Three new q-supercongruences from Jackson's summation and Watson's transformation

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Abstract. We present three q-supercongruences modulo the fifth power of a cyclotomic polynomial by using Jackson's ${}_8\phi_7$ summation and Watson's ${}_8\phi_7$ transformation, together with the creative microscoping method introduced in [Adv. Math. 346 (2019), 329–358]. As conclusions, we give a partial q-analogue of a supercongruence of Barman and Saikia, and a complete q-analogue of the supercongruence:

$$\sum_{k=0}^{(p-1)/4}(16k+1)\frac{(\frac{1}{8})_k(\frac{1}{4})_k^5}{k!(\frac{7}{8})_k^5}\equiv -\frac{5p^3}{64}\Gamma_p(\tfrac{7}{8})^6\Gamma_p(\tfrac{3}{8})^{10} \pmod{p^5},$$

where $p\equiv 5 \pmod 8$ is a prime, $(x)_k$ is the Pochhammer symbol, and $\Gamma_p(x)$ is the p-adic Gamma function.

Keywords: q-supercongruences; creative microscoping; Chinese remainder theorem for polynomials; p-adic Gamma function; Jackson's ${}_8\phi_7$ summation; Watson's ${}_8\phi_7$ transformation

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1. Introduction

For any complex number x, the *Pochhammer symbol* is defined by $(x)_0=1$ and $(x)_k=x(x+1)\cdots(x+k-1)$ for $k\geqslant 1$. It is well known that $(x)_k=\Gamma(x+k)/\Gamma(x)$, where $\Gamma(x)$ denotes the classical *Gamma function*. For any odd prime p, let $\mathbb{Z}_p$ be the ring of p-adic integers. The p-adic Gamma function Γ_p is defined as $\Gamma_p(0)=1$ and

$$\Gamma_p(n)=(-1)^n\prod_{0<k<n;\ p\nmid k}k.$$

This function can be uniquely extended to a continuous function $\Gamma_p\colon \mathbb{Z}_p\to\mathbb{Z}_p^{\times}$. For any $x\in\mathbb{Z}_p$ and $x\neq 0$, we define

$$\Gamma_p(x)=\lim_{x_n\to x}\Gamma_p(x_n),$$

where x_n ranges over any sequence of positive integers that p-adically approximate x.

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In 1997, Van Hamme [12, (D.2)] observed the following supercongruence: for any prime $p \equiv 1 \pmod 6$,

$$\sum_{k=0}^{(p-1)/3} (6k+1)\frac{(\frac{1}{3})_k^6}{k!^6} \equiv -p\Gamma_p(\tfrac{1}{3})^9 \pmod{p^4}, \tag{1.1}$$

In 2006, Long and Ramakrishna [10, Theorem 2] proved that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p(\frac{1}{3})^9 \pmod{p^6}, & \text{if } p \equiv 1 \pmod 6, \\ -\frac{10}{27}p^4\Gamma_p(\frac{1}{3})^9 \pmod{p^6}, & \text{if } p \equiv 5 \pmod 6, \end{cases} \tag{1.2}$$

thus confirming Van Hamme's supercongruence (1.1). Recently, some authors have given different generalizations of (1.2) (see [5, 8, 14, 16, 17]). In particular, by making use of Jackson's ${}_8\phi_7$ summation formula, the method of "creative microscoping" devised by the first author and Zudilin [7], and the Chinese remainder theorem for polynomials, Wei [16] gave a q-analogue of the second part in (1.2): for any positive integer $n \equiv 2 \pmod 3$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^6}{(q^3;q^3)_k^6}q^{3k} \equiv 5[2n]\frac{(q^2;q^3)_{(2n-1)/3}^3}{(q^3;q^3)_{(2n-1)/3}^3} \pmod{[n]\Phi_n(q)^5}. \tag{1.3}$$

Meanwhile, he also gave a q-analogue of a weaker form of the first part in (1.2), where the modulus p^6 is replaced by p^5. At the moment being, we need to recall the standard q-notation. The *q-shifted factorial* is defined as $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for all positive integers n, the *q-integer* is defined by $[n] = (1-q^n)/(1-q)$. For convenience, we shall also adopt the abbreviated notation for products of q-shifted factorials: $(a_1,\dots,a_m;q)_n = (a_1;q)_n\dots(a_m;q)_n$. Moreover, let $\Phi_n(q)$ be the n-th *cyclotomic polynomial*, which can be factorized as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ denotes an nth primitive root of unity. It is clear that $\Phi_p(q) = [p]$ for any prime p. For more recent results on q-congruences, see [3, 4, 11, 13, 15].

In 2020, motivated by Long and Ramakrishna's work, Barman and Saikia [1, Theorem 1.4] proved that, for any prime $p \equiv 1 \pmod 8$,

$$\sum_{k=0}^{7(p-1)/8}(16k+1)\frac{(\frac{1}{8})_k(\frac{1}{4})_k^5}{k!(\frac{7}{8})_k^5} \equiv -p\Gamma_p(\tfrac{7}{8})^6\Gamma_p(\tfrac{3}{8})^{10} \pmod{p^6}. \tag{1.4}$$

Note that we may truncate the left-hand side of (1.4) at $k = (p-1)/4$, since the p-adic order of $(\frac{1}{8})_k(\frac{1}{4})_k^5/(k!(\frac{7}{8})_k^5)$ is 6 for $(p-1)/4 < k \leqslant 7(p-1)/8$.

Inspired by Wei's work [16], we shall establish the following q-supercongruence, which is a q-analogue of (1.4) modulo p^5.

Theorem 1.1. *Let $n \equiv 1 \pmod 8$ be a positive integer. Then, modulo $\Phi_n(q)^5$,*

$$\sum_{k=0}^{(n-1)/4} [16k+1] \frac{(q;q^8)_k (q^2;q^8)_k^5}{(q^7;q^8)_k^5 (q^8;q^8)_k} q^{8k}$$
$$\equiv \frac{(q^9;q^8)_{(n-1)/4} (q^5;q^8)_{(n-1)/4}^3}{(q^3;q^8)_{(n-1)/4} (q^7;q^8)_{(n-1)/4}^3} \left\{ 1 + [2n]^2 (2-q^{2n}) \sum_{j=1}^{(n-1)/4} \left(\frac{q^{8j-3}}{[8j-3]^2} - \frac{q^{8j-1}}{[8j-1]^2} \right) \right\}. \tag{1.5}$$

Letting $n = p^r$ be a prime power with $p \equiv 1 \pmod 8$ and taking the limits as $q \to 1$ in (1.5), we obtain the following conclusion.

Corollary 1.2. *Let $p \equiv 1 \pmod 8$ be a prime and r a positive integer. Then*

$$\sum_{k=0}^{(p^r-1)/4} (16k+1) \frac{(\frac{1}{8})_k (\frac{1}{4})_k^5}{k! (\frac{7}{8})_k^5}$$
$$\equiv \frac{(\frac{9}{8})_{(p^r-1)/4} (\frac{5}{8})_{(p^r-1)/4}^3}{(\frac{3}{8})_{(p^r-1)/4} (\frac{7}{8})_{(p^r-1)/4}^3} \left\{ 1 + 4p^{2r} \sum_{j=1}^{(p^r-1)/4} \left(\frac{1}{(8j-3)^2} - \frac{1}{(8j-1)^2} \right) \right\} \pmod{p^5}. \tag{1.6}$$

Combining the supercongruence (1.4) modulo p^5 and the $r=1$ case of (1.6), we are led to the following corollary.

Corollary 1.3. *Let $p \equiv 1 \pmod 8$ be a prime. Then, modulo p^5,*

$$\frac{(\frac{9}{8})_{(p-1)/4} (\frac{5}{8})_{(p-1)/4}^3}{(\frac{3}{8})_{(p-1)/4} (\frac{7}{8})_{(p-1)/4}^3} \left\{ 1 + 4p^2 \sum_{j=1}^{(p-1)/4} \left(\frac{1}{(8j-3)^2} - \frac{1}{(8j-1)^2} \right) \right\} \equiv -p\Gamma_p(\tfrac{7}{8})^6 \Gamma_p(\tfrac{3}{8})^{10}.$$

It should be pointed out that the q-supercongruence (1.5) is also true for $n \equiv 5 \pmod 8$. However, in this case the result can be simplified as follows.

Theorem 1.4. *Let $n \equiv 5 \pmod 8$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/4} [16k+1] \frac{(q;q^8)_k (q^2;q^8)_k^5}{(q^7;q^8)_k^5 (q^8;q^8)_k} q^{8k} \equiv 5 \frac{(q^9;q^8)_{(n-1)/4} (q^5;q^8)_{(n-1)/4}^3}{(q^3;q^8)_{(n-1)/4} (q^7;q^8)_{(n-1)/4}^3} \pmod{\Phi_n(q)^5}. \tag{1.7}$$

From the above result we shall deduce the following conclusion.

Corollary 1.5. *Let $p \equiv 5 \pmod 8$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/4} (16k+1) \frac{(\frac{1}{8})_k (\frac{1}{4})_k^5}{k! (\frac{7}{8})_k^5} \equiv -\frac{5p^3}{64} \Gamma_p(\tfrac{7}{8})^6 \Gamma_p(\tfrac{3}{8})^{10} \pmod{p^5}. \tag{1.8}$$

Not like (1.4), numerical computations imply that the supercongruence (1.8) does not hold modulo p^6 in general.

We shall also give the following companion of (1.5), which seems a little complicated.

Theorem 1.6. *Let $n \equiv 3 \pmod 4$ be an integer with $n > 3$. Then, modulo $\Phi_n(q)^5$,*

$$\sum_{k=0}^{(n+1)/4} [16k-1] \frac{(q^{-1};q^8)_k (q^{-2};q^8)_k^5}{(q^9;q^8)_k^5 (q^8;q^8)_k} q^{24k}$$

$$\equiv -q^{(n-1)/2} \frac{(q^7, q^{-5};q^8)_{(n+1)/4}}{(q^9, q^{-3};q^8)_{(n+1)/4}} \sum_{k=0}^{(n+1)/4} \frac{(q^{11};q^8)_k (q^{-2};q^8)_k^3}{(q^8, q^{-5};q^8)_k (q^9;q^8)_k^2} q^{8k}$$

$$\times \left\{ 1 - [2n]^2 (2-q^{2n}) \sum_{j=1}^{k} \left(\frac{q^{8j-10}}{[8j-10]^2} + \frac{q^{8j+1}}{[8j+1]^2} \right) \right\}. \qquad (1.9)$$

It is not difficult to see that the left-hand side of (1.7) is congruent to 0 modulo $\Phi_n(q)^3$. From Theorem 1.6 we can derive a similar result as follows.

Corollary 1.7. *Let $n \equiv 3 \pmod 8$ be an integer with $n > 3$. Then*

$$\sum_{k=0}^{(n+1)/4} [16k-1] \frac{(q^{-1};q^8)_k (q^{-2};q^8)_k^5}{(q^9;q^8)_k^5 (q^8;q^8)_k} q^{24k} \equiv 0 \pmod{\Phi_n(q)^3}. \qquad (1.10)$$

The paper is organized as follows. In the next section, we shall first give a parametric version of Theorems 1.1 and 1.4 by employing the creative microscoping method introduced in [7], together with the Chinese remainder theorem for polynomials; then we deduce Theorems 1.1 and 1.4 from this parametric version. We shall prove Corollary 1.5 in Section 3 by utilizing some basic properties of the p-adic Gamma function. The proof of Theorem 1.6 will be presented in Section 4. Finally, in Section 5, we will give a proof of Corollary 1.7. Note that Jackson's ${}_8\phi_7$ summation and Watson's ${}_8\phi_7$ transformation will also play important roles in this paper.

2. Proof of Theorems 1.1 and 1.4

Recall that the basic hypergeometric ${}_{r+1}\phi_r$ series (see [2]) is defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_0, a_1, \dots, a_r;q)_k}{(q, b_1, \dots, b_r;q)_k} z^k.$$

Then Jackson's ${}_8\phi_7$ summation [2, Appendix (II.22)] can be stated as follows:

$${}_8\phi_7 \left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, q \right]$$

$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \tag{2.1}$$

where $a^2q = bcdeq^{-n}$.

In order to prove Theorem 1.1, we need to give the following identity.

Lemma 2.1. *Let $n \equiv 1 \pmod 4$ be a positive integer with $n > 1$. Then*

$$\sum_{k=0}^{(n-1)/4} \frac{1-q^{16k+1-n}}{1-q^{1-n}} \frac{(aq^2, q^2/a, bq^2, q^2/b, q^{1-n}, q^{2-2n}; q^8)_k}{(q^{7-n}/a, aq^{7-n}, q^{7-n}/b, bq^{7-n}, q^{7+n}, q^8; q^8)_k} q^{8k} = 0. \tag{2.2}$$

Proof. Letting $q \mapsto q^8$, and taking $a = q^{1-n}$, $b = aq^2$, $c = q^2/a$, $d = bq^2$, $e = q^2/b$ and $n \mapsto (n-1)/4$ in (2.1), we see that the left-hand side of (2.2) is equal to

$$\frac{(q^{9-n}, q^{5-n}, q^{5-n}/ab, aq^{5-n}/b; q^8)_{(n-1)/4}}{(q^{7-n}/a, aq^{7-n}, q^{7-n}/b, q^{3-n}/b; q^8)_{(n-1)/4}} = 0.$$

The is because $(q^{9-n}, q^{5-n}; q^8)_{(n-1)/4}$ in the numerator vanishes, while the denominator is not equal to 0. □

With the help of Lemma 2.1, we can give a parametric version of Theorems 1.1 and 1.4.

Lemma 2.2. *Let $n \equiv 1 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-aq^{2n})(a-q^{2n})(1-bq^{2n})(b-q^{2n})$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/4} [16k+1] \frac{(aq^2, q^2/a, bq^2, q^2/b, q^2, q; q^8)_k}{(q^7/a, aq^7, q^7/b, bq^7, q^7, q^8; q^8)_k} q^{8k} \\
&\quad \equiv \frac{(1-bq^{2n})(b-q^{2n})(-1-a^2+aq^{2n})}{(a-b)(1-ab)} \frac{(bq^5, q^5/b, q^5, q^9; q^8)_{(n-1)/4}}{(q^7/b, bq^7, q^7, q^3; q^8)_{(n-1)/4}} \\
&\qquad + \frac{(1-aq^{2n})(a-q^{2n})(-1-b^2+bq^{2n})}{(b-a)(1-ba)} \frac{(aq^5, q^5/a, q^5, q^9; q^8)_{(n-1)/4}}{(q^7/a, aq^7, q^7, q^3; q^8)_{(n-1)/4}}.
\end{aligned} \tag{2.3}$$

Proof. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, the identity (2.2) immediately leads to

$$\sum_{k=0}^{(n-1)/4} [16k+1] \frac{(aq^2, q^2/a, bq^2, q^2/b, q^2, q; q^8)_k}{(q^7/a, aq^7, q^7/b, bq^7, q^7, q^8; q^8)_k} q^{8k} \equiv 0 \pmod{\Phi_n(q)}.$$

Moreover, the right-hand side of (2.3) is also congruent to 0 because $(q^5, q^9; q^8)_{(n-1)/4}$ in the numerator contains the factor $1-q^n$, while the denominator is coprime with $\Phi_n(q)$. This implies that the q-congruence (2.3) holds modulo $\Phi_n(q)$.

For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (2.3) can be written as

$$\sum_{k=0}^{(n-1)/4}[16k+1]\frac{(q^{2-2n},q^{2+2n},bq^2,q^2/b,q^2,q;q^8)_k}{(q^{7+2n},q^{7-2n},q^7/b,bq^7,q^7,q^8;q^8)_k}q^{8k}$$
$$={}_8\phi_7\left[\begin{matrix} q, & q^{\frac{17}{2}}, & -q^{\frac{17}{2}}, & q^2, & bq^2, & q^2/b, & q^{2+2n}, & q^{2-2n} \\ & q^{\frac{1}{2}}, & -q^{\frac{1}{2}}, & q^7, & q^7/b, & bq^7, & q^{7-2n}, & q^{7+2n} \end{matrix};q^8,\ q^8\right]. \qquad (2.4)$$

By Jackson's summation (2.1), the right-hand side of (2.4) is equal to

$$\frac{(bq^5,q^5/b,q^5,q^9;q^8)_{(n-1)/4}}{(q^7/b,bq^7,q^7,q^3;q^8)_{(n-1)/4}}.$$

Since the polynomial $1-aq^{2n}$ is coprime with the polynomial $a-q^{2n}$, we immediately obtain the following q-congruence: modulo $(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n-1)/4}[16k+1]\frac{(aq^2,q^2/a,bq^2,q^2/b,q^2,q;q^8)_k}{(q^7/a,aq^7,q^7/b,bq^7,q^7,q^8;q^8)_k}q^{8k}\equiv\frac{(bq^5,q^5/b,q^5,q^9;q^8)_{(n-1)/4}}{(q^7/b,bq^7,q^7,q^3;q^8)_{(n-1)/4}}. \qquad (2.5)$$

Noting that the left-hand side of (2.5) is symmetric in a and b, we deduce from (2.5) that, modulo $(1-bq^{2n})(b-q^{2n})$,

$$\sum_{k=0}^{(n-1)/4}[16k+1]\frac{(aq^2,q^2/a,bq^2,q^2/b,q^2,q;q^8)_k}{(q^7/a,aq^7,q^7/b,bq^7,q^7,q^8;q^8)_k}q^{8k}\equiv\frac{(aq^5,q^5/a,q^5,q^9;q^8)_{(n-1)/4}}{(q^7/a,aq^7,q^7,q^3;q^8)_{(n-1)/4}}. \qquad (2.6)$$

It is obvious that $\Phi_n(q)$, $(1-aq^{2n})(a-q^{2n})$, and $(1-bq^{2n})(b-q^{2n})$ are pairwise coprime polynomials in q. Furthermore, we have the following relation

$$\frac{(1-bq^{2n})(b-q^{2n})(-1-a^2+aq^{2n})}{(a-b)(1-ab)}\equiv 1 \pmod{(1-aq^{2n})(a-q^{2n})}. \qquad (2.7)$$

By making use of the Chinese remainder theorem for polynomials, from (2.5), (2.6), (2.7) and its dual form ($a\leftrightarrow b$), we are led to the q-congruence (2.3). □

We are now able to prove Theorems 1.1 and 1.4.

Proof of Theorem 1.1. Since $1-q^{2n}$ contains the factor $\Phi_n(q)$, letting $b=1$ in (2.3), we have the following q-congruence: modulo $\Phi_n(q)^3(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n-1)/4}[16k+1]\frac{(aq^2,q^2/a,q^2,q^2,q^2,q;q^8)_k}{(q^7/a,aq^7,q^7,q^7,q^7,q^8;q^8)_k}q^{8k}$$
$$\equiv\frac{(1-q^{2n})^2(1+a^2-aq^{2n})}{(1-a)^2}\frac{(q^5;q^8)^3_{(n-1)/4}(q^9;q^8)_{(n-1)/4}}{(q^7;q^8)^3_{(n-1)/4}(q^3;q^8)_{(n-1)/4}}$$

$$
\begin{aligned}
&-\frac{(1-aq^{2n})(a-q^{2n})(2-q^{2n})}{(1-a)^2}\frac{(aq^5,q^5/a,q^5,q^9;q^8)_{(n-1)/4}}{(q^7/a,aq^7,q^7,q^3;q^8)_{(n-1)/4}}\\
&=(1-q^{2n})^2\frac{(q^5;q^8)^3_{(n-1)/4}(q^9;q^8)_{(n-1)/4}}{(q^7;q^8)^3_{(n-1)/4}(q^3;q^8)_{(n-1)/4}}\\
&\quad+\frac{a(1-q^{2n})^2(2-q^{2n})}{(1-a)^2}\frac{(q^5;q^8)^3_{(n-1)/4}(q^9;q^8)_{(n-1)/4}}{(q^7;q^8)^3_{(n-1)/4}(q^3;q^8)_{(n-1)/4}}\\
&\quad-\frac{(1-aq^{2n})(a-q^{2n})(2-q^{2n})}{(1-a)^2}\frac{(aq^5,q^5/a,q^5,q^9;q^8)_{(n-1)/4}}{(q^7/a,aq^7,q^7,q^3;q^8)_{(n-1)/4}}. \qquad (2.8)
\end{aligned}
$$

By L'Hôpital's rule, we get

$$
\begin{aligned}
&\lim_{a\to 1}\left\{\frac{a(1-q^{2n})^2}{(1-a)^2}\frac{(q^5;q^8)^2_{(n-1)/4}}{(q^7;q^8)^2_{(n-1)/4}}-\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\frac{(aq^5,q^5/a;q^8)_{(n-1)/4}}{(q^7/a,aq^7;q^8)_{(n-1)/4}}\right\}\\
&=\frac{(q^5;q^8)^2_{(n-1)/4}}{(q^7;q^8)^2_{(n-1)/4}}\left\{q^{2n}+[2n]^2\sum_{j=1}^{(n-1)/4}\left(\frac{q^{8j-3}}{[8j-3]^2}-\frac{q^{8j-1}}{[8j-1]^2}\right)\right\}.
\end{aligned}
$$

Therefore, taking $a\to 1$ in (2.8) and applying the above limit, we obtain the q-supercongruence (1.5). □

Proof of Theorem 1.4. In the proof of Theorem 1.1, we see that (1.5) is true for $n\equiv 1 \pmod 4$. Now, for $n\equiv 5 \pmod 8$, we have $(q^5;q^8)_{(n-1)/4}\equiv 0 \pmod{\Phi_n(q)}$ and

$$
\begin{aligned}
&1+[2n]^2(2-q^{2n})\sum_{j=1}^{(n-1)/4}\left(\frac{q^{8j-3}}{[8j-3]^2}-\frac{q^{8j-1}}{[8j-1]^2}\right)\\
&\equiv 1+[2n]^2(2-q^{2n})\frac{q^n}{[n]^2}\equiv 5 \pmod{\Phi_n(q)^2}.
\end{aligned}
$$

This completes the proof. □

3. Proof of Corollary 1.5

Let p be an odd prime. We first give some fundamental properties of the p-adic Gamma function. By the definition of p-adic Gamma function, it is easy to see that

$$
\frac{\Gamma_p(x+1)}{\Gamma_p(x)}=\begin{cases}-x, & p\nmid x,\\ -1, & p\mid x.\end{cases} \qquad (3.1)
$$

Furthermore, for any $x\in\mathbb{Z}_p$, we have

$$
\Gamma_p(x)\Gamma_p(1-x)=(-1)^{p-\langle -x\rangle_p}, \qquad (3.2)
$$

where $\langle x\rangle_p$ represents the least nonnegative residue of x modulo p, and for any $a, m \in \mathbb{Z}_p$,

$$\Gamma_p(a+mp) \equiv \Gamma_p(a) + \Gamma_p'(a)mp \pmod{p^2} \tag{3.3}$$

(see, for example, [10, Theorem 14]).

Proof of Corollary 1.5. Letting $n = p$ and taking the limits as $q \to 1$ in (1.7), we get

$$\sum_{k=0}^{(p-1)/4} (16k+1)\frac{(\frac{1}{8})_k(\frac{1}{4})_k^5}{k!(\frac{7}{8})_k^5} \equiv 5\frac{(\frac{9}{8})_{(p-1)/4}(\frac{5}{8})_{(p-1)/4}^3}{(\frac{3}{8})_{(p-1)/4}(\frac{7}{8})_{(p-1)/4}^3} \pmod{p^5}. \tag{3.4}$$

In view of (3.1), we have

$$\begin{aligned}\frac{(\frac{9}{8})_{(p-1)/4}(\frac{5}{8})_{(p-1)/4}^3}{(\frac{3}{8})_{(p-1)/4}(\frac{7}{8})_{(p-1)/4}^3} &= \frac{\Gamma(\frac{3}{8})\Gamma(\frac{2p+7}{8})\Gamma(\frac{7}{8})^3\Gamma(\frac{2p+3}{8})^3}{\Gamma(\frac{2p+1}{8})\Gamma(\frac{9}{8})\Gamma(\frac{5}{8})^3\Gamma(\frac{2p+5}{8})^3} \\ &= \frac{p^3}{8^3}\cdot\frac{\Gamma_p(\frac{3}{8})\Gamma_p(\frac{2p+7}{8})\Gamma_p(\frac{7}{8})^3\Gamma_p(\frac{2p+3}{8})^3}{\Gamma_p(\frac{2p+1}{8})\Gamma_p(\frac{9}{8})\Gamma_p(\frac{5}{8})^3\Gamma_p(\frac{2p+5}{8})^3}.\end{aligned} \tag{3.5}$$

Since $p \equiv 5 \pmod 8$, in light of (3.2) and (3.3), we obtain

$$\frac{\Gamma_p(\frac{2p+7}{8})}{\Gamma_p(\frac{2p+1}{8})} = (-1)^{(3p+1)/8}\Gamma_p(\tfrac{7+2p}{8})\Gamma_p(\tfrac{7-2p}{8}) \equiv (-1)^{(3p+1)/8}\Gamma_p(\tfrac{7}{8})^2 \pmod{p^2}, \tag{3.6}$$

$$\frac{\Gamma_p(\frac{2p+3}{8})}{\Gamma_p(\frac{2p+5}{8})} = (-1)^{(p+3)/8}\Gamma_p(\tfrac{3+2p}{8})\Gamma_p(\tfrac{3-2p}{8}) \equiv (-1)^{(p+3)/8}\Gamma_p(\tfrac{3}{8})^2 \pmod{p^2}. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) and using the identities $1/\Gamma_p(\frac{9}{8}) = -8/\Gamma_p(\frac{1}{8}) = -(-1)^{(3p+1)/8}8\Gamma_p(\frac{7}{8})$ and $1/\Gamma_p(\frac{5}{8}) \equiv (-1)^{(p+3)/8}\Gamma_p(\frac{3}{8})$, we deduce that

$$\frac{(\frac{9}{8})_{(p-1)/4}(\frac{5}{8})_{(p-1)/4}^3}{(\frac{3}{8})_{(p-1)/4}(\frac{7}{8})_{(p-1)/4}^3} \equiv -\frac{p^3}{64}\Gamma_p(\tfrac{7}{8})^6\Gamma_p(\tfrac{3}{8})^{10} \pmod{p^5}. \tag{3.8}$$

The proof then follows from (3.4) and (3.8). □

4. Proof of Theorem 1.6

The proof is similar to that of Theorem 1.1. We first give the following q-identity.

Lemma 4.1. *Let $n \equiv 3 \pmod 4$ be an integer with $n > 3$. Then*

$$\sum_{k=0}^{(n+1)/4} \frac{1-q^{16k-1-n}}{1-q^{-1-n}}\frac{(aq^{-2}, q^{-2}/a, bq^{-2}, q^{-2}/b, q^{-1-n}, q^{-2-2n}; q^8)_k}{(q^{9-n}/a, aq^{9-n}, q^{9-n}/b, bq^{9-n}, q^{9+n}, q^8; q^8)_k}q^{24k} = 0. \tag{4.1}$$

Proof. Recall that Watson's ${}_8\phi_7$ transformation (see [2, Appendix (III.18)]) can be written as follows:

$$
{}_8\phi_7\left[\begin{matrix} a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n} \\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q,\, \frac{a^2q^{n+2}}{bcde}\right]
$$

$$
= \frac{(aq, aq/de;q)_n}{(aq/d, aq/e;q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/bc,\ d,\ e,\ q^{-n} \\ aq/b,\ aq/c,\ deq^{-n}/a \end{matrix}; q,\, q\right]. \tag{4.2}
$$

Letting $q\to q^8$, and taking $a=q^{-1-n}$, $b=aq^{-2}$, $c=q^{-2}/a$, $d=bq^{-2}$, $e=q^{-2}/b$ and $n\mapsto (n+1)/4$ in (4.2), we obtain

$$
\sum_{k=0}^{(n+1)/4} \frac{1-q^{16k-1-n}}{1-q^{-1-n}} \frac{(aq^{-2}, q^{-2}/a, bq^{-2}, q^{-2}/b;q^8)_k (q^{-1-n}, q^{-2-2n};q^8)_k}{(q^{9-n}/a, aq^{9-n}, q^{9-n}/b, bq^{9-n};q^8)_k (q^{9+n}, q^8;q^8)_k} q^{24k}
$$

$$
= \frac{(q^{7-n}, q^{11-n};q^8)_{(n+1)/4}}{(q^{9-n}/b, bq^{9-n};q^8)_{(n+1)/4}} \sum_{k=0}^{(n+1)/4} \frac{(q^{11-n}, bq^{-2}, q^{-2}/b, q^{-2n-2};q^8)_k}{(q^8, q^{9-n}/a, aq^{9-n}, q^{-n-5};q^8)_k} q^{8k}. \tag{4.3}
$$

It is easy to see that the right-hand side of (4.3) vanishes, because of the factor $(q^{7-n}, q^{11-n};q^8)_{(n+1)/4}$ in the numerator. □

On the basis of Lemma 4.1, we can present a parametric version of Theorem 1.6.

Lemma 4.2. *Let $n\equiv 3 \pmod 4$ be an integer with $n>3$. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-aq^{2n})(a-q^{2n})(1-bq^{2n})(b-q^{2n})$,*

$$
\sum_{k=0}^{(n+1)/4} [16k-1] \frac{(aq^{-2}, q^{-2}/a, bq^{-2}, q^{-2}/b, q^{-2}, q^{-1};q^8)_k}{(q^9/a, aq^9, q^9/b, bq^9, q^9, q^8;q^8)_k} q^{24k}
$$

$$
\equiv -q^{(n-1)/2} \frac{(q^7, q^{-5};q^8)_{(n+1)/4}}{(q^9, q^{-3};q^8)_{(n+1)/4}}
$$

$$
\times \left\{ \frac{(1-bq^{2n})(b-q^{2n})(-1-a^2+aq^{2n})}{(a-b)(1-ab)} \sum_{k=0}^{(n+1)/4} \frac{(q^{11}, q^{-2}, q^{-2}/a, aq^{-2};q^8)_k}{(q^8, q^9/b, bq^9, q^{-5};q^8)_k} q^{8k} \right.
$$

$$
\left. + \frac{(1-aq^{2n})(a-q^{2n})(-1-b^2+bq^{2n})}{(b-a)(1-ba)} \sum_{k=0}^{(n+1)/4} \frac{(q^{11}, q^{-2}, q^{-2}/b, bq^{-2};q^8)_k}{(q^8, q^9/a, aq^9, q^{-5};q^8)_k} q^{8k} \right\}. \tag{4.4}
$$

Proof. Since $q^n\equiv 1 \pmod{\Phi_n(q)}$, the identity (4.1) indicates that

$$
\sum_{k=0}^{(n+1)/4} [16k-1] \frac{(aq^{-2}, q^{-2}/a, bq^{-2}, q^{-2}/b, q^{-2}, q^{-1};q^8)_k}{(q^9/a, aq^9, q^9/b, bq^9, q^9, q^8;q^8)_k} q^{24k} \equiv 0 \pmod{\Phi_n(q)}.
$$

Moreover, the right-hand side of (4.4) is also congruent to 0 because $(q^7, q^{-5};q^8)_{(n+1)/4}$ has the factor $1-q^n$, while the denominator is coprime with $\Phi_n(q)$. This means that the q-congruence (4.4) holds modulo $\Phi_n(q)$.

For $a=q^{-2n}$ or $a=q^{2n}$, the left-hand side of (4.4) may be written as

$$\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(q^{-2-2n},q^{-2+2n},bq^{-2},q^{-2}/b,q^{-2},q^{-1};q^8)_k}{(q^{9+2n},q^{9-2n},q^9/b,bq^9,q^9,q^8;q^8)_k}q^{24k}$$
$$=-\frac{1}{q}\,{}_8\phi_7\left[\begin{matrix} q^{-1}, & q^{\frac{15}{2}}, & -q^{\frac{15}{2}}, & bq^{-2}, & q^{-2}/b, & q^{-2}, & q^{-2+2n}, & q^{-2-2n} \\ & q^{-\frac12}, & -q^{-\frac12}, & q^9/b, & bq^9, & q^9, & q^{9-2n}, & q^{9+2n}\end{matrix};q^8,\,q^{24}\right]. \tag{4.5}$$

By Watson's transformation (4.2), the right-hand side of (4.5) is equal to

$$-\frac{(q^7,q^{11-2n};q^8)_{(n+1)/4}}{q(q^9,q^{9-2n};q^8)_{(n+1)/4}}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2+2n},q^{-2-2n};q^8)_k}{(q^8,q^9/b,bq^9,q^{-5};q^8)_k}q^{8k}$$
$$=-q^{(n-1)/2}\frac{(q^7,q^{-5};q^8)_{(n+1)/4}}{(q^9,q^{-3};q^8)_{(n+1)/4}}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2+2n},q^{-2-2n};q^8)_k}{(q^8,q^9/b,bq^9,q^{-5};q^8)_k}q^{8k}. \tag{4.6}$$

Since the polynomial $1-aq^{2n}$ is coprime with $a-q^{2n}$, the above identity yields the following q-congruence: modulo $(1-aq^{2n})(a-q^{2n})$,

$$\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(aq^{-2},q^{-2}/a,bq^{-2},q^{-2}/b,q^{-1},q^{-2};q^8)_k}{(q^9/a,aq^9,q^9/b,bq^9,q^9,q^8;q^8)_k}q^{24k}$$
$$\equiv -q^{(n-1)/2}\frac{(q^7,q^{-5};q^8)_{(n+1)/4}}{(q^9,q^{-3};q^8)_{(n+1)/4}}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2}/a,aq^{-2};q^8)_k}{(q^8,q^9/b,bq^9,q^{-5};q^8)_k}q^{8k}. \tag{4.7}$$

Exchanging the indeterminates a and b in (4.7) leads to another q-congruence: modulo $(1-bq^{2n})(b-q^{2n})$,

$$\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(aq^{-2},q^{-2}/a,bq^{-2},q^{-2}/b,q^{-1},q^{-2};q^8)_k}{(q^9/a,aq^9,q^9/b,bq^9,q^9,q^8;q^8)_k}q^{24k}$$
$$\equiv -q^{(n-1)/2}\frac{(q^7,q^{-5};q^8)_{(n+1)/4}}{(q^9,q^{-3};q^8)_{(n+1)/4}}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2}/b,bq^{-2};q^8)_k}{(q^8,q^9/a,aq^9,q^{-5};q^8)_k}q^{8k}. \tag{4.8}$$

Applying the Chinese remainder theorem for polynomials, from (4.7), (4.8), (2.7) and its dual form, we arrive at the desired q-congruence (4.4). □

Proof of Theorem 1.4. Letting $b=1$ in (4.4), and using the identity

$$\frac{(1-q^{2n})^2(1+a^2-aq^{2n})}{(1-a)^2}=(1-q^{2n})^2+\frac{a(1-q^{2n})^2(2-q^{2n})}{(1-a)^2},$$

we get the following q-congruence: modulo $\Phi_n(q)^3(1-aq^{2n})(a-q^{2n})$,

$$
\begin{aligned}
&\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(aq^{-2},q^{-2}/a,bq^{-2},q^{-2}/b,q^{-1},q^{-2};q^8)_k}{(q^9/a,aq^9,q^9/b,bq^9,q^9,q^8;q^8)_k}q^{24k}\\
&\quad\equiv -q^{(n-1)/2}\frac{(q^7,q^{-5};q^8)_{(n+1)/4}}{(q^9,q^{-3};q^8)_{(n+1)/4}}\bigg\{(1-q^{2n})^2\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2}/a,aq^{-2};q^8)_k}{(q^8,q^{-5};q^8)_k(q^9;q^8)_k^2}q^{8k}\\
&\qquad+\frac{a(1-q^{2n})^2(2-q^{2n})}{(1-a)^2}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2}/a,aq^{-2};q^8)_k}{(q^8,q^{-5};q^8)_k(q^9;q^8)_k^2}q^{8k}\\
&\qquad-\frac{(1-aq^{2n})(a-q^{2n})(2-q^{2n})}{(1-a)^2}\sum_{k=0}^{(n+1)/4}\frac{(q^{11};q^8)_k(q^{-2};q^8)_k^3}{(q^8,q^{-5},q^9/a,aq^9;q^8)_k}q^{8k}\bigg\}. \qquad (4.9)
\end{aligned}
$$

By the L'Hôpital rule, we have

$$
\begin{aligned}
&\lim_{a\to 1}\bigg\{\frac{a(1-q^{2n})^2}{(1-a)^2}\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2}/a,aq^{-2};q^8)_k}{(q^8,q^{-5};q^8)_k(q^9;q^8)_k^2}q^{8k}\\
&\qquad-\frac{(1-aq^{2n})(a-q^{2n})}{(1-a)^2}\sum_{k=0}^{(n+1)/4}\frac{(q^{11};q^8)_k(q^{-2};q^8)_k^3}{(q^8,q^{-5},q^9/a,aq^9;q^8)_k}q^{8k}\bigg\}\\
&=\sum_{k=0}^{(n+1)/4}\frac{(q^{11};q^8)_k(q^{-2};q^8)_k^3}{(q^8,q^{-5};q^8)_k(q^9;q^8)_k^2}q^{8k}\bigg\{q^{2n}-[2n]^2\sum_{j=1}^{k}\bigg(\frac{q^{8j-10}}{[8j-10]^2}+\frac{q^{8j+1}}{[8j+1]^2}\bigg)\bigg\}.
\end{aligned}
$$

Hence, taking $a\to 1$ in (4.4) and using the above limit, we get (1.5). □

5. Proof of Corollary 1.7

We first establish the following identity.

Lemma 5.1. *Let $n\equiv 3 \pmod 8$ be an integer with $n>3$. Then*

$$
\sum_{k=0}^{(n+1)/4}\frac{(q^{11},q^{-2},q^{-2+2n},q^{-2-2n};q^8)_k}{(q^8,q^{9-n},q^{9+n},q^{-5};q^8)_k}q^{8k}=0. \qquad (5.1)
$$

Proof. When $b=q^n$, the left-hand side of (4.5) is equal to

$$
-q\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(q^{-2-2n},q^{-2+2n},q^{-2+n},q^{-2-n},q^{-1},q^{-2};q^8)_k}{(q^{9+2n},q^{9-2n},q^{9-n},q^{9+n},q^9,q^8;q^8)_k}q^{24k}
$$

$$= {}_8\phi_7\left[\begin{matrix} q^{-1},\ q^{\frac{15}{2}},\ -q^{\frac{15}{2}},\ q^{-2+n},\ q^{-2},\ q^{-2-n},\ q^{-2+2n},\ q^{-2-2n} \\ q^{-\frac12},\ -q^{-\frac12},\ q^{9-n},\ \ q^{9},\ \ q^{9+n},\ \ q^{9-2n},\ \ q^{9+2n} \end{matrix}; q^8,\ q^{24}\right].$$

$$= \lim_{x\to 1} {}_8\phi_7\left[\begin{matrix} q^{-1}, q^{\frac{15}{2}},\ -q^{\frac{15}{2}}, xq^{-2+n}, q^{-2}, q^{-2-n}/x, x^2q^{-2+2n}, q^{-2-2n}/x^2 \\ q^{-\frac12}, -q^{-\frac12}, q^{9-n}/x,\ q^9,\ \ xq^{9+n},\ \ q^{9-2n}/x^2,\ \ x^2q^{9+2n} \end{matrix}; q^8,\ q^{24}\right]. \quad (5.2)$$

In view of Watson's ${}_8\phi_7$ transformation (4.2), the above limit is equal to

$$\lim_{x\to 1}\frac{(q^7, q^{11-n}/x; q^8)_{(n+1)/4}}{(xq^{9+n}, q^{9-2n}/x^2; q^8)_{(n+1)/4}} \times \sum_{k=0}^{(n+1)/4}\frac{(q^{11-n}/x, q^{-2-n}/x, x^2q^{-2+2n}, q^{-2-2n}/x^2; q^8)_k}{(q^8, q^{9-n}/x, q^9, q^{-5-n}/x; q^8)_k} q^{8k}. \quad (5.3)$$

Since the limit of $(q^{11-n}/x; q^8)_{(n+1)/4}$ as $x \to 1$ has the factor $1-q^0$, one sees that (5.3) vanishes. Namely, the left-hand side of (5.2) is equal to 0. The proof of (5.1) then follows from (4.6) with $b = q^n$ and the fact that $(q^7, q^{-5}; q^8)_{(n+1)/4}/(q^9, q^{-3}; q^8)_{(n+1)/4} \neq 0$. □

We also need a q-congruence modulo $\Phi_n(q)^2$, which was already utilized in [6].

Lemma 5.2. *Let α and r be integers and n a positive integer. Then, for $k \geqslant 0$,*

$$(q^{r-\alpha n}, q^{r+\alpha n}; q^d)_k \equiv (q^r; q^d)_k^2 \pmod{\Phi_n(q)^2}.$$

Proof of Corollary 1.7. Since $n \equiv 3 \pmod 8$ and $n > 3$, we have $(q^{-5}; q^8)_{(n+1)/4} \equiv [2n] \equiv 0 \pmod{\Phi_n(q)}$. Furthermore, for $1 \leqslant k \leqslant (n+1)/4$, the denominators in

$$\sum_{j=1}^{k}\left(\frac{q^{8j-10}}{[8j-10]^2} + \frac{q^{8j+1}}{[8j+1]^2}\right)$$

are coprime with $\Phi_n(q)$. It follows that, modulo $\Phi_n(q)^3$,

$$\sum_{k=0}^{(n+1)/4}[16k-1]\frac{(q^{-1}; q^8)_k (q^{-2}; q^8)_k^5}{(q^9; q^8)_k^5 (q^8; q^8)_k} q^{24k}$$

$$\equiv -q^{(n-1)/2}\frac{(q^7, q^{-5}; q^8)_{(n+1)/4}}{(q^9, q^{-3}; q^8)_{(n+1)/4}} \sum_{k=0}^{(n+1)/4}\frac{(q^{11}; q^8)_k (q^{-2}; q^8)_k^3}{(q^8, q^{-5}; q^8)_k (q^9; q^8)_k^2} q^{8k}. \quad (5.4)$$

By firstly applying Lemma 5.2 twice and then applying Lemma 5.1, we have

$$\sum_{k=0}^{(n+1)/4}\frac{(q^{11}; q^8)_k (q^{-2}; q^8)_k^3}{(q^8, q^{-5}; q^8)_k (q^9; q^8)_k^2} q^{8k} \equiv \sum_{k=0}^{(n+1)/4}\frac{(q^{11}, q^{-2}, q^{-2+2n}, q^{-2-2n}; q^8)_k}{(q^8, q^{9-n}, q^{9+n}, q^{-5}; q^8)_k} q^{8k}$$
$$= 0 \pmod{\Phi_n(q)^2}.$$

Substituting the above q-congruence into (5.4) and noticing $(q^{-5}; q^8)_{(n+1)/4} \equiv 0 \pmod{\Phi_n(q)}$ again, we complete the proof of the corollary. □

Declarations

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