New q-supercongruences from Rahman's and Gasper and Rahman's transformations

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Abstract. We give two q-supercongruences. One is modulo the fifth power of a cyclotomic polynomial, and the other is a q-analogue of the supercongruence: for odd primes p,

$$\sum_{k=0}^{p-1} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p \pmod{p^4},$$

which was first proved by Guillera and Zudilin in the modulus p^3 case. Our proof employs Rahman's and Gasper and Rahman's quadratic transformations, the creative microscoping method devised by the first author in joint work with Zudilin, along with the Chinese remainder theorem for polynomials.

Keywords: q-supercongruences; creative microscoping; Rahman's transformation; Gasper and Rahman's transformation; Chinese remainder theorem for polynomials

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1. Introduction

In 2012, Guillera and Zudilin [3] established the following two supercongruences: for odd primes p,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)4^k \equiv p \pmod{p^3}, \tag{1.1}$$

$$\sum_{k=0}^{(p-1)/2} (-1)^k \frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)8^k \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.2}$$

where the *Pochhammer symbol* is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geqslant 1$. Clearly, we may also sum over k up to $p-1$ on the left-hand sides of (1.1) and (1.2), since the p-adic order of $(\frac{1}{2})_k/k!$ is 1 for k in the range $(p+1)/2 \leqslant k \leqslant p-1$.

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The first author [4] gave the following q-analogues of (1.1) and (1.2): for any positive integer n,

$$\sum_{k=0}^{n-1}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k} \equiv [n]q^{(1-n)/2} \pmod{[n]\Phi_n(q)^2}, \tag{1.3}$$

$$\sum_{k=0}^{n-1}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv [n](-q)^{(n-1)^2/4} \pmod{[n]\Phi_n(q)^2}. \tag{1.4}$$

Here and in what follows, the q-*integer* is defined as $[n] = [n]_q = (1-q^n)/(1-q)$, and the q-*shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$. For simplicity, we also use the abbreviated notation $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$ for products of shifted factorials. Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ can be written as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. We know that the polynomial $\Phi_n(q)$ is irreducible over the integers, and $\Phi_{p^r}(q) = [p]_{q^{p^{r-1}}}$ for all primes p and positive integers r.

In 2019, the first author and Zudilin [10] devised a method called "creative microscoping" to give a new proof of (1.3) and (1.4). In addition, they also obtained the following q-supercongruence: for $n \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}[3k+1]_{q^2}\frac{(q;q^2)_k^2(q^2;q^4)_k^3 q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k(q^5;q^4)_k^2} \equiv 0 \pmod{\Phi_n(q)^3}. \tag{1.5}$$

In particular, for primes $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}(3k+1)\frac{(\frac{1}{2})_k^5}{(1)_k^3(\frac{5}{4})_k^2} \equiv 0 \pmod{p^3}. \tag{1.6}$$

For more recent work on q-supercongruences, see [1, 9, 13, 17–19].

In this paper, we shall give the following generalization of (1.5).

Theorem 1.1. *Let n be a positive integer with $n \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{(n-1)/2}[3k+1]_{q^2}\frac{(q;q^2)_k^2(q^2;q^4)_k^3 q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k(q^5;q^4)_k^2} \equiv \frac{(q^3;q^4)_{(n-1)/2}^2}{(q^5;q^4)_{(n-1)/2}^2}[n]_{q^2} \pmod{\Phi_n(q)^5}. \tag{1.7}$$

Letting $n = p^r$ be a prime power and taking the limits as $q \to 1$ in (1.7), we obtain the following supercongruence, which is also a generalization of (1.6).

Corollary 1.2. *Let $p \equiv 3 \pmod 4$ be a prime and r a positive odd integer. Then*

$$\sum_{k=0}^{(p^r-1)/2} (6k+1)\frac{(\frac{1}{2})_k^5}{(1)_k^3(\frac{5}{4})_k^2} \equiv \frac{(\frac{3}{4})_{(p^r-1)/2}^2}{(\frac{5}{4})_{(p^r-1)/2}^2} p^r \pmod{p^5}.$$

Employing the method of creative microscoping together with the Chinese remainder theorem for polynomials, the first author [5] gave the following generalization of (1.3): for positive integers n, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{n-1}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k} \equiv [n]q^{(1-n)/2}\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \tag{1.8}$$

For a prime power $n = p^r$ with $p > 3$, the limiting case $q \to 1$ of the above q-supercongruence reduces to

$$\sum_{k=0}^{p^r-1} \frac{(\frac{1}{2})_k^3}{k!^3}(3k+1)4^k \equiv p^r \pmod{p^{r+3}}. \tag{1.9}$$

A refinement of (1.9) modulo p^{r+4} was conjectured by Sun [14, Conjecture 5.1(ii)], and has been confirmed by Wang and Hu [16].

It is known that supercongruences may have different q-analogues. For instance, the choice $q \to q^2$, $b = -q$ and $a = 1$ of [10, Theorem 4.8] yields that, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(-q;q^2)_k^2(q^2;q^4)_k^3 q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k(-q^5;q^4)_k^2} \equiv \frac{(-q^3;q^4)_{(n-1)/2}^2}{(-q^5;q^4)_{(n-1)/2}^2}[n]_{q^2}, \tag{1.10}$$

which is another q-analogue of (1.1). In this paper, we shall give the following generalization of (1.10).

Theorem 1.3. *Let n be a positive odd integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(-q;q^2)_k^2(q^2;q^4)_k^3 q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k(-q^5;q^4)_k^2}$$
$$\equiv \frac{(-q^3;q^4)_{(n-1)/2}^2}{(-q^5;q^4)_{(n-1)/2}^2}[n]_{q^2}\left\{1-(1-q^{2n})^2\left(\frac{1-n^2}{24}+\sum_{k=0}^{(n-1)/2}\frac{q^{4k-1}}{(1+q^{4k-1})^2}\right)\right\}. \tag{1.11}$$

Letting $n = p^r$ be a prime power and taking the limits as $q \to 1$ in (1.11), we get the supercongruence (1.9) again.

The rest of the paper is organized as follows. In the next section, we give the definition of basic hypergeometric series and state Rahman's and Gasper and Rahman's quadratic transformations. We shall prove Theorems 1.1 and 1.3 in Sections 3 and 4, respectively, by making use of the creative microscoping method. In Section 5, we propose five related conjectures for further study.

2. Preliminaries

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ involving $r+1$ upper parameters $a_1, \dots, a_{r+1}$, r lower parameters $b_1, \dots, b_r$, base q, and argument z can be defined as

$$
{}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, \dots, b_r; q)_k}.
$$

Then a quadratic transformation of Rahman [2, eq. (3.8.13)] can be stated as follows:

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(1-aq^{3k})(a, d, aq/d; q^2)_k (b, c, aq/bc; q)_k}{(1-a)(q, aq/d, d; q)_k (aq^2/b, aq^2/c, bcq; q^2)_k} q^k \\
&\quad = \frac{(aq^2, bq, cq, aq^2/bc; q^2)_\infty}{(q, aq^2/b, aq^2/c, bcq; q^2)_\infty} {}_3\phi_2\left[\begin{matrix} b, & c, & aq/bc \\ & dq, & aq^2/d \end{matrix}; q^2, q^2\right], \qquad (2.1)
\end{aligned}
$$

and a quadratic transformation of Gasper and Rahman (see [2, (3.8.14)]) can be written as

$$
\begin{aligned}
&\sum_{k=0}^{n} \frac{1-acq^{3k}}{1-ac} \frac{(a, b, cq/b; q)_k (f, a^2c^2q^{2n+1}/f, q^{-2n}; q^2)_k}{(cq^2, acq^2/b, abq; q^2)_k (acq/f, f/acq^{2n}, acq^{2n+1}; q)_k} q^k \\
&\quad = \frac{(acq; q)_{2n} (ac^2q^2/bf, abq/f; q^2)_n}{(acq/f; q)_{2n} (abq, ac^2q^2/b; q^2)_n} \\
&\qquad \times {}_{10}W_9(ac^2/b; f, ac/b, c, cq/b, cq^2/b, a^2c^2q^{2n+1}/f, q^{-2n}; q^2, q^2), \qquad (2.2)
\end{aligned}
$$

where

$$
{}_{r+3}W_{r+2}(a_0; a_1, a_2, ..., a_r; q, z) = \sum_{k=0}^{\infty} \frac{(1-a_0q^{2k})(a_0, a_1, \dots, a_r; q)_k z^k}{(1-a_0)(q, a_0q/a_1, \dots, a_0q/a_r; q)_k}.
$$

3. Proof of Theorem 1.1

The following result is a common generalization of (1.3) and (1.4). By making use of Gasper and Rahman's transformation (2.2), the first author and Zudilin [10, Theorem 4.7] proved that it holds modulo $(1-aq^n)(a-q^n)$, while employing Rahman's transformation (2.1), the first author and Schlosser [7, Theorem 6.1] confirmed the truth of it modulo $[n]$.

Lemma 3.1. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$
\sum_{k=0}^{N} [3k+1] \frac{(aq, q/a, q; q^2)_k (q/b, q/c, bc; q)_k}{(aq, q/a, q; q)_k (bq^2, cq^2, q^3/bc; q^2)_k} q^k \equiv \frac{(bcq, q^2/b, q^2/c; q^2)_{(n-1)/2}}{(q^3/bc, bq^2, cq^2; q^2)_{(n-1)/2}} [n], \qquad (3.1)
$$

where $N = n-1$ or $(n-1)/2$.

We now present a generalization of Theorem 1.1 with an additional parameter a.

Theorem 3.2. *Let n be a positive integer with $n \equiv 3 \pmod 4$. Then, modulo $[n]_{q^2}(1-aq^{2n})(a-q^{2n})(1-bq^n)(b-q^n)$,*

$$\sum_{k=0}^{(n-1)/2}[3k+1]_{q^2}\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(bq;q^2)_k(q/b;q^2)_k(q^2;q^4)_k q^{2k}}{(aq^2;q^2)_k(q^2/a;q^2)_k(bq^5;q^4)_k(q^5/b;q^4)_k(q^4;q^4)_k}$$
$$\equiv \frac{(bq^3,q^3/b;q^4)_{(n-1)/2}}{(bq^5,q^5/b;q^4)_{(n-1)/2}}[n]_{q^2}. \tag{3.2}$$

Proof. It is easy to see that the modulus $[n]_{q^2}(1-aq^{2n})(a-q^{2n})$ case of q-congruence (3.2) follows from (3.1) by performing the parameter substitutions $N=(n-1)/2$, $q\mapsto q^2$, $b\mapsto bq$, and $c=1$.

In what follows, we shall prove the validity of (3.2) modulo $(1-bq^n)(b-q^n)$. In other words, we need to show that the two sides of (3.2) are equal for $b=q^n$ or $b=q^{-n}$. To this end, taking $a=q^2$, $d=aq^2$, $b=q^{1+n}$, $c=q^{1-n}$, and $q\mapsto q^2$ in Rahman's transformation (2.1), we obtain

$$\sum_{k=0}^{(n-1)/2}[3k+1]_{q^2}\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^{1+n};q^2)_k(q^{1-n};q^2)_k(q^2;q^4)_k(q^2;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^{5+n};q^4)_k(q^{5-n};q^4)_k(q^2;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$=\frac{(q^6,q^{3+n},q^{3-n},q^4;q^4)_\infty}{(q^2,q^{5-n},q^{5+n},q^4;q^4)_\infty}\,{}_3\phi_2\left[\begin{matrix}q^{1+n}, & q^{1-n}, & q^2\\ & aq^4, & q^4/a\end{matrix};q^4,q^4\right]$$
$$=0, \tag{3.3}$$

where we have used $(q^{1-n};q^2)_k=0$ for $k>(n+1)/2$, and $(q^{3-n};q^4)_\infty=0$. It is easy to see that

$$\frac{(q^{3+n},q^{3-n};q^4)_{(n-1)/2}}{(q^{5+n},q^{5-n};q^4)_{(n-1)/2}}[n]_{q^2}=0.$$

This proves that the q-congruence (3.2) holds modulo $1-bq^n$ and $b-q^n$.

Since $[n]_{q^2}(1-aq^{2n})(a-q^{2n})$, $1-bq^n$, and $b-q^n$ are pairwise coprime polynomials in q, we complete the proof of the theorem. □

Proof of Theorem 1.1. When $a=b=1$ the polynomial $(1-aq^{2n})(a-q^{2n})(1-bq^n)(b-q^n)=(1-q^{2n})^2(1-q^n)^2$ has the factor $\Phi_n(q)^4$, which is coprime with the denominators on both sides of (3.2). Thus, putting $a=b=1$ in (3.2), we are led to the desired q-supercongruence (1.7). □

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need three lemmas.

Lemma 4.1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=1}^{(n-1)/2}\frac{q^k}{[k]^2}\equiv\frac{1-n^2}{24}(1-q)^2 \pmod{\Phi_n(q)}. \tag{4.1}$$

Proof. Using the same technique in the proof of [12, Lemma 2], the first author [4, Lemma 2.5] established the following result:

$$\sum_{k=1}^{n-1}\frac{q^k}{[k]^2}\equiv\frac{1-n^2}{12}(1-q)^2\pmod{\Phi_n(q)}.$$

It is easy to check that, for $1\leqslant k\leqslant n-1$,

$$\frac{q^{n-k}}{[n-k]^2}\equiv\frac{q^k}{[k]^2}\pmod{\Phi_n(q)}.$$

This implies that

$$2\sum_{k=1}^{(n-1)/2}\frac{q^k}{[k]^2}\equiv\sum_{k=1}^{n-1}\frac{q^k}{[k]^2}\pmod{\Phi_n(q)},$$

and so the q-congruence (4.1) holds. □

Lemma 4.2. *Let n be a positive odd integer. Then, modulo $[n](1-aq^{2n})(a-q^{2n})$,*

$$\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(aq^2,q^2/a,q^2;q^4)_k(-q,-bq,q^2/b;q^2)_kq^{2k}}{(aq^2,q^2/a,q^2;q^2)_k(-q^5,-q^5/b,bq^4;q^4)_k}$$
$$\equiv\frac{(-bq^3,q^4/b,-q^3;q^4)_{(n-1)/2}}{(-q^5/b,bq^4,-q^5;q^4)_{(n-1)/2}}[n]_{q^2}. \tag{4.2}$$

Proof. Letting $N=n-1$, $q\mapsto q^2$, and $c=-q$ in (3.1), we arrive at (4.2). □

Lemma 4.3. *Let n be a positive odd integer. Then, modulo $b-q^{2n}$,*

$$\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(aq^2,q^2/a,q^2;q^4)_k(-q,-bq,q^2/b;q^2)_kq^{2k}}{(aq^2,q^2/a,q^2;q^2)_k(-q^5,-q^5/b,bq^4;q^4)_k}$$
$$\equiv\frac{(q^4/b,q^4,-aq^3,-q^3/a;q^4)_{(n-1)/2}}{(-q^5,-q^5/b,aq^4,q^4/a;q^4)_{(n-1)/2}}[n]_{q^2}. \tag{4.3}$$

Proof. For $b=q^{2n}$, the left-hand side of (4.3) is equal to

$$\sum_{k=0}^{n-1}\frac{1-q^{6k+2}}{1-q^2}\frac{(q^2,aq^2,q^2/a;q^4)_k(-q,-q^{2n+1},q^{2-2n};q^2)_k}{(q^2,q^2/a,aq^2;q^2)_k(-q^5,-q^{5-2n},q^{2n+4};q^4)_k}q^{2k}. \tag{4.4}$$

Taking $a=q^2$, $d=aq^2$, $b=-q$, $c=-q^{2n+1}$, and $q=q^2$ in Rahman's transformation (2.1), we can write (4.4) as

$$\frac{(q^6,-q^3,-q^{2n+3},q^{4-2n};q^4)_\infty}{(q^2,-q^5,-q^{5-2n},q^4;q^4)_\infty}{}_3\phi_2\left[\begin{matrix}-q, & -q^{2n+1}, & q^{2-2n}\\ & aq^4, & q^4/a\end{matrix};q^4,q^4\right]$$
$$=\frac{(q^6,-q^3,-q^{2n+3},q^{4-2n};q^4)_\infty}{(q^2,-q^5,-q^{5-2n},q^4;q^4)_\infty}\frac{(-aq^3,-aq^{3-2n};q^4)_{(n-1)/2}}{(aq^4,aq^{2-2n};q^4)_{(n-1)/2}}$$
$$=\frac{(q^{4-2n},q^4,-aq^3,-q^3/a;q^4)_{(n-1)/2}}{(-q^5,-q^{5-2n},aq^4,q^4/a;q^4)_{(n-1)/2}}[n]_{q^2}, \tag{4.5}$$

where we have used the q-Pfaff–Saalschütz summation (see [2, Appendix (II.12)]):

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, a, b \\ c, q^{1-n}ab/c \end{matrix}; q,\, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.
$$

The right-hand side of (4.5) is just the $b = q^{2n}$ case of that of (4.3). This proves the q-congruence (4.3). □

Proof of Theorem 1.3. Since $[n](1-aq^{2n})(a-q^{2n})$ and $b-q^{2n}$ are coprime polynomials, we may determine the remainder of the left-hand side of (4.2) modulo $[n](1-aq^{2n})(a-q^{2n})(b-q^{2n})$ from (4.2) and (4.3) by the Chinese reminder theorem for polynomials. In fact, using the following two q-congruences:

$$
\begin{aligned}
\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)} &\equiv 1 \pmod{(1-aq^{2n})(a-q^{2n})}, \\
\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)} &\equiv 1 \pmod{b-q^{2n}}.
\end{aligned}
$$

we deduce from (4.2) and (4.3) that, modulo $[n](1-aq^{2n})(a-q^{2n})(b-q^{2n})$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(aq^2, q^2/a, q^2; q^4)_k(-q, -bq, q^2/b; q^2)_k q^{2k}}{(aq^2, q^2/a, q^2; q^2)_k(-q^5, -q^5/b, bq^4; q^4)_k} \\
&\quad\equiv \frac{(-bq^3, q^4/b, -q^3; q^4)_{(n-1)/2}}{(-q^5/b, bq^4, -q^5; q^4)_{(n-1)/2}}\frac{(b-q^{2n})(ab-1-a^2+aq^{2n})}{(a-b)(1-ab)}[n]_{q^2} \\
&\qquad+\frac{(q^4/b, q^4, -aq^3, -q^3/a; q^4)_{(n-1)/2}}{(-q^5, -q^5/b, aq^4, q^4/a; q^4)_{(n-1)/2}}\frac{(1-aq^{2n})(a-q^{2n})}{(a-b)(1-ab)}[n]_{q^2}.
\end{aligned}\tag{4.6}
$$

Letting $b=1$ in (4.6) and using the following identity:

$$
(1-q^{2n})(1+a^2-a-aq^{2n}) = (1-a)^2 + (1-aq^{2n})(a-q^{2n}),
$$

we acquire the following q-congruence: modulo $\Phi_n(q)^2(1-aq^{2n})(a-q^{2n})$,

$$
\begin{aligned}
&\sum_{k=0}^{n-1}[3k+1]_{q^2}\frac{(aq^2, q^2/a, q^2; q^4)_k(-q, -q, q^2; q^2)_k q^{2k}}{(aq^2, q^2/a, q^2; q^2)_k(-q^5, -q^5, q^4; q^4)_k} \\
&\quad\equiv \frac{(-q^3; q^4)_{(n-1)/2}^2}{(-q^5; q^4)_{(n-1)/2}^2}[n]_{q^2} + [n]_{q^2}\frac{(q^4, q^4)_{(n-1)/2}^2}{(-q^5, q^4)_{(n-1)/2}^2} \\
&\qquad\times\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\left\{\frac{(-q^3, q^4)_{(n-1)/2}^2}{(q^4, q^4)_{(n-1)/2}^2} - \frac{(-aq^3, -q^3/a; q^4)_{(n-1)/2}}{(aq^4, q^4/a; q^4)_{(n-1)/2}}\right\}.
\end{aligned}\tag{4.7}
$$

By L'Hôpital's rule, we have

$$\lim_{a\to 1}\frac{1}{(1-a)^2}\left\{\frac{(-q^3,q^4)^2_{(n-1)/2}}{(q^4,q^4)^2_{(n-1)/2}}-\frac{(-aq^3,-q^3/a;q^4)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}}\right\}$$
$$=-\frac{(-q^3,q^4)^2_{(n-1)/2}}{(q^4,q^4)^2_{(n-1)/2}}\sum_{k=0}^{(n-1)/2}\left\{\frac{q^{4k-1}}{(1+q^{4k-1})^2}+\frac{q^{4k}}{(1-q^{4k})^2}\right\}.$$

In view of Lemma 4.1, there holds

$$\sum_{k=0}^{(n-1)/2}\frac{q^{4k}}{(1-q^{4k})^2}\equiv\frac{1-n^2}{24}\pmod{\Phi_n(q)}.$$

Hence, taking the limits of the two sides of (4.7) as $a\to 1$, and using the above q-congruence, we conclude that (1.11) holds modulo $\Phi_n(q)^4$. But the proof of [7, Theorem 6.1] indicates that it also holds modulo $[n]$. The proof then follows from the fact that the least common multiple of the polynomials $\Phi_n(q)^4$ and $[n]$ is just $[n]\Phi_n(q)^3$. □

5. Concluding remarks and open problems

As the reader might see, the method of creative microscoping is very useful in dealing with q-supercongruences. Nevertheless, there are still many q-supercongruences that cannot be settled by this method. We believe that the following conjecture on generalizations of (1.5) modulo $\Phi_n(q)^2$ is such an example.

Conjecture 5.1. *Let m and n be positive integers with $n\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{mn-1}[3k+1]_{q^2}\frac{(q;q^2)_k^2(q^2;q^4)_k^3}{(q^2;q^2)_k^2(q^4;q^4)_k(q^5;q^4)_k^2}q^{2k}\equiv 0\pmod{\Phi_n(q)^2},\tag{5.1}$$
$$\sum_{k=0}^{mn+(n-1)/2}[3k+1]_{q^2}\frac{(q;q^2)_k^2(q^2;q^4)_k^3}{(q^2;q^2)_k^2(q^4;q^4)_k(q^5;q^4)_k^2}q^{2k}\equiv 0\pmod{\Phi_n(q)^2}.\tag{5.2}$$

For more similar conjectures, we refer the reader to [6].

On the basis of numerical computation, we believe that the following supercongruence related to (1.6) should be true.

Conjecture 5.2. *Let $p\equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{(p^2-1)/2}(3k+1)\frac{(\frac{1}{2})_k^5}{(1)_k^3(\frac{5}{4})_k^2}\equiv p^2\pmod{p^6}.\tag{5.3}$$

Perhaps, in order to prove (5.3), we need to establish a q-analogue of it. However, we did not find any q-analogue of the supercongruence (5.3) even in the modulus p^3 case.

Motivated by Swisher's work [15], the first author [4] conjectured that, for any odd prime p and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)4^k\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}\frac{(\frac12)_k^3}{k!^3}(3k+1)4^k \pmod{p^{3r}}, \tag{5.4}$$

$$\sum_{k=0}^{p^r-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)4^k\equiv p\sum_{k=0}^{p^{r-1}-1}\frac{(\frac12)_k^3}{k!^3}(3k+1)4^k \pmod{p^{4r-\delta_{p,3}}}, \tag{5.5}$$

where δ is the Kronecker delta with $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise. Supercongruences of the form (5.4) or (5.5) are called Dwork-type supercongruences. The supercongruence (5.5) still remains open so far. The first author and Zudilin [11, Theorems 1.2 and 3.1] have proved (5.4) and (5.5) modulo p^{3r} by establishing q-analogues of them. Numerical computation implies that the following new q-analogue of (5.4) and (5.5) modulo p^{3r}, which is also a generalization of (1.10), should be true.

Conjecture 5.3. *Let $n>1$ be an odd integer and let $r\geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r}\Phi_{n^j}(q)^2$,*

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/d}[3k+1]_{q^2}\frac{(-q;q^2)_k^2(q^2;q^4)_k^3}{(q^2;q^2)_k^2(q^4;q^4)_k(-q^5;q^4)_k^2}q^{2k}\\
&\quad\equiv\frac{(-q^3;q^4)_{(n^r-1)/2}^2(-q^{5n};q^{4n})_{(n^{r-1}-1)/2}}{(-q^5;q^4)_{(n^r-1)/2}^2(-q^{3n};q^{4n})_{(n^{r-1}-1)/2}}[n]_{q^2}\\
&\quad\quad\times\sum_{k=0}^{(n^{r-1}-1)/d}[3k+1]_{q^{2n}}\frac{(-q^n;q^{2n})_k^2(q^{2n};q^{4n})_k^3}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k(-q^{5n};q^{4n})_k^2}q^{2nk},
\end{aligned}$$

where $d=1,2$.

We believe that the q-supercongruence (1.8) can be extended as follows.

Conjecture 5.4. *Let m and n be positive integers with n odd. Then, modulo $\Phi_n(q)^4$,*

$$\begin{aligned}
\sum_{k=0}^{mn-1}[3k+1]\frac{(q;q^2)_k^3q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k}&\equiv[n]q^{(1-n)/2}\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}\\
&\quad\times\sum_{k=0}^{m-1}[3k+1]_{q^{n^2}}\frac{(q^{n^2};q^{2n^2})_k^3q^{-\binom{k+1}{2}n^2}}{(q^{n^2};q^{n^2})_k^2(q^{2n^2};q^{2n^2})_k}.
\end{aligned}$$

Similarly, the q-supercongruence (1.11) modulo $\Phi_n(q)^4$ should possess the following extension too.

Conjecture 5.5. *Let m and n be positive integers with n odd. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=0}^{mn-1}[3k+1]_{q^2}\frac{(-q;q^2)_k^2(q^2;q^4)_k^3}{(q^2;q^2)_k^2(q^4;q^4)_k(-q^5;q^4)_k^2}q^{2k}$$
$$\equiv \frac{(-q^3;q^4)_{(n-1)/2}^2}{(-q^5;q^4)_{(n-1)/2}^2}[n]_{q^2}\left\{1-(1-q^{2n})^2\left(\frac{1-n^2}{24}+\sum_{k=1}^{(n-1)/2}\frac{q^{4k-1}}{(1+q^{4k-1})^2}\right)\right\}$$
$$\times\sum_{k=0}^{m-1}[3k+1]_{q^{2n^2}}\frac{(-q^{n^2};q^{2n^2})_k^2(q^{2n^2};q^{4n^2})_k^3}{(q^{2n^2};q^{2n^2})_k^2(q^{4n^2};q^{4n^2})_k(-q^{5n^2};q^{4n^2})_k^2}q^{2n^2k}.$$

It would be very interesting if someone can confirm or make progress on any of these five conjectures in this section.

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6. Declarations

Conflicts of interest. The authors declare no conflict of interest.

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