Two q-supercongruences from a terminating very-well-poised ${}_6\phi_5$ summation

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Abstract. We prove two q-supercongruences modulo the cube of a cyclotomic polynomial. One is a generalization of a supercongruence of the first author and Schlosser, and the other is a generalization of a result of Hu. Meanwhile, these two results may be deemed q-analogues of two supercongruences of Wang and Sun. Our proof makes use of a terminating very-well-poised ${}_6\phi_5$ summation, the method of creative microscoping devised by the first author and Zudilin, and the Chinese remainder theorem for polynomials.

Keywords: q-supercongruences; creative microscoping; ${}_6\phi_5$ summation; Chinese remainder theorem for polynomials.

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1. Introduction

In 1997, Van Hamme [11] listed 13 interesting supercongruences corresponding to Ramanujan-type formulae for $1/\pi$. For example, the following two infinite series

$$\sum_{k=0}^{\infty}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}=\frac{3\sqrt{3}}{2\pi},$$
$$\sum_{k=0}^{\infty}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}=\frac{2\sqrt{2}}{\pi},$$

have the p-adic analogues:

$$\sum_{k=0}^{(p-1)/3}(-1)^k(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3}\equiv p \pmod{p^3}, \quad \text{if} \quad p\equiv 1 \pmod{3}, \tag{1.1}$$
$$\sum_{k=0}^{(p-1)/4}(-1)^k(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3}, \quad \text{if} \quad p\equiv 1 \pmod{4}, \tag{1.2}$$

where p is an odd prime, $(x)_n=x(x+1)\cdots(x+n-1)$ is the *Pochhammer symbol*, and $\left(\frac{\cdot}{p}\right)$ denotes the *Legendre symbol modulo p*. The supercongruences (1.1) and (1.2) were

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first proved by Swisher [10] and He [6] using Long's method [9]. q-Analogues of (1.1) and (1.2) were given by the first author [2]. He [7] proved that, for any odd prime $p \equiv 2 \pmod 3$,

$$\sum_{k=0}^{p-1}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv -6\Gamma_p(\tfrac{2}{3})^3 \pmod{p^2}, \tag{1.3}$$

where $\Gamma_p(x)$ denotes the *p-adic Gamma function*. The first author and Schlosser [4] presented a new q-analogue of (1.1) and the following q-analogue of (1.3): for any positive integer $n \equiv 2 \pmod 3$,

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv q^{2(1-2n)/3}[2n]\frac{(q^3;q^3)_{(2n-1)/3}}{(q^2;q^3)_{(2n-1)/3}} \pmod{\Phi_n(q)^2}. \tag{1.4}$$

From now on, we have to familiarize ourselves with the classical q-notation. We always assume that q is inside the disc, $|q|<1$, and define

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

Then the *q-shifted factorial* is given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{j=0}^{n-1}(1-aq^j)$$

for any non-negative integer n. For simplicity, we will also adopt the abbreviated notation: $(a_1,\dots,a_m;q)_n = (a_1;q)_n \dots (a_m;q)_n$. The relevant q-notation also includes the *q-integer* $[n]=[n]_q=(1-q^n)/(1-q)$ and the n-th *cyclotomic polynomial*, which can be written as

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ denotes any n-th primitive root of unity.

Recently, a result of Wang and Sun [12, Theorem 1.2] implies that (1.3) is also true modulo p^3. In this paper, we shall give a q-analogue of this result, which is also a generalization of (1.4).

Theorem 1.1. *Let $n \equiv 2 \pmod 3$ be a positive integer. Then, modulo $\Phi_n(q)^3$,*

$$\sum_{k=0}^{n-1}[6k+1]\frac{(q;q^3)_k^3}{(q^3;q^3)_k^3} \equiv q^{2(1-2n)/3}[2n]\frac{(q^3;q^3)_{(2n-1)/3}}{(q^2;q^3)_{(2n-1)/3}}\left(1-4[n]^2\sum_{k=1}^{(n-2)/3}\frac{q^{3k}}{[3k]^2}\right). \tag{1.5}$$

Note that a similar result can be found in [13]. Letting $n=p^r$ be a prime power subject to $p^r \equiv 2 \pmod 3$ and taking $q \to 1$ in (1.5), we obtain the following supercongruence.

Corollary 1.2. *Let $p \equiv 2 \pmod 3$ be a prime and r a positive odd integer. Then*

$$\sum_{k=0}^{p^r-1}(6k+1)\frac{(\frac{1}{3})_k^3}{k!^3} \equiv 2p^r \frac{(1)_{(2p^r-1)/3}}{(\frac{2}{3})_{(2p^r-1)/3}}\left(1-\frac{4}{9}p^{2r}\sum_{k=1}^{(p^r-2)/3}\frac{1}{k^2}\right) \pmod{p^3}. \tag{1.6}$$

Combining the $r=1$ case of (1.6) and Wang and Sun's supercongruence, we arrive at the following conclusion.

Corollary 1.3. *Let $p \equiv 2 \pmod 3$ be an odd prime. Then*

$$p\frac{(1)_{(2p-1)/3}}{(\frac{2}{3})_{(2p-1)/3}}\left(1-\frac{4}{9}p^{2}\sum_{k=1}^{(p-2)/3}\frac{1}{k^2}\right) \equiv -3\Gamma_p(\tfrac{2}{3})^3 \pmod{p^3}.$$

Motivated by the work of [4], Hu [8] gave a new q-analogue of (1.2), as well as the following q-supercongruence: for any positive integer $n \equiv 3 \pmod 4$, modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{n-1}[8k+1]_{q^2}\frac{(q^2;q^8)_k^3}{(q^8;q^8)_k^3}q^k \equiv q^{-5(3n-1)/4}[3n]_{q^2}\frac{(q^7;q^8)_{(3n-1)/4}}{(q^5;q^8)_{(3n-1)/4}}. \tag{1.7}$$

In this paper, we shall prove that (1.7) also holds modulo $\Phi_n(q)^3$. Namely, we have the following q-supercongruence.

Theorem 1.4. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{n-1}[8k+1]_{q^2}\frac{(q^2;q^8)_k^3}{(q^8;q^8)_k^3}q^k \equiv q^{-5(3n-1)/4}[3n]_{q^2}\frac{(q^7;q^8)_{(3n-1)/4}}{(q^5;q^8)_{(3n-1)/4}} \pmod{\Phi_n(q)^3}. \tag{1.8}$$

Similarly as before, we can deduce a supercongruence from (1.8) as follows.

Corollary 1.5. *Let $p \equiv 3 \pmod 4$ be a prime and r a positive odd integer. Then*

$$\sum_{k=0}^{p^r-1}(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv 3p^r\frac{(\frac{7}{8})_{(3p^r-1)/4}}{(\frac{5}{8})_{(3p^r-1)/4}} \pmod{p^3}.$$

Note that Wang and Sun [12, Theorem 1.2] also obtained the following result: for primes $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{p-1}(8k+1)\frac{(\frac{1}{4})_k^3}{k!^3} \equiv \begin{cases} -\frac{15}{8}p^2\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 \pmod{p^3}, & \text{if } p \equiv 3 \pmod 8, \\ p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3 \pmod{p^3}, & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Comparing the above two supercongruences, we are led to the following result.

Corollary 1.6. *Let $p \equiv 3 \pmod 4$ be a prime. Then, modulo p^2,*

$$\frac{(\frac{7}{8})_{(3p-1)/4}}{(\frac{5}{8})_{(3p-1)/4}} \equiv \begin{cases} -\frac{5}{8}p\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3, & \text{if } p \equiv 3 \pmod 8, \\ \frac{1}{3}\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5}{8})^3, & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Recall that a terminating very-well-poised ${}_6\phi_5$ summation (see [1, Appendix (II.21)]) can be stated as follows:

$${}_6\phi_5\left[\begin{matrix} a, & qa^{\frac{1}{2}}, & -qa^{\frac{1}{2}}, & b, & c, & q^{-n} \\ & a^{\frac{1}{2}}, & -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc}\right] = \frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}, \tag{1.9}$$

where the *basic hypergeometric ${}_{r+1}\phi_r$ series* (see [1]) is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k.$$

We shall prove Theorems 1.1 and 1.4 in Sections 2 and 3, respectively. Our proof mainly employs the ${}_6\phi_5$ summation (1.9) and the method of creative microscoping introduced by the first author in collaboration with Zudilin [5].

2. Proof of Theorem 1.1

We need to build two simple q-congruences on a parametric version of the left-hand side of (1.5). The first one is a q-congruence modulo $(1-aq^n)(a-q^n)$.

Lemma 2.1. *Let $n \equiv 2 \pmod 3$ be a positive integer. Let a and b be indeterminates. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{n-1} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(a^2q, q/a^2, q/b^2; q^3)_k}{(a^2q^3/b^2, q^3/a^2b^2, q^3; q^3)_k} b^{-k} \equiv \frac{(q^4/b^2, q^{1-2n}/b; q^3)_{(2n-1)/3}}{(q^{3-2n}/b^2, q^2/b; q^3)_{(2n-1)/3}}. \tag{2.1}$$

Proof. For $a = q^{-n}$ or $a = q^n$, the left-hand side of (2.1) becomes

$$\sum_{k=0}^{(2n-1)/3} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(q^{1-2n}, q^{1+2n}, q/b^2; q^3)_k}{(q^{3-2n}/b^2, q^{3+2n}/b^2, q^3; q^3)_k} b^{-k}, \tag{2.2}$$

where we have used the fact that $(q^{1-2n}; q^3)_k = 0$ for $k > (2n-1)/3$. Performing the parameter substitutions $q \mapsto q^3$, $a = q/b^2$, $b = q^{1+2n}$, $c = q^2/b$ and $n = (2n-1)/3$ in (1.9), we conclude that (2.2) is equal to

$$\frac{(q^4/b^2, q^{1-2n}/b; q^3)_{(2n-1)/3}}{(q^{3-2n}/b^2, q^2/b; q^3)_{(2n-1)/3}}.$$

This implies that the q-congruence (2.1) holds modulo $1-aq^n$ and $a-q^n$. Since $1-aq^n$ and $a-q^n$ are coprime polynomials in q, we obtain the q-congruence (2.1). □

The second one is a q-congruence modulo $b-q^n$.

Lemma 2.2. *Let $n \equiv 2 \pmod 3$ be a positive integer. Let a and b be indeterminates. Then, modulo $b-q^n$,*

$$\sum_{k=0}^{n-1} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(a^2q, q/a^2, q/b^2; q^3)_k}{(a^2q^3/b^2, q^3/a^2b^2, q^3; q^3)_k} b^{-k} \equiv \frac{2(q^4/b^2, q^2/b^2; q^3)_{(n-2)/3}}{(q^3/a^2b^2, a^2q^3/b^2; q^3)_{(n-2)/3}}. \tag{2.3}$$

Proof. Clearly, the q-congruence (2.3) is equivalent to the following identity.

$$\sum_{k=0}^{n-1} \frac{q^{2n}-q^{6k+1}}{q^{2n}-q} \frac{(a^2q, q/a^2, q^{1-2n}; q^3)_k}{(a^2q^{3-2n}, q^{3-2n}/a^2, q^3; q^3)_k} q^{-nk} = \frac{2(q^{4-2n}, q^{2-2n}; q^3)_{(n-2)/3}}{(q^{3-2n}/a^2, a^2q^{3-2n}; q^3)_{(n-2)/3}}. \tag{2.4}$$

Let $m \equiv 2 \pmod 3$ be any integer with $m \geqslant n$. Then, in view of (2.1), we have

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(q^{1-2m}, q^{1+2m}, q/b^2; q^3)_k}{(q^{3-2m}/b^2, q^{3+2m}/b^2, q^3; q^3)_k} b^{-k} \\
&\quad = \frac{(q^4/b^2, q^{1-2m}/b; q^3)_{(2m-1)/3}}{(q^2/b, q^{3-2m}/b^2; q^3)_{(2m-1)/3}} \\
&\quad = (1+q^n/b) \frac{(q^4/b^2; q^3)_{(2n-4)/3} (q^{2n+3}/b^2; q^3)_{(2m-2n)/3} (q^{1-2m}/b; q^3)_{(2m-1)/3}}{(q^2/b; q^3)_{(n-2)/3} (q^{n+3}/b; q^3)_{(2m-n-2)/3} (q^{3-2m}/b^2; q^3)_{(2m-1)/3}}.
\end{aligned} \tag{2.5}$$

Note that both sides of (2.4) are rational functions in a. It suffices to show that (2.4) holds for sufficiently many a's. To this end, we take $a = q^{-m}$, where $m = n, n+3, n+6, \dots$. Then the left-hand side of (2.4) is equal to the $b = q^n$ case of the left-hand side of (2.5), which can be written as

$$\begin{aligned}
&\frac{2(q^{4-2n}; q^3)_{(2n-4)/3} (q^3; q^3)_{(2m-2n)/3} (q^{1-2m-n}; q^3)_{(2m-1)/3}}{(q^{2-n}; q^3)_{(n-2)/3} (q^3; q^3)_{(2m-n-2)/3} (q^{3-2m-2n}; q^3)_{(2m-1)/3}} \\
&\quad = \frac{2(q^{4-2n}; q^3)_{(n-2)/3} (q^{1-2m-n}; q^3)_{(2m-1)/3}}{(q^{3+2m-2n}; q^3)_{(n-2)/3} (q^{3-2m-2n}; q^3)_{(2m-1)/3}}.
\end{aligned} \tag{2.6}$$

Using the following relation

$$\frac{(aq^k; q)_j}{(a; q)_j} = \frac{(aq^j; q)_k}{(a; q)_k}, \tag{2.7}$$

we can write the expression (2.6) as

$$\frac{2(q^{4-2n}, q^{2-2n}; q^3)_{(n-2)/3}}{(q^{3+2m-2n}, q^{3-2m-2n}; q^3)_{(n-2)/3}},$$

which is just the $a = q^{-m}$ case of the right-hand side of (2.4). This completes the proof. □

It is worth mentioning that the q-congruence (2.3) has a companion as follows. For any positive integer $n \equiv 2 \pmod 3$, modulo $b-q^n$,

$$\sum_{k=0}^{(n-2)/3} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(a^2q, q/a^2, q/b^2; q^3)_k}{(a^2q^3/b^2, q^3/a^2b^2, q^3; q^3)_k} b^{-k} \equiv \frac{(q^4/b^2, q^2/b^2; q^3)_{(n-2)/3}}{(q^3/a^2b^2, a^2q^3/b^2; q^3)_{(n-2)/3}}.$$

This can be proved by letting $q \mapsto q^3$, and taking $a = q^{1-2n}$, $b = a^2q$, $c = q/a^2$ and $n = (n-2)/3$ in (1.9).

We now proceed to prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that the polynomials $(1-aq^n)(a-q^n)$ and $b-q^n$ are coprime with each other. The following two easily proved q-congruences:

$$\frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \equiv 1 \pmod{(1-aq^n)(a-q^n)}, \tag{2.8}$$

$$\frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)} \equiv 1 \pmod{b-q^n} \tag{2.9}$$

can be found in [3]. Therefore, by making use of the Chinese remainder theorem for polynomials, we get a new q-congruence from (2.1) and (2.3): modulo $(1-aq^n)(a-q^n)(b-q^n)$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{b^2-q^{6k+1}}{b^2-q} \frac{(a^2q, q/a^2, q/b^2; q^3)_k}{(a^2q^3/b^2, q^3/a^2b^2, q^3; q^3)_k} b^{-k} \\
&\quad \equiv \frac{(q^4/b^2, q^{1-2n}/b; q^3)_{(2n-1)/3}}{(q^{3-2n}/b^2, q^2/b; q^3)_{(2n-1)/3}} \frac{(b-q^n)(ab-1-a^2+aq^n)}{(a-b)(1-ab)} \\
&\qquad + \frac{2(q^4/b^2, q^2/b^2; q^3)_{(n-2)/3}}{(q^3/a^2b^2, a^2q^3/b^2; q^3)_{(n-2)/3}} \frac{(1-aq^n)(a-q^n)}{(a-b)(1-ab)}.
\end{aligned} \tag{2.10}$$

It is easy to see that

$$(1-q^n)(1+a^2-a-aq^n) = (1-a)^2 + (1-aq^n)(a-q^n). \tag{2.11}$$

Furthermore, the polynomial $1-q^n$ has the factor $\Phi_n(q)$. Letting $b=1$ in (2.10), we obtain the q-congruence: modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{1-q^{6k+1}}{1-q} \frac{(a^2q, q/a^2, q; q^3)_k}{(a^2q^3, q^3/a^2, q^3; q^3)_k} \\
&\quad \equiv \frac{(q^4, q^{1-2n}; q^3)_{(2n-1)/3}}{(q^{3-2n}, q^2; q^3)_{(2n-1)/3}} \left(1 + \frac{(1-aq^n)(a-q^n)}{(1-a)^2}\right) \\
&\qquad - \frac{2(q^4, q^2; q^3)_{(n-2)/3}}{(q^3/a^2, a^2q^3; q^3)_{(n-2)/3}} \frac{(1-aq^n)(a-q^n)}{(1-a)^2}.
\end{aligned} \tag{2.12}$$

Since $q^n \equiv 1 \pmod{\Phi_n(q)}$, we have

$$\frac{(q^4, q^{1-2n}; q^3)_{(2n-1)/3}}{(q^{3-2n}, q^2; q^3)_{(2n-1)/3}} = \frac{(q^4, q^{n+2}, q^{1-2n}; q^3)_{(n-2)/3}(q^{-1-n}; q^3)_{(n+1)/3}(1-q^{2n})}{(q^2, q^{n+3}, q^{3-2n}; q^3)_{(n-2)/3}(q^{1-n}; q^3)_{(n+1)/3}(1-q^n)}$$
$$\equiv 2\frac{(q^4, q^2; q^3)_{(n-2)/3}}{(q^3, q^3; q^3)_{(n-2)/3}} \pmod{\Phi_n(q)}. \tag{2.13}$$

Hence, the q-congruence (2.12) can be written as follows: modulo $\Phi_n(q)(1-aq^n)(a-q^n)$,

$$\sum_{k=0}^{n-1} \frac{1-q^{6k+1}}{1-q} \frac{(a^2q, q/a^2, q; q^3)_k}{(a^2q^3, q^3/a^2, q^3; q^3)_k}$$
$$\equiv \frac{(q^4, q^{1-2n}; q^3)_{(2n-1)/3}}{(q^{3-2n}, q^2; q^3)_{(2n-1)/3}}$$
$$+ 2\frac{(1-aq^n)(a-q^n)}{(1-a)^2} \left\{ \frac{(q^4, q^2; q^3)_{(n-2)/3}}{(q^3, q^3; q^3)_{(n-2)/3}} - \frac{(q^4, q^2; q^3)_{(n-2)/3}}{(q^3/a^2, a^2q^3; q^3)_{(n-2)/3}} \right\}. \tag{2.14}$$

Letting $a \to 1$ in (2.14), and noticing that, by the L'Hôpital rule,

$$\lim_{a\to 1} \frac{(1-aq^n)(a-q^n)}{(1-a)^2} \left\{ \frac{1}{(q^3, q^3; q^3)_{(n-2)/3}} - \frac{1}{(q^3/a^2, a^2q^3; q^3)_{(n-2)/3}} \right\}$$
$$= -\frac{4[n]^2}{(q^3, q^3; q^3)_{(n-2)/3}} \sum_{k=1}^{(n-2)/3} \frac{q^{3k}}{[3k]^2},$$

and that the left-hand side of (2.13) is equal to

$$q^{2(1-2n)/3}[2n] \frac{(q^3; q^3)_{(2n-1)/3}}{(q^2; q^3)_{(2n-1)/3}},$$

we arrive at the q-congruence (1.5).

□

3. Proof of Theorem 1.4

Like the proof of Theorem 1.1, we first give two q-congruences on a parametric version of the left-hand side of (1.8). The first one is a q-congruence modulo $(1-aq^{3n})(a-q^{3n})$.

Lemma 3.1. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $(1-aq^{3n})(a-q^{3n})$,*

$$\sum_{k=0}^{n-1} \frac{b^2-q^{16k+2}}{b^2-q^2} \frac{(a^2q^2, q^2/a^2, q^2/b^2; q^8)_k}{(a^2q^8/b^2, q^8/a^2b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k \equiv \frac{(q^{10}/b^2, q^{3-6n}/b; q^8)_{(3n-1)/4}}{(q^{8-6n}/b^2, q^5/b; q^8)_{(3n-1)/4}}. \tag{3.1}$$

Proof. For $a = q^{-3n}$ or $a = q^{3n}$, the left-hand side of (3.1) becomes

$$\sum_{k=0}^{(3n-1)/4} \frac{b^2 - q^{16k+2}}{b^2 - q^2} \frac{(q^{2-6n}, q^{2+6n}, q^2/b^2; q^8)_k}{(q^{8-6n}/b^2, q^{8+6n}/b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k, \tag{3.2}$$

where we have used $(q^{2-6n}; q^8)_k = 0$ for $k > (3n-1)/4$. Substituting $q \mapsto q^8$, and setting $a = q^2/b^2$, $b = q^{2+6n}$, $c = q^5/b$ and $n = (3n-1)/4$ in (1.9), we see that (3.2) is equal to

$$\frac{(q^{10}/b^2, q^{3-6n}/b; q^8)_{(3n-1)/4}}{(q^{8-6n}/b^2, q^5/b; q^8)_{(3n-1)/4}}.$$

Since $1 - aq^{3n}$ is coprime with $a - q^{3n}$, we get the q-congruence (3.1). □

The second one is a q-congruence modulo $b - q^{3n}$.

Lemma 3.2. *Let $n \equiv 3 \pmod 4$ be a positive integer. Let a and b be indeterminates. Then, modulo $b - q^{3n}$,*

$$\sum_{k=0}^{n-1} \frac{b^2 - q^{16k+2}}{b^2 - q^2} \frac{(a^2q^2, q^2/a^2, q^2/b^2; q^8)_k}{(a^2q^8/b^2, q^8/a^2b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k \equiv \frac{2(q^{10}/b^2, q^6/b^2; q^8)_{(\delta n-5)/8}}{(q^8/a^2b^2, a^2q^8/b^2; q^8)_{(\delta n-5)/8}}, \tag{3.3}$$

where $\delta = 3$ if $n \equiv 7 \pmod 8$, and $\delta = 7$ if $n \equiv 3 \pmod 8$.

Proof. We first consider the $n \equiv 7 \pmod 8$ case. The q-congruence (3.3) is equivalent to

$$\sum_{k=0}^{(3n-1)/4} \frac{q^{6n} - q^{16k+2}}{q^{6n} - q^2} \frac{(a^2q^2, q^2/a^2, q^{2-6n}; q^8)_k}{(a^2q^{8-6n}, q^{8-6n}/a^2, q^8; q^8)_k} \left(q^{1-3n}\right)^k = \frac{2(q^{10-6n}, q^{6-6n}; q^8)_{(3n-5)/8}}{(q^{8-6n}/a^2, a^2q^{8-6n}; q^8)_{(3n-5)/8}}. \tag{3.4}$$

Let $m \equiv 3 \pmod 4$ be any integer such that $m \geqslant n$. Then, in light of (3.1), we obtain

$$\begin{aligned}
&\sum_{k=0}^{m-1} \frac{b^2 - q^{16k+2}}{b^2 - q^2} \frac{(q^{2-6m}, q^{2+6m}, q^2/b^2; q^8)_k}{(q^{8-6m}/b^2, q^{8+6m}/b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k \\
&\quad = \frac{(q^{10}/b^2, q^{3-6m}/b; q^8)_{(3m-1)/4}}{(q^5/b, q^{8-6m}/b^2; q^8)_{(3m-1)/4}} \\
&\quad = (1 + q^{3n}/b) \frac{(q^{10}/b^2; q^8)_{(3n-5)/4} (q^{6n+8}/b^2; q^8)_{(3m-3n)/4} (q^{3-6m}/b; q^8)_{(3m-1)/4}}{(q^5/b; q^8)_{(3n-5)/8} (q^{3n+8}/b; q^8)_{(6m-3n-5)/8} (q^{8-6m}/b^2; q^8)_{(3m-1)/4}}.
\end{aligned} \tag{3.5}$$

Since both sides of (3.4) are rational functions in a, we need only to prove that (2.4) is true for sufficiently many a's. For this purpose, we take $a = q^{-3m}$, where $m = n, n+4, n+8, \dots$. Then the left-hand side of (3.4) is equal to the $b = q^{3n}$ case of the left-hand side of (3.5), and is therefore given by

$$\frac{2(q^{10-6n}; q^8)_{(3n-5)/4} (q^8; q^8)_{(3m-3n)/4} (q^{3-6m-3n}; q^8)_{(3m-1)/4}}{(q^{5-3n}; q^8)_{(3n-5)/8} (q^8; q^8)_{(6m-3n-5)/8} (q^{8-6m-6n}; q^8)_{(3m-1)/4}}$$

$$= \frac{2(q^{10-6n};q^8)_{(3n-5)/8}(q^{3-6m-3n};q^8)_{(3m-1)/4}}{(q^{8+6m-6n};q^8)_{(3n-5)/8}(q^{8-6n-6m};q^8)_{(3m-1)/4}}.$$

Furthermore, in view of (2.7), the above expression can be written as

$$\frac{2(q^{10-6n},q^{6-6n};q^8)_{(3n-5)/8}}{(q^{8+6m-6n},q^{8-6m-6n};q^8)_{(3n-5)/8}},$$

which is the right-hand side of (3.4) with $a = q^{-3m}$. This proves the $n \equiv 7 \pmod 8$ case of the lemma. Similarly, we can prove the $n \equiv 3 \pmod 8$ case. □

We remark that the q-congruence (3.1) has a companion as follows. For example, for any positive integer $n \equiv 7 \pmod 8$, modulo $b - q^{3n}$,

$$\sum_{k=0}^{(3n-5)/8} \frac{b^2 - q^{16k+2}}{b^2 - q^2} \frac{(a^2q^2, q^2/a^2, q^2/b^2; q^8)_k}{(a^2q^8/b^2, q^8/a^2b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k \equiv \frac{(q^{10}/b^2, q^6/b^2; q^8)_{(3n-5)/8}}{(q^8/a^2b^2, a^2q^8/b^2; q^8)_{(3n-5)/8}}.$$

This can be proved by specializing $q \mapsto q^8$, $a = q^{2-6n}$, $b = a^2q^2$, $c = q^2/a^2$ and $n = (3n-5)/8$ in (1.9).

We now focus on proving Theorem 1.4.

Proof of Theorem 1.4. We first assume $n \equiv 3 \pmod 8$. It is apparent that the polynomials $(1 - aq^{3n})(a - q^{3n})$ and $b - q^{3n}$ are coprime. Hence, employing the Chinese remainder theorem for polynomials, using the $n \mapsto 3n$ case of the relations (2.8) and (2.9), we get a new q-congruence from (3.1) and (3.3): modulo $(1 - aq^{3n})(a - q^{3n})(b - q^{3n})$,

$$\begin{aligned} &\sum_{k=0}^{n-1} \frac{b^2 - q^{16k+2}}{b^2 - q^2} \frac{(a^2q^2, q^2/a^2, q^2/b^2; q^8)_k}{(a^2q^8/b^2, q^8/a^2b^2, q^8; q^8)_k} \left(\frac{q}{b}\right)^k \\ &\quad \equiv \frac{(q^{10}/b^2, q^{3-6n}/b; q^8)_{(3n-1)/4}}{(q^{8-6n}/b^2, q^5/b; q^8)_{(3n-1)/4}} \frac{(b - q^{3n})(ab - 1 - a^2 + aq^{3n})}{(a-b)(1-ab)} \\ &\quad\quad + \frac{2(q^{10}/b^2, q^6/b^2; q^8)_{(3n-5)/8}}{(q^8/a^2b^2, a^2q^8/b^2; q^8)_{(3n-5)/8}} \frac{(1 - aq^{3n})(a - q^{3n})}{(a-b)(1-ab)}. \end{aligned} \tag{3.6}$$

Note that the polynomial $1 - q^{3n}$ has the factor $\Phi_n(q)$, and

$$\frac{(q^{10}, q^{3-6n}; q^8)_{(3n-1)/4}}{(q^{8-6n}, q^5; q^8)_{(3n-1)/4}} \equiv 0 \pmod{\Phi_n(q)}.$$

Letting $b = 1$ and using (2.11) with $n \mapsto 3n$, we see that (3.6) reduces to

$$\sum_{k=0}^{n-1} \frac{1 - q^{16k+2}}{1 - q^2} \frac{(a^2q^2, q^2/a^2, q^2; q^8)_k}{(a^2q^8, q^8/a^2, q^8; q^8)_k} q^k$$

$$\equiv \frac{(q^{10}, q^{3-6n}; q^8)_{(3n-1)/4}}{(q^{8-6n}, q^5; q^8)_{(3n-1)/4}} \pmod{\Phi_n(q)(1-aq^{3n})(a-q^{3n})}. \tag{3.7}$$

Finally, putting $a = 1$ in (3.7) and noticing that

$$\frac{(q^{10}, q^{3-6n}; q^8)_{(3n-1)/4}}{(q^{8-6n}, q^5; q^8)_{(3n-1)/4}} = q^{-5(3n-1)/4}[3n]_{q^2} \frac{(q^7; q^8)_{(3n-1)/4}}{(q^5; q^8)_{(3n-1)/4}},$$

we complete the proof of the theorem for $n \equiv 7 \pmod 8$. Similarly, we can prove the $n \equiv 3 \pmod 8$ case. □

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