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Two new q-congruences from Gasper's Karlsson–Minton type summation

Victor J. W. GUO, Xin ZHAO

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
jwguo@hznu.edu.cn, zhaoxin1@stu.hznu.edu.cn (corresponding author)

Abstract. We give two new q-congruences by using the method of "creative microscoping" and Gaspers Karlsson–Minton type summation. In particular, we present a q-analogue of a congruence of Barman and Saikia.

***Keywords**:* supercongruence; p-adic Gamma function; cyclotomic polynomials; creative microscoping.
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1. INTRODUCTION

Rodriguez-Villegas [12] studied hypergeometric families of Calabi–Yau manifolds, and found a number of possible supercongruences. For instance, he observed that, for any prime $p > 2$,

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{1}$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ $(n \geqslant 1)$ is the *rising factorial*. Mortenson [11] first confirmed the congruence (1). Later, the first author and Zeng [4] obtained a q-analogue of (1):

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} \equiv (-1)^{(p-1)/2} q^{(p^2-1)/4} \pmod{[p]^2} \quad \text{for any odd prime } p. \tag{2}$$

Here and throughout the paper, $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ $(n \geqslant 1)$ is the *q-shifted factorial*, and $[n] = 1+q+\cdots+q^{n-1}$ is the *q-integer*. For convenience, we will also adopt the condensed notation $(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$.

In 2020, Barman and Saikia [1] gave a generalization of (1) as follows: for $d \geqslant 1$ and any prime p satisfying $p \equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv (-1)^{d+1} \Gamma_p(\tfrac{1}{d})^d \Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}, \tag{3}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function (see [8]).

Let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, which can be written as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}} (q-\zeta^k),$$

where ζ is an n-th primitive root of unity. The first aim of this note is to give the following q-analogue of Barman and Saikia's congruence (3).

THEOREM 1. *Let d and n be positive integers with $n\equiv 1 \pmod{d^2+d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d;q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1};q^{d^2+d})_k^d (q^{d^2+d};q^{d^2+d})_k}$$
$$\equiv \frac{(-1)^{(n-1)/(1+d)} (q^{d^2+d};q^{d^2+d})_{(n-1)/(1+d)} q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}}. \tag{4}$$

For n prime, letting $q\to 1$ in Theorem 1, we arrive the following congruence: for $d\geqslant 1$ and any prime $p\equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!}{(\frac{1}{d})^d_{(p-1)/(d^2+d)}} \pmod{p^2}. \tag{5}$$

In view of properties of p-adic Gamma functions (see [10, Section 2]), it is not hard to show that

$$\frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!}{(\frac{1}{d})^d_{(p-1)/(d^2+d)}} \equiv (-1)^{d+1}\Gamma_p(\tfrac{1}{d})^d\Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}. \tag{6}$$

Hence, the congruence (5) is equivalent to (3).

We shall also establish the following congruence similar to (3).

THEOREM 2. *Let $d\geqslant 1$ and let p be a prime with $p\equiv 1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{k(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{d+2}}{2(d^2+d)}\Gamma_p(\tfrac{1}{d})^d\Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}. \tag{7}$$

Since $\Gamma_p(1)=-1$ and $\Gamma_p(\frac12)^2=(-1)^{(p+1)/2}$, for $d=1$, the congruence (7) reduces to

$$\sum_{k=0}^{p-1} \frac{k(\frac12)_k^2}{k!^2} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2}, \tag{8}$$

of which a generalization modulo p^3 for $p>3$ has already been given by Sun [13, Theorem 1.2, (1.8) and (1.10)].

It is easy to see that, for any prime $p>2$,

$$\sum_{k=0}^{(p+1)/2} \frac{(-\frac12)_k^2}{k!^2} = \frac{2p+3}{4^{p+1}}\binom{p+1}{(p+1)/2}^2 \equiv 0 \pmod{p^2}. \tag{9}$$

The last aim of this note is to give the following generalization of (9).

THEOREM 3. *Let d and n be positive integers with $n\equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(n+1)/(d+1)} \frac{(q^{-d};q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1};q^{d^2+d})_k^d (q^{d^2+d};q^{d^2+d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{10}$$

In particular, letting n be prime and taking $q\to 1$ in Theorem 3, we are led to the conclusion.

COROLLARY 1. *Let $d\geqslant 1$ and let p be a prime with $p\equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{(-\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv 0 \pmod{p^2}.$$

Similarly to the proof of Theorem 2, we can also deduce the following congruence from Theorem 3.

COROLLARY 2. *Let $d\geqslant 1$ and let p be a prime with $p\equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{k(-\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv 0 \pmod{p^2}.$$

2. PROOF OF THEOREM 1

We will make use of Gasper's Karlsson–Minton type summation (see [2, (1.9.9)]; and see [3, (5.13)] for a generalization): for all non-negative integers $n_1,\dots,n_m$,

$$\sum_{k=0}^{N} \frac{(q^{-N},b_1q^{n_1},\dots,b_mq^{n_m};q)_k}{(q,b_1,\dots,b_m;q)_k} q^k = (-1)^N \frac{(q;q)_N b_1^{n_1}\cdots b_m^{n_m}}{(b_1;q)_{n_1}\cdots(b_m;q)_{n_m}} q^{\binom{n_1}{2}+\cdots+\binom{n_m}{2}}, \tag{11}$$

where $N=n_1+\cdots+n_m$. For some recent congruences and q-congruences related to (11), see [5, 7, 9].

We first build the following generalization of Theorem 1 with an extra parameter a by employing the "creative microscoping" method devised in [6].

THEOREM 4. *Let $d,n>1$ be integers with $n\equiv 1 \pmod{d^2+d}$. Let a be an indeterminate. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/(d+1)} \frac{(a^dq^d,a^{d-2}q^d,\dots,aq^d;q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{d+1};q^{d^2+d})_k}\\
&\quad\times \frac{(a^{-d}q^d,a^{2-d}q^d,\dots,a^{-1}q^d;q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-2}q^{d+1};q^{d^2+d})_k (q^{d^2+d};q^{d^2+d})_k}\\
&\quad\equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}}\\
&\qquad\times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-2}q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}}
\end{aligned} \tag{12}$$

if d is odd, and

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(a^d q^d, a^{d-2} q^d, \ldots, a^2 q^d, q^d; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \ldots, a q^{d+1}; q^{d^2+d})_k}$$
$$\times \frac{(a^{-d} q^d, a^{2-d} q^d, \ldots, a^{-2} q^d; q^{d^2+d}) q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \ldots, a^{-1} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}$$
$$\equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \ldots, a q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}}$$
$$\times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \ldots, a^{-1} q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \tag{13}$$

if d is even.

Proof. It is obvious that $\gcd(d,n)=1$, and therefore none of the numbers $d, 2d, \ldots (n-1)d$ are divisible by n. This indicates that the denominators on the left-hand side of (12) do not have the factor $1-aq^n$ nor $1-a^{-1}q^n$. Thus, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (12) may be written as

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d}, q^{-(n-1)d+2n}, \ldots, q^{-n+d}; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \ldots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_k}$$
$$\times \frac{(q^{(n+1)d}, q^{(n+1)d-2n}, \ldots, q^{n+d}; q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \ldots, q^{2n+d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \tag{14}$$

Letting $q \mapsto q^{d^2+d}$, $N=(n-1)/(1+d)$, $m=d$, $b_j = q^{-(d-1)n+d+1+(j-1)2n}$ and $n_j=(n-1)/(d^2+d)$ $(1 \leqslant j \leqslant d)$ in (11), we conclude that (12) is equal to

$$\frac{(-1)^{(n-1)/(1+d)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(1+d)}}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \ldots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}}$$
$$\times \frac{q^{(n-1)+d(d^2+d)\binom{(n-1)/(d^2+d)}{2}}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \ldots, q^{2n+d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}}. \tag{15}$$

which is just the $a=q^{-n}$ or $a=q^n$ case of the right-hand side (12). This proves the q-congruence (12).

Similarly, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (13) may be expressed as

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d}, q^{-(n-1)d+2n}, \ldots, q^{-2n+d}, q^d; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \ldots, q^{-n+d+1}; q^{d^2+d})_k}$$
$$\times \frac{(q^{(n+1)d}, q^{(n+1)d-2n}, \ldots, q^{2n+d}; q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \ldots, q^{n+d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \tag{16}$$

Letting $q \mapsto q^{d(1+d)}$, $N=(n-1)/(1+d)$, $m=d$, $b_j = q^{-(d-1)n+d+1+(j-1)2n}$ and $n_j=(n-1)/(d^2+d)$ $(1 \leqslant j \leqslant d)$ in (11), we deduce that (16) is equal to the $a=q^{-n}$ or $a=q^n$ case of the right-hand side (13). This establishes (13). □

Proof of Theorem 1. Note that $\Phi_n(q)$ is a factor of $1-q^m$ if and only if m is divisible by n. Hence, when $a=1$ the denominators of (12) are all coprime with $\Phi_n(q)$. Meanwhile, when $a=1$ the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ incorporates the factor $\Phi_n(q)^2$. The proof of (4) then follows immediately from the $a=1$ case of (12) and (13). □

3. PROOF OF THEOREM 2

Let $n>1$ be an integer with $n\equiv 1 \pmod{d^2+d}$. Performing the substitution $q\mapsto q^{-1}$ in (4), we get its dual form: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/(d+1)}\frac{(q^d;q^{d^2+d})_k^{d+1}}{(q^{d+1};q^{d^2+d})_k^d(q^{d^2+d};q^{d^2+d})_k}\equiv\frac{(-1)^{(n-1)/(1+d)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(1+d)}q^{\frac{(1-n)(nd+d^2)}{2(d+1)}}}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}}. \tag{17}$$

Subtracting (4) from (17) and dividing both sides by $1-q$, we are led to

$$\begin{aligned}&\sum_{k=0}^{(n-1)/(d+1)}\frac{(q^d;q^{d^2+d})_k^{d+1}(1-q^{(d^2+d)k})}{(q^{d+1};q^{d^2+d})_k^d(1-q)}\\&\quad\equiv\frac{(-1)^{(n-1)/(1+d)}(q^d;q^{d^2+d})_{(n-1)/(1+d)}q^{\frac{(1-n)(nd+d^2)}{2(d+1)}}(1-q^{\frac{(n-1)(nd+d+n+1)}{2(d+1)}})}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}(1-q)}\pmod{\Phi_n(q)^2}.\end{aligned}$$

Letting $n=p$ be a prime and taking the limit as $q\to 1$ in the above q-supercongruence, we obtain the following result: for any positive integer d and prime $p\equiv 1 \pmod{d^2+d}$,

$$\begin{aligned}\sum_{k=0}^{(p-1)/(d+1)}\frac{k(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!}&\equiv\frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!(p^2-1)}{2(\frac{1}{d})^d_{(p-1)/(d^2+d)}(d+1)(d^2+d)}\\&\equiv\frac{(-1)^{(p+d)/(d+1)}(\frac{p-1}{d+1})!}{2(\frac{1}{d})^d_{(p-1)/(d^2+d)}(d^2+d)}.\end{aligned}$$

The proof then follows from the congruence (6).

4. PROOF OF THEOREM 3

We will utilize another Karlsson–Minton type summation due to Gasper (see [2, (1.9.11)]): for all non-negative integers $n_1,\ldots,n_m$,

$$\sum_{k=0}^{N}\frac{(q^{-N},b_1q^{n_1},\ldots,b_mq^{n_m};q)_k}{(q,b_1,\ldots,b_m;q)_k}q^k=0. \tag{18}$$

where $N>n_1+\cdots+n_m$.

We first establish the following parametric generalization of Theorem 3.

THEOREM 5. *Let $d,n>1$ be integers with $n\equiv 2d+1 \pmod{d^2+d}$. Let a be an indeterminate. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\begin{aligned}&\sum_{k=0}^{(n+1)/(d+1)}\frac{(a^dq^{-d},a^{d-2}q^{-d},\ldots,aq^{-d};q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\ldots,a^2q^{d+1},q^{d+1};q^{d^2+d})_k}\\&\quad\times\frac{(a^{-d}q^{-d},a^{2-d}q^{-d},\ldots,a^{-1}q^{-d};q^{d^2+d})_kq^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\ldots,a^{-2}q^{d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}=0\end{aligned}\tag{19}$$

if d is odd, and

$$\sum_{k=0}^{(n+1)/(d+1)} \frac{(a^d q^{-d}, a^{d-2} q^{-d}, \dots, a^2 q^{-d}, q^{-d}; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a q^{d+1}; q^{d^2+d})_k} \times \frac{(a^{-d} q^{-d}, a^{2-d} q^{-d}, \dots, a^{-2} q^{-d}; q^{d^2+d}) q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-1} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k} = 0 \qquad (20)$$

if d is even.

Proof. It is easy to see that $\gcd(d,n)=1$ and so none of the numbers $d, 2d, \dots (n-1)d$ are multiples of n. This implies that the denominators of the left-hand sides of (19) have no factors $1-aq^n$ and $1-a^{-1}q^n$. Therefore, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (19) can be expressed as

$$\sum_{k=0}^{(n+1)/(d+1)} \frac{(q^{-(n+1)d}, q^{-(n+1)d+2n}, \dots, q^{-n+d}; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \dots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_k} \times \frac{(q^{(n-1)d}, q^{(n-1)d-2n}, \dots, q^{n-d}; q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \dots, q^{2n+d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \qquad (21)$$

Letting $q \mapsto q^{d^2+d}$, $N = nd+d$, $m=d$, $b_j = q^{-(d-1)n+d+1+(j-1)2n}$ and $n_j = (n-2d-1)/(d^2+d)$ $(1 \leqslant j \leqslant d)$ in (18), we conclude that (21) is equal to 0, which is just the $a=q^{-n}$ or $a=q^n$ case of the right-hand side of (19). Namely, the congruence (19) holds. Exactly in the same way, we can prove the q-congruence (20). ☐

Proof of Theorem 3. When $a=1$, the polynomial $(1-aq^n)(a-q^n)$ contains the factor $\Phi_n(q)^2$, which is coprime with the denominators of the left-hand sides of (19) and (20). Hence, the congruence (10) immediately follows from the $a=1$ case of (19) and (20). ☐

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