Some new q-supercongruences arising from a quadratic summation of Gasper and Rahman

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Abstract. Employing Gasper and Rahman's quadratic summation, the method of "creative microscoping" developed by the first author and Zudilin, and the Chinese remainder theorem for coprime polynomials, we prove some new q-supercongruences modulo the third and fourth powers of a cyclotomic polynomial. As a conclusion, we obtain some new supercongruences, such as: for primes $p \geqslant 13$ with $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(3p+1)/4} (6k+1)\frac{(\frac{1}{2})_k^3(-\frac{1}{4})_k}{(k+1)!k!^3 4^k} \equiv 0 \pmod{p^4},$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geqslant 1$.

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1 Introduction

In 1914, Ramanujan [16] discovered some series for $1/\pi$ (see also [1, p. 352]), such as

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k} = \frac{4}{\pi}, \tag{1.1}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the *Pochhammer symbol*. The formula (1.1) was first proved by J.M. Borwein and P.B. Borwein in their monograph [2, pp. 177–187]. In 1997, Van Hamme [17] proposed 13 p-adic analogues of Ramanujan-type series. For example, he conjectured that the following supercongruence holds for primes $p > 3$:

$$\sum_{k=0}^{(p-1)/2} (6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k} \equiv (-1)^{(p-1)/2} p \pmod{p^4},$$

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which was later confirmed by Long [14]. In 2017, He [10] gave the following two similar supercongruences: for any odd prime p,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k}{k!^4 4^k} \equiv \begin{cases} (-1)^{(p+3)/4}p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2 \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p\equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

where $\Gamma_p(x)$ is Morita's *p-adic Gamma function* [15], and conjectured that (1.2) also holds modulo p^4 for $p \equiv 3 \pmod 4$. Liu and Wang [13] observed that the supercongruence (1.2) modulo p^3 is a consequence of the following q-supercongruence: for any positive odd integer n, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^3(q;q^4)_k q^{k^2+k}}{(q^2;q^2)_k(q^4;q^4)_k^3} \equiv \begin{cases} \dfrac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4} & \text{if } n\equiv 1 \pmod 4, \\ 0 & \text{if } n\equiv 3 \pmod 4, \end{cases}$$

which is just the $(a,b)=(1,q)$ case of [8, Theorem 4.5]. Applying the "creative microscoping" method introduced by the first author and Zudilin [8], Wei [18] extended the above q-supercongruence to the modulus $[n]\Phi_n(q)^3$, thus confirming He's conjecture. Here and in what follows, for all complex numbers x, q and nonnegative integers n, the *q-shifted factorials* are defined as

$$(x;q)_\infty=\prod_{k=1}^{\infty}(1-xq^k), \quad \text{and} \quad (x;q)_n=\frac{(x;q)_\infty}{(xq^n;q)_\infty}.$$

For simplicity, we shall also adopt the compact notation $(x_1,\dots,x_m;q)_n=(x_1;q)_n\cdots(x_m;q)_n$ for $n=0,1,2,\dots,$ or $n=\infty$. Moreover, let $[n]=(1-q^n)/(1-q)$ be the q-integer, and let $\Phi_n(q)$ denote the n-th cyclotomic polynomial, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

Inspired by the aforementioned work, we shall establish the following q-supercongruence.

Theorem 1.1. *Let n be an integer with $n\equiv 1 \pmod 4$ and $n\geqslant 9$. Then*

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^2)_k(q^4;q^4)_k^3}q^{k^2+3k} \equiv 0 \pmod{[n]\Phi_n(q)^3}. \tag{1.3}$$

Note that the q-supercongruence (1.3) modulo $[n]\Phi_n(q)^2$ follows from the $(a,b)=(1,q^{-1})$ case of [8, Theorem 4.5]. For n prime, letting $q\to 1$ in (1.3), we arrive at the following result: for any prime $p\geqslant 13$ with $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(3p+1)/4}(6k+1)\frac{(\frac{1}{2})_k^3(-\frac{1}{4})_k}{(k+1)!k!^3 4^k} \equiv 0 \pmod{p^4}.$$

We shall also give a variation of Theorem 1.1 as follows.

Theorem 1.2. *Let n be an integer with $n \equiv 1 \pmod 4$ and $n \geqslant 5$. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^2)_k(q^4;q^4)_k^3}q^{k^2+3k} \equiv -\frac{[n]^3[3n](1+q)}{[2n]}q^{(9n^2-5n)/4+1}. \tag{1.4}$$

Similarly as before, we can deduce the following conclusion from (1.4): for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(-\frac{1}{4})_k}{(k+1)!k!^3 4^k} \equiv -3p^3 \pmod{p^4}. \tag{1.5}$$

We have the following q-supercongruence, which is a companion of (1.3).

Theorem 1.3. *Let n be a positive integer with $n \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^2)_k(q^4;q^4)_k^3}q^{k^2+3k} \equiv -[n]q^{(n+5)/4}\frac{(q^{-2};q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}\left(1-[n]^2\sum_{j=1}^{(n+1)/4}\frac{q^{4j}}{[4j]^2}\right) \pmod{[n]\Phi_n(q)^3}. \tag{1.6}$$

For n prime, letting $q \to 1$ in (1.6), we get the following conclusion: for any prime $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3(-\frac{1}{4})_k}{(k+1)!k!^3 4^k} \equiv -p\frac{(-\frac{1}{2})_{(p+1)/4}}{(1)_{(p+1)/4}}\left(1-\frac{p^2}{16}\sum_{j=1}^{(p+1)/4}\frac{1}{j^2}\right) \pmod{p^4}. \tag{1.7}$$

It seems that the following generalization of (1.5) modulo p^3 is true.

Conjecture 1.4. *Let $p \equiv 1 \pmod 4$ be a prime and $r \geqslant 1$. Then*

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac{1}{2})_k^3(-\frac{1}{4})_k}{(k+1)!k!^3 4^k} \equiv 0 \pmod{p^{3r}}. \tag{1.8}$$

The supercongruence (1.8) may be regarded as a reduced Dwork-type supercongruence (see [3]). A number of Dwork-type supercongruences are proved in [5, 9] by an upgraded version of the creative microscoping method. Perhaps the method therein can be utilized to tackle the above conjecture.

He [10] also proved the following supercongruence:

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{2})_k^3(\frac{1}{4})_k^2}{k!^5} \equiv \begin{cases} -p\Gamma_p(\frac{1}{4})^4 \pmod{p^2}, & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.9}$$

and conjectured that it is also true modulo p^3 for $p \equiv 3 \pmod 4$. Liu [12] confirmed this conjecture of He. Wei [18] further generalized He's supercongruence (1.9) to the modulus p^5 case by establishing a q-analogue of it.

Motivated by Wei's work, we shall build the following q-supercongruences.

Theorem 1.5. *Let n be an integer with $n \equiv 1 \pmod 4$ and $n \geqslant 9$. Then*

$$\sum_{k=0}^{(3n+1)/4} [6k+1] \frac{(q;q^2)_k^3 (q^{-1},q;q^4)_k}{(q^4,q^2,q^{-1};q^2)_k (q^4;q^4)_k^2} q^{2k} \equiv 0 \pmod{[n]\Phi_n(q)^2}. \tag{1.10}$$

Likewise, we may derive the following supercongruence from (1.10): for any prime $p \geqslant 13$ with $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(3p+1)/4} (6k+1) \frac{(\frac{1}{2})_k^3 (-\frac{1}{4})_k (\frac{1}{4})_k}{(k+1)!k!^3(-\frac{1}{2})_k} \equiv 0 \pmod{p^3}.$$

We believe that the following extension of Theorem 1.5 should be true.

Conjecture 1.6. *Let n be an integer with $n \equiv 1 \pmod 4$ and $n \geqslant 9$. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{(3n+1)/4} [6k+1] \frac{(q;q^2)_k^3 (q^{-1},q;q^4)_k}{(q^4,q^2,q^{-1};q^2)_k (q^4;q^4)_k^2} q^{2k} \equiv 0, \tag{1.11}$$

$$\sum_{k=0}^{n-1} [6k+1] \frac{(q;q^2)_k^3 (q^{-1},q;q^4)_k}{(q^4,q^2,q^{-1};q^2)_k (q^4;q^4)_k^2} q^{2k} \equiv -\frac{[n]^3[3n](1+q)}{[2n]} q^{(n^2-n)/2+1}. \tag{1.12}$$

Furthermore, can we generalize the q-supercongruences (1.11) and (1.12) to the modulus $[n]\Phi_n(q)^4$ case? Note that there exists a q-supercongruence modulo $[n]\Phi_n(q)^4$ in [18], which is also deduced from Gasper and Rahman's summation (1.13).

Recall that a quadratic summation of Gasper and Rahman (see [4, eq. (3.8.12)]) can be stated as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a,b,q/b;q)_k (d,f,a^2q/df;q^2)_k q^k}{(aq/d,aq/f,df/a;q)_k (q^2,aq^2/b,abq;q^2)_k} \\
&\quad + \frac{(aq,f/a,b,q/b;q)_\infty (d,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty (aq^2/b,abq,fq/ab,bf/a;q^2)_\infty} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(f,bf/a,fq/ab;q^2)_k q^{2k}}{(q^2,fq^2/d,df^2q/a^2;q^2)_k} \\
&\quad = \frac{(aq,f/a;q)_\infty (aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty (aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}.
\end{aligned} \tag{1.13}$$

Gasper and Rahman's summation (1.13) is very important in the investigate of q-congruences. See [7, 11, 18]. We shall prove Theorem 1.1–1.5 by applying Gasper and Rahman's summation (1.13) in Sections 2–5.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we first give the following lemma.

Lemma 2.1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b^3;q^4)_k q^{(k^2+3k)}b^{3k}}{(q^4,q^4/a,aq^4;q^4)_k(b^3q^4;q^2)_k}\equiv 0 \pmod{\Phi_n(q)}. \tag{2.1}$$

Proof. Putting $d=q^{-2n}$ and then taking $n\to\infty$ in (1.13), we obtain

$$\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(f;q^2)_k q^{(k^2+k)/2}}{(q^2,aq^2/b,abq;q^2)_k(aq/f;q)_k}\left(\frac{a}{f}\right)^k=\frac{(aq,aq^2,aq^2/bf,abq/f;q^2)_\infty}{(aq/f,aq^2/f,aq^2/b,abq;q^2)_\infty}, \tag{2.2}$$

which was already noticed by [18, eq. (2.2)]. Letting $q\mapsto q^2$, $a=q^{1-n}$, $b=aq$, and $f=q^{-1}/b^3$ in (2.2), we find that

$$\begin{aligned}\sum_{k=0}^{(n-1)/2}&\frac{1-q^{6k+1-n}}{1-q^{1-n}}\frac{(q^{1-n},aq,q/a;q^2)_k(q^{-1}/b^3;q^4)_k q^{(k^2+k)}}{(q^4,q^{4-n}/a,aq^{4-n};q^4)_k(b^3q^{4-n};q^2)_k}(b^3q^{2-n})^k\\&=\frac{(q^{3-n},q^{5-n},b^3q^{5-n}/a,ab^3q^{5-n};q^4)_\infty}{(aq^{4-n},q^{4-n}/a,b^3q^{4-n},b^3q^{6-n};q^4)_\infty}=0\end{aligned}$$

because of the factor $(q^{3-n},q^{5-n};q^4)_\infty=0$ in the numerator. The proof then follows from the fact $q^n\equiv 1 \pmod{\Phi_n(q)}$. □

Lemma 2.2. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{M}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^4)_k^3(q^4;q^2)_k}q^{k^2+3k}\equiv 0 \pmod{[n]}, \tag{2.3}$$

where $M=(3n+1)/4$ if $n\equiv 1 \pmod 4$, and $M=(n-1)/2$ if $n\equiv 3 \pmod 4$.

Proof. For $n\equiv 1 \pmod 4$, let $c_q(k)$ be the k-th term on the left-hand side of (2.1) with $a=b=1$, i.e.,

$$c_q(k)=[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^4)_k^3(q^4;q^2)_k}q^{k^2+3k}.$$

Let $\zeta\neq 1$ be an n-th root of unity. Namely, ζ is a primitive root of unity of degree $m\mid n$. The $a=b=1$ case of q-congruence (2.1) with $n=m$ indicates that: if $m\equiv 1 \pmod 4$, then

$$\sum_{k=0}^{m-1}c_\zeta(k)=\sum_{k=0}^{(3m+1)/4}c_\zeta(k)=0.$$

In terms of the relation

$$\frac{c_\zeta(jm+k)}{c_\zeta(jm)}=\lim_{q\to\zeta}\frac{c_q(jm+k)}{c_q(jm)}=c_\zeta(k),$$

we have

$$\sum_{k=0}^{(3n+1)/4}c_\zeta(k)=\sum_{j=0}^{(3n/m-7)/4}c_\zeta(jm)\sum_{k=0}^{m-1}c_\zeta(k)+\sum_{k=0}^{(3m+1)/4}c_\zeta(k+3(n-m)/4)=0. \qquad (2.4)$$

If $m\equiv 3 \pmod 4$, then

$$\sum_{k=0}^{m-1}c_\zeta(k)=\sum_{k=0}^{(m+1)/4}c_\zeta(k)=0,$$

and so

$$\sum_{k=0}^{(3n+1)/4}c_\zeta(k)=\sum_{j=0}^{(3n/m-5)/4}c_\zeta(jm)\sum_{k=0}^{m-1}c_\zeta(k)+\sum_{k=0}^{(m+1)/4}c_\zeta(k+(3n-m)/4)=0. \qquad (2.5)$$

The identities (2.4) and (2.5) imply that $\sum_{k=0}^{(3n+1)/4}c_q(k)$ is divisible by the cyclotomic polynomials $\Phi_m(q)$. Since this is true for all divisors $m\geqslant 1$ of n, we conclude that it is divisible by $\prod_{m|n,m>1}\Phi_m(q)=[n]$. This proves (2.3) for the first case.

Similarly we can prove (2.3) for the second case. □

We now present a parametric extension of Theorem 1.1.

Theorem 2.3. *Let $n>1$ be an integer with $n\equiv 1 \pmod 4$. Then modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,*

$$\begin{aligned}
&\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b^3;q^4)_k}{(q^4,q^4/a,aq^4;q^4)_k(b^3q^4;q^2)_k}q^{k^2+3k}b^{3k}\\
&\quad\equiv\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}\frac{(q^5,b^{-3};q^4)_{(n-1)/4}(b^3q)^{(n-1)/4}}{(b^3q^6,q;q^4)_{(n-1)/4}}\\
&\qquad+\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\frac{(q^5,q^3;q^4)_{(3n+1)/4}}{(aq^4,q^4/a;q^4)_{(3n+1)/4}}. \qquad (2.6)
\end{aligned}$$

Proof. For $a=q^n$ or $a=q^{-n}$, the left-hand side of (2.6) can be written as

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,q^{1-n},q^{1+n};q^2)_k(q^{-1}/b^3;q^4)_k}{(q^4,q^{4+n},q^{4-n};q^4)_k(b^3q^4;q^2)_k}q^{k^2+3k}b^{3k}, \qquad (2.7)$$

where we used $(q^{1-n};q^2)_k=0$ for $k>(n-1)/2$. Letting $q\mapsto q^2$, and taking $a=q$, $b=q^{1-n}$, $f=q^{-1}/b^3$ in (2.2), the expression (2.7) is equal to

$$\frac{(q^3,q^5,b^3q^{5+n},b^3q^{5-n};q^4)_\infty}{(b^3q^4,b^3q^6,q^{4+n},q^{4-n};q^4)_\infty}=\frac{(b^3q^{5-n};q^4)_{(n-1)/4}(q^5;q^4)_{(n-1)/4}}{(q^{4-n};q^4)_{(n-1)/4}(b^3q^6;q^4)_{(n-1)/4}}$$
$$=\frac{(q^5,b^{-3};q^4)_{(n-1)/4}(b^3q)^{(n-1)/4}}{(b^3q^6,q;q^4)_{(n-1)/4}}.$$

Since $1-aq^n$ and $a-q^n$ are coprime polynomials, we deduce that, modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b^3;q^4)_k}{(q^4,q^4/a,aq^4;q^4)_k(b^3q^4;q^2)_k}q^{k^2+3k}b^{3k}\equiv\frac{(q^5,b^{-3};q^4)_{(n-1)/4}(b^3q)^{(n-1)/4}}{(b^3q^6,q;q^4)_{(n-1)/4}}. \tag{2.8}$$

For $b=q^n$, the left-hand side of (2.6) can be written as

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1-3n};q^4)_kq^{k^2+3k+3nk}}{(q^4,aq^4,q^4/a;q^4)_k(q^{4+3n};q^2)_k}. \tag{2.9}$$

Putting $q\mapsto q^2$, $a=q$, $b=aq$, and $f=q^{-1-3n}$ in (2.2), the summation (2.9) is equal to

$$\frac{(q^3,q^5,q^{5+3n}/a,aq^{5+3n};q^4)_\infty}{(q^{4+3n},q^{6+3n},q^4/a,aq^4;q^4)_\infty}=\frac{(q^5,q^3;q^4)_{(3n+1)/4}}{(aq^4,q^4/a;q^4)_{(3n+1)/4}}.$$

This establishes the q-congruence: modulo $b-q^n$,

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b^3;q^4)_k}{(q^4,q^4/a,aq^4;q^4)_k(b^3q^4;q^2)_k}q^{k^2+3k}b^{3k}\equiv\frac{(q^5,q^3;q^4)_{(3n+1)/4}}{(aq^4,q^4/a;q^4)_{(3n+1)/4}}. \tag{2.10}$$

It is clear that $\Phi_n(q)(1-aq^n)(a-q^n)$ is coprime with $b-q^n$. Noting the relations

$$\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}\equiv 1\pmod{(1-aq^n)(a-q^n)}, \tag{2.11}$$

$$\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\equiv 1\pmod{(b-q^n)}, \tag{2.12}$$

and employing the Chinese reminder theorem for coprime polynomials, we obtain (2.6) from (2.1), (2.8) and (2.10). □

Proof of Theorem 1.1. The $b=1$ case of (2.6) produces the q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(3n+1)/4}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1};q^4)_k}{(q^4,q^4/a,aq^4;q^4)_k(q^4;q^2)_k}q^{k^2+3k}$$
$$\equiv-\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(q^5,q^3;q^4)_{(3n+1)/4}}{(aq^4,q^4/a;q^4)_{(3n+1)/4}}. \tag{2.13}$$

Clearly, the q-shifted factorial $(q^3, q^5; q^4)_{(3n+1)/4}$ contains the factor $(1-q^{3n})(1-q^n)$. The right-hand side of (2.13) vanishes modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$.

Letting $a \to 1$ in (2.13), we see that (1.3) holds modulo $\Phi_n(q)^4$. By (2.3), the q-congruence (1.3) also holds modulo $[n]$. Thus, the proof follows from the fact that the least common multiple of $\Phi_n(q)^4$ and $[n]$ is $[n]\Phi_n(q)^3$. □

3 Proof of Theorem 1.2

It is easy to check that Theorem 1.2 is true for $n = 5$. We now assume that $n \geqslant 9$. Consider the summation

$$\sum_{k=0}^{n-2}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^2)_k(q^4;q^4)_k^3}q^{k^2+3k}. \tag{3.1}$$

For $(3n+1)/4 < k \leqslant n-2$, the numerator $(q;q^2)_k^3(q^{-1};q^4)_k$ has the factor $(1-q^n)^3(1-q^{3n})$, and the denominator $(q^4;q^2)_k(q^4;q^4)_k^3$ is coprime with $\Phi_n(q)$. It follows that

$$\sum_{k=(3n+5)/4}^{n-2}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^2)_k(q^4;q^4)_k^3}q^{k^2+3k} \equiv 0 \pmod{\Phi_n(q)^4}.$$

This, together with (1.3), implies that (3.1) is congruent to 0 modulo $\Phi_n(q)^4$, and therefore the left-hand side of (1.4) is congruent to

$$[6n-5]\frac{(q;q^2)_{n-1}^3(q^{-1};q^4)_{n-1}}{(q^4;q^2)_{n-1}(q^4;q^4)_{n-1}^3}q^{n^2+n-2} \pmod{\Phi_n(q)^4}.$$

It is easy to see that

$$\begin{aligned}
(q^{-1};q^4)_{n-1} &= (-1)^{(3n+1)/4}q^{(3n-5)(3n+1)/8}(q^{4-3n};q^4)_{(3n+1)/4}(1-q^{3n})(q^{3n+4};q^4)_{(n-9)/4},\\
(q;q^2)_{n-1} &= (q;q^2)_{(n-1)/2}(1-q^n)(q^{n+2};q^2)_{(n-3)/2},\\
(q^4;q^4)_{n-1} &= (q^2;q^2)_{(n-1)/2}(q^{n+1};q^2)_{(n-1)/2}(-q^2;q^2)_{n-1},\\
(q^4;q^2)_{n-1} &= (q^2;q^2)_{n-1}[n]_{q^2}.
\end{aligned}$$

Thus, by making use of $q^n \equiv 1 \pmod{\Phi_n(q)}$ and the following two q-congruences (see [6, eq. (2.3)]):

$$(q;q)_{n-1} \equiv n \pmod{\Phi_n(q)}, \quad \text{and} \quad (-q;q)_{n-1} \equiv 1 \pmod{\Phi_n(q)},$$

we have

$$\begin{aligned}
[6n-5]\frac{(q;q^2)_{n-1}^3(q^{-1};q^4)_{n-1}}{(q^4;q^2)_{n-1}(q^4;q^4)_{n-1}^3}q^{n^2+n-2} &\equiv \frac{(1-q^n)^3(1-q^{3n})(1+q)}{(1-q^{-1})^3(1-q^{2n})}q^{(9n^2-5n)/4-2}\\
&\equiv \frac{-[n]^3[3n](1+q)}{[2n]}q^{(9n^2-5n)/4+1} \pmod{\Phi_n(q)^4}.
\end{aligned}$$

This proves that (1.4) is true modulo $\Phi_n(q)^4$. Similarly as before, we can prove it is true modulo $[n]$.

4 Proof of Theorem 1.3

We first construct a parametric version of Theorem 1.3.

Theorem 4.1. *Let n be a positive integer with $n \equiv 3 \pmod 4$. Then modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b;q^4)_k}{(q^4,q^4/a,aq^4;q^4)_k(bq^4;q^2)_k}q^{k^2+3k}b^k$$
$$\equiv \frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)}\frac{(q^3,1/bq^2;q^4)_{(n+1)/4}(bq)^{(n+1)/4}}{(bq^4,q^{-1};q^4)_{(n+1)/4}}$$
$$+\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}\frac{(q^5,q^3;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}. \tag{4.1}$$

Proof. Letting $q\mapsto q^2$, and taking $a=q$, $b=q^{1-n}$, and $f=q^{-1}/b$ in (2.2), we obtain the following identity:

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,q^{1-n},q^{1+n};q^2)_k(q^{-1}/b;q^4)_k}{(q^4,q^{4+n},q^{4-n};q^4)_k(bq^4;q^2)_k}q^{k^2+3k}b^k=\frac{(q^3,1/bq^2;q^4)_{(n+1)/4}}{(bq^4,q^{-1};q^4)_{(n+1)/4}}(bq)^{(n+1)/4}.$$

This means that, modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1}/b;q^4)_k}{(q^4,aq^4,q^4/a;q^4)_k(bq^4;q^2)_k}q^{k^2+3k}b^k\equiv\frac{(q^3,1/bq^2;q^4)_{(n+1)/4}}{(bq^4,q^{-1};q^4)_{(n+1)/4}}(bq)^{(n+1)/4}. \tag{4.2}$$

On the other hand, putting $q\mapsto q^2$, $a=q$, $b=aq$, and $f=q^{-1-n}$ in (2.2), we get

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1-n};q^4)_k}{(q^4,aq^4,q^4/a;q^4)_k(q^{4+n};q^2)_k}q^{k^2+3k+nk}=\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}.$$

In other words, we have the q-congruence: modulo $b-q^n$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1-n}/b;q^4)_k}{(q^4,aq^4,q^4/a;q^4)_k(q^{4+n};q^2)_k}q^{k^2+3k}b^k\equiv\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}. \tag{4.3}$$

Applying (2.11), (2.12), and the Chinese reminder theorem for polynomials, we conclude (4.1) from (2.1), (4.2), and (4.3). □

Proof of Theorem 1.3. Letting $b=1$ in (4.1), we obtain the q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q,aq,q/a;q^2)_k(q^{-1};q^4)_k}{(q^4,aq^4,q^4/a;q^4)_k(q^4;q^2)_k}q^{k^2+3k}\\
&\equiv -[n]q^{(n+5)/4}\frac{(q^{-2};q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}\\
&\quad+\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(\frac{(q^3,q^{-2};q^4)_{(n+1)/4}}{(q^4,q^{-1};q^4)_{(n+1)/4}}q^{(n+1)/4}-\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}\right)\\
&\equiv -[n]q^{(n+5)/4}\frac{(q^{-2};q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}\\
&\quad+\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(q^4,q^4;q^4)_{(n+1)/4}}-\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}\right).
\end{aligned}
\tag{4.4}
$$

By the L'Hôpital rule, there holds

$$
\begin{aligned}
&\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(q^4,q^4;q^4)_{(n+1)/4}}-\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(aq^4,q^4/a;q^4)_{(n+1)/4}}\right)\\
&=-[n]^2\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(q^4,q^4;q^4)_{(n+1)/4}}\sum_{j=1}^{(n+1)/4}\frac{q^{4j}}{[4j]^2}.
\end{aligned}
$$

Letting $a\to 1$ in (4.4) and using the above limit, we obtain the q-congruence: modulo $\Phi_n(q)^4$,

$$
\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^3(q^{-1};q^4)_k}{(q^4;q^4)_k^3(q^4;q^2)_k}q^{k^2+3k}\\
&\equiv -[n]q^{(n+5)/4}\frac{(q^{-2};q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}-[n]^2\frac{(q^3,q^5;q^4)_{(n+1)/4}}{(q^4,q^4;q^4)_{(n+1)/4}}\sum_{j=1}^{(n+1)/4}\frac{q^{4j}}{[4j]^2}\\
&\equiv -[n]q^{(n+5)/4}\frac{(q^{-2};q^4)_{(n+1)/4}}{(q^4;q^4)_{(n+1)/4}}\left(1-[n]^2\sum_{j=1}^{(n+1)/4}\frac{q^{4j}}{[4j]^2}\right).
\end{aligned}
$$

In light of (2.3), we compete the proof the theorem. □

5 Proof of Theorem 1.5

We have the following parametric version of Theorem 1.5.

Theorem 5.1. *Let n be a positive integer with $n \equiv 1 \pmod 4$ and $n \geqslant 9$. Then modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(3n+1)/4} [6k+1] \frac{(q, aq, q/a; q^2)_k (q, q^{-1}; q^4)_k}{(aq^4, q^4/a; q^4)_k (q^4, q^2, q^{-1}; q^2)_k} q^{2k} \equiv 0. \tag{5.1}$$

Proof. Letting $a = q^{1-n}$, $b = aq$, $d = q$, $f = q^{-1}$, and $q \mapsto q^2$ in (1.13), we obtain

$$\begin{aligned} &\sum_{k=0}^{(3n+1)/4} \frac{1-q^{1+6k-n}}{1-q^{1-n}} \frac{(q^{1-n}, aq, q/a; q^2)_k (q^{4-2n}, q, q^{-1}; q^4)_k}{(aq^{4-n}, q^{4-n}/a, q^4; q^4)_k (q^{4-n}, q^{2-n}, q^{n-1}; q^2)_k} q^{2k} \\ &= \frac{(q^{3-n}, q^{n-2}; q^2)_\infty (q^{3-n}/a, aq^{3-n}, aq^n, q^n/a; q^4)_\infty}{(q^{2-n}, q^{n-1}; q^2)_\infty (q^{4-n}/a, aq^{4-n}, aq^{n-1}, q^{n-1}/a; q^4)_\infty} \\ &= 0, \end{aligned}$$

where we have used the fact that $(q^{3-n}; q^2)_\infty = 0$. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, we conclude that (5.1) is true modulo $\Phi_n(q)$. On the other hand, taking $a = q$, $b = q^{1+n}$, $d = q$, $f = q^{-1}$ and $q \mapsto q^2$ in (1.13), we have

$$\sum_{k=0}^{(3n+1)/4} [6k+1] \frac{(q, q^{1+n}, q^{1-n}; q^2)_k (q, q^{-1}; q^4)_k}{(q^{4+n}, q^{4+n}; q^4)_k (q^4, q^2, q^{-1}; q^2)_k} q^{2k} = 0.$$

This proves the truth of (5.1) modulo $(1-aq^n)(a-q^n)$. Therefore, the q-congruence (5.1) holds. □

Proof of Theorem 1.5. Letting $a = 1$ in (5.1), we see that (1.10) holds modulo $\Phi_n(q)^3$. Along the same lines of the proof of Lemma 2.2, we can prove that it also holds modulo $[n]$. □

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6 Declarations

Conflicts of interest: No potential conflict of interest was reported by the authors.
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