Shifted Power Method for H-eigenvalue of Symmetric Tensors

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Outline

1. Tensor and Eigenvalue
2. NQZ Algorithm for largest Eigenvalue
3. HOPM Algorithm for Z-eigenvalue
4. Shifted Power Method for H-eigenvalue

Tensor

The real-valued tensor of order m and dimension n is defined as follows

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \cdots, i_m \leq n.$$

- Symmetric Tensor: $\mathcal{A}$ is called symmetric if its entries do not change under any permutation of its m indices.
- Nonnegative Tensor: $\mathcal{A} =$ is called nonnegative if $a_{i_1 \cdots i_m} \geq 0$.
- Irreducible Tensor [CPZ '08]: $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 \cdots i_m} = 0 \quad \text{for all } i_1 \in I \quad \text{and } i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

Tensor-vector product

Let r be an integer such that $0 \leq r \leq m-1$.

The $(m-r)$-times product of a symmetric tensor $\mathcal{A}$ with a vector x is denoted by $\mathcal{A}x^{m-r}$ and defined as

$$(\mathcal{A}x^{m-r})_{i_1 \cdots i_r} := \sum_{i_{r+1},\ldots,i_m} a_{i_1 \cdots i_m} x_{i_{r+1}} \cdots x_{i_m}$$

for all $i_1, \ldots, i_r \in \{1, \ldots, n\}$.

In particular, $\mathcal{A}x^m$ is a scalar and $\mathcal{A}x^{m-1}$ is a vector

Eigenvalue and H-eigenvalue

Definition (Eigenvalue, [Qi '05, Lim '05])
Let $\mathcal{A}$ be a tensor of order m and dimension n. Then $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = [x_1^{m-1}, \cdots, x_n^{m-1}]^T$. The vector x is the corresponding eigenvector.

H-eigenvalue

If, in addition, both λ and x are real, then they are called the H-eigenvalue and H-eigenvector, respectively.

NQZ algorithm

NQZ algorithm [NQZ '09] : an iterative method for finding the largest eigenvalue of irreducible nonnegative tensors.

Choose a positive vector $x^{(0)}$ and compute $y^{(0)} = \mathcal{A}(x^{(0)})^{m-1}$

for $k = 1, 2, \ldots,$ until convergence **do**

$$x^{(k)} = \frac{\left(y^{(k-1)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\| \left(y^{(k-1)}\right)^{\left[\frac{1}{m-1}\right]} \right\|}$$

$$y^{(k)} = \mathcal{A}(x^{(k)})^{m-1}$$

$$\lambda_k^- = \min_{x_i^{(k)}>0} \frac{y_i^{(k)}}{\left(x_i^{(k)}\right)^{m-1}}$$

$$\lambda_k^+ = \max_{x_i^{(k)}>0} \frac{y_i^{(k)}}{\left(x_i^{(k)}\right)^{m-1}}$$

end for

Convergence of NQZ

Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. Then

$$\lambda_k^- \le \lambda_{k+1}^- \quad \text{and} \quad \lim_{k\to\infty} \lambda_k^- = \lambda^-$$

and

$$\lambda_k^+ \ge \lambda_{k+1}^+ \quad \text{and} \quad \lim_{k\to\infty} \lambda_k^+ = \lambda^+$$

Moreover,

$$\lambda^- \le \rho(\mathcal{A}) \le \lambda^+$$

where $\rho(\mathcal{A})$ is the spectral radius of $\mathcal{A}$.

In general, the convergence of NQZ for irreducible nonnegative tensors is not guaranteed.

HOPM algorithm

HOPM [LMV '00] : Higher-Order Power Method

1: Choose a vector $x^{(0)} \in \mathbb{R}^n$ with $\|x^{(0)}\| = 1$
2: Compute $\lambda_0 = \mathcal{A}(x^{(0)})^m$
3: **for** $k = 1, 2, \ldots,$ until convergence **do**
4: $\quad y^{(k)} = \mathcal{A}(x^{(k-1)})^{m-1}$
5: $\quad x^{(k)} = y^{(k)} / \|y^{(k)}\|$
6: $\quad \lambda_k = \mathcal{A}(x^{(k)})^m$
7: **end for**

- HOPM is proposed for the low-rank tensor approximation and predates the definition of tensor eigenvalue problem
- The initial vector is not required to be positive
- HOPM can be used to compute Z-eigenvalue of symmetric tensors

SS-HOPM algorithm

SS-HOPM [KM '11] : Shifted Symmetric Higher-Order Power Method

Choose a vector $x^{(0)} \in \mathbb{R}^n$ with $\|x^{(0)}\| = 1$ and a shift α
Compute $\lambda_0 = \mathcal{A}(x^{(0)})^m$
for $k = 1, 2, \ldots$, until convergence **do**
  if $\alpha \geq 0$ **then**
    $y^{(k)} = \mathcal{A}(x^{(k-1)})^{m-1} + \boldsymbol{\alpha x^{(k-1)}}$
  else
    $y^{(k)} = -\left(\mathcal{A}(x^{(k-1)})^{m-1} + \boldsymbol{\alpha x^{(k-1)}}\right)$
  end if
  $x^{(k)} = y^{(k)} / \|y^{(k)}\|$
  $\lambda_k = \mathcal{A}(x^{(k)})^m$
end for

Power method for H-eigenvalue (S-HOPM-H)

$\lambda \in \mathbb{R}$ is an H-eigenvalue if there exists a nonzero vector $x \in \mathbb{R}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$$

Symmetric High Order Power Method for H-eigenvalue (S-HOPM-H)

1: Given a symmetric even-order tensor $\mathcal{A}$
2: choose $x^{(0)} \in \mathbb{R}^n$ with $\|x^{(0)}\|_m = 1$ and compute $\lambda_0 = \mathcal{A}\left(x^{(0)}\right)^m$
3: **for** $k = 1, 2, \ldots$, until convergence **do**
4: $\quad y^{(k)} = \mathcal{A}\left(x^{(k-1)}\right)^{m-1}$
5: $\quad z^{(k)} = \left(y^{(k)}\right)^{\left[\frac{1}{m-1}\right]}$
6: $\quad x^{(k)} = z^{(k)}/\|z^{(k)}\|_m$
7: $\quad \lambda_k = \mathcal{A}\left(x^{(k)}\right)^m$
8: **end for**

Example 1

Example ([KR '02])

Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 3, whose entries are defined by

$$\begin{array}{llll} a_{1111} = 0.2883, & a_{1112} = -0.0031, & a_{1113} = 0.1973, & a_{1122} = -0.2485, \\ a_{1123} = -0.2939, & a_{1133} = 0.3847, & a_{1222} = 0.2972, & a_{1223} = 0.1862, \\ a_{1233} = 0.0919, & a_{1333} = -0.3619, & a_{2222} = 0.1241, & a_{2223} = -0.3420, \\ a_{2233} = 0.2127, & a_{2333} = 0.2727, & a_{3333} = -0.3054. & \end{array}$$

- There are totally 27 different eigenvalues, among which 11 are real

Example

Table: All H-eigenpairs generated by `Mathematica`

λ	x^T		
2.3129	[0.7875	0.6483	-0.8138]
1.9316	[0.8749	-0.6536	0.6936]
0.9780	[0.1474	-0.9540	0.6432]
0.8944	[0.5223	0.8048	0.8434]
0.7228	[0.8526	0.4939	0.8012]
0.4108	[0.2035	-0.5145	-0.9816]
0.2528	[1.0000	0.1020	-0.0868]
0.2499	[0.4178	0.9917	0.2184]
-0.0887	[0.9158	0.7376	0.1559]
-0.6665	[0.2291	-0.2579	0.9982]
-2.6841	[0.7793	-0.8675	-0.5044]

Thank you!