A $q$-analogue of Zhang’s binomial coefficient identities

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Abstract. In this paper, we prove some identities for the alternating sums of squares and cubes of the partial sum of the $q$-binomial coefficients. Our proof also leads to a $q$-analogue of the sum of the first $n$ squares due to Schlosser.

Keywords: binomial coefficient; $q$-binomial coefficient; partial sum

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1 Introduction

Calkin [4] proved a curious identity of sums of 3-powers of the partial sum of binomial coefficients:

$$\sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^3 = n2^{3n-1} + 2^{3n} - 3n \left( \begin{array}{c} 2n \\ n \end{array} \right) 2^{n-2}. \quad (1.1)$$

Hirschhorn [6] established some recurrence relations of sums of powers of the partial sum of binomial coefficients, and obtained

$$\sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j} = n2^{n-1} + 2^n, \quad (1.2)$$
$$\sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^2 = n2^{2n-1} + 2^{2n} - \frac{n}{2} \left( \begin{array}{c} 2n \\ n \end{array} \right), \quad (1.3)$$

and Calkin’s identity. Zhang [12, 13] considered the alternating forms and proved

$$\sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{k} \binom{n}{j} \right)^2 = \begin{cases} 1, & \text{if } n = 0, \\ 2^{2n-1}, & \text{if } n \text{ is even and } n \neq 0, \\ -2^{2n-1} - (-1)^{(n-1)/2} \left( \begin{array}{c} n-1 \\ (n-1)/2 \end{array} \right), & \text{if } n \text{ is odd}, \end{cases} \quad (1.4)$$
$$\sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{k} \binom{n}{j} \right)^3 = -2^{3n-1} - 3(-1)^{(n-1)/2} 2^{n-1} \left( \begin{array}{c} n-1 \\ (n-1)/2 \end{array} \right), \quad (1.5)$$

Some other proofs can be found in [2, 5]. Several generalizations of (1.1)–(1.5) are given in [9, 15, 16]. The aim of this paper is to give some $q$-analogues of (1.4)–(1.5).
Recall that the \( q \)-binomial coefficient \( \binom{n}{k}_q \) is defined by
\[
\binom{n}{k}_q = \begin{cases} 
\frac{(q:q)_n}{(q; q)_k(q:q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\
0, & \text{otherwise},
\end{cases}
\]
where \((a;q)_N = (1-a)(1-aq) \cdots (1-aq^{N-1})\). Our main results may be stated as follows.

**Theorem 1.1.** For \( n \geq 0 \), we have
\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^{k} \binom{2n}{j}_q \right)^2 = \left( \sum_{k=0}^{2n} \binom{2n}{k}_q \right) \left( \sum_{k=0}^{2n} \binom{2n}{2k}_q \right), 
\]
\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^{k} (-1)^j \binom{2n}{j}_q \right)^2 = (q; q^n)_n \left( \sum_{k=0}^{2n} \binom{2n}{2k}_q \right). 
\]

**Theorem 1.2.** For \( n \geq 0 \), we have
\[
\sum_{k=0}^{2n+1} (-1)^k \left( \sum_{j=0}^{k} \binom{2n+1}{j}_q \right)^2 = -\left( \sum_{k=0}^{n} \binom{2n+1}{2k}_q \right) \left( \sum_{k=0}^{n} \binom{2n+1}{k}_q \right)
- \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q - 2 \sum_{0 \leq i < j \leq n} (-1)^i \binom{2n+1}{i}_q \binom{2n+1}{j}_q. 
\]

**Theorem 1.3.** For \( n \geq 0 \), we have
\[
\sum_{k=0}^{2n+1} (-1)^k \left( \sum_{j=0}^{k} \binom{2n+1}{j}_q \right)^3
= -\left( \sum_{k=0}^{n} \binom{2n+1}{2k}_q \right) \left( \sum_{j=0}^{n} \binom{2n+1}{j}_q \right)^2
- \frac{3}{2} \binom{2n+1}{2n+1}_q \left( \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q \right)^2 + 2 \sum_{0 \leq i < j \leq n} (-1)^i \binom{2n+1}{i}_q \binom{2n+1}{j}_q. 
\]

Since
\[
\sum_{k=0}^{n} \binom{n}{k} = 2 \sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^n,
\]
when \( q \to 1 \) the identity (1.6) reduces to the even case of (1.4). To see that (1.8) is a generalization of the odd case of (1.4), first note that
\[
\sum_{i=0}^{j} (-1)^i \binom{n}{i} = (-1)^j \binom{n-1}{j}. 
\]
Hence, when \( q \to 1 \) the right-hand of (1.8) reduces to

\[
-2^{4n+1} - \sum_{k=0}^{n} (-1)^{k} \binom{2n+1}{k}^2 \left( -2 \sum_{k=1}^{n} (-1)^{k-1} \binom{2n+1}{k} \binom{2n}{k-1} \right) - 2^{4n} - \sum_{k=0}^{n} (-1)^{k} \binom{2n+1}{k} \binom{2n}{k} - \sum_{k=1}^{n} (-1)^{k-1} \binom{2n+1}{k} \binom{2n}{k} - 2^{4n} - (-1)^{n} \binom{2n}{n} = -2^{4n+1} - \sum_{k=0}^{n} (-1)^{k} \binom{2n+1}{k} \binom{2n}{k} - 2^{4n} - \sum_{k=0}^{n} (-1)^{k} \binom{2n+1}{k} \binom{2n}{k} - 2^{4n} - (-1)^{n} \binom{2n}{n}.
\]

The last equality follows from equating the coefficients of \( x^{2n+1} \) of

\[
(1 + x)^{2n+1}(1 - x)^{2n} = (1 + x)(1 - x)^{2n}.
\]

For the same reason, when \( q \to 1 \) the identity (1.9) reduces to (1.5).

Nowadays, Zeilberger’s algorithm \([7, 11]\) is a powerful tool to prove hypergeometric identities. Furthermore, almost every terminating \( q \)-series identities, such as the \( q \)-Dixon identity (see \([3]\))

\[
\sum_{k=-n}^{n} (-1)^{k} q^{(3k^2+k)/2} \left[ \frac{2n}{n+k} \right]_{q}^{3} = \frac{(q;q)_{3n}}{(q;q)_{n}},
\]

could be proved by the \( q \)-analogue of Zeilberger’s algorithm (see \([10]\)). However, the \( q \)-series identities in Theorems 1.1–1.3 do not fall into the scope of the \( q \)-analogue of Zeilberger’s algorithm. Thus, we have to prove them technically, and the proof of Theorem 1.3 is a bit long. We would like to mention that in the last step of our proof of Theorem 1.3, we need to use an interesting fact, from which we may obtain \( q \)-analogues of the sums of the first \( n \) squares and odd squares due to Schlosser \([8]\).

### 2 Proof of Theorems 1.1 and 1.2

**Proof of (1.6).** Let \( a_k = \binom{2n}{k} q^{k} \). Then the left-hand side of (1.6) is equal to

\[
\sum_{k=0}^{2n} (-1)^{k} \left( \sum_{j=0}^{k} a_j \right)^2 = \sum_{k=0}^{n} \left( \left( \sum_{j=0}^{2k} a_j \right)^2 - \left( \sum_{j=0}^{2k-1} a_j \right)^2 \right) = \sum_{k=0}^{n} \left( \left( \sum_{j=0}^{2k} a_j + \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=0}^{2k} a_j - \sum_{j=0}^{2k-1} a_j \right) \right) = \sum_{k=0}^{n} \sum_{j=0}^{2k} a_{2k} \left( a_{2k} + 2 \sum_{j=0}^{2k-1} a_j \right).
\]

(2.1)
Replacing $k$ by $n-k$ in the right-hand side of (2.1) and noticing that $a_{2n-j} = a_j$, we obtain

\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k a_j \right)^2 = \sum_{k=0}^n a_{2k} \left( a_{2k} + 2 \sum_{j=2k+1}^{2n} a_j \right) \tag{2.2}
\]

Combining (2.1) and (2.2) yields

\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k a_j \right)^2 = \frac{1}{2} \sum_{k=0}^n a_{2k} \left( 2a_{2k} + 2 \sum_{j=0}^{2k-1} a_j + 2 \sum_{j=2k+1}^{2n} a_j \right)
\]

\[
= \left( \sum_{k=0}^n a_{2k} \right) \left( \sum_{j=0}^{2n} a_j \right), \tag{2.3}
\]

as desired. \hfill \blacksquare

**Proof of (1.7).** Similarly, we have

\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k (-1)^j \left( \begin{array}{c} 2n \\ j \end{array} \right) \right)^2 = \left( \sum_{k=0}^n (-1)^k \left( \begin{array}{c} 2n \\ k \end{array} \right) \right) \left( \sum_{k=0}^n \left( \begin{array}{c} 2n \\ 2k \end{array} \right) \right).
\]

The proof then follows from Gauss' theorem (see, for example, [1, (3.3.8)]). \hfill \blacksquare

Note that (2.3) holds on condition that $a_k = a_{2n-k}$ ($0 \leq k \leq n$). We may take other values of $a_k$ to obtain more interesting identities. For example, letting $a_k = 1$, we obtain the alternating sum of squares:

\[
\sum_{k=1}^{2n+1} (-1)^{k-1} k^2 = \left( \begin{array}{c} 2n+2 \\ 2 \end{array} \right),
\]

while letting $a_k = \left( \begin{array}{c} 2n \\ k \end{array} \right)^2$ or $(-1)^k \left( \begin{array}{c} 2n \\ k \end{array} \right)^2$ in (2.3) and using the following two well-known identities

\[
\sum_{k=0}^n \left( \begin{array}{c} 2n \\ k \end{array} \right)^2 = \left( \begin{array}{c} 4n \\ 2n \end{array} \right),
\]

\[
\sum_{k=0}^n (-1)^k \left( \begin{array}{c} 2n \\ k \end{array} \right)^2 = (-1)^n \left( \begin{array}{c} 2n \\ n \end{array} \right),
\]

we get the following result.

**Corollary 2.1.** For $n \geq 0$, we have

\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k \left( \begin{array}{c} 2n \\ j \end{array} \right)^2 \right)^2 = \frac{1}{2} \left( \begin{array}{c} 4n \\ 2n \end{array} \right) \left( \begin{array}{c} 4n \\ 2n \end{array} \right) + (-1)^n \left( \begin{array}{c} 2n \\ n \end{array} \right),
\]

\[
\sum_{k=0}^{2n} (-1)^k \left( \sum_{j=0}^k (-1)^j \left( \begin{array}{c} 2n \\ j \end{array} \right)^2 \right)^2 = \frac{(-1)^n}{2} \left( \begin{array}{c} 2n \\ n \end{array} \right) \left( \begin{array}{c} 4n \\ 2n \end{array} \right) + (-1)^n \left( \begin{array}{c} 2n \\ n \end{array} \right).
\]
Proof of (1.8). Let \( a_k = \binom{2n+1}{k, q} \). Then the left-hand side of (1.8) is equal to
\[
\sum_{k=0}^{n} \left( \sum_{j=0}^{2k} a_j - \sum_{j=0}^{2k+1} a_j \right)^2 = \sum_{k=0}^{n} \left( \sum_{j=0}^{2k} a_j + \sum_{j=0}^{2k+1} a_j \right) \left( \sum_{j=0}^{2k} a_j - \sum_{j=0}^{2k+1} a_j \right)
\]
\[
= - \sum_{k=0}^{n} a_{2k+1} \left( a_{2k+1} + 2 \sum_{j=0}^{k} a_j \right)
\]
\[
= - \sum_{k=0}^{n} a_{2k+1} \left( \sum_{j=2k+1}^{2n+1} a_j - \sum_{j=2k+2}^{2n+2} a_j \right)
\]
\[
= \begin{cases} 
- a_{2k+1} \left( \sum_{j=2k+1}^{2n+1} a_j \right), & \text{if } 2k \leq n - 1, \\
 a_{2k+1} \left( \sum_{j=2n-2k}^{2k} a_j \right), & \text{if } 2k \geq n,
\end{cases}
\]
(2.5)

Noticing \( a_k = a_{2n+1-k} \), we have
\[
a_{2k+1} \left( \sum_{j=0}^{2k} a_j - \sum_{j=2k+2}^{2n+1} a_j \right) = a_{2k+1} \left( \sum_{j=0}^{2k} a_j - \sum_{j=0}^{2n-2k} a_j \right)
\]
\[
= \begin{cases} 
- a_{2k+1} \left( \sum_{j=2k+1}^{2n+1} a_j \right), & \text{if } 2k \leq n - 1, \\
 a_{2k+1} \left( \sum_{j=2n-2k}^{2k} a_j \right), & \text{if } 2k \geq n,
\end{cases}
\]
(2.6)

Combining (2.4) and (2.6), we complete the proof.

Motivated by Corollary 2.1, we would like to propose the following problem.

**Problem 2.2.** Is there a simple formula for the expression
\[
\sum_{k=0}^{2n+1} (-1)^k \left( \sum_{j=0}^{k} \binom{2n+1}{j} \right)^2.
\]
(2.7)
Note that, by the proof of Theorem 1.2 and using the Chu-Vandermonde convolution formula, (2.7) is equal to
\[-\frac{1}{2} \binom{4n + 2}{2n + 1}^2 - \sum_{k=0}^{n} (-1)^k \binom{2n + 1}{k}^4 - 2 \sum_{0 \leq i < j \leq n} (-1)^i \binom{2n + 1}{i}^2 \binom{2n + 1}{j}^2.

3 Proof of Theorem 1.3

Let \( a_k = \binom{2n+1}{k} q \). Our idea is to rewrite the left-hand side of (1.9) as closely as to its right-hand side. Like Zhang and Wang [16], the computation is a bit long. We will use the binomial theorem and the symmetry \( a_k = a_{2n+1-k} \) several times. It is natural that we will also use (2.6) in what follows.

It is clear that the left-hand side of (1.9) is equal to
\[\sum_{k=0}^{n} \left( \sum_{j=0}^{2k} a_j \right)^3 - \left( \sum_{j=0}^{2k+1} a_j \right)^3.\]
\[= \sum_{k=0}^{n} a_{2k+1} \left( -a_{2k+1}^2 - 3a_{2k+1} \left( \sum_{j=0}^{2k} a_j \right) - 3 \left( \sum_{j=0}^{2k} a_j \right)^2 \right). \quad (3.1)\]

Since \( a_k = a_{2n+1-k} \), the right-hand side of (3.1) is equal to
\[\sum_{k=0}^{n} a_{2k} \left( -a_{2k}^2 - 3a_{2k} \left( \sum_{j=0}^{2n-2k} a_j \right) - 3 \left( \sum_{j=0}^{2n-2k} a_j \right)^2 \right)\]
\[= - \left( \sum_{k=0}^{n} a_{2k} \right) \left( \sum_{j=0}^{2n+1} a_j \right)^2 + \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2n+1} a_j \right)^2 - a_{2k}^2 - 3a_{2k} \left( \sum_{j=0}^{2n-2k} a_j \right) - 3 \left( \sum_{j=0}^{2n-2k} a_j \right)^2. \quad (3.2)\]

Noticing that
\[\left( \sum_{j=0}^{2n+1} a_j \right)^2 = \left( \sum_{j=0}^{2k-1} a_j \right)^2 + a_{2k}^2 + \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2a_{2k} \left( \sum_{j=0}^{2k+1} a_j \right) + 2a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right),\]

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we have

\[
\sum_{k=0}^{n} a_{2k} \left( \left( \sum_{j=0}^{2n+1} a_j \right)^2 - a_{2k}^2 - 3a_{2k} \left( \sum_{j=0}^{2n-2k} a_j \right) - 3 \left( \sum_{j=0}^{2n-2k} a_j \right)^2 \right)
\]

\[
= \sum_{k=0}^{n} a_{2k} \left( \left( \sum_{j=0}^{2k-1} a_j \right)^2 - 2 \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) \right)
\]

\[
- a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right) + 2 \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right)
\]

\[
= \sum_{k=0}^{n} a_{2k} \left( \left( \sum_{j=0}^{2k-1} a_j \right)^2 - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) \right)
\]

\[
+ \sum_{k=0}^{n} a_{2k} \left( a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right) - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2 \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) \right)
\]

\[
= \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j - \sum_{j=2k+1}^{2n+1} a_j \right) \left( \sum_{j=0}^{2n+1} a_j \right)
\]

\[
+ \sum_{k=0}^{n} a_{2k} \left( a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right) - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2 \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) \right) \right). \tag{3.3}
\]

Replacing \( k \) by \( n - k \) and using \( a_k = a_{2n+1-k} \), we have

\[
\sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j - \sum_{j=2k+1}^{2n+1} a_j \right) = -\sum_{k=0}^{n} a_{2k+1} \left( \sum_{j=0}^{2k} a_j - \sum_{j=2k+2}^{2n+1} a_j \right)
\]

\[
= -\left( \sum_{k=0}^{n} (-1)^k a_k^2 + 2 \sum_{0 \leq i < j \leq n} (-1)^i a_i a_j \right) \tag{3.4}
\]

by (2.6).

Combining (3.1)–(3.4), we have proved

\[
\sum_{k=0}^{n} \left( \left( \sum_{j=0}^{2k} a_j \right)^3 - \left( \sum_{j=0}^{2k+1} a_j \right)^3 \right)
\]

\[
= -\sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2n+1} a_j \right)^2 - \left( \sum_{j=0}^{2n+1} a_j \right) \sum_{k=0}^{n} (-1)^k a_k^2 + 2 \sum_{0 \leq i < j \leq n} (-1)^i a_i a_j
\]

\[
+ \sum_{k=0}^{n} a_{2k} \left( a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right) - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2 \left( \sum_{j=0}^{2k-1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) \right). \tag{3.5}
\]

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It remains to prove
\[
\sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j - \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2 \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right)^2 \left( \sum_{j=2k+1}^{2n+1} a_j \right).
\]

By (3.4), the right-hand side of (3.6) is equal to
\[
= \frac{1}{2} \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right)^2 - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right)^2 \left( \sum_{j=2k+1}^{2n+1} a_j \right).
\]

It follows that (3.6) is equivalent to
\[
2 \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j - \sum_{j=2k+1}^{2n+1} a_j \right)^2
= \sum_{k=0}^{n} a_{2k} \left( \sum_{j=0}^{2k-1} a_j \right)^2 - \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + \left( \sum_{j=0}^{2k-1} a_j \right)^2 \left( \sum_{j=2k+1}^{2n+1} a_j \right).
\]

Since \(a_k = a_{2n+1-k}\), the left-hand side of (3.7) may be written as
\[
2 \sum_{k=0}^{\lfloor n/2 \rfloor} a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2 + 2 \sum_{k=\lfloor n/2 \rfloor + 1}^{n} a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2
= 2 \sum_{k=0}^{n} a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right)^2
= \sum_{k=0}^{n} \left( a_k + a_{2n+1-k} \right) \left( \sum_{j=0}^{2n+1-k} a_j \right) \left( \sum_{j=2n+1-k}^{2n+1} a_j \right),
\]

while the right-hand side of (3.7) is equal to
\[
\sum_{k=0}^{n} a_{2k+1} \left( \sum_{j=2k+1}^{2n+1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right) + \sum_{k=0}^{n} a_{2k} \left( \sum_{j=2k+1}^{2n+1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right)
= \sum_{k=0}^{n} a_k \left( \sum_{j=2k+1}^{2n+1} a_j \right) \left( \sum_{j=2k+1}^{2n+1} a_j \right)
\]

It is easy to see that both right-hand sides of (3.8) and (3.9) can be expanded as
\[
\sum_{0 \leq i < j \leq 2n+1} a_i a_j (a_i + a_j) + 2 \sum_{0 \leq i < j < k \leq 2n+1} a_i a_j a_k.
\]
Namely,
\[
\sum_{k=0}^{n}(a_k + a_{2n+1-k}) \left( \sum_{j=k}^{2n-k} a_j \right) \left( \sum_{j=k+1}^{2n+1-k} a_j \right) = \sum_{k=0}^{2n+1} a_k \left( \sum_{j=k}^{2n+1} a_j \right) \left( \sum_{j=k+1}^{2n+1} a_j \right).
\] (3.10)

This completes the proof.

**Remark.** Note that the odd integer \(2n + 1\) in (3.10) can be replaced by any natural number. Indeed, we have
\[
\sum_{k=0}^{[n/2]} (a_k + a_{n-k}) \left( \sum_{j=k}^{n-k-1} a_j \right) \left( \sum_{j=k+1}^{n-k} a_j \right) = \sum_{k=0}^{n} a_k \left( \sum_{j=k}^{n} a_j \right) \left( \sum_{j=k+1}^{n} a_j \right).
\] (3.11)

Letting \(a_k = q^k\) \((0 \leq k \leq n)\) in (3.11), we obtain
\[
\sum_{k=0}^{[n/2]} q^{3k}(1 - q^{2n-4k})(1 - q^{n-k}) = \sum_{k=0}^{n} q^{3k}(1 - q^{n-k})(1 - q^{n-k+1})
\] (3.12)

Replacing \(k\) by \(n - k\), it is easy to see that the right-hand side of (3.12) is equal to
\[
\frac{(1 - q^n)(1 - q^{n+1})(1 - q^{n+2})}{(1 - q^2)}
\]
(by induction on \(n\)). Thus, from (3.12) we deduce that
\[
\sum_{k=0}^{[n/2]} q^{3k}(1 - q^{2n-4k})(1 - q^{n-k}) = \frac{(1 - q^n)(1 - q^{n+1})(1 - q^{n+2})}{(1 - q^2)}.
\] (3.13)

Substituting \(n \rightarrow 2n\), \(k \rightarrow n - k\), \(q \rightarrow q^{1/2}\) into (3.13) and dividing both sides by \((1 - q)(1 - q^2)\), we get
\[
\sum_{k=1}^{n} q^{3(n-k)/2} \frac{(1 - q^{2k})(1 - q^{k})}{(1 - q^2)(1 - q)} = \frac{(1 - q^n)(1 - q^{n+1/2})(1 - q^{n+1})}{(1 - q)(1 - q^{3/2})},
\] (3.14)

which is a \(q\)-analogue of the sum of the first \(n\) squares:
\[
\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

On the other hand, substituting \(n \rightarrow 2n - 1\), \(k \rightarrow n - k\), \(q \rightarrow q^{1/2}\) into (3.13) and dividing both sides by \((1 - q)(1 - q^{1/2})\), we get
\[
\sum_{k=1}^{n} q^{3(n-k)/2} \frac{(1 - q^{2k-1})(1 - q^{k-1/2})}{(1 - q)(1 - q^{1/2})} = \frac{(1 - q^n)(1 - q^{n-1/2})(1 - q^{n+1/2})}{(1 - q)(1 - q^{3/2})},
\] (3.15)

which is a \(q\)-analogue of the sum of odd squares:
\[
\sum_{k=1}^{n} (2k - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}.
\] 9
Both (3.14) and (3.15) were first obtained by Schlosser [8].

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References