alpha_0$, we have

$$k(w', -\alpha_0) = -k(w, -\alpha_0) + 2 + k(\overline{w}, -\alpha_0) + k(\overline{w} s_{\alpha_0}, -\alpha_0).$$

Since, $(-\alpha_0) \overline{w}^{-1} \in \Phi^+$ if and only if $(-\alpha_0) s_{\alpha_0} \overline{w}^{-1} \in \Phi^-$, we have

$$k(\overline{w}, -\alpha_0) + k(\overline{w}_{\alpha_0}, -\alpha_0) = -1$$

and hence $k(w', -\alpha_0) = -k(w, -\alpha_0) + 1$.

To sum up, we get the following, by using (3.3.1).

PROPOSITION 4.2. *Let $w' = w s_j$ with $w \in W_a$ and $s_j \in \Delta$. Then for any $\alpha \in \Phi$, we have* $k(w', \alpha) = k(w, (\alpha) s_j) + k(s_j, \alpha)$.

For any $w \in W_a$, we associate two subsets of Δ:

$$\mathscr{L}(w) = \{s \in \Delta \mid sw < w\},$$

$$\mathscr{R}(w) = \{s \in \Delta \mid ws < w\}.$$

Now we can describe an element w of W_a in terms of the $k(w, \alpha)$.

PROPOSITION 4.3. *Suppose that $A_w = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha; k(w, \alpha)}$ for $w \in W_a$.*

(i) *The length $l(w)$ of w is equal to $\sum_{\alpha \in \Phi^+} |k(w, \alpha)|$.*

(ii) $\mathscr{R}(w) = \{s_j \in \Delta \mid k(w, \alpha_j) < 0\}$.

(iii) $\mathscr{L}(w) = \{s_j \in \Delta \mid k(w, (\alpha_j) \overline{w}) > 0\}$.

The proof of this result is the same as that in [3, Chapter 7].

5. *The map* $\phi: W_a \to F_0$

In this section, we shall characterize any element of W_a in terms of a Φ-tuple over $\mathbb{Z}$.

Let F be the set of Φ-tuples $(k_\alpha)_{\alpha\in\Phi}$ satisfying $k_\alpha = -k_{-\alpha}$ for any $\alpha\in\Phi$. Let F_0 be the subset of F consisting of all $(k_\alpha)_{\alpha\in\Phi}$ such that for any $\alpha, \beta\in\Phi^+$ with $\alpha+\beta\in\Phi^+$, the inequality

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leqslant |\alpha+\beta|^2 (k_{\alpha+\beta}+1) \leqslant |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1$$

holds. Any Φ-tuple in F_0 is called a special Φ-tuple.

Define a map $\phi: W_a \to F$ be sending $A_w = \bigcap_{\alpha\in\Phi} H^1_{\alpha;k(w,\alpha)}$ to $(k(w,\alpha))_{\alpha\in\Phi}$. Then from Lemma 1.2, we see that the image of ϕ is in F_0 and hence ϕ can be regarded as a map from W_a to F_0. Now we shall show that ϕ is bijective. It is obvious that ϕ is injective. So it is enough to show that ϕ is also surjective.

Define a right action of W_a on F as follows: for $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi}\in F$, $s_t\in\Delta$ and $x, y\in W_a$,

(i) $(\mathbf{k})s_t = (k'_\alpha)_{\alpha\in\Phi}$ with $k'_\alpha = k_{(\alpha)s_t} + \varepsilon_{\alpha,t}$, where

$$\varepsilon_{\alpha,t} = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_t, \\ -1 & \text{if } \alpha = \alpha_t, \\ 1 & \text{if } \alpha = -\alpha_t. \end{cases}$$

(ii) $(\mathbf{k})xy = ((\mathbf{k})x)y$.

One can easily check that this action of W_a on F is well defined. By noting that $k(s_t, \alpha) = \varepsilon_{\alpha,t}$ for $0 \leqslant t \leqslant l$ and $\alpha\in\Phi$, we see that the map $\phi: W_a \to F$ is W_a-equivariant and so $\phi(W_a)$ is a W_a-orbit of F in F_0.

Now we are ready to show the following.

PROPOSITION 5.1. *The map* $\phi: W_a \to F_0$ *is bijective.*

Proof. It is sufficient to show that F_0 is a single W_a-orbit. Call $\mathbf{k} = (k_\alpha)_{\alpha\in\Phi}\in F$ a minimal element if, for any $s\in\Delta$,

$$\sum_{\alpha\in\Phi^+} |k_\alpha| < \sum_{\alpha\in\Phi^+} |k'_\alpha|,$$

where $\mathbf{k}' = (\mathbf{k})s = (k'_\alpha)_{\alpha\in\Phi}$. It is clear that $\mathbf{k}$ is minimal if and only if $k_{\alpha_t} \geqslant 0$ for all t, with $0 \leqslant t \leqslant l$. It is not difficult to show that F_0 contains a unique minimal element, namely $\phi(A_1)$. So it suffices to show that if $\mathbf{k}' = (\mathbf{k})s_r$ for some $\mathbf{k}\in F_0$ and $s_r\in\Delta$ then $\mathbf{k}'\in F_0$; that is, we must show that for any $\alpha, \beta\in\Phi^+$ with $\alpha+\beta\in\Phi^+$, the inequality

$$|\alpha|^2 k'_\alpha + |\beta|^2 k'_\beta + 1 \leqslant |\alpha+\beta|^2 (k'_{\alpha+\beta}+1) \leqslant |\alpha|^2 k'_\alpha + |\beta|^2 k'_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1$$

holds, or equivalently, the inequality

$$\begin{aligned} |\alpha|^2 k_{(\alpha)s_r} + |\beta|^2 k_{(\beta)s_r} + |\alpha|^2 \varepsilon_{\alpha,r} + |\beta|^2 \varepsilon_{\beta,r} + 1 &\leqslant |\alpha+\beta|^2 (k_{(\alpha+\beta)s_r} + \varepsilon_{\alpha+\beta,r} + 1) \\ &\leqslant |\alpha|^2 k_{(\alpha)s_r} + |\beta|^2 k_{(\beta)s_r} + |\alpha|^2 \varepsilon_{\alpha,r} + |\beta|^2 \varepsilon_{\beta,r} + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1 \quad (1) \end{aligned}$$

holds.

When $r \neq 0$ and $\alpha_r \neq \alpha, \beta$, the result is obvious. When $r \neq 0$ and $\alpha_r \in \{\alpha, \beta\}$, say $\alpha_r = \alpha$, (1) becomes

$$|\alpha|^2 k_\alpha + |(\beta)\bar{s}_r - \alpha|^2 k_{(\beta)\bar{s}_r-\alpha} + 1 \leqslant |(\beta)\bar{s}_r|^2 (k_{(\beta)\bar{s}_r} + 1)$$
$$\leqslant |\alpha|^2 k_\alpha + |(\beta)\bar{s}_r - \alpha|^2 k_{(\beta)\bar{s}_r-\alpha} + |\alpha|^2 + |(\beta)\bar{s}_r - \alpha|^2 + |(\beta)\bar{s}_r|^2 - 1,$$

which holds since $\mathbf{k} \in F_0$ and α, $(\beta)\bar{s}_r$, $(\beta)\bar{s}_r - \alpha \in \Phi^+$.

Now assume that $r = 0$. Then one of the following cases must occur:

(i) $-\alpha_0 = \alpha + \beta$; (ii) $-\alpha_0 \in \{\alpha, \beta\}$; (iii) $-\alpha_0 \notin \{\alpha, \beta, \alpha+\beta\}$.

First assume that we are in case (i). Then $\varepsilon_{\alpha,0} = \varepsilon_{\beta,0} = 0$, $\varepsilon_{\alpha+\beta,0} = 1$ and $\alpha+\beta$ is a short root. By Lemma 2.2, one of α, β, say α, must be a short root. If β is also a short root then (1) becomes

$$|-(\alpha)\bar{s}_0|^2 k_{-(\alpha)\bar{s}_0} + |-(\beta)\bar{s}_0|^2 k_{-(\beta)\bar{s}_0} + 1 \leqslant |-\alpha_0|^2 (k_{-\alpha_0} + 1)$$
$$\leqslant |-(\alpha)\bar{s}_0|^2 k_{-(\alpha)\bar{s}_0} + |-(\beta)\bar{s}_0|^2 k_{-(\beta)\bar{s}_0} + |-(\alpha)\bar{s}_0|^2 + |-(\beta)\bar{s}_0|^2 + |-\alpha_0|^2 - 1$$

which holds since $\mathbf{k} \in F_0$ and $-(\alpha)\bar{s}_0$, $-(\beta)\bar{s}_0$, $-\alpha_0 \in \Phi^+$. If β is a long root, (1) becomes

$$|\alpha|^2 k_\alpha + |-\alpha_0|^2 k_{-\alpha_0} + 1 \leqslant |-(\beta)\bar{s}_0|^2 (k_{-(\beta)\bar{s}_0} + 1)$$
$$\leqslant |\alpha|^2 k_\alpha + |-\alpha_0|^2 k_{-\alpha_0} + |\alpha|^2 + |-\alpha_0|^2 + |-(\beta)\bar{s}_0|^2 - 1$$

which holds because $\mathbf{k} \in F_0$ and $-(\beta)\bar{s}_0 = \alpha + (-\alpha_0)$.

Next assume that we are in case (ii), say $\alpha = -\alpha_0$. Then $\varepsilon_{\alpha,0} = 1$, $\varepsilon_{\beta,0} = \varepsilon_{\alpha+\beta,0} = 0$. Let Φ'^+ be the positive subsystem of Φ spanned by α, β over $\mathbb{Z}$. Then Φ'^+ is either of type B_2 or G_2 and α is the highest short root of Φ'^+. By the condition that $\alpha+\beta \in \Phi'^+$ and by Lemma 2.2, β is a positive short root of Φ' and $\alpha+\beta$ is a positive long root of Φ'. If Φ'^+ has type B_2 then (1) becomes

$$|\beta|^2 k_\beta + |-\alpha_0 - \beta|^2 k_{-\alpha_0-\beta} + 1 \leqslant |-\alpha_0|^2 (k_{-\alpha_0} + 1)$$
$$\leqslant |\beta|^2 k_\beta + |-\alpha_0 - \beta|^2 k_{-\alpha_0-\beta} + |\beta|^2 + |-\alpha_0 - \beta|^2 + |-\alpha_0|^2 - 1.$$

This holds because β, $-\alpha_0 - \beta$, $-\alpha_0 \in \Phi^+$ and $\mathbf{k} \in F_0$. If Φ'^+ has type G_2 then (1) becomes

$$|-\alpha_0|^2 k_{-\alpha_0} + |-(\beta)\bar{s}_0|^2 k_{-(\beta)\bar{s}_0} + 1 \leqslant |-\alpha_0 - (\beta)\bar{s}_0|^2 (k_{-\alpha_0-(\beta)\bar{s}_0} + 1)$$
$$\leqslant |-\alpha_0|^2 k_{-\alpha_0} + |-(\beta)\bar{s}_0|^2 k_{-(\beta)\bar{s}_0} + |-\alpha_0|^2 + |-(\beta)\bar{s}_0|^2 + |-\alpha_0 - (\beta)\bar{s}_0|^2 - 1.$$

This holds because $-\alpha_0$, $-(\beta)\bar{s}_0$, $-\alpha_0 - (\beta)\bar{s}_0 \in \Phi^+$ and $\mathbf{k} \in F_0$. So (1) holds when one of α, β is equal to $-\alpha_0$.

Finally assume that we are in case (iii). Then $\varepsilon_{\alpha,0} = \varepsilon_{\beta,0} = \varepsilon_{\alpha+\beta,0} = 0$ and one of the following cases must occur:

(a) $(\alpha)\bar{s}_0$, $(\beta)\bar{s}_0 \in \Phi^+$; (b) $(\alpha)\bar{s}_0$, $(\beta)\bar{s}_0 \in \Phi^-$;

(c) $(\alpha)\bar{s}_0 \in \Phi^+$, $(\beta)\bar{s}_0 \in \Phi^-$; (d) $(\alpha)\bar{s}_0 \in \Phi^-$, $(\beta)\bar{s}_0 \in \Phi^+$.

One can verify (1) in case (a) by Lemma 2.3, in case (b) by Lemma 2.4 and in cases (c), (d) by Lemma 2.5.

An immediate consequence of the above proposition is the following.

THEOREM 5.2. *Let $A_{\mathbf{k}} = \bigcap_{\alpha\in\Phi^+} H^1_{\alpha;k_\alpha}$ with $k_\alpha \in \mathbb{Z}$. Then $A_{\mathbf{k}}$ is an alcove of E if and only if for any $\alpha, \beta \in \Phi^+$ with $\alpha+\beta \in \Phi^+$, the inequality*

$$|\alpha|^2 k_\alpha + |\beta|^2 k_\beta + 1 \leqslant |\alpha+\beta|^2 (k_{\alpha+\beta}+1) \leqslant |\alpha|^2 k_\alpha + |\beta|^2 k_\beta + |\alpha|^2 + |\beta|^2 + |\alpha+\beta|^2 - 1$$

holds.

This theorem characterizes an alcove of E by a special Φ-tuple.

6. *The facets of an alcove*

We know that each alcove of E has the form $(A_1)w$ for some $w \in W_a$. So by Lemma 1.1, any alcove of E has $l+1$ facets. We know that the right action of W_a on E induces a permutation on the set of facets of all alcoves of E. It is well known that each W_a-orbit of such facets intersects the closure of any alcove in a unique facet. So we can label any facet of an alcove by an element $s \in \Delta$ if it is in the W_a-orbit of facets containing the common facet of A_1 and A_s.

LEMMA 6.1. *If $w, w' \in W_a$ have the relation $w' = s_t w$ for some $s_t \in \Delta$ then the alcoves A_w and $A_{w'}$ share the common s_t-facet. Conversely, if A_w and $A_{w'}$ are two alcoves of E which share a common facet then the labelling of this facet for A_w is the same as for $A_{w'}$, say s_t-facet. We have $w' = s_t w$.*

Proof. First assume that $w' = s_t w$. We have $A_w = (A_1)w$, $A_{w'} = (A_{s_t})w$, and A_1 and A_{s_t} share the common s_t-facet, this implies that A_w and $A_{w'}$ share the common s_t-facet.

Conversely, assume that A_w and $A_{w'}$ share a common facet. Then by the definition, the labelling of this facet for A_w and for $A_{w'}$ must be the same, say s_t-facet. Let $y = s_t w$. Then by the above argument, A_y and A_w share the common s_t-facet. This forces $A_y = A_{w'}$ and hence $w' = s_t w$.

Now we can give another description of the length function $l(w)$ on W_a which is a direct consequence of Lemma 6.1.

COROLLARY 6.2. *For any $w \in W_a$, $l(w)$ is the minimum number of facets of alcoves of E which separate the alcove A_w from A_1. In other words, $l(w)$ is the smallest number r such that there exists a sequence of alcoves $A_0 = A_w, A_1, \ldots, A_r = A_1$ where any two consecutive alcoves in this sequence share a common facet.*

Recall that in §1 we assumed that

$$H^1_{-\alpha;k} = H^1_{\alpha;-k} = \{v \in E \mid -k < \langle v, \alpha^\vee\rangle < -k+1\} \quad \text{for } \alpha \in \Phi^+.$$

So $H^1_{\alpha;k}$ (respectively $H^1_{-\alpha;k}$) is bounded by two parallel hyperplanes $H_{\alpha;k}$ and $H_{\alpha;k+1}$ (respectively $H_{\alpha;-k}$ and $H_{\alpha;-k+1}$). We define $H_{-\alpha;h}$ by $H_{\alpha;h+1}$ for any integer h and any positive root $\alpha \in \Phi^+$. Then $H^1_{-\alpha;k}$ is also bounded by $H_{-\alpha;k}$ and $H_{-\alpha;k+1}$. So we can say that for any integer k and any root $\alpha \in \Phi$, $H^1_{\alpha;k}$ is bounded by $H_{\alpha;k}$ and $H_{\alpha;k+1}$.

For any $w \in W_a$ and integer t, $0 \leqslant t \leqslant l$, we denote $k_t = k(w, (\alpha_t)\bar{w})$ in the remainder of this section. Let $H_t(w)$ be the hyperplane of E supporting the s_t-facet of the alcove A_w. Then we have $H_t(w) = (H_t(1))\,w = (H_{\alpha_t;0})\,w$. So $H_t(w)$ has the form $H_{(\alpha_t)\varpi;k}$ for some $k \in \{k_t, k_t \pm 1\}$.

Now we wish to decide $H_t(w)$.

By Corollary 6.2 and Lemma 6.1, we see that $s_t \in \mathscr{L}(w)$ if and only if A_w and A_1 are on different sides of $H_t(w)$, where $\mathscr{L}(x) = \{s \in \Delta \mid sx < x\}$ for any $x \in W_a$. On the other hand, by Proposition 4.3 (iii), we see that $s_t \in \mathscr{L}(w)$ if and only if $k_t > 0$, or equivalently, $s_t \notin \mathscr{L}(w)$ if and only if $k_t \leqslant 0$.

First assume that $s_t \in \mathscr{L}(w)$. Then $H^1_{(\alpha_t)\varpi;k_t}$ and A_1 are on different sides of $H_t(w)$. So $H_t(w) = H_{(\alpha_t)\varpi;k_t}$ by the fact that $k_t > 0$.

Next assume that $s_t \notin \mathscr{L}(w)$. Then $H^1_{(\alpha_t)\varpi;k_t}$ and A_1 are on the same side of $H_t(w)$. In that case we have $k_t \leqslant 0$. If $k_t < 0$ then $H_t(w) = H_{(\alpha_t)\varpi;k_t}$. If $k_t = 0$ then $H_t(w) = H_{(\alpha_t)\varpi;1}$.

So we can summarize the above results as follows.

Proposition 6.3. *For any $w \in W_a$ and integer t, $0 \leqslant t \leqslant l$, let $H_t(w)$ be the hyperplane of E supporting the s_t-facet of the alcove A_w. Let $k_t = k(w, (\alpha_t)\varpi)$. Then*

$$H_t(w) = \begin{cases} H_{(\alpha_t)\varpi;k_t} & \text{if } k_t \neq 0, \\ H_{(\alpha_t)\varpi;1} & \text{if } k_t = 0, \end{cases}$$

and so $A_w = \bigcap_{0 \leqslant t \leqslant l} H^1_{(\alpha_t)\varpi;k_t}$.

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