New q-congruences from a quadratic summation of Gasper and Rahman

Victor J. W. Guo and Xing-Ye Zhu*

School of Mathematics, Hangzhou Normal University, Hangzhou 311121, People's Republic of China
jwguo@hznu.edu.cn, xyzhu@stu.hznu.edu.cn

Abstract. By making use of Gasper and Rahman's quadratic summation, the creative microscoping method introduced by the first author and Zudilin, and the Chinese remainder theorem for polynomials, we present two new q-congruences modulo the fourth power of a cyclotomic polynomial, along with a Dwork-type q-congruence. Our results are generalizations of two recent q-congruences due to He and Wang [Proc. Amer. Math. Soc. (2024), 4775–4784]. We also propose four related conjectures on congruences and q-congruences.

Keywords: q-congruences; creative microscoping; Gasper and Rahman's quadratic summation; Chinese remainder theorem for polynomials.

AMS Subject Classifications: 33D15; 11A07, 11B65

1. Introduction

More than one hundred years ago, Ramanujan discovered 17 remarkable infinite series for $1/\pi$ (see [1, p. 352]), such as

$$\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ is the Pochhammer symbol. Formulae of the form (1.1) were later utilized to evaluate π more exactly. In 1997, affected by Ramanujan's work, Van Hamme [14] numerically observed 13 neat p-adic analogues of Ramanujan-type series, including the following one: for any prime $p > 3$,

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv(-1)^{(p-1)/2}p \pmod{p^4}. \tag{1.2}$$

Nowadays, a number of authors are interested in finding q-analogues of supercongruences. For example, the first author and Zudilin [7] devised a method called "creative microscoping" to build q-analogues of many supercongruences modulo p^3. The first author [5] then applied this method and the Chinese remainder theorem for polynomials to

*Corresponding author.

give a q-analogue of (1.2) as follows: for any positive odd integer n, modulo $[n]\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}q^{k^2}\equiv(-q)^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \tag{1.3}$$

Here and in what follows, $[n]=(1-q^n)/(1-q)$ denotes the *q-integer*, and $(x;q)_n=(1-x)(1-xq)\cdots(1-xq^{n-1})$ $(n\geqslant 0)$ denotes the *q-shifted factorial*. For convenience, we will also adopt the abbreviated notation: $(x_1,\dots,x_m;q)_n=(x_1;q)_n\dots(x_m;q)_n$. Let $\Phi_n(q)$ stand for the n-th *cyclotomic polynomial*, which can be written as

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

with ζ being an n-th primitive root of unity. It is well known that $\Phi_p(q)=[p]$ for primes p.

Recently, using the method of creative microscoping and a quadratic summation of Gasper and Rahman, He and Wang [9, Theorem 2.4] proved the following q-congruence: for any positive integer n with $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n]\pmod{[n]\Phi_n(q)^2}. \tag{1.4}$$

The first purpose of this paper is to establish a generalization of (1.4).

Theorem 1.1. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{-k^2}\equiv q^{(1-n)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \tag{1.5}$$

Letting $n=p^r$ be a prime power with $p^r\equiv 1 \pmod 4$ and $p>3$, and then taking the limits as $q\to 1$ in (1.5), we deduce the following result.

Corollary 1.2. *Let p be an odd prime and r a positive integer with $p^r\equiv 1 \pmod 4$ and $p>3$. Then*

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\equiv p^r\pmod{p^{r+3}}. \tag{1.6}$$

Note that the supercongruence (1.6) modulo p^{r+2} follows from (1.4) and was already mentioned in [9]. He and Wang [9, Theorem 2.4] also proved that, for any positive integer n with $n\equiv 3 \pmod 4$,

$$\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k-k^2}\equiv q^{-(n+1)/2}[n]\pmod{[n]\Phi_n(q)^2}. \tag{1.7}$$

The second purpose of this paper is to give a generalization of (1.7).

Theorem 1.3. *Let $n\equiv 3 \pmod 4$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}q^{2k-k^2}\equiv q^{-(n+1)/2}[n]\left\{1+\frac{(n^2-1)(1-q)^2}{24}[n]^2\right\}. \quad (1.8)$$

Similarly, letting $n=p^r$ be a prime power with $p^r\equiv 3 \pmod 4$, and then taking the limits as $q\to 1$ in (1.8), we get the following result.

Corollary 1.4. *Let p be an odd prime and r a positive integer with $p^r\equiv 3 \pmod 4$ and $p>3$. Then*

$$\sum_{k=0}^{(p^r+1)/2}(6k-1)\frac{(\frac{1}{2})_k(-\frac{1}{4})_k^2}{k!^3}4^k\equiv p^r \pmod{p^{r+3}}. \quad (1.9)$$

Note that the congruence (1.9) modulo p^{r+2} also follows from (1.7) and is due to He and Wang [9].

The rest of the paper is arranged as follows. We shall prove Theorems 1.1 and 1.3 in Sections 2 and 3, respectively. The proofs make uses of the creative microscoping method, Gasper and Rahman's quadratic summation (see (2.1)), and the Chinese remainder theorem for polynomials. In Section 4, we shall give a Dwork-type generalization of He and Wang's q-congruence (1.4). Finally, in Section 5, we propose some relevant conjectures on congruences and q-congruences for further study.

2. Proof of Theorem 1.1

Recall that Gasper and Rahman's quadratic summation (see [3, eq. (3.8.12)]) can be stated as follows:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1-aq^{3k}}{1-a}\frac{(a,b,q/b;q)_k(d,f,a^2q/df;q^2)_kq^k}{(aq/d,aq/f,df/a;q)_k(q^2,aq^2/b,abq;q^2)_k}\\&\quad+\frac{(aq,f/a,b,q/b;q)_\infty(d,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty(aq^2/b,abq,fq/ab,bf/a;q^2)_\infty}\\&\quad\times\sum_{k=0}^{\infty}\frac{(f,bf/a,fq/ab;q^2)_kq^{2k}}{(q^2,fq^2/d,df^2q/a^2;q^2)_k}\\&\quad=\frac{(aq,f/a;q)_\infty(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq/d,df/a;q)_\infty(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}.\end{aligned} \quad (2.1)$$

Gasper and Rahman's summation (2.1) plays an important role in the study of q-congruences. Wei [15] first employed (2.1) to give some q-congruences. The first author [6] then deduced more q-congruences modulo $\Phi_n(q)^2$ or $\Phi_n(q)^3$ from (2.1). As mentioned before, Liu and Wang's proofs of (1.4) and (1.7) are also based on Gasper and Rahman's summation (2.1).

We first establish the following identity.

Lemma 2.1. *Let $n \equiv 1 \pmod 4$ be a positive integer with $n > 1$. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{1-q^{6k+1-n}}{1-q} \frac{(q^{1-n};q^2)_k (ab^2q, q/a, q^{2-2n}/b^2;q^4)_k}{(q^4;q^4)_k (q^{2-n}/ab^2, aq^{2-n}, b^2q^{1+n};q^2)_k} q^{-k^2+nk} = 0. \tag{2.2}$$

Proof. Putting $b = q^{-2n}$ and $n \to \infty$ in (2.1), we obtain

$$\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \frac{(a;q)_k (d,f,a^2q/df;q^2)_k q^{(k-k^2)/2}}{(q^2;q^2)_k (aq/d, aq/f, df/a;q)_k} a^{-k} = \frac{(aq^2, dq/a, fq/a, aq^2/df;q^2)_\infty}{(q/a, aq^2/d, aq^2/f, dfq/a;q^2)_\infty}, \tag{2.3}$$

as was already noticed by He and Wang [9].

Letting $q \mapsto q^2$, and taking $a = q^{1-n}$, $d = ab^2q$, and $f = q/a$ in (2.3), we see that the left-hand side of (2.2) is equal to

$$\frac{(q^{5-n}, ab^2q^{2+n}, q^{2+n}/a, q^{3-n}/b^2;q^4)_\infty}{(q^{1+n}, q^{4-n}/ab^2, aq^{4-n}, b^2q^{3+n};q^4)_\infty} = 0.$$

This is because of the factor $(q^{5-n};q^4)_\infty = 0$ in the numerator. □

We need the following parametric version of (1.4).

Lemma 2.2. *Let $n \equiv 1 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-ab^2q^n)(a-q^n)$,*

$$\sum_{k=0}^{(n-1)/2} [6k+1] \frac{(q;q^2)_k (ab^2q, q/a, q^2/b^2;q^4)_k}{(q^4;q^4)_k (q^2/ab^2, aq^2, b^2q;q^2)_k} q^{-k^2} \equiv \frac{(q^3;q^2)_{(n-1)/2}}{(b^2q;q^2)_{(n-1)/2}} b^{(n-1)/2} q^{(1-n)/2}. \tag{2.4}$$

Proof. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, the identity (2.2) immediately implies that

$$\sum_{k=0}^{(n-1)/2} [6k+1] \frac{(q;q^2)_k (ab^2q, q/a, q^2/b^2;q^4)_k}{(q^4;q^4)_k (q^2/ab^2, aq^2, b^2q;q^2)_k} q^{-k^2} \equiv 0 \pmod{\Phi_n(q)}.$$

Moreover, the right-hand side of (2.4) is also congruent to 0 because $(q^3;q^2)_{(n-1)/2}$ contains the factor $1-q^n$, while $(b^2q;q^2)_{(n-1)/2}$ is coprime with $\Phi_n(q)$. This implies that the q-congruence (2.4) holds modulo $\Phi_n(q)$.

For $a = q^n$ or $a = q^{-n}/b^2$, the left-hand side of (2.4) can be written as

$$\sum_{k=0}^{(n-1)/2} \frac{1-q^{6k+1}}{1-q} \frac{(q,q^2)_k (q^{1-n}, b^2q^{1+n}, q^2/b^2;q^4)_k}{(q^4,q^4)_k (q^{2+n}, q^{2-n}/b^2, b^2q,;q^2)_k} q^{-k^2}. \tag{2.5}$$

Performing the parameter substitutions $q \mapsto q^2$, $a = q$, $d = q^{1-n}$, and $f = b^2q^{1+n}$ in (2.3), we deduce that (2.5) is equal to

$$\frac{(q^5, q^{2-n}, b^2q^{2+n}, q^3/b^2;q^4)_\infty}{(q, q^{4+n}, q^{4-n}/b^2, b^2q^3;q^4)_\infty} = \frac{(q^5,q^3;q^4)_{(n-1)/4}}{(b^2q, b^2q^3;q^4)_{(n-1)/4}} b^{(n-1)/2} q^{(1-n)/2}$$

$$= \frac{(q^3;q^2)_{(n-1)/2}}{(b^2q;q^2)_{(n-1)/2}} b^{(n-1)/2} q^{(1-n)/2}.$$

This implies that the q-congruence (2.4) holds modulo $1-ab^2q^n$ and $a-q^n$. Since the polynomials $\Phi_n(q)$, $1-ab^2q^n$, and $a-q^n$ are coprime with one another, we arrive at the q-congruence (2.4). □

We also need the following q-congruence.

Lemma 2.3. *Let $n \equiv 1 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $b-q^n$,*

$$\sum_{k=0}^{(n-1)/2} [6k+1] \frac{(q,q^2)_k (ab^2q, q/a, q^2/b^2; q^4)_k}{(q^4,q^4)_k (q^2/ab^2, aq^2, b^2q; q^2)_k} q^{-k^2} \equiv \frac{(q^5, q^3/b^2; q^4)_{(n-1)/2}}{(aq^4, q^4/ab^2; q^4)_{(n-1)/2}}. \tag{2.6}$$

Proof. Letting $q \mapsto q^2$, and taking $a=q$, $d=q/a$, and $f=aq^{2n+1}$ in (2.3), we get

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2} \frac{1-q^{6k+1}}{1-q} \frac{(q,q^2)_k (q/a, aq^{2n+1}, q^{2-2n}; q^4)_k}{(q^4,q^4)_k (aq^2, q^{2-2n}/a, q^{2n+1}; q^2)_k} q^{-k^2} \\
&= \frac{(q^5, q^2/a, aq^{2n+2}, q^{3-2n}; q^4)_\infty}{(q, aq^4, q^{4-2n}/a, q^{2n+3}; q^4)_\infty} \\
&= \frac{(q^5, q^{3-2n}; q^4)_{(n-1)/2}}{(aq^4, q^{4-2n}/a; q^4)_{(n-1)/2}}.
\end{aligned}$$

This means that both sides of (2.6) are equal for $b=q^n$, and so the q-congruence (2.6) holds. □

Finally, we require the following easily proved lemma. For a short proof of it, see [5, Lemma 2.1].

Lemma 2.4. *Let n be a positive odd integer. Then*

$$(aq^2,q^2)_{(n-1)/2} (q^2/a, q^2)_{(n-1)/2} \equiv (-1)^{(n-1)/2} \frac{(1-a^n) q^{-(n-1)^2/4}}{(1-a)a^{(n-1)/2}} \pmod{\Phi_n(q)}, \tag{2.7}$$

$$(q;q)_{n-1} \equiv n \pmod{\Phi_n(q)}. \tag{2.8}$$

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. It is obvious that $\Phi_n(q)(1-ab^2q^n)(a-q^n)$ and $b-q^n$ are coprime polynomials. By the Chinese reminder theorem for coprime polynomials, we may determine the remainder of the left-hand side of (2.4) modulo $\Phi_n(q)(1-ab^2q^n)(a-q^n)(b-q^n)$ from (2.4) and (2.6). To this end, we require the following two q-congruences:

$$\frac{(b-q^n)(ab^2q^n + ab^3 - a^2b^2 - 1)}{(a-b)(1-ab^3)} \equiv 1 \pmod{(1-ab^2q^n)(a-q^n)}, \tag{2.9}$$

$$\frac{(1-ab^2q^n)(a-q^n)}{(a-b)(1-ab^3)} \equiv 1 \pmod{b-q^n}. \tag{2.10}$$

Therefore, combining (2.4) and (2.6) we obtain

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k(ab^2q,q/a,q^2/b^2;q^4)_k}{(q^4;q^4)_k(q^2/ab^2,aq^2,b^2q;q^2)_k}q^{-k^2} \\
&\quad\equiv \frac{(q^3;q^2)_{(n-1)/2}}{(b^2q;q^2)_{(n-1)/2}}b^{(n-1)/2}q^{(1-n)/2}\frac{(b-q^n)(ab^2q^n+ab^3-a^2b^2-1)}{(a-b)(1-ab^3)} \\
&\qquad+\frac{(q^5,q^3/b^2;q^4)_{(n-1)/2}}{(aq^4,q^4/ab^2;q^4)_{(n-1)/2}}\frac{(1-ab^2q^n)(a-q^n)}{(a-b)(1-ab^3)}
\end{aligned} \tag{2.11}$$

modulo $\Phi_n(q)(1-ab^2q^n)(a-q^n)(b-q^n)$.

Moreover, since $(q^5,q^3;q^4)_{(n-1)/2}=(q^3;q^2)_{n-1}=[n](q;q^2)_{(n-1)/2}(q^{n+2};q^2)_{(n-1)/2}$, by (2.7) and (2.8), we have

$$\frac{(q^5,q^3;q^4)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{[n](q,q^2;q^2)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{[n]n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-1)/2}} \pmod{\Phi_n(q)^2}. \tag{2.12}$$

Also, it is easy to see that

$$(1-q^n)(1+a^2-a-aq^n)=(1-a)^2+(1-aq^n)(a-q^n), \tag{2.13}$$

and when $b=1$ the polynomial $b-q^n=1-q^n$ has the factor $\Phi_n(q)$. Thus, letting $b=1$ in (2.11) and applying (2.12) and (2.13), we are led to the following q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q;q^2)_k(aq,q/a,q^2;q^4)_k}{(q^4;q^4)_k(q^2/a,aq^2,q;q^2)_k}q^{-k^2} \\
&\quad\equiv q^{(1-n)/2}[n]+q^{(1-n)/2}[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right).
\end{aligned} \tag{2.14}$$

By the L'Hôpital rule, there holds

$$\lim_{a\to 1}\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\frac{(1-a^n-n(1-a)a^{(n-1)/2})}{(1-a^n)}=\frac{(n^2-1)(1-q)^2}{24}[n]^2, \tag{2.15}$$

which was already used in [5]. Hence, taking $a\to 1$ in (2.14) and making use of the above limit, we conclude that (1.5) is true modulo $\Phi_n(q)^4$. In view of (1.4), the q-congruence (1.5) is also true modulo $[n]$. The proof then follows from the fact that the least common multiple of $\Phi_n(q)^4$ and $[n]$ is $[n]\Phi_n(q)^3$. □

3. Proof of Theorem 1.3

The proof is similar to that of Theorem 1.1. We need to establish three related lemmas.

Lemma 3.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n+1)/2} \frac{1-q^{6k-1-n}}{1-q} \frac{(q^{-1-n};q^2)_k (ab^2q^{-1}, q^{-1}/a, q^{2-2n}/b^2;q^4)_k}{(q^4;q^4)_k (q^{2-n}/ab^2, aq^{2-n}, b^2q^{-1+n};q^2)_k} q^{-k^2+(n+2)k} = 0. \tag{3.1}$$

Proof. Letting $q \mapsto q^2$, and taking $a = q^{-1-n}$, $d = ab^2q^{-1}$ and $f = q^{-1}/a$ in (2.3), we see that the left-hand side of (3.1) is equal to

$$\frac{(q^{3-n}, ab^2q^{2+n}, q^{2+n}/a, q^{5-n}/b^2;q^4)_\infty}{(q^{3+n}, q^{4-n}/ab^2, aq^{4-n}, b^2q^{1+n};q^4)_\infty} = 0, \tag{3.2}$$

where we have used the fact that $(q^{3-n};q^4)_\infty = 0$. □

Lemma 3.2. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $\Phi_n(q)(1-ab^2q^n)(a-q^n)$,*

$$\begin{aligned}\sum_{k=0}^{(n+1)/2} [6k-1] &\frac{(q^{-1};q^2)_k (ab^2q^{-1}, q^{-1}/a, q^2/b^2;q^4)_k}{(q^4;q^4)_k (q^2/ab^2, aq^2, b^2q^{-1};q^2)_k} q^{2k-k^2} \\ &\equiv \frac{-(q;q^2)_{(n+1)/2}}{q(b^2q^{-1};q^2)_{(n+1)/2}} b^{(n+1)/2} q^{-(n+1)/2}.\end{aligned} \tag{3.3}$$

Proof. Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, from (3.1) we deduce that

$$\sum_{k=0}^{(n+1)/2} [6k-1] \frac{(q^{-1};q^2)_k (ab^2q^{-1}, q^{-1}/a, q^2/b^2;q^4)_k}{(q^4;q^4)_k (q^2/ab^2, aq^2, b^2q^{-1};q^2)_k} q^{2k-k^2} \equiv 0 \pmod{\Phi_n(q)}.$$

Moreover, the right-hand side of (2.4) is also congruent to 0 because $(q;q^2)_{(n+1)/4}$ in the numerator contains the factor $1-q^n$. This implies that (3.3) holds modulo $\Phi_n(q)$.

For $a = q^n$ or $a = q^{-n}/b^2$, the left-hand side of (3.3) can be written as

$$\sum_{k=0}^{(n+1)/2} \frac{1-q^{6k-1}}{1-q} \frac{(q^{-1},q^2)_k (q^{-1-n}, b^2q^{-1+n}, q^2/b^2;q^4)_k}{(q^4,q^4)_k (q^{2+n}, q^{2-n}/b^2, b^2q^{-1},;q^2)_k} q^{2k-k^2}. \tag{3.4}$$

Making the parameter substitutions $q \mapsto q^2$, $a = q^{-1}$, $d = q^{-1-n}$ and $f = b^2q^{-1+n}$ in (2.3), we conclude that (3.4) is equal to

$$\begin{aligned}\frac{-(q^3, q^{2-n}, b^2q^{2+n}, q^5/b^2;q^4)_\infty}{q(q^3, q^{4+n}, q^{4-n}/b^2, b^2q;q^4)_\infty} &= \frac{-(q^3,q;q^4)_{(n+1)/4}}{q(b^2q^{-1}, b^2q;q^4)_{(n+1)/4}} b^{(n+1)/2} q^{-(n+1)/2} \\ &= \frac{-(q;q^2)_{(n+1)/2}}{q(b^2q^{-1};q^2)_{(n+1)/2}} b^{(n+1)/2} q^{-(n+1)/2}.\end{aligned}$$

This implies that (3.3) holds modulo $1-ab^2q^n$ and $a-q^n$. Since $\Phi_n(q)$, $1-ab^2q^n$, and $a-q^n$ are pairwise coprime polynomials, we get the desired q-congruence (3.3). □

Lemma 3.3. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $b-q^n$,*

$$\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^2)_k(ab^2q^{-1},q^{-1}/a,q^2/b^2;q^4)_k}{(q^4;q^4)_k(q^2/ab^2,aq^2,b^2q^{-1};q^2)_k}q^{2k-k^2} \equiv \frac{-(q^3,q^5/b^2;q^4)_{(n-1)/2}}{q(aq^4,q^4/ab^2;q^4)_{(n-1)/2}}. \quad (3.5)$$

Proof. Letting $q \mapsto q^2$, and taking $a=q^{-1}$, $d=q^{-1}/a$ and $f=aq^{2n-1}$ in (2.3), we have

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^2)_k(ab^2q^{-1},q^{-1}/a,q^2/b^2;q^4)_k}{(q^4;q^4)_k(q^2/ab^2,aq^2,b^2q^{-1};q^2)_k}q^{2k-k^2}\\
&\quad=-\frac{(q^3,q^2/a,aq^{2n+2},q^{5-2n};q^4)_\infty}{q(q^3,aq^4,q^{4-2n}/a,q^{2n+1};q^4)_\infty}\\
&\quad=-\frac{(q^3,q^{5-2n};q^4)_{(n-1)/2}}{q(aq^4,q^{4-2n}/a;q^4)_{(n-1)/2}}.
\end{aligned}$$

This indicates that the two sides of (3.5) are equal for $b=q^n$. Namely, the q-congruence (3.5) holds. □

Proof of Theorem 1.3. Applying the q-congruences (2.9) and (2.10), from (3.3) and (3.5) we deduce that, modulo $\Phi_n(q)(1-ab^2q^n)(a-q^n)(b-q^n)$,

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^2)_k(ab^2q^{-1},q^{-1}/a,q^2/b^2;q^4)_k}{(q^4;q^4)_k(q^2/ab^2,aq^2,b^2q^{-1};q^2)_k}q^{2k-k^2}\\
&\equiv-\frac{(q;q^2)_{(n+1)/2}}{q(b^2q^{-1};q^2)_{(n+1)/2}}b^{(n+1)/2}q^{-(n+1)/2}\frac{(b-q^n)(ab^2q^n+ab^3-a^2b^2-1)}{(a-b)(1-ab^3)}\\
&\quad-\frac{(q^3,q^5/b^2;q^4)_{(n-1)/2}}{q(aq^4,q^4/ab^2;q^4)_{(n-1)/2}}\frac{(1-ab^2q^n)(a-q^n)}{(a-b)(1-ab^3)}.
\end{aligned} \quad (3.6)$$

Since $n\equiv 3 \pmod 4$, by (2.7) and (2.8), we have

$$\frac{(q^5,q^3;q^4)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv \frac{[n](q,q^2;q^2)_{(n-1)/2}}{(aq^4,q^4/a;q^4)_{(n-1)/2}} \equiv -\frac{[n]n(1-a)a^{(n-1)/2}}{(1-a^n)q^{(n-1)/2}} \pmod{\Phi_n(q)^2}. \quad (3.7)$$

Thus, letting $b=1$ in (3.6) and applying (3.7) and (2.13), we are led to the following q-congruence: modulo $\Phi_n(q)^2(1-aq^n)(a-q^n)$,

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[6k-1]\frac{(q^{-1};q^2)_k(aq^{-1},q^{-1}/a,q^2;q^4)_k}{(q^4;q^4)_k(q^2/a,aq^2,q^{-1};q^2)_k}q^{2k-k^2}\\
&\equiv q^{-(1+n)/2}[n]+q^{-(1+n)/2}[n]\frac{(1-aq^n)(a-q^n)}{(1-a)^2}\left(1-\frac{n(1-a)a^{(n-1)/2}}{1-a^n}\right).
\end{aligned} \quad (3.8)$$

Therefore, taking $a\to 1$ in (3.8) and applying the limit (2.15), we are led to (1.8). □

4. A Dwork-type generalization of (1.4)

Swisher [13, (J.3)] conjectured that Van Hamme's congruence (1.2) can be generalized as follows: for any prime $p>3$ and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac12)_k^3}{k!^3 4^k}\equiv(-1)^{(p-1)/2}p\sum_{k=0}^{(p^{r-1}-1)/2}(6k+1)\frac{(\frac12)_k^3}{k!^3 4^k}\pmod{p^{4r}}. \tag{4.1}$$

Note that Swisher's conjecture (4.1) may be viewed as a particular instance of Dwork-type congruences [2,11]. The first author [4] proved the modulus p^{3r} case of (4.1) by building a q-analogue of it. For more Dwork-type congruences, we refer the reader to [8].

In this section, we shall establish the following Dwork-type congruence: for any odd prime p and positive integer r,

$$\sum_{k=0}^{(p^r-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\equiv p\sum_{k=0}^{(p^{r-1}-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\pmod{p^{3r}}, \tag{4.2}$$

which is obviously a generalization of (1.6) modulo p^{r+2}.

We first give the following result, which is due to He and Wang [9, Lemma 5.2].

Lemma 4.1. *Let $n\equiv 1\pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^4;q^4)_k}q^{-k^2}\equiv 0\pmod{[n]}. \tag{4.3}$$

We also need another two q-congruences with a parameter a.

Lemma 4.2. *Let $n\equiv 1\pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}[6k+1]\frac{(q^{1-n};q^4)_k(q^{1+n};q^4)_k(q^2;q^4)_k}{(q^{2+n};q^2)_k,q^{2-n};q^2)_k(q^4;q^4)_k}q^{-k^2}=q^{(1-n)/2}[n]. \tag{4.4}$$

Proof. Letting $q\mapsto q^2$, $a=q$, $d=q^{1-n}$, and $f=q^{1+n}$ in (2.3) and making some simplifications, we obtain (4.4). This identity also follows from the $b=1$ case of (2.4). □

Lemma 4.3. *Let $n\equiv 1\pmod 4$ be an integer with $n>1$ and let $r\geqslant 1$. Then, modulo*

$$[n^r]\prod_{j=0}^{(n^{r-1}-1)/2}(1-aq^{(4j+1)n})(a-q^{(4j+1)n}),$$

we have

$$\sum_{k=0}^{(n^r-1)/2}[6k+1]\frac{(aq;q^4)_k(q/a;q^4)_k(q^2;q^4)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^4;q^4)_k}q^{-k^2}$$
$$\equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/2}q^{-nk^2}[6k+1]_{q^n}\frac{(aq^n;q^{4n})_k(q^n/a;q^{4n})_k(q^{2n};q^{4n})_k}{(aq^{2n};q^{2n})_k(q^{2n}/a;q^{2n})_k(q^{4n};q^{4n})_k}. \tag{4.5}$$

Proof. Replacing n by n^r in (4.3), we know that the left-hand side of (4.5) is congruent to 0 modulo $[n^r]$. Meanwhile, making the parameter substitutions $r \mapsto r-1$ and $q \mapsto q^n$ in (4.3), we derive that the sum on the right-hand side of (4.5) without the pre-factor is congruent to 0 modulo $[n^{r-1}]_{q^n}$. Further, it is not hard to see that, for any positive odd integer n, the q-integer $[n]$ is coprime with $1+q^k$ for all positive integers k. Therefore, $[n]$ is also coprime with the denominators of the sum on the right-hand side of (4.5) because of the identity

$$\frac{(q^{2n};q^{4n})_k}{(q^{4n};q^{4n})_k} = \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^{2n}} \frac{1}{(-q^{2n};q^{2n})_k^2},$$

where ${2k \brack k}_{q^{2n}} = (q^{2n};q^{2n})_{2k}/(q^{2n};q^{2n})_k^2$ is the q-binomial coefficient (a polynomial in q^{2n} with integer coefficients). This implies that the right-hand side of (4.5) is congruent to 0 modulo $[n][n^{r-1}]_{q^n} = [n^r]$. Thus, we have proved the q-congruence (4.5) modulo $[n^r]$.

In order to prove (4.5) modulo

$$\prod_{j=0}^{(n^{r-1}-1)/2} (1-aq^{(4j+1)n})(a-q^{(4j+1)n}), \tag{4.6}$$

it suffices to show that both sides of (4.5) are equal when $a = q^{-(4j+1)n}$ or $a = q^{(4j+1)n}$ for all $0 \leqslant j \leqslant (n^{r-1}-1)/2$. Namely, we need to prove that, for these j,

$$\begin{aligned} &\sum_{k=0}^{(n^r-1)/2} [6k+1] \frac{(q^{1-(4j+1)n};q^4)_k (q^{1+(4j+1)n};q^4)_k (q^2;q^4)_k}{(q^{2-(4j+1)n};q^2)_k (q^{2+(4j+1)n};q^2)_k (q^4;q^4)_k} q^{-k^2} \\ &\quad = q^{(1-n)/2}[n] \sum_{k=0}^{(n^{r-1}-1)/2} q^{-nk^2}[6k+1]_{q^n} \frac{(q^{-4jn};q^{4n})_k (q^{(4j+2)n};q^{4n})_k (q^{2n};q^{4n})_k}{(q^{(1-4j)n};q^{2n})_k (q^{(4j+3)n};q^{2n})_k (q^{4n};q^{4n})_k}. \end{aligned} \tag{4.7}$$

First notice that $(n^r-1)/2 \geqslant ((4j+1)n-1)/4$ for $0 \leqslant j \leqslant (n^{r-1}-1)/2$, and $(q^{1-(4j+1)n};q^4)_k = 0$ for $k > ((4j+1)n-1)/4$. Then, in light of Lemma 4.2, the left-hand side of (4.7) equals $q^{(1-(4j+1)n)/2}[(4j+1)n]$, and the right-hand side of (4.7) equals

$$q^{(1-n)/2}[n] \cdot q^{-2jn}[4j+1]_{q^n} = q^{(1-(4j+1)n)/2}[(4j+1)n].$$

This establishes the identity (4.7). Consequently, the q-congruence (4.5) is true modulo (4.6). Since $[n^r]$ and (4.6) are coprime polynomials, we finish the proof of (4.5). □

We are now able to prove (4.2) by establishing the following q-congruence, which may also be deemed a Dwork-type generalization of (1.4).

Theorem 4.4. *Let $n \equiv 1 \pmod 4$ be an integer with $n > 1$ and let $r \geqslant 1$. Then, modulo $[n^r]\prod_{j=1}^{r} \Phi_{n^j}(q)^2$,*

$$\sum_{k=0}^{(n^r-1)/2} q^{-k^2}[6k+1] \frac{(q;q^4)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k}$$

$$\equiv q^{(1-n)/2}[n]\sum_{k=0}^{(n^{r-1}-1)/2} q^{-nk^2}[6k+1]_{q^n}\frac{(q^n;q^{4n})_k^2(q^{2n};q^{4n})_k}{(q^{2n};q^{2n})_k^2(q^{4n};q^{4n})_k}. \tag{4.8}$$

Proof. Clearly, the limit of (4.6) as $a\to 1$ contains the factor $\prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}+1}$. On the other hand, the denominator of the left-hand side of (4.5) is divisible that of the right-hand side of (4.5). The factor of the former involving a is $(aq^2;q^2)_{(n^r-1)/2}(q^2/a;q^2)_{(n^r-1)/2}$, the limit of which as a tends to 1 merely owns the factor $\prod_{j=1}^r \Phi_{n^j}(q)^{n^{r-j}-1}$ that is related to $\Phi_n(q),\Phi_{n^2}(q),\dots,\Phi_{n^r}(q)$. Therefore, taking $a\to 1$ in (4.5), we conclude that (4.8) holds modulo $\prod_{j=1}^r \Phi_{n^j}(q)^3$, one factor $\prod_{j=1}^r \Phi_{n^j}(q)$ arising from $[n^r]$.

Moreover, in view of (1.4), we have

$$\sum_{k=0}^{(n-1)/2} q^{-k^2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}\equiv 0 \pmod{[n]}. \tag{4.9}$$

Substituting $n\mapsto n^r$ into the above q-congruence, we deduce that the left-hand side of (4.8) is congruent to 0 modulo $[n^r]$, while putting $q\mapsto q^n$ and $n\mapsto n^{r-1}$ in (4.9), we see that the right-hand side of (4.8) is congruent to 0 modulo $[n][n^{r-1}]_{q^n}=[n^r]$. This means that (4.8) is true modulo $[n^r]$. Since the least common multiple of $\prod_{j=1}^r \Phi_{n^j}(q)^3$ and $[n^r]$ is $[n^r]\prod_{j=1}^r \Phi_{n^j}(q)^2$, we finish the proof. □

Letting $n=p$ be a prime and taking $q\to 1$ in (4.8), we immediately obtain (4.2).

From Theorem 4.4, we can also deduce the following conclusion.

Corollary 4.5. *Let $p\equiv 3 \pmod 4$ be a prime and r a positive integer. Then*

$$\sum_{k=0}^{(p^{2r}-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k\equiv p^2\sum_{k=0}^{(p^{2r-2}-1)/2}(6k+1)\frac{(\frac12)_k(\frac14)_k^2}{k!^3}4^k \pmod{p^{4r}}. \tag{4.10}$$

Proof. For any positive integer n with $n\equiv 3 \pmod 4$, we have $n^2\equiv 1 \pmod 4$. Replacing n by n^2 in (4.8), we arrive at the following q-congruence: modulo $[n^{2r}]\prod_{j=1}^r \Phi_{n^{2j}}(q)^2$,

$$\begin{aligned}&\sum_{k=0}^{(n^{2r}-1)/2} q^{-k^2}[6k+1]\frac{(q;q^4)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}\\ &\quad\equiv q^{(1-n^2)/2}[n^2]\sum_{k=0}^{(n^{2r-2}-1)/2} q^{-n^2k^2}[6k+1]_{q^{n^2}}\frac{(q^{n^2};q^{4n^2})_k^2(q^{2n^2};q^{4n^2})_k}{(q^{2n^2};q^{2n^2})_k^2(q^{4n^2};q^{4n^2})_k}.\end{aligned} \tag{4.11}$$

Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (4.11), we get the desired congruence (4.10). □

5. Concluding remarks and open problems

Recall that the *Euler numbers* E_n are defined by

$$E_0 = 1, \quad \text{and } E_n = -\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} E_{n-2k} \quad \text{for } n = 1, 2, \ldots.$$

In 2021, Mao and Wen [10] proved that, for any prime $p > 3$,

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k^3}{k!^3 4^k} \equiv (-1)^{\frac{p-1}{2}} p - p^3 E_{p-3} \pmod{p^4}, \tag{5.1}$$

which was originally conjectured by Sun [12].

Numerical calculation implies that the following similar congruences seems to be true.

Conjecture 5.1. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k (\frac{1}{4})_k^2}{k!^3} 4^k \equiv p - p^3 E_{p-3} \pmod{p^4}. \tag{5.2}$$

If the above conjecture is true, then combining (5.1) and (5.2) yields that, for any prime $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k^3}{k!^3 4^k} \equiv \sum_{k=0}^{p-1} (6k+1) \frac{(\frac{1}{2})_k (\frac{1}{4})_k^2}{k!^3} 4^k \pmod{p^4}. \tag{5.3}$$

In fact, much more should be true. We conjecture that the following q-analogue of (5.3) holds.

Conjecture 5.2. *Let $n \equiv 1 \pmod 4$ be a positive integer. Then, modulo $[n]\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1} [6k+1] \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} q^{k^2} \equiv \sum_{k=0}^{n-1} [6k+1] \frac{(q;q^4)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{-k^2}. \tag{5.4}$$

In view of (1.3) and (1.8), we see that the q-congruence (5.4) is true when both sides are truncated at $k = (n-1)/2$.

On the basis of numerical calculation, we believe that the following generalizations of (4.2) and (4.10) should be true.

Conjecture 5.3. *Let $p \equiv 1 \pmod 4$ be a prime and r a positive integer. Then* (4.2) *holds modulo p^{4r}.*

Conjecture 5.4. *Let $p \equiv 3 \pmod 4$ be a prime and r a positive integer with $p^r > 3$. Then* (4.10) *holds modulo p^{5r}.*

Note that all the proved Dwork-type congruences in [4, 8] are modulo p^{3r} or p^{2r}. For this reason, we think that Conjectures 5.3 and 5.4 are rather challenging. We hope that an interested reader can make progress on these two conjectures.

References

[1] B. Berndt, Ramanujans Notebooks Part IV, Springer-Verlag, New York, 1994.

[2] B. Dwork, p-adic cycles, Publ. Math. Inst. Hautes Etudes Sci. 37 (1969), 27–115.

[3] G. Gasper and M. Rahman, Basic Hypergeometric Series, Second edition, Cambridge University Press, Cambridge, 2004.

[4] V.J.W. Guo, q-Analogues of Dwork-type supercongruences, J. Math. Anal. Appl. 487 (2020), Art. 124022.

[5] V.J.W. Guo, q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping, Adv. Appl. Math. 120 (2020), Art. 102078.

[6] V.J.W. Guo, Some q-supercongruences from the Gasper and Rahman quadratic summation, Rev. Mat. Complut. 36 (2023), 993–1002.

[7] V.J.W. Guo and W. Zudilin, A q-microscope for supercongruences, Adv. Math. 346 (2019), 329–358.

[8] V.J.W. Guo and W. Zudilin, Dwork-type supercongruences through a creative q-microscope, J. Combin. Theory, Ser. A 178 (2021), Art. 105362.

[9] H. He and X. Wang, Some parametric q-supercongruences from a summation of Gasper and Rahman, Proc. Amer. Math. Soc. 152 (2024), 4775–4784.

[10] G.-S. Mao and C.-W. Wen, On two supercongruences of truncated hypergeometric series ${}_4F_3$, Ramanujan J. 56 (2021), 597–612.

[11] A. Mellit and M. Vlasenko, Dwork's congruences for the constant terms of powers of a Laurent polynomial, Int. J. Number Theory 12 (2016), 313–321.

[12] Z.-W. Sun, Super congruences and Euler numbers, Sci. China Math. 54 (2011), 2509–2535.

[13] H. Swisher, On the supercongruence conjectures of van Hamme, Res. Math. Sci. 2 (2015), Art. 18.

[14] L. Van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-Adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math. 192, Dekker, New York, 1997, pp. 223–236.

[15] C. Wei, q-Supercongruences from Gasper and Rahman's summation formula, Adv. Appl. Math. 139 (2022), Art. 102376.