Some q-supercongruences from a very-well-poised ${}_6\phi_5$ summation

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Abstract. By making use of a very-well-poised ${}_6\phi_5$ summation, the creative microscoping method, and the Chinese remainder theorem for coprime polynomials, we establish some new q-supercongruences, including some Dwork-type q-supercongruences. As a consequence, we obtain the following result: for any prime $p\equiv 1 \pmod 4$ and integer $r\geqslant 1$,

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv p^r \pmod{p^{r+3}},$$

where $(x)_0=1$ and $(x)_n=x(x+1)\cdots(x+n-1)$ for $n\geqslant 1$.

Keywords: q-supercongruences; p-adic Gamma function; cyclotomic polynomials; creative microscoping; ${}_6\phi_5$ summation

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1. Introduction

India's mathematician Ramanujan discovered a number of interesting infinite series identities, such as

$$\sum_{k=0}^{\infty}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}=\frac{2\sqrt{2}}{\Gamma(\frac{3}{4})^2\sqrt{\pi}} \tag{1.1}$$

(see [2]). Here $\Gamma(x)$ is the classical *Gamma function* and $(a)_n=a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol*. The identity (1.1) was first proved by Hardy [13]. In 1997, Van Hamme [20] observed that Ramanujan's series have amazing p-adic analogues. For instance, the series (1.1) corresponds to the following result: for any prime $p\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p-1)/4}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}\equiv p\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})}{\Gamma_p(\frac{3}{4})} \pmod{p^3}, \tag{1.2}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function. Some generalizations of (1.2) are given by Barman and Saikia [1] and Pan, Tauraso, and Wang [18].

Liu and Wang [15] proved that (1.2) can be deduced from the following q-supercongruence: for positive integers n with $n \equiv 1 \pmod 4$, modulo $[n]\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k} \equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4} \tag{1.3}$$

(see [9, Theorem 4.3] for a generalization). Here and in what follows, $[n]=(1-q^n)/(1-q)$ denotes the q-integer, the *q-shifted factorials* are defined as follows:

$$(x;q)_\infty=\prod_{k=1}^{\infty}(1-xq^k), \quad \text{and} \quad (x;q)_n=\frac{(x;q)_\infty}{(xq^n;q)_\infty}.$$

For convenience, we shall also use the condensed notation $(x_1,\dots,x_m;q)_n=(x_1;q)_n\cdots(x_m;q)_n$ for $n=0,1,\dots$ or $n=\infty$. Moreover, we let $\Phi_n(q)$ stand for the n-th cyclotomic polynomial, i.e.,

$$\Phi_n(q)=\prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Employing the creative microscoping method [10] and the Chinese remainder theorem for polynomials [6], Liu and Wang [16] further proved the following refinement of (1.3): for $n\equiv 1 \pmod 4$, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/4}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k}$$
$$\equiv \frac{(q^2;q^4)_{(n-1)/4}}{(q^4;q^4)_{(n-1)/4}}[n]q^{(1-n)/4}\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2+[n]^2\sum_{k=0}^{(n-1)/4}\frac{q^{4k-2}}{[4k-2]^2}\right\}. \tag{1.4}$$

Recently, using the same method along with a terminating very-well-poised ${}_6\phi_5$ summation (see [4, Appendix (II.21)]), the first author [7, Theorem 1.1] gave an extension of (1.4). For some other interesting q-supercongruences, see [21, 22].

The first aim of this paper is to establish the following new q-supercongruence.

Theorem 1.1. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then, modulo $\Phi_n(q)^4$,*

$$\sum_{k=0}^{(n-1)/2}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \tag{1.5}$$

We shall also give a more general form of Theorem 1.1. Then we will take the $n=p^r$ and $q\to 1$ case to deduce the following result.

Theorem 1.2. *Let $p\equiv 1 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac14)_k^3(\frac12)_k}{k!^3(\frac34)_k}\equiv p^r \pmod{p^{r+3}}. \tag{1.6}$$

Swisher [19] built the following supercongruence similar to (1.2): for primes $p \equiv 3 \pmod 4$ and $p > 3$,

$$\sum_{k=0}^{p-1}(8k+1)\frac{(\frac14)_k^4}{k!^4} \equiv -\frac{3p^2\Gamma_p(\frac12)\Gamma_p(\frac14)}{2\Gamma_p(\frac34)} \pmod{p^4}. \tag{1.7}$$

The first author and Schlosser [9] gave the following q-analogue of (1.7): for positive integers $n \equiv 3 \pmod 4$, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^4}{(q^4;q^4)_k^4}q^{2k} \equiv \frac{(q^2;q^4)_{(3n-1)/4}}{(q^4;q^4)_{(3n-1)/4}}[3n]q^{(1-3n)/4}.$$

The second aim of this paper is to establish the following result, which is a companion of (1.5) modulo $\Phi_n(q)^3$.

Theorem 1.3. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \equiv q^{(1-3n)/2}[3n] \pmod{\Phi_n(q)^3}. \tag{1.8}$$

Similarly, we can deduce the following conclusion from Theorem 1.3.

Theorem 1.4. *Let $p \equiv 3 \pmod 4$ be a prime and let $r \geqslant 1$ be an odd integer. Then*

$$\sum_{k=0}^{p^r-1}(8k+1)\frac{(\frac14)_k^3(\frac12)_k}{k!^3(\frac34)_k} \equiv 3p^r \pmod{p^{r+2}}. \tag{1.9}$$

Swisher [19, (G.3)] conjectured that the supercongruence (1.2) can be extended as follows: for primes $p \equiv 1 \pmod 4$ and positive integers r,

$$\sum_{k=0}^{(p^r-1)/4}(8k+1)\frac{(\frac14)_k^4}{k!^4} \equiv (-1)^{(p^2+7)/8}p\Gamma_p(\tfrac12)\Gamma_p(\tfrac14)^2\sum_{k=0}^{(p^{r-1}-1)/4}(8k+1)\frac{(\frac14)_k^4}{k!^4} \pmod{p^{4r}}. \tag{1.10}$$

This conjecture can be considered as a particular example of Dwork-type supercongruences [3, 17]. The supercongruence (1.10) remains open so far. Some other Dwork-type supercongruences are given in [5, 8, 11, 12, 14, 23].

The third aim of this paper is to establish the following Dwork-type supercongruences: for any prime $p \equiv 1 \pmod 4$ and positive r,

$$\sum_{k=0}^{(p^r-1)/d}(8k+1)\frac{(\frac14)_k^3(\frac12)_k}{k!^3(\frac34)_k} \equiv p\sum_{k=0}^{(p^{r-1}-1)/d}(8k+1)\frac{(\frac14)_k^3(\frac12)_k}{k!^3(\frac34)_k} \pmod{p^{3r}}, \tag{1.11}$$

where $d=1,2$, and for any prime $p\equiv 3 \pmod 4$ with $p>3$ and integer $r\geqslant 2$,

$$\sum_{k=0}^{p^r-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv p^2\sum_{k=0}^{p^{r-2}-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k} \pmod{p^{3r-2}}. \qquad (1.12)$$

We shall prove (1.11) and (1.12) by building the following two Dwork-type q-supercongruences.

Theorem 1.5. *Let $n\equiv 1 \pmod 4$ be an integer with $n>1$ and let $r\geqslant 1$. Then, modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)^3$,*

$$\sum_{k=0}^{(n^r-1)/d}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d}q^{nk}[8k+1]_{q^n}\frac{(q^n;q^{4n})_k^3(q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k^3(q^{3n};q^{4n})_k}, \qquad (1.13)$$

where $d=1,2$.

Theorem 1.6. *Let $n\equiv 3 \pmod 4$ be a positive integer and let $r\geqslant 2$. Then, modulo $\Phi_n(q)\prod_{j=2}^{r}\Phi_{n^j}(q)^3$,*

$$\sum_{k=0}^{n^r-1}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{n^{r-2}-1}q^{n^2k}[8k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^3(q^{2n^2};q^{4n^2})_k}{(q^{4n^2};q^{4n^2})_k^3(q^{3n^2};q^{4n^2})_k}. \qquad (1.14)$$

If $n\geqslant 7$, or $n=3$ and $r\geqslant 3$ is odd, then the denominators on the two sides of (1.14) are coprime with $\Phi_{n^j}(q)$ for any $j>r$. Further, assuming that these n are primes and taking the limits as $q\to 1$ in (1.14), we get the supercongruence (1.12) and for odd $r\geqslant 3$,

$$\sum_{k=0}^{3^r-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv 9\sum_{k=0}^{3^{r-2}-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k} \pmod{3^{3r-2}}.$$

Similarly, the supercongruence (1.11) can be deduced from the limiting case of (1.13).

From Theorem 1.5, we can deduce the following conclusion.

Theorem 1.7. *Let $p\equiv 3 \pmod 4$ be a prime and $r\geqslant 2$ even. Then*

$$\sum_{k=0}^{(p^r-1)/d}(8k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^3}{k!^3(\frac{3}{4})_k}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/d}(8k+1)\frac{(\frac{1}{2})_k(\frac{1}{4})_k^3}{k!^3(\frac{3}{4})_k} \pmod{p^{3r/2}}, \qquad (1.15)$$

where $d=1,2$.

The paper is arranged as follows. We shall prove Theorems 1.1–1.4 by using a very-well-poised ${}_6\phi_5$ summation in Sections 2–5, respectively. The proof of Theorems 1.5–1.7 will be given in Section 6. In the final Section 7, we raise several related conjectures on supercongruences and q-congruences.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first give the following lemma.

Lemma 2.1. *Let n be a positive odd integer. Let a and b be indeterminates. Then*

$$\sum_{k=0}^{M}[8k+1]\frac{(q,aq,q/a,q^2/b^2;q^4)_k}{(q^4,q^4/a,aq^4,b^2q^3;q^4)_k}b^{2k}q^k \equiv 0 \pmod{\Phi_n(q)},$$

where $M=(n-1)/2$ if $n\equiv 1 \pmod 4$, and $M=n-1$ if $n\equiv 3 \pmod 4$.

Proof. Recall that a very-well-poised ${}_6\phi_5$ summation (see [4, Appendix (II.20)]) can be stated as follows:

$$\sum_{k=0}^{\infty}\frac{1-aq^{2k}}{1-a}\frac{(a,b,c,d;q)_k}{(q,aq/b,aq/c,aq/d;q)_k}\left(\frac{aq}{bcd}\right)^k = \frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/bcd,aq/d;q)_\infty}. \tag{2.1}$$

For $n\equiv 1 \pmod 4$, letting $q\mapsto q^4$, $a=q^{1-n}$, $b=aq$, $c=q/a$, and $d=q^{2-n}/b^2$ in (2.1), we obtain

$$\sum_{k=0}^{(n-1)/2}\frac{1-q^{8k+1-n}}{1-q^{1-n}}\frac{(q^{1-n},aq,q/a,q^{2-n}/b^2;q^4)_k}{(q^4,q^{4-n}/a,aq^{4-n},b^2q^3;q^4)_k}b^{2k}q^k=0.$$

For $n\equiv 3 \pmod 4$, letting $q\mapsto q^4$, $a=q^{1-3n}$, $b=aq$, $c=q/a$, and $d=q^{2-3n}/b^2$ in (2.1), we get

$$\sum_{k=0}^{n-1}\frac{1-q^{8k+1-3n}}{1-q^{1-3n}}\frac{(q^{1-3n},aq,q/a,q^{2-3n}/b^2;q^4)_k}{(q^4,q^{4-3n}/a,aq^{4-3n},b^2q^3;q^4)_k}b^{2k}q^k=0.$$

The proof then follows from the q-congruence $q^n\equiv 1 \pmod{\Phi_n(q)}$. □

We also require the following easily proved lemma. For a short proof of it, see [6, Lemma 2.1].

Lemma 2.2. *Let n be a positive odd integer. Then*

$$(aq^2,q^2/a;q^2)_{(n-1)/2}\equiv(-1)^{(n-1)/2}\frac{(1-a^n)q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}, \tag{2.2}$$

$$(q;q)_{n-1}\equiv n \pmod{\Phi_n(q)}. \tag{2.3}$$

We now present a parametric extension of Theorem 1.1.

Theorem 2.3. *Let $n > 1$ be an integer with $n \equiv 1 \pmod 4$. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,*

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, aq, q/a, q^2/b^2; q^4)_k}{(q^4, q^4/a, aq^4, b^2q^3; q^4)_k} b^{2k} q^k$$
$$\equiv \frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \frac{(q^5, q^3/b^2; q^4)_{(n-1)/4} (b/q)^{(n-1)/2}}{(b^2q^3, q; q^4)_{(n-1)/4}}$$
$$+ \frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \frac{(q^5, q^3; q^4)_{(n-1)/2}}{(aq^4, q^4/a; q^4)_{(n-1)/2}}. \tag{2.4}$$

Proof. For $a = q^n$ or $a = q^{-n}$, the left-hand side of (2.4) can be written as

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, q^{1+n}, q^{1-n}, q^2/b^2; q^4)_k}{(q^4, q^{4-n}, q^{4+n}, b^2q^3; q^4)_k} b^{2k} q^k. \tag{2.5}$$

Performing the parameter substitutions $q \mapsto q^4$, $a = q$, $b = q^{1+n}$, $c = q^{1-n}$, and $d = q^2/b^2$ in (2.1), we deduce that (2.5) is equal to

$$\frac{(q^5, q^3, b^2q^{2-n}, b^2q^{2+n}; q^4)_\infty}{(q^{4-n}, q^{4+n}, b^2q^3, b^2q; q^4)_\infty} = \frac{(q^5, q^3/b^2; q^4)_{(n-1)/4}}{(q, b^2q^3; q^4)_{(n-1)/4}} b^{(n-1)/2} q^{(1-n)/2}.$$

Since the polynomials $1-aq^n$ and $a-q^n$ are coprime with each other, we obtain the q-congruence: modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, aq, q/a, q^2/b^2; q^4)_k}{(q^4, q^4/a, aq^4, b^2q^3; q^4)_k} b^{2k} q^k$$
$$\equiv \frac{(q^5, q^3/b^2; q^4)_{(n-1)/4}}{(q, b^2q^3; q^4)_{(n-1)/4}} b^{(n-1)/2} q^{(1-n)/2}. \tag{2.6}$$

For $b = q^n$, the left-hand side of (2.4) can be written as

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, aq, q/a, q^{2-2n}; q^4)_k q^{k+2nk}}{(q^4, aq^4, q^4/a, q^{3+2n}; q^4)_k}. \tag{2.7}$$

Putting $q \mapsto q^4$, $a = q$, $b = aq$, $c = q/a$ and $f = q^{2-2n}$ in (2.1), we see that (2.7) is equal to

$$\frac{(q^5, q^3, q^{2+2n}/a, aq^{2+2n}; q^4)_\infty}{(q^4/a, aq^4, q^{3+2n}, q^{1+2n}; q^4)_\infty} = \frac{(q^5, q^3; q^4)_{(n-1)/2}}{(aq^4, q^4/a; q^4)_{(n-1)/2}}.$$

This establishes the q-congruence: modulo $b - q^n$,

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, aq, q/a, q^{2-2n}; q^4)_k q^{k+2nk}}{(q^4, aq^4, q^4/a, q^{3+2n}; q^4)_k} \equiv \frac{(q^5, q^3; q^4)_{(n-1)/2}}{(aq^4, q^4/a; q^4)_{(n-1)/2}}. \tag{2.8}$$

It is clear that $\Phi_n(q)(1-aq^n)(a-q^n)$ and $b-q^n$ are coprime polynomials. In view of the Chinese reminder theorem for coprime polynomials, we can determine the remainder of the left-hand side of (2.4) modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$ from (2.6) and (2.8). For this purpose, we need the following two q-congruences:

$$\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^n)(a-q^n)}, \tag{2.9}$$

$$\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \equiv 1 \pmod{b-q^n}. \tag{2.10}$$

The q-congruence (2.4) then follows from (2.6), (2.8), (2.9) and (2.10) immediately. □

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Since

$$(q^5,q^3;q^4)_{(n-1)/2} = (q^3;q^2)_{n-1} = [n](q;q^2)_{(n-1)/2}(q^{n+2};q^2)_{(n-1)/2},$$

in light of(2.2) and (2.3), we get

$$\frac{(q^5,q^3;q^4)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{[n](q,q^2;q^2)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{[n]n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-1)/2}} \pmod{\Phi_n(q)^2}. \tag{2.11}$$

Moreover, we have the identity:

$$(1-q^n)(1+a^2-a-aq^n) = (1-a)^2 + (1-aq^n)(a-q^n). \tag{2.12}$$

And, when $b=1$ the polynomial $b-q^n = 1-q^n$ has the factor $\Phi_n(q)$. Hence, letting $b=1$ in (2.4) and making use of (2.11) and (2.12), we arrive at the following q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{(n-1)/2} [8k+1]\frac{(q,aq,q/a,q^2;q^4)_k}{(q^4,q^4/a,aq^4,q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n)/2}[n] + q^{(1-n)/2}[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left\{1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right\}. \tag{2.13}$$

By the L'Hôpital rule, we have

$$\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)} = \frac{(n^2-1)(1-q)^2}{24}[n]^2,$$

which first appeared in [6]. Hence, taking $a\to 1$ in (2.13) and applying the above limit, we get the desired q-supercongruence (1.5). □

3. Proof of Theorem 1.2

We first prove the following lemma.

Lemma 3.1. *Let $n \equiv 1 \pmod 4$ be a positive integer and $r \geqslant 1$. Then*

$$\sum_{k=0}^{(n^r-1)/d} [8k+1] \frac{(q, aq, q/a, q^2; q^4)_k}{(q^4, q^4/a, aq^4, q^3; q^4)_k} q^k \equiv 0 \pmod{\prod_{j=1}^{r} \Phi_{n^j}(q)}, \tag{3.1}$$

where $d = 1, 2$.

Proof. From Lemma 2.1, we know that

$$\sum_{k=0}^{(n-1)/2} [8k+1] \frac{(q, aq, q/a, q^2/b^2; q^4)_k}{(q^4, q^4/a, aq^4, b^2q^3; q^4)_k} b^{2k} q^k \equiv 0 \pmod{\Phi_n(q)}. \tag{3.2}$$

We now assume that $r \geqslant 2$ and $1 \leqslant s \leqslant r$. Since $n \equiv 1 \pmod 4$, we have $n^s \equiv 1 \pmod 4$. Consequently, replacing $n \mapsto n^s$ and $b = 1$ in (3.2) yields the q-congruence:

$$\sum_{k=0}^{(n^s-1)/d} [8k+1] \frac{(q, aq, q/a, q^2; q^4)_k}{(q^4, q^4/a, aq^4, q^3; q^4)_k} q^k \equiv 0 \pmod{\Phi_{n^s}(q)}, \tag{3.3}$$

where $d = 1, 2$. Let ζ be an n^s-th primitive root of unity and $c_q(k)$ the k-th term on the left-hand side of (3.3), i.e.,

$$c_q(k) = [8k+1] \frac{(q, aq, q/a, q^2; q^4)_k}{(q^4, q^4/a, aq^4, q^3; q^4)_k} q^k.$$

Thus, the q-congruence (3.3) means that

$$\sum_{k=0}^{n^s-1} c_\zeta(k) = \sum_{k=0}^{(n^s-1)/2} c_\zeta(k) = 0.$$

It is easy to see that, for any non-negative integers l and k,

$$\frac{c_\zeta(ln^s+k)}{c_\zeta(ln^s)} = \lim_{q\to\zeta} \frac{c_q(ln^s+k)}{c_q(ln^s)} = c_\zeta(k).$$

Therefore,

$$\sum_{k=0}^{n^r-1} c_\zeta(k) = \sum_{l=0}^{n^{r-s}-1} \sum_{k=0}^{n^s-1} c_\zeta(ln^s+k) = \sum_{l=0}^{n^{r-s}-1} c_\zeta(ln^s) \sum_{k=0}^{n^s-1} c_\zeta(k) = 0,$$

$$\sum_{k=0}^{(n^r-1)/2} c_\zeta(k) = \sum_{l=0}^{(n^{r-s}-3)/2} c_\zeta(ln^s) \sum_{k=0}^{n^s-1} c_\zeta(k) + \sum_{k=0}^{(n^s-1)/2} c_\zeta((n^r-n^s)/2+k) = 0.$$

This indicates that the sums $\sum_{k=0}^{n^r-1} c_q(k)$ and $\sum_{k=0}^{(n^r-1)/2} c_q(k)$ are both congruent to 0 modulo $\Phi_{n^s}(q)$ for $1 \leqslant j \leqslant r$. Since $\Phi_n(q), \Phi_{n^2}(q), \dots, \Phi_{n^r}(q)$ are pairwise coprime polynomials in $\mathbb{Z}[q]$, we conclude that the q-congruence (3.1) holds. ☐

Proof of Theorem 1.2. Replacing n by n^r in Theorem 1.1, we obtain

$$\sum_{k=0}^{(n^r-1)/2} [8k+1] \frac{(q;q^4)_k^3 (q^2;q^4)_k}{(q^4;q^4)_k^3 (q^3;q^4)_k} q^k \\ \equiv q^{(1-n^r)/2} [n^r] \left\{ 1 + \frac{(n^{2r}-1)(1-q)^2}{24} [n^r]^2 \right\} \pmod{\Phi_{n^r}(q)^4}. \tag{3.4}$$

Since $[n^r]$ is divisible by $\prod_{j=1}^r \Phi_{n^j}(q)$, we conclude from Lemma 3.1 that (3.4) is true modulo $\Phi_{n^r}(q)^4 \prod_{j=1}^{r-1} \Phi_{n^j}(q)$. Letting $n = p$ be a prime and taking the limits as $q \to 1$ in this q-congruence, we obtain the supercongruence (1.6). ☐

4. Proof of Theorem 1.3

Like the proof of Theorem 1.1, we first present a parametric version of Theorem 1.3.

Theorem 4.1. *Let n be a positive integer with $n \equiv 3 \pmod 4$. Let a and b be indeterminates. Then modulo $\Phi_n(q)(1-aq^n)(a-q^n)(b-q^n)$,*

$$\sum_{k=0}^{n-1} [8k+1] \frac{(q, a^3 q, q/a^3, q^2/b^2; q^4)_k}{(a^3 q^4, q^4/a^3, q^4, b^2 q^3; q^4)_k} b^{2k} q^k \\ \equiv \frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \frac{(q^5, q^3/b^2; q^4)_{(3n-1)/4} (b/q)^{(3n-1)/2}}{(b^2 q^3, q; q^4)_{(3n-1)/4}} \\ + \frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \frac{(q^5, q^3; q^4)_{(n-1)/2}}{(a^3 q^4, q^4/a^3; q^4)_{(n-1)/2}}. \tag{4.1}$$

Proof. For $a = q^n$ or $a = q^{-n}$, the left-hand side of (4.1) can be written as

$$\sum_{k=0}^{n-1} [8k+1] \frac{(q, q^{1+3n}, q^{1-3n}, q^2/b^2; q^4)_k}{(q^4, q^{4-3n}, q^{4+3n}, b^2 q^3; q^4)_k} b^{2k} q^k. \tag{4.2}$$

Making the substitutions $q \mapsto q^4$, $a = q$, $b = q^{1+3n}$, $c = q^{1-3n}$, and $d = q^2/b^2$ in (2.1), we see that (4.2) is equal to

$$\frac{(q^5, q^3, b^2 q^{2-3n}, b^2 q^{2+3n}; q^4)_\infty}{(q^{4-3n}, q^{4+3n}, b^2 q^3, b^2 q; q^4)_\infty} = \frac{(q^5, q^3/b^2; q^4)_{(3n-1)/4}}{(q, b^2 q^3; q^4)_{(3n-1)/4}} (b/q)^{(3n-1)/2}.$$

This implies that, modulo $(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q,a^3q,q/a^3,q^2/b^2;q^4)_k}{(q^4,q^4/a^3,a^3q^4,b^2q^3;q^4)_k}b^{2k}q^k \equiv \frac{(q^5,q^3/b^2;q^4)_{(3n-1)/4}}{(q,b^2q^3;q^4)_{(3n-1)/4}}(b/q)^{(3n-1)/2}. \tag{4.3}$$

For $b=q^n$, the left-hand side of (4.1) can be written as

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q,a^3q,q/a^3,q^{2-2n};q^4)_k q^{k+2nk}}{(q^4,a^3q^4,q^4/a^3,q^{3+2n};q^4)_k}. \tag{4.4}$$

Putting $q\mapsto q^4$, $a=q$, $b=a^3q$, $c=q/a^3$ and $d=q^{2-2n}$ in (2.1), we conclude that (4.4) is equal to

$$\frac{(q^5,q^3,q^{2+2n}/a^3,a^3q^{2+2n};q^4)_\infty}{(q^4/a^3,a^3q^4,q^{3+2n},q^{1+2n};q^4)_\infty} = \frac{(q^5,q^3;q^4)_{(n-1)/2}}{(a^3q^4,q^4/a^3;q^4)_{(n-1)/2}}.$$

This establishes the q-congruence: modulo $b-q^n$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q,a^3q,q/a^3,q^{2-2n};q^4)_k q^{k+2nk}}{(q^4,a^3q^4,q^4/a^3,q^{3+2n};q^4)_k} \equiv \frac{(q^5,q^3;q^4)_{(n-1)/2}}{(a^3q^4,q^4/a^3;q^4)_{(n-1)/2}}. \tag{4.5}$$

By the Chinese reminder theorem for coprime polynomials, we obtain the q-congruence (4.1) from (2.9), (2.10), (4.3) and (4.5). □

Proof of Theorem 1.3. In view of (2.12), the $b=1$ case of (4.1) reduces to the following q-congruence: modulo $\Phi_n(q)(1-aq^n)(a-q^n)$

$$\begin{aligned}
&\sum_{k=0}^{n-1}[8k+1]\frac{(q,a^3q,q/a^3,q^2;q^4)_k}{(q^4,q^4/a^3,a^3q^4,q^3;q^4)_k}q^k \\
&\quad\equiv q^{(1-3n)/2}[3n]+\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(q^{(1-3n)/2}[3n]-\frac{(q^5,q^3;q^4)_{(n-1)/2}}{(a^3q^4,q^4/a^3;q^4)_{(n-1)/2}}\right) \\
&\quad\equiv q^{(1-3n)/2}[3n].
\end{aligned} \tag{4.6}$$

This is because $(q^3;q^4)_{n-1}$ only contains the factor $\Phi_n(q)$, but does not contain the square of $\Phi_n(q)$. Finally, letting $n=p$ and $q\to 1$ in (4.6), we finish the proof of (1.8). □

5. Proof of Theorem 1.4

We can prove that (3.1) also holds for $n\equiv 3 \pmod 4$ and $d=1$, and so we can show that (6.5) holds modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)$. This, together with the $n\mapsto n^r$ case of Theorem 1.3, implies that

$$\sum_{k=0}^{n^r-1}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \equiv q^{(1-3n^r)/2}[3n^r] \pmod{\Phi_{n^r}(q)^3\prod_{j=1}^{r-1}\Phi_{n^j}(q)}. \tag{5.1}$$

The proof then follows from (5.1) by taking $n=p$ and $q\to 1$.

6. Proof of Theorems 1.5, 1.6 and Theorem 1.7

To prove Theorem 1.5, we need to establish the following parametric q-congruence.

Lemma 6.1. *Let $n \equiv 1 \pmod 4$ be an integer with $n > 1$ and let $r \geqslant 1$. Let a be an indeterminate. Then, modulo*

$$\left(\prod_{j=1}^{r} \Phi_{n^j}(q)\right) \left(\prod_{j=0}^{(n^{r-1}-1)/d} (1-aq^{(4j+1)n})(a-q^{(4j+1)n})\right),$$

we have

$$\begin{aligned}&\sum_{k=0}^{(n^r-1)/d} [8k+1]\frac{(aq, q/a, q, q^2; q^4)_k}{(aq^4, q^4/a, q^4, q^3; q^4)_k} q^k \\ &\quad\equiv q^{(1-n)/2}[n] \sum_{k=0}^{(n^{r-1}-1)/d} q^{nk}[8k+1]_{q^n} \frac{(aq^n, q^n/a, q^n, q^{2n}; q^{4n})_k}{(aq^{4n}, q^{4n}/a, q^{4n}, q^{3n}; q^{4n})_k},\end{aligned} \tag{6.1}$$

where $d = 1, 2$.

Proof. In view of (3.1), the right-hand side of (6.1) is congruent to 0 modulo $\Phi_n(q)\prod_{j=1}^{r-1}\Phi_{n^j}(q^n)$. Since $\Phi_{n^j}(q^n)$ is divisible by $\Phi_{n^{j+1}}(q)$, we see that both sides of (6.1) are congruent to 0 modulo $\prod_{j=1}^{r}\Phi_{n^j}(q)$.

In order to prove (6.1) modulo

$$\prod_{j=0}^{(n^{r-1}-1)/d} (1-aq^{(4j+1)n})(a-q^{(4j+1)n}), \tag{6.2}$$

we need the following identity: for $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/4} [8k+1]\frac{(q^{1-n}, q^{1+n}, q, q^2; q^4)_k}{(q^{4+n}, q^{4-n}, q^4, q^3; q^4)_k} q^k = q^{(1-n)/2}[n], \tag{6.3}$$

which is equivalent to the $b = 1$ case of (2.6).

For $a = q^{-(4j+1)n}$ or $a = q^{(4j+1)n}$ with $0 \leqslant j \leqslant (n^{r-1}-1)/d$, in view of (6.3), the left-hand side of (6.1) is equal to

$$\sum_{k=0}^{(n^r-1)/d} [8k+1]\frac{(q^{1-(4j+1)n}, q^{1+(4j+1)n}, q, q^2; q^4)_k}{(q^{4-(4j+1)n}, q^{4+(4j+1)n}, q^4, q^3; q^4)_k} q^k = q^{(1-(4j+1)n)/2}[(4j+1)n], \tag{6.4}$$

where we have used the facts that $(n^r-1)/d \geqslant ((4j+1)n-1)/4$ for $0 \leqslant j \leqslant (n^{r-1}-1)/d$, and $(q^{1-(4j+1)n}; q^4)_k = 0$ for $k > ((4j+1)n-1)/4$. Similarly, in this case the right-hand side of (6.1) is equal to

$$q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/d} q^{nk}[8k+1]_{q^n}\frac{(q^{-4jn},q^{(4j+2)n},q^n,q^{2n};q^{4n})_k}{(q^{(3-4j)n},q^{(4j+5)n},q^{4n},q^{3n};q^{4n})_k}$$
$$=q^{(1-n)/2}[n]q^{-2jn}[4j+1]_{q^n},$$

which is the same as the right-hand side of (6.4). This proves that both sides of (6.1) are equal for $a=q^{\pm(4j+1)n}$ with $0\leqslant j\leqslant (n^{r-1}-1)/d$. Namely, the q-congruence (6.1) is true modulo (6.2). □

Proof of Theorem 1.5. It is not hard to see that the limit of (6.2) as $a\to 1$ incorporates the factor

$$\begin{cases}\prod_{j=1}^r \Phi_{n^j}(q)^{2n^{r-j}} & \text{if } d=1,\\ \prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}+1}, & \text{if } d=2.\end{cases}$$

Furthermore, the denominator of the right-hand side of (6.1) divides that of the left-hand side of (6.1). The factor involving a in the latter is $(aq^4,q^4/a;q^4)_{(n^r-1)/d}$, the limit of which as a tends to 1 merely owns the following factor

$$\begin{cases}\prod_{j=1}^r \Phi_{n^j}(q)^{2n^{r-j}-2} & \text{if } d=1,\\ \prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}-1}, & \text{if } d=2.\end{cases}$$

related to $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^r}(q)$. Therefore, taking $a\to 1$ in (6.1), we conclude that (1.13) holds modulo $\prod_{j=1}^r \Phi_{n^j}(q)^3$, as desired. □

The proof of Theorem 1.6 is analogous, and we need to construct a parametric q-congruence as follows.

Lemma 6.2. *Let $n\equiv 3 \pmod 4$ be a positive integer and let $r\geqslant 1$. Let a be an indeterminate. Then, modulo*

$$\left(\prod_{j=1}^r \Phi_{n^j}(q)\right)\left(\prod_{j=0}^{n^{r-2}-1}(1-aq^{(4j+1)n^2})(a-q^{(4j+1)n^2})\right),$$

we have

$$\sum_{k=0}^{n^r-1}[8k+1]\frac{(aq,q/a,q,q^2;q^4)_k}{(aq^4,q^4/a,q^4,q^3;q^4)_k}q^k$$
$$\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{n^{r-2}-1}q^{n^2k}[8k+1]_{q^{n^2}}\frac{(aq^{n^2},q^{n^2}/a,q^{n^2},q^{2n^2};q^{4n^2})_k}{(aq^{4n^2},q^{4n^2}/a,q^{4n^2},q^{3n^2};q^{4n^2})_k}. \tag{6.5}$$

Proof. We have already mentioned in Section 5, the q-congruence (3.1) is also true for $n \equiv 3 \pmod 4$ and $d = 1$. This enables us to show that (6.5) holds modulo $\prod_{j=1}^{r} \Phi_{n^j}(q)$.

In order to prove (6.5) holds modulo

$$\prod_{j=0}^{n^{r-2}-1} (1 - aq^{(4j+1)n^2})(a - q^{(4j+1)n^2}), \tag{6.6}$$

it suffices to show that both sides of (6.5) are equal when $a = q^{-(4j+1)n^2}$ or $a = q^{(4j+1)n^2}$ for all $0 \leqslant j \leqslant n^{r-2} - 1$, namely,

$$\begin{aligned}
&\sum_{k=0}^{n^r-1} [8k+1] \frac{(q^{1-(4j+1)n^2}, q^{1+(4j+1)n^2}, q, q^2; q^4)_k}{(q^{4-(4j+1)n^2}, q^{4+(4j+1)n^2}, q^4, q^3; q^4)_k} q^k \\
&\quad = q^{(1-n^2)/2} [n^2] \sum_{k=0}^{n^{r-2}-1} q^{n^2 k} [8k+1]_{q^{n^2}} \frac{(q^{-4jn^2}, q^{(4j+2)n^2}, q^{n^2}, q^{2n^2}; q^{4n^2})_k}{(q^{(3-4j)n^2}, q^{(4j+5)n^2}, q^{4n^2}, q^{3n^2}; q^{4n^2})_k}.
\end{aligned} \tag{6.7}$$

It is easy to see that $n^r - 1 \geqslant ((4j+1)n^2 - 1)/4$ for $0 \leqslant j \leqslant n^{r-2} - 1$, and $(q^{1-(4j+1)n^2}; q^4)_k = 0$ for $k > ((4j+1)n^2 - 1)/4$. In light of (6.3), the two sides of (6.7) are both equal to

$$q^{(1-(4j+1)n^2)/2} [(4j+1)n^2] = q^{(1-n^2)/2} [n^2] \cdot q^{-2jn^2} [4j+1]_{q^{n^2}}.$$

This completes the proof of (6.5) modulo (6.6). □

Proof of Theorem 1.6. Obviously, the limit of (6.6) as $a \to 1$ contains the factor

$$\prod_{j=2}^{r} \Phi_{n^j}(q)^{2n^{r-j}}.$$

On the other hand, the denominator of the left-hand side of (6.5) is divisible by that of the right-hand side of (6.5). The factor of the former containing the indeterminate a is $(aq^4, q^4/a; q^4)_{n^r-1}$. Its limit as a tends to 1 only has the factor

$$\prod_{j=2}^{r} \Phi_{n^j}(q)^{2n^{r-j}-2}$$

that is relevant to $\Phi_{n^2}(q), \Phi_{n^3}(q), \ldots, \Phi_{n^r}(q)$. Hence, letting $a \to 1$ in (6.5), we complete the proof of (1.14). □

Proof of Theorem 1.7. For any positive integer n with $n \equiv 3 \pmod 4$, we have $n^2 \equiv 1 \pmod 4$. Putting $n \mapsto n^2$ and $r \mapsto r/2$ in (1.13), we acquire the following q-congruence: modulo $\prod_{j=1}^{r/2} \Phi_{n^{2j}}(q)^3$,

$$\begin{aligned}
&\sum_{k=0}^{(n^r-1)/d} [8k+1] \frac{(q; q^4)_k^3 (q^2; q^4)_k}{(q^4; q^4)_k^3 (q^3; q^4)_k} q^k \\
&\quad \equiv q^{(1-n^2)/2} [n^2] \sum_{k=0}^{(n^{r-2}-1)/d} q^{n^2 k} [8k+1]_{q^{n^2}} \frac{(q^{n^2}; q^{4n^2})_k^3 (q^{2n^2}; q^{4n^2})_k}{(q^{4n^2}; q^{4n^2})_k^3 (q^{3n^2}; q^{4n^2})_k},
\end{aligned} \tag{6.8}$$

where $d=1,2$. Letting $n=p$ be a prime and taking $q\to 1$ in (6.8), we get the desired supercongruence (1.15). $\square$

7. Concluding remarks and open problems

In this section, we put forward several related conjectures for further investigation. We first propose the following generalization of Theorem 1.2.

Conjecture 7.1. *Let $p\equiv 1 \pmod 4$ be a prime with $p>5$ and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv p^r \pmod{p^{r+5}}.$$

A natural idea to tackle Conjecture 7.1 is to give a generalization of (1.5) modulo $\Phi_n(q)^6$. However, it seems rather difficult to provide such a generalization, even in the modulus $\Phi_n(q)^5$ case.

Although we did not find a generalization of (1.8) modulo $\Phi_n(q)^4$, on the basis of numerical calculation, we believe that the following extension of (1.9) modulo p^{r+3} should be true.

Conjecture 7.2. *Let $p\equiv 3 \pmod 4$ be a prime and $r\geqslant 1$ an odd integer. Then*

$$\sum_{k=0}^{p^r-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv 3p^r+\frac{27p^{3r}}{4}\sum_{j=1}^{(p^r-3)/4}\frac{1}{j^2} \pmod{p^{r+3}}.$$

We have the following generalization of (1.11) for $d=2$. Note that the $r=1$ case is also open.

Conjecture 7.3. *Let $p\equiv 1 \pmod 4$ be a prime and let $r\geqslant 1$. Then*

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k} \pmod{p^{6r-\delta_{p,5}}},$$

where δ is the Kronecker delta with $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ otherwise.

We also observe that (1.12) has the following companion.

Conjecture 7.4. *Let $p\equiv 3 \pmod 4$ be a prime and let $r\geqslant 2$. Then*

$$\sum_{k=0}^{(p^r-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/2}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k} \pmod{p^{3r-1}}. \tag{7.1}$$

Although there exists a parametric q-congruence similar to (6.5) when the left-hand side is truncated at $(n^r-1)/2$ and the right-hand side is truncated at $(n^{r-2}-1)/2$, we cannot use this q-congruence to get a q-analogue of (7.1). Therefore, in order to confirm (7.1) we perhaps need to employ new methods and techniques.

It is easy to see that Theorem 1.1 implies the following q-supercongruence: for $n \equiv 1 \pmod 4$, modulo $\Phi_n(q)^3$,

$$\sum_{k=0}^{n-1}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \equiv q^{(1-n)/2}[n].$$

We end this paper with the following two challenging conjectures.

Conjecture 7.5. *Let m and n be positive integers with $n \equiv 1 \pmod 4$ and $n>1$. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{mn-1}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \equiv q^{(1-n)/2}[n]\sum_{k=0}^{m-1}(8k+1)\frac{(\frac{1}{4})_k^3(\frac{1}{2})_k}{k!^3(\frac{3}{4})_k}.$$

Conjecture 7.6. *Let m and n be positive odd integers with $n \equiv 1 \pmod 4$ and $n>1$. Then, modulo $\Phi_n(q)^4$,*

$$\begin{aligned}
&\sum_{k=0}^{(mn-1)/2}[8k+1]\frac{(q;q^4)_k^3(q^2;q^4)_k}{(q^4;q^4)_k^3(q^3;q^4)_k}q^k \\
&\quad\equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\} \\
&\quad\quad\times\sum_{k=0}^{(m-1)/2}q^{n^2k}[8k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^3(q^{2n^2};q^{4n^2})_k}{(q^{4n^2};q^{4n^2})_k^3(q^{3n^2};q^{4n^2})_k}.
\end{aligned}$$

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