# 第一讲　分数阶微分方程

主要参考资料: [4, 5, 7].

## 1.1　分数阶导数

分数阶导数 (Fractional derivatives) 有多种定义方式, 常用的有 Riemann-Liouville 分数阶导数, Caputo 分数阶导数, Grünwald-Letnikov 分数阶导数, 等等. 下面我们就对以上三种定义进行分别介绍, 更多定义可参见 [3].

我们首先以幂函数 $f(x)=x^m$ 为例. 考虑它的整数阶导数, 有

$$
\begin{aligned}
&f'(x)=mx^{m-1}\\
&f''(x)=m(m-1)x^{m-2}\\
&\cdots\ \ \cdots\\
&f^{(n)}(x)=m\cdots(m-n+1)x^{m-n}=\frac{m!}{(m-n)!}x^{m-n}.
\end{aligned}
$$

由于阶乘只对正整数有定义, 因此无法将上面的定义方式直接推广到分数情形. 此时我们可以借助 $\Gamma$ 函数, 将 $f^{(n)}(x)$ 改写成

$$f^{(n)}(x)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}. \tag{1.1}$$

注意到 $\Gamma$ 函数是在复平面的整个右半平面上都有定义的, 因此可以将 (1.1) 推广到 $n$ 是正实数的情形, 即

$$f^{(\alpha)}(x)=\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{m-\alpha}. \tag{1.2}$$

对于一般情形, 分数阶导数可以通过整数阶导数和分数阶积分来表示. 下面我们首先给出分数阶积分的概念.

### 1.1.1　分数阶积分

记导数算子为 D, 即

$$\mathrm{D}\,f(x)\triangleq\frac{\mathrm{d}f}{\mathrm{d}x},\quad \mathrm{D}^2\,f(x)\triangleq\frac{\mathrm{d}^2f}{\mathrm{d}x^2},\quad\cdots,\quad \mathrm{D}^n\,f(x)\triangleq\frac{\mathrm{d}^nf}{\mathrm{d}x^n}.$$

记 ${}_a\mathrm{D}_x^{-1}$ 表示变上限积分, 即

$$
\begin{aligned}
&{}_a\mathrm{D}_x^{-1}\,f(x)\triangleq\int_a^x f(\tau)\,\mathrm{d}\tau,\\
&{}_a\mathrm{D}_x^{-2}\,f(x)\triangleq{}_a\mathrm{D}_x^{-1}\left({}_a\mathrm{D}_x^{-1}\,f(x)\right),\\
&\qquad\cdots\cdots\\
&{}_a\mathrm{D}_x^{-n}\,f(x)\triangleq{}_a\mathrm{D}_x^{-1}\left({}_a\mathrm{D}_x^{-(n-1)}\,f(x)\right),\quad x>a.
\end{aligned}
$$

易知, 对任意正整数 $n$, 都有

$$\mathrm{D}^n\left({}_a\mathrm{D}_x^{-n} f(x)\right) = f(x),$$

即积分算子 ${}_a\mathrm{D}_x^{-n}$ 可以看作是整数阶导数算子的左逆. 但反之不成立. 事实上, 我们有下面的结论.

**引理 1.1** 设 $f(x) \in C^n[a,b]$, 则

$${}_a\mathrm{D}_x^{-n}(\mathrm{D}^n f(x)) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a^+), \quad a < x < b.$$

下面讨论 $n$ 重变上限积分 ${}_a\mathrm{D}_x^{-n} f(x)$ 的表达形式. 根据定义,

$${}_a\mathrm{D}_x^{-n} f(x) = \underbrace{\int_a^x \mathrm{d}\tau_1 \int_a^{\tau_1} \mathrm{d}\tau_2 \cdots \int_a^{\tau_{n-1}} f(\tau_n)\,\mathrm{d}\tau_n}_{n\text{-fold}}.$$

由 Cauchy 公式可知

$${}_a\mathrm{D}_x^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x (x-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau = \frac{1}{\Gamma(n)} \int_a^x (x-\tau)^{n-1} f(\tau)\,\mathrm{d}t. \tag{1.3}$$

由于 Gamma 函数对任意正实数都有定义, 因此我们可以将公式 (1.3) 推广到任意正实数情形.

**定义 1.1 (分数阶积分)** 设 $\alpha > 0$, $f(x) \in L_1[a,b]$. 则 $f(x)$ 的 $\alpha$ 阶积分为

$${}_a\mathrm{D}_x^{-\alpha} f(x) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \tag{1.4}$$

这就是 **分数阶积分**.

☞ 虽然名称中是 “分数阶”, 但事实上是对任意正实数都有定义.

显然, 当 $\alpha$ 是正整数时, ${}_a\mathrm{D}_x^{-\alpha}$ 就是通常意义下的整数阶导数.

如果 $f(x)$ 在 $[a,b]$ 上连续, 则可以证明

$$\lim_{\alpha\to 0} {}_a\mathrm{D}_x^{-\alpha} f(x) = f(x).$$

因此, 为了操作方便, 我们通常记

$${}_a\mathrm{D}_x^{0} f(x) = f(x).$$

### 分数阶积分的存在性

分数阶积分的定义是否合理, 只要看积分 (1.4) 是否存在. 如果 $\alpha \geq 1$, 则 $(x-a)^{\alpha-1}$ 是连续的, 因此对任意 $f(x) \in L_1[a,b]$, 积分 (1.4) 是显然存在的. 当 $0 < \alpha < 1$ 时, 结论并不是显然的.

**定理 1.1** 设 $\alpha > 0$, $f(x) \in L_1[a,b]$, 则对任意 $x \in [a,b]$, 积分 (1.4) 几乎处处存在, 且 ${}_a\mathrm{D}_x^{-\alpha} f(x) \in L_1[a,b]$.

**证明.** 参见 [4, page 13]. □

**分数阶积分的性质**

设 $f(x) \in C[a,b]$, 由于函数 $(x-t)^{\alpha-1}$ 在区间 $[a,x]$ 内不变号, 因此由积分中值定理可知, 存在 $\xi \in (a,x)$ 使得

$$
\begin{aligned}
{}_a\mathrm{D}_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t &= \frac{1}{\Gamma(\alpha)} f(\xi) \int_a^x (x-t)^{\alpha-1}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha+1)} f(\xi)(x-a)^{\alpha}.
\end{aligned}
$$

因此

$$
\lim_{x\to a} {}_a\mathrm{D}_x^{-\alpha} f(x) = 0. \tag{1.5}
$$

☞ 如果 $f(x)$ 不是 $[a,b]$ 上的连续函数, 仅仅使得 (1.4) 可积, 则 (1.5) 不一定成立.

**例 1.1** 设 $\alpha > 0$, $\beta > -1$, 函数 $f(x)$ 定义为

$$
f(x) = \begin{cases} x^{\beta}, & x > 0 \\ 0, & x = 0, \end{cases}
$$

则

$$
\lim_{x\to 0} {}_0\mathrm{D}_x^{-\alpha} x^{\beta} = \begin{cases} 0, & \alpha+\beta > 0 \\ \Gamma(\beta+1), & \alpha+\beta = 0 \\ \infty, & \alpha+\beta < 0. \end{cases}
$$

**定理 1.2** [4, page 14] 设 $\alpha > 0$, $\beta > 0$, $f(x) \in L_1[a,b]$, 则等式

$$
{}_a\mathrm{D}_x^{-\alpha}\left({}_a\mathrm{D}_x^{-\beta} f(x)\right) = {}_a\mathrm{D}_x^{-(\alpha+\beta)} f(x)
$$

在 $[a,b]$ 上几乎处处成立. 如果 $f(x) \in C[a,b]$ 或 $\alpha+\beta \geq 1$ 则上式对任意 $x \in [a,b]$ 都成立.

### 1.1.2 Riemann-Liouville 分数阶导数

Riemann-Liouville 分数阶导数是历史上最早的分数阶导数定义, 也是目前理论研究相对较完善的分数阶导数.

设 $\alpha > 0$ 是任意正实数, $n$ 是大于 $\alpha$ 的最小正整数, 即 $n-1 \leq \alpha < n$, 则 R-L 分数阶导数 定义为

$$
{}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha} f(x) \triangleq \mathrm{D}^n\left({}_a\mathrm{D}_x^{\alpha-n} f(x)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\,\mathrm{d}t, \tag{1.6}
$$

即先做 $n-\alpha$ 次分数阶积分, 然后再求 $n$ 次导数. 我们注意到 $0 < n-\alpha \leq 1$.

**例 1.2** 设 $f(x)=(x-a)^{\gamma}, x\in[a,b], \gamma>-1$, 计算 $f(x)$ 的 R-L 型分数阶导数.

**解.** 令 $\beta=n-\alpha$. 首先计算 $f(x)$ 的分数阶积分, 作变量代换 $\tau=\dfrac{t-a}{x-a}$, 可得:

$$\begin{aligned}{}_a\mathrm{D}_x^{-\beta} f(x) &= \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}(t-a)^{\gamma}\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\beta)}(x-a)^{\beta+\gamma}\int_0^1(1-\tau)^{\beta-1}\tau^{\gamma}\,\mathrm{d}\tau\\ &= \frac{1}{\Gamma(\beta)}(x-a)^{\beta+\gamma}\mathcal{B}(\beta,\gamma+1)\\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)}(t-a)^{\beta+\gamma},\end{aligned}$$

其中 $\mathcal{B}$ 是 Beta 函数. 因此, R-L 型分数阶导数为

$$\begin{aligned}{}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha} f(x) = \mathrm{D}^n\left({}_a\mathrm{D}_x^{-\beta} f(x)\right) &= \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(t-a)^{\beta+\gamma}\\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma-n+1)}(t-a)^{\beta+\gamma-n}\\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}.\end{aligned}$$

□

**引理 1.2** 分数阶积分是可交换的, 即对任意 $\beta,\gamma>0$, 有

$${}_a\mathrm{D}_x^{-\beta}\,{}_a\mathrm{D}_x^{-\gamma} f(x) = {}_a\mathrm{D}_x^{-\beta-\gamma} f(x)$$

**证明.** 由二重积分的性质, 交换积分次序可得

$$\begin{aligned}{}_a\mathrm{D}_x^{-\beta}\,{}_a\mathrm{D}_x^{-\gamma} f(x) &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x (x-t)^{\beta-1}\int_a^t (t-\tau)^{\gamma-1} f(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x\int_\tau^x (x-t)^{\beta-1}(t-\tau)^{\gamma-1} f(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x f(\tau)\int_\tau^x (x-t)^{\beta-1}(t-\tau)^{\gamma-1}\,\mathrm{d}t\,\mathrm{d}\tau.\end{aligned}$$

作变量代换 $\xi=\dfrac{t-\tau}{x-\tau}$, 代入后得

$$\begin{aligned}{}_a\mathrm{D}_x^{-\beta}\,{}_a\mathrm{D}_x^{-\gamma} f(x) &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x (x-\tau)^{\beta+\gamma-1} f(\tau)\int_0^1 (1-\xi)^{\beta-1}\xi^{\gamma-1}\,\mathrm{d}\xi\,\mathrm{d}\tau\\ &= \frac{\mathcal{B}(\gamma,\beta)}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x (x-\tau)^{\beta+\gamma-1} f(\tau)\,\mathrm{d}\tau\\ &= {}_a\mathrm{D}_x^{-\beta-\gamma} f(x).\end{aligned}$$

□

与整数阶微积分基本定理类似, 我们有下面的性质.

**定理 1.3** *R-L* 分数阶微分算子是分数阶积分算子的左逆, 即对任意 $\alpha > 0$ 有

$$ {}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha}\, {}_a\mathrm{D}_x^{-\alpha} f(x) = f(x). $$

**证明.** 由整数阶微积分基本定理和引理 1.4 可知

$$ {}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha}\, {}_a\mathrm{D}_x^{-\alpha} f(x) = \mathrm{D}^n\, {}_a\mathrm{D}_x^{\alpha-n}\, {}_a\mathrm{D}_x^{-\alpha} f(x) = \mathrm{D}^n\, {}_a\mathrm{D}_x^{-n} f(x) = f(x). $$

□

如果将次序反过来的话, 则有下面的复合公式.

**定理 1.4** 设 $\alpha > 0$, 且 $n-1 \le \alpha < n$, 则

$$ {}_a\mathrm{D}_x^{-\alpha}\, {}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha} f(x) = f(x) - \sum_{i=1}^{n} \frac{[{}_a\mathrm{D}_x^{\alpha-i} f(t)]_{t=a}}{\Gamma(\alpha-i+1)} (x-a)^{\alpha-i}. $$

**证明.** 证明参见相关文献, 如: [4]. □

分数阶积分算子在 $L^2(a,b)$ 内积意义下是互为伴随的, 即

$$ ({}_a\mathrm{D}_x^{-\alpha} f, g) = (f, {}_x\mathrm{D}_b^{-\alpha} g), \quad \alpha > 0. $$

**例 1.3 (常数的分数阶导数)** 设常值函数 $f(x) \equiv 1$, 则

$$ {}_a\mathrm{D}_x^{\alpha} f(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \ge 0,\ t > 0. $$

这里 $\alpha$ 不是正整数. 即**在 R-L 意义下, 常数的分数阶导数不等于 0**.

☞ 当 $\alpha$ 是正整数时, $\Gamma(1-\alpha) = \infty$, 所以 ${}_a\mathrm{D}_x^{\alpha} f(x) = 0$.

**与整数阶导数之间的关系**

如果 $\alpha = n$, 则由定义可知

$$ {}_a^{\mathrm{RL}}\mathrm{D}_x^{\alpha} f(x) = \mathrm{D}^{n+1}\left({}_a\mathrm{D}_x^{-1} f(x)\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x), $$

因此, 当 $\alpha$ 是正整数时, R-L 分数阶导数与整数阶导数的定义是一致的. 所以, R-L 分数阶导数在整数阶导数之间架起了 “桥”.

### 1.1.3 Caputo 分数阶导数

R-L 分数阶导数是最先提出来的, 理论分析也相对完善. 但与实际应用却存在一定的困难和障碍 [4, page 4]. 一个比较好的解决方法就是由 Caputo [1, 2] 提出来的 Caputo 分数阶导数.

☞ The Riemann-Liouville derivative is historically the first (developed in works of Abel, Riemann and Liouville in the first half of the nineteenth century) and the one for which the mathematical theory has been established quite well by now, but it has certain features that lead to difficulties when applying it to “realworld” problems [4, page 4].

Caputo **分数阶导数**的定义方式与 R-L 分数阶导数基本相同, 只是微分和积分的顺序相反. R-L 分数阶导数是先积分后微分, 而 Caputo 分数阶导数则是先微分后积分.

**定义 1.2** 设 $\alpha > 0$ 且 $n-1 < \alpha < n$. 则 *Caputo* 左分数阶导数为

$$ {}_a^{\mathrm{C}}\mathrm{D}_x^{\alpha} f(x) \triangleq \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} \,\mathrm{d}t. \tag{1.7} $$

☞ 需要注意的是, 与 R-L 定义不同, 这里 $\alpha$ 必须大于 $n-1$, 即当 $\alpha$ 是正整数时, Caputo 分数阶导数没有定义.

☞ 从定义可以看出, Caputo 定义中对 $f(x)$ 要求比较高, 即 $f(x)$ 至少要 $n$ 阶可微.

☞ Caputo 分数阶导数经常用于时间导数 [6], 特别是在初 (边) 值问题中.

**例 1.4 (常值函数的 Caputo 分数阶导数)** 设常值函数 $f(x) \equiv 1$, $\alpha > 0$, 则

$$ {}_a^{\mathrm{C}}\mathrm{D}_x^{\alpha} f(x) = 0. $$

### 与整数阶导数之间的关系

根据定义, 当 $\alpha \to n^-$ 时, 有

$$ \begin{aligned} \lim_{\alpha\to n^-} {}_a^{\mathrm{C}}\mathrm{D}_x^{\alpha} f(x) &= \lim_{\alpha\to n^-} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau \\ &= \lim_{\alpha\to n^-} \left( \frac{f^{(n)}(a^+) t^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^x (x-\tau)^{n-\alpha} f^{(n+1)}(\tau)\,\mathrm{d}\tau \right) \\ &= f^{(n)}(a^+) + \int_a^x f^{(n+1)}(\tau)\,\mathrm{d}\tau \\ &= f^{(n)}(x). \end{aligned} $$

而当 $\alpha \to (n-1)^+$ 时, 有

$$ \begin{aligned} \lim_{\alpha\to (n-1)^+} {}_a^{\mathrm{C}}\mathrm{D}_x^{\alpha} f(x) &= \lim_{\alpha\to (n-1)^+} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau \\ &= \int_a^x f^{(n)}(\tau)\,\mathrm{d}\tau \\ &= f^{(n-1)}(x) - f^{(n-1)}(a^+). \end{aligned} $$

因此, 当 $\alpha$ 趋向于正整数时, Caputo 定义的左导数与整数阶导数是一致的, 但右导数却有差别.

**与 R-L 定义之间的关系**

一般来说, 两者是不等价的. 事实上, 我们有下面的结论.

**定理 1.5** 设 $\alpha > 0$, $n-1 < \alpha < n$, 即 $\alpha$ 是正实数, 但不是正整数. 则对 $\tau > a$, 有

$$ {}_{a}^{\mathrm{RL}}\mathrm{D}_{x}^{\alpha} f(x) = {}_{a}^{\mathrm{C}}\mathrm{D}_{x}^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a). $$

**证明.** 由于 $n-\alpha>0$, 根据分部积分法

$$ \begin{aligned} \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau &= -\frac{1}{n-\alpha}(x-\tau)^{n-\alpha} f(\tau)\Big|_a^x + \int_a^x \frac{1}{n-\alpha}(x-\tau)^{n-\alpha} f'(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{n-\alpha}(x-a)^{n-\alpha} f(a) + \int_a^x \frac{1}{n-\alpha}(x-\tau)^{n-\alpha} f'(\tau)\,\mathrm{d}\tau. \end{aligned} $$

因此

$$ \begin{aligned} {}_{a}^{\mathrm{RL}}\mathrm{D}_{x}^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{1-\alpha}(x-a)^{-\alpha} f(a) + \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \int_a^x (x-\tau)^{n-\alpha-1} f'(\tau)\,\mathrm{d}\tau. \end{aligned} $$

依此类推, 可得

$$ \begin{aligned} {}_{a}^{\mathrm{RL}}\mathrm{D}_{x}^{\alpha} f(x) &= \frac{1}{1-\alpha}(x-a)^{-\alpha} f(a) + \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \int_a^x (x-\tau)^{n-\alpha-1} f'(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{1-\alpha}(x-a)^{-\alpha} f(a) + \frac{1}{2-\alpha}(x-a)^{1-\alpha} f'(a) \\ &\quad + \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} \int_a^x (x-\tau)^{n-\alpha-1} f''(\tau)\,\mathrm{d}\tau \\ &= \cdots\cdots \\ &= \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) + \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau. \end{aligned} $$

所以, 结论成立. □

### 1.1.4 Grünwald-Letnikov 分数阶导数

**G-L 分数阶导数**是对导数的极限定义的推广. 普通导数的定义为

$$ f'(x) = \lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = \lim_{h\to 0} \frac{(I-T_h)f(x))}{h}, $$

其中 $T_h$ 是位移算子, 即

$$ T_h f = f(x-h), \quad T_h^2 f = T_h(T_h f) = f(x-2h), \quad \ldots. $$

当 $h>0$ 时是向后移位, 当 $h<0$ 时是向前移位. 因此,

$$ f^{(n)}(x) = \lim_{h\to 0} \frac{\Delta_h^n f(x)}{h^n} = \lim_{h\to 0} h^{-n}(I-T_h)^n f(x) $$

$$= \lim_{h\to 0} h^{-n} \sum_{k=0}^{n} \binom{n}{k} (-T_h)^k I^{n-k} f(x)$$
$$= \lim_{h\to 0} h^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x-kh),$$

其中 $\Delta_h$ 是步长为 $h$ 的差分算子. 注意到当 $k > n$ 时, $\binom{n}{k} = 0$. 因此我们可以往远处延伸, 将点 $x - kh$ $(k = n+1, n+2, \dots)$ 也包含进来 (直到边界). 于是

$$f^{(n)}(x) = \lim_{h\to 0} h^{-n} \sum_{k=0}^{\infty} \binom{n}{k} (-1)^k f(x-kh).$$

如果是有界区域的话, 则上式求和的上限是到边界为止. 由 Gamma 函数的性质 (2.3) 可得

$$\binom{n}{k}(-1)^k = (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!}$$
$$= (-1)^k \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$
$$= \frac{\Gamma(k-n)}{\Gamma(k+1)\Gamma(-n)}.$$

注意, 上面最后一个式子只是一个形式 (该等式是假定 $n$ 不是一个整数, 然后运用性质 (2.3)), 并没有实际意义 (Gamma 函数在 0 和负整数上没有定义).

推广到正实数情形即为 G-L 分数阶导数.

**定义 1.3** 设 $\alpha > 0$, 则 *G-L* 分数阶导数定义为

$${}^{\mathrm{GL}}_{\ a}\mathrm{D}_x^{\alpha} \triangleq \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} f(x-kh)$$
$$= \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} f(x-kh). \tag{1.8}$$

## 1.2 相关性质

一般情况下,

$${}^{\mathrm{RL}}_{\ a}\mathrm{D}_x^{\alpha} f(x) \neq {}^{\mathrm{C}}_{a}\mathrm{D}_x^{\alpha} f(x).$$

当 $f(a^+) = f'(a^+) = \cdots = f^{(n-1)}(a^+) = 0$ 时, 两者相等. 事实上, 我们有

$${}^{\mathrm{RL}}_{\ a}\mathrm{D}_x^{\alpha} f(x) = {}^{\mathrm{C}}_{a}\mathrm{D}_x^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+).$$

其中 $n-1 < \alpha < n$, $x > a$.

具有相同分数阶导数的函数.

**引理 1.3** 设 $n-1<\alpha\le n$, $x>0$, $a=0$, 则

$$
{}_{a}^{\mathrm{RL}}\mathrm{D}_x^{\alpha} f(x) = {}_{a}^{\mathrm{RL}}\mathrm{D}_x^{\alpha} g(x) \iff f(x) = g(x) + \sum_{k=1}^{n} c_k x^{\alpha-k}
$$

$$
{}_{a}^{\mathrm{C}}\mathrm{D}_x^{\alpha} f(x) = {}_{a}^{\mathrm{C}}\mathrm{D}_x^{\alpha} g(x) \iff f(x) = g(x) + \sum_{k=1}^{n} c_k x^{n-k}
$$

## 1.3 右分数阶积分和右分数阶导数

前面的分数阶导数通常称为 **左分数阶导数**, 这是因为 $f(x)$ 在 $x$ 点的分数阶导数是通过 $f(x)$ 在区间 $[a,x]$ 中的值来表示的. 相应地, 我们也可以利用 $f(x)$ 在区间 $[x,b]$ 中的值来定义 $f(x)$ 则 $x$ 点的分数阶导数. 通过这种方法定义的分数阶导数就称为**右分数阶导数**. 一个代表 “过去”, 而另一个代表 “未来”.

### 1.3.1 右分数阶积分

首先介绍右分数阶积分.

**右分数阶积分**定义为

$$
{}_{x}\mathrm{D}_b^{-\alpha} f(x) \triangleq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t,
$$

其中 $\alpha$ 是任意的正分数.

相应地, 将 **右整数阶积分** 记为

$$
{}_{x}\mathrm{D}_b^{-n} f(x) \triangleq \underbrace{\int_x^b \cdots \int_x^b}_{n} f = \frac{1}{\Gamma(n)} \int_x^b (t-x)^{n-1} f(t)\,\mathrm{d}t.
$$

### 1.3.2 R-L 右分数阶导数

相类似地, R-L **右分数阶导数**定义为 [5, page 89]

$$
{}_{x}^{\mathrm{RL}}\mathrm{D}_b^{\alpha} f(x) \triangleq \mathrm{D}^n \left({}_{x}\mathrm{D}_b^{\alpha-n} f(x)\right) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}\,\mathrm{d}t. \tag{1.9}
$$

**例 1.5** 设 $f(x)=(b-x)^{\gamma}$, $x\in[a,b]$, $\gamma>-1$, 则 $f(x)$ 的 R-L 型右分数阶导数为

$$
{}_{x}^{\mathrm{RL}}\mathrm{D}_b^{\alpha} f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (b-t)^{\gamma-\alpha}.
$$

**引理 1.4** 分数阶积分是可交换的, 即对任意 $\alpha, \gamma > 0$, 有

$$ {}_x\mathrm{D}_b^{-\alpha}\, {}_x\mathrm{D}_b^{-\gamma} f(x) = {}_x\mathrm{D}_b^{-\alpha-\gamma} f(x). $$

**定理 1.6** *R-L* 右分数阶微分算子是右分数阶积分算子的左逆, 即对任意 $\alpha > 0$ 有

$$ {}^{\mathrm{RL}}_{\;x}\mathrm{D}_b^{\alpha}\, {}_x\mathrm{D}_b^{-\alpha} f(x) = f(x). $$

**定理 1.7** 设 $\alpha > 0$, 且 $n-1 \le \alpha < n$, 则

$$ {}_x\mathrm{D}_b^{-\alpha}\, {}^{\mathrm{RL}}_{\;x}\mathrm{D}_b^{\alpha} f(x) = f(x) - \sum_{i=1}^{n} \frac{[{}_x\mathrm{D}_b^{\alpha-i} f(t)]_{t=b}}{\Gamma(\alpha-i+1)} (b-x)^{\alpha-i}. $$

### 1.3.3 Caputo 右分数阶导数

相应地, Caputo 右分数阶导数定义为

$$ {}^{\mathrm{C}}_{x}\mathrm{D}_b^{\alpha} f(x) \triangleq \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{(-1)^n f^{(n)}(t)}{(t-x)^{\alpha+1-n}} \,\mathrm{d}t. \tag{1.10} $$

### 1.3.4 G-L 右分数阶导数

相应地, G-L 右分数阶导数定义为

$$ \begin{aligned} {}^{\mathrm{GL}}_{\;x}\mathrm{D}_b^{\alpha} &\triangleq \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} f(x+kh) \\ &= \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\lceil \frac{b-x}{h} \rceil} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} f(x+kh). \end{aligned} \tag{1.11} $$

### 1.3.5 对称 Riesz 分数阶导数

将 R-L 左右分数阶导数相加后做平均, 即可得 **对称 Riesz 分数阶导数**:

$$ \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}|t|^{\alpha}} = {}_t\mathrm{D}_R^{\alpha} f(x) \triangleq \frac{1}{2}\left( {}^{\mathrm{RL}}_{\;a}\mathrm{D}_x^{\alpha} + {}^{\mathrm{RL}}_{\;x}\mathrm{D}_b^{\alpha} f(x) \right). $$

## 1.4 分数阶扩散方程 Fractional Diffusion Processes

The field of fractional (more generally anomalous) diffusion processes in recent decades has won more and more interest in applications in the sciences, in physics and chemistry, and even in finance.

First consider the Cauchy problem for the classical diffusion equation

$$ \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0^+) = f(x), x \in \mathbb{R}, t \ge 0. \tag{1.12} $$

In fractional diffusion equations the differentiations with respect to $t$ and $x$ are replaced by differentiations of non-integer order. For (1.12) it is well known that its solution $u(x,t)$ has the essential properties we expect from a diffusion process, that is a process of re-distribution in space $x$ and time $t$. Considering $u(x,t)$ as the spatial density of an extensive quantity, e.g. mass, charge, or probability, we have

(1) conservation of the total quantity:

$$\int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x = \int_{-\infty}^{+\infty} u(x,0^+)\,\mathrm{d}x, \forall\, t > 0.$$

(2) preservation of non-negativity:

$$u(x,0^+) \geq 0, \forall\, x \in \mathbb{R} \quad \text{implies} \quad u(x,t) \geq 0, \forall\, x \in \mathbb{R}, t > 0.$$

(3) Another essential characteristic of problem (1.12) concerns the law of spreading (or dispersion) of the quantity. With the special initial condition $u(x,0^+) = \delta(x)$ (the Dirac generalized function), the variance grows linearly in time, that is $\sigma^2(t) \triangleq \int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x = 2t$. More generally, in a classical diffusion process the variance, which is a natural and common quadratic measure of the spread of a diffusing substance, grows linearly in time, that is, if we allow a drift, we have $\sigma^2(t) \triangleq \int_{-\infty}^{+\infty} x^2(u(x,t) - m(t))\,\mathrm{d}x \sim Ct$ as $t \to \infty$ with $m(t) \triangleq \int_{-\infty}^{+\infty} xu(x,t)\,\mathrm{d}x$, for some constant $C > 0$.

The above properties (1), (2) and (3) are indeed shared by many processes governed by second-order linear parabolic equations. Usurping the term diffusion for processes having properties (1) and (2) but not necessarily (c), we follow the custom to call processes of anomalous diffusion those in which, for initial condition $u(x,0^+) = \delta(x)$, the variance does not exhibit essentially linear grow with $t \to \infty$. Among these processes we single out the sub-diffusive ones for which the variance grows (for large $t$) more slowly than linearly, and the super-diffusive for which it grows (for large $t$) faster than linearly, or even does not exist (i.e is infinite).

Consider the Cauchy problem for the (spatially one-dimensional) space-time fractional diffusion equation

$${}_t\mathrm{D}_0^{\alpha}\, u(x,t) = {}_x\mathrm{D}_0^{\alpha}\, u(x,t), \quad u(x,0^+) = \delta(x), x \in \mathbb{R}, t \geq 0, \tag{1.13}$$

where

$$0 < \alpha \leq 2 \quad \text{and} \quad 0 < \alpha \leq 1.$$

Here ${}_t\mathrm{D}_0^{\alpha}\, u(x,t)$ denotes the Caputo fractional derivative of order $\alpha$

$${}_t\mathrm{D}_0^{\alpha}\, u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,t)}{\partial t} \frac{\mathrm{d}s}{(t-s)^{\alpha}},$$

and ${}_x\mathrm{D}_0^{\alpha}\, u(x,t)$ denotes the symmetric Riesz–Feller fractional derivative of order $\alpha$

$${}_x\mathrm{D}_0^{\alpha}\, u(x,t) = \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^{\infty} \frac{u(x+s) - 2u(x,t) + u(x-s,t)}{s^{1+\alpha}}\,\mathrm{d}s.$$

The above representations of the space fractional derivatives are based on a suitable regularization of hyper-singular integrals. In the limits $\alpha = 1$ and $\alpha = 2$ we recover the first time derivative $\frac{\partial u(x,t)}{\partial t}$ and the second space derivative $\frac{\partial^2 u(x,t)}{\partial x^2}$, respectively.

These representations mirror the fact that time-fractional (for $0 < \alpha < 1$) processes are processes with long memory whereas space fractional (for $0 < \alpha < 2$) are processes with spatial long-range interactions.

☞ 上面的介绍主要参考: R. Gorenfloa and F. Mainardi, Some recent advances in theory and simulation of fractional diffusion processes, Journal of Computational and Applied Mathematics, 229 (2009), 400–415.

# 第二讲　附录

## 2.1　特殊函数

### 2.1.1　Gamma 函数

**定义 2.1**　*Gamma 函数定义为*

$$\Gamma(x) \triangleq \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, \quad \mathrm{Re}(x) > 0, \tag{2.1}$$

其中 $t^{x-1} \triangleq e^{(x-1)\ln(t)}$.

☞ Gamma 函数也称为**第二类 Euler 积分** (Euler integral of the second kind )

我们这里只考虑实数情形, 即 $x \in \mathbb{R}$.

通过直接计算, 易知

$$\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = \lim_{z\to\infty}[-e^{-t}]_0^z = 1.$$

对任意 $x > 0$, 由分部积分法可得

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t = t^x(-e^{-t})\Big|_0^\infty - \int_0^\infty xt^{x-1}(-e^{-t})\,\mathrm{d}t = x\Gamma(x).$$

**引理 2.1**　*Gamma 函数满足下面的关系式:*

$$\Gamma(x+1) = x\Gamma(x). \tag{2.2}$$

该性质的一个直接推广就是

$$\Gamma(n+1) = n!$$

因此, Gamma 函数是阶乘概念的推广.

下面考虑 $x < 0$ 时的情形. 注意到 Gamma 函数的原始定义 (2.1) 对负数是没有定义的, 因为当 $x < 0$ 时该积分不存在. 但我们可以通过关系式 (2.2) 来定义. 首先将 (2.2) 改写为

$$\Gamma(x) = \frac{\Gamma(x)}{x}.$$

由于 $\Gamma(x)$ 关于 $x > 0$ 都有定义, 因此可以在开区间 $(-1, 0)$ 上定义 $\Gamma(x)$, 然后再延伸到 $(-2, -1)$ 上. 依此类推, 可以将 Gamma 函数延伸到整个实数域, 除了 0 和负的整数点, 即 $\mathbb{R} \setminus \{0, -1, -2, \ldots\}$. 事实上, Gamma 函数可以延伸到除了 0 和负整数点之外的所有复数域上.

☞ Gamma 函数的另外一个定义是

$$\Gamma(x)=\lim_{n\to\infty}\frac{n!\,n^x}{x(x+1)\cdots(x+n)},\quad x\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

该表达式是由 Gauss 给出的.

通过上面的定义方式, 我们可以得到下面的性质.

**定理 2.1** 设 $k$ 是正整数, $\alpha\in\mathbb{R}$ 不是整数, 则

$$(-1)^k\Gamma(\alpha-k+1)\Gamma(k-\alpha)=\Gamma(-\alpha)\Gamma(\alpha+1). \tag{2.3}$$

Gamma 函数的另一个重要性质是

**定理 2.2 (Reflection Formula)** 设 $0<x<1$, 则

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}.$$

由此可知

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Gamma 函数有下面的渐进表达式

**定理 2.3 (Stirling Asymptotic Formula)** 设 $x\in\mathbb{R}$, 则

$$\Gamma(x)=\sqrt{2\pi}\,x^{x-1/2}\,e^{-x}\left(1+O\left(\frac{1}{x}\right)\right),\quad x\to+\infty.$$

Gamma 函数的一些其他性质:

(1) 当 $z\to0^+$ 时, $\Gamma(z)\to+\infty$

(2) $\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$

(3) $\Gamma(z)\Gamma(z+\frac{1}{n})\Gamma(z+\frac{2}{n})\cdots\Gamma(z+\frac{n-1}{n})=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz)$

(4) $\Gamma\left(n+\frac{1}{2}\right)=\dfrac{(2n)!\sqrt{\pi}}{4^n n!}$

**例 2.1** 一些简单的 Gamma 函数值:

$$\Gamma(0)=\infty,\ \Gamma(1)=1,\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\ \Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}.$$

### 2.1.2 Beta 函数

Beta 函数是二项式系数的推广, 其定义为

$$\mathcal{B}(z,w)\triangleq\int_0^1\tau^{z-1}(1-\tau)^{w-1}\,\mathrm{d}\tau,\quad \mathrm{Re}(z)>0,\mathrm{Re}(w)>0. \tag{2.4}$$

易知 $\mathcal{B}(w,z)=\mathcal{B}(z,w)$, 且 Beta 函数可以用 Gamma 函数来表示, 即

$$\mathcal{B}(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

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